Departamento de Álgebra, Geometría y Topología
Facultad de ciencias
Universidad de Málaga

Ph.D. dissertation

Jordan elements in Lie algebras and inner ideals in the skew elements of prime rings with involution

Jose Ramón Brox López

Supervised by

Antonio Fernández López
and
Miguel Gómez Lozano

December 2014

Publicaciones y Divulgación Científica
UNIVERSIDAD DE MÁLAGA

AUTOR: José Ramón Brox López
EDITA: Publicaciones y Divulgación Científica. Universidad de Málaga

Contents

Agradecimientos

Sin lugar a dudas, no podría haber llegado hasta donde estoy ahora ni haber desarrollado de la misma manera este trabajo sin la colaboración y la ayuda de muchas personas. Quiero agradecer en primer lugar su labor de mantenimiento al **PAS de la Universidad de Málaga**, llevada a cabo con tal eficiencia que paradójicamente puede llegar a pasar desapercibida, pero sin la cual mi trabajo habría sido mucho más engorroso. También doy las gracias a los dos Antonios que han ejercido como directores de departamento a lo largo de mi doctorado (**Antonio Viruel** y **Antonio Sánchez**) por interceder y ayudarme en los asuntos espinosos que surgieron durante el mismo. I am also grateful to **Gene Abrams** and **Alexander Baranov** who were, independently, so kind to accept tutorship me at their universities, although sadly those short stays couldn't be materialized for different reasons. Estoy agradecido a los **miembros del tribunal** de la tesis por su disposición y prestanza a la hora de entregar documentos, que les pedimos con urgencia. Va también mi más sincero agradecimiento a:

- **María Luisa Gil**, que vale igual para un roto que para un descosío y cuya simpatía es una fuerza cohesiva del departamento.
- Mi alumno olímpico **Pablo Báez**, porque sus ganas de aprender y de superarse resolviendo problemas me han obligado a prepararme más y mejor, y eso ha tenido un impacto directo en mi investigación.
- **Alberto Elduque**, que tuvo la amabilidad de perder una tarde orientándome con su dilatada experiencia, que me sirvió para decidir mi futuro inmediato.
- **Cristina Draper** y **Paco Villatoro**, por ofrecerse estoicamente para dirigirme con

tal de que no me quedara varado.

- **Cándido Martín** y **Alberto Castellón**, que me abrieron desde el principio un hueco en su despacho, aun a sabiendas de que compartir dos mesas entre tres personas nos iba a causar más de un quebradero de cabeza. ¡Y sin rechistar ni una sola vez! También he de agradecerles las ocasionales pero iluminantes explicaciones matemáticas y las estimulantes conversaciones sobre temas variados, que demuestran que la universidad, además de ser un establecimiento para la libre circulación del papeleo, también puede llegar a ser un centro de desarrollo de ideas.
- **Esther García**, por aceptarme con naturalidad y disposición en el contubernio investigador que hasta el momento compartía con Miguel en exclusividad, lo que me ha permitido ser partícipe de trabajos que considero muy sólidos y elegantes.
- **Mercedes Siles** y **Gonzalo Aranda**, que durante mi primer proyecto de tesis me ayudaron de muchas formas cruciales y distintas, no todas académicas, sin las cuales no habría tenido la oportunidad de terminar un doctorado. **Juan Luis Galeas** tiene mi agradecimiento eterno por dejar lo demás para echarme una mano en el peor momento de mi vida.
- **Antonio Fernández**, erudito moderno y tejedor de telarañas densas e iridiscentes: por empezar a mostrarme las grandes vías y grandes maneras del álgebra con mayúsculas (que tanto ansiaba y ansío), por mantenerse firme en sus principios ante la adversidad y la hipocresía imperantes, por aplicarme su *laissez-faire* para dejarme independencia confiando en mi madurez (¡arriesgada apuesta! ;P), por reflexionar en frío tras discutir en caliente, por saber siempre cómo y hacia dónde orientarme, por hacer valer la profundidad de los resultados por sobre las modas (lo cual, desgraciadamente, tiene un coste elevado), por su perenne trato familiar y acogedor, y por su sentido del humor incandescente, que decora cada día el mundo con saludable sorna.
- **Miguel Gómez**, semipolítico-semihumano y manipulaidentidades genial (a su pesar). Por rescatarme y por haber querido rescatarme: sin su convincente intervención seguramente no habría entrado en el estudio del álgebra no asociativa, que en retrospectiva creo que era lo que más me convenía, y no habría formado parte de este grupo

de investigación, que sin duda es lo más importante que ha ocurrido en mi formación. Por esperar (casi) siempre con paciencia a que descubra por mí mismo cosas para él ya bien conocidas. Por su salero contando anécdotas chistosas (preguntadle cómo suena un PI-anillo), con las que es imposible no partirse de risa. Por compartir conmigo ideas en ciernes y proyectos ambiciosos. Por estar pendiente con constancia de que los asuntos salgan adelante (a veces a pesar de los más interesados en ellos). Y por su capacidad magistral para hilar con naturalidad demostraciones asombrosamente complejas, que me muestra cada día que hay niveles insospechados que alcanzar y me espolea para intentar alcanzarlos.

- **Mis padres**, que toda la vida me han cuidado, alimentado, mimado, educado y apoyado sin reservas, y aún hoy lo siguen haciendo. Si mi madre no me hubiera intentado responder todos los *por qués* y mi padre no me hubiera regalado mis primeros libros de programación (por no hablar de la biblioteca de ciencia ficción presente en casa), tal vez hoy no sería matemático. **Mis suegros**, que también me acogen siempre con cariño haciendo que me sienta como en casa, me han dado un sustento esencial en los múltiples días de duro trabajo doctoral.
- **Patry**, mi mujercita adorada: por confiar tuertamente en mí, cuidarme literalmente sin descanso en los meses de trabajo a jornada completa, acicatearme en los lapsos de vagancia, animarme en los de depresión, alegrarse con mis alegrías, estar dispuesta a hacer la mona en cualquie instante, compartir conmigo sus grandes victorias y pequeños pesares y, en definitiva, hacer mi tránsito por la vida más llevadero y memorable.

Introduction

The Lie and Jordan structures coming from associative algebras (with and without involution) play an essential role in the classification of simple algebras in both nonassociative settings. In the 1950's, Herstein gave the start to a program of analysis of these structures, based on a careful study of elementary identities that are constructed in a step-by-step formulation (what we will call, rather diffusely, 'a combinatorial approach'). He began by showing ([Herstein'55]) that for a simple ring R the Jordan ring R^+ is simple, while the ideals of the Lie ring R^- either are inside $Z(R)$ or contain $[R, R]$. He also showed ([Herstein'55(2)]) that any Lie ideal of $[R, R]$ lies inside $Z(R)$, so that $[R, R]/([R, R] \cap Z(R))$ is always simple whenever R is simple with $\mathrm{char}(R) \neq 2, 3$, proving that this fact, already observed in the classification theorems, is not due to any finiteness condition. Following the same motivation, in [Herstein'56] he proved that if R has an involution and $\mathrm{char}(R) \neq 2$, then the Lie subring of skew elements K and the Jordan subring of symmetric elements H are simple except if R has dimension 4 or less over its center. This program has been followed by Baxter (who studied the cases of characteristic 2 and 3 in [Baxter'56]), Erickson (who generalized the results to prime rings with involution), Benkart (who determined the Lie inner structure of R and K for simple rings in [Benkart'76]), Martindale (who introduced in the program the powerful tools of GPIs theory, as exemplified by [Martindale&Miers'86]) and Fernández López (who determined the Lie inner structure of centrally closed prime rings in [Fernández'14]) among others, and was also extensively developed in Herstein's books [TopicsRingTheory] and [RingsInvolution]. Herstein theory produces a beautiful interplay between the associative properties of a ring and its nonassociative structures. So, for example, when R is simple with involution with $\mathrm{char}(R) \neq 2$ and is more than 4-dimensional over its center, then the associative subrings generated by H and K are equal to R.

A good part of this dissertation can be ascribed to Herstein program. One of our main objectives is the determination of the Lie inner structure of K when R is a centrally closed prime ring with involution, i.e., the classification of its (abelian) Lie inner ideals, which we achieve in Chapter 3. These inner ideals turn out to appear in classes analogous to those of R^- (called isotropic, standard and special), plus a kind of inner ideal exclusive to K but that already appears when R is simple with socle, the Clifford inner ideal. Clifford inner ideals were described by Benkart in terms of bases of the algebra, and geometrically in [Fernández,García&Gómez'06(2)] in terms of hyperbolic planes. A ring-theoretic description of these inner ideals was yet lacking in the literature; we include one in the mentioned chapter, in terms of minimal $*$-orthogonal idempotents. To determine the classification of inner ideals of K when R is simple, Benkart resorted to the Lie structure of R and K as developed by Herstein's theory. The same could be done in the prime context, building on results of Lanski and Martindale but, although with the same combinatorial spirit, we have preferred to reduce the case of K to the case of R^-. For this we prove in Chapter 1 (entirely dedicated to the structure of K), mostly by a recollection and interrelation of previously known results, that $\langle K \rangle$, the subring generated by K inside R, is prime and contains an ideal of R except if $[K, K] = 0$ or, equivalently, except if R is commutative or its central closure is a quaternion algebra over the extended centroid with an involution of the first kind and transpose type (to prove this we will need to introduce some concepts from PIs theory, following Erickson). The existence of the mentioned ideal allows to transport associative properties from R to $\langle K \rangle$ (for example, their extended centroid is the same). In addition, the structure of $\langle K \rangle$ happens to be very near to the structure of K (we just need to add the sums of squares of K) and therefore many properties of $\langle K \rangle$ can be translated to K, after a twist. With this tool in hand we are able to prove Herstein Lemma for K, a known result that states that any adnilpotent element is the sum of a nilpotent element and a central one, from Herstein Lemma for R, and is this result the one which opens the gates to the classification of abelian inner ideals (as was already recognized by Benkart in the simple context), since every element of an abelian inner ideal is a *Jordan element*, i.e., an adnilpotent element of index at most 3. As an aside, in Chapter 1 we also show, by a combinatorial approach which avoids the fundamental theorems of PIs theory, that if $[K, K] = 0$ then R satisfies Hall Identity, a polynomial identity of degree 5 that is satisfied by quaternion algebras. This we do for R arbitrary (except for $\mathrm{char}(R) \neq 2$).

Clifford inner ideals of K, being the ones which have no analogous counterpart in R^-, are the inner ideals which behave more differently. The main reason for this is that they are the only ones which contain Jordan elements c such that $c^2 \neq 0$ but $c^3 = 0$, called *Clifford elements* by us. This produces an special case in Herstein Lemma which guarantees that R has socle and involution of orthogonal type. The study of Clifford elements is of independent interest, so we carry it out in Chapter 4. Jordan elements are called that way because associated to any Jordan element $a \in L$ there exists a Jordan algebra L_a which behaves as a local algebra for L. If L is a nondegenerate Lie algebra over a field F with $\mathrm{char}(F) > 5$ which has a Jordan element $c \in L$ such that L_c is a Clifford Jordan algebra, then it can be shown via a grading of L and the TKK of the Jordan pair of its extremes, which is a finitary orthogonal algebra, that c actually lives in the skew elements of a simple ring with socle and involution of orthogonal type, and that in addition verifies $c^2 \neq 0$ and $c^3 = 0$, i.e., c is a Clifford element. Conversely, we are able to show that if R is a centrally closed prime ring with involution such that $\mathrm{char}(R) > 5$ and with a Clifford element c, then K_c is a Clifford Jordan algebra. To prove this we find that any symmetric von Neumann regular element of zero square can be paired with another element of the same qualities, apply this result to c^2 (which in addition is a reduced element) to get a partner d, and develop a series of identities not afar from Herstein's theory which culminate by showing that $cKc = \mathcal{C}c$ (with $\mathcal{C}$ the extended centroid) and the fact that the element $\sqrt{d} := dc + cd$ is a regular partner for c with very nice properties, in addition to being a square root of d. All these properties are used to build a trace and a bilinear form on K, which then serve to prove the main result about K_c.

In another order of things, the development of a general theory of Lie algebras without finiteness conditions (e.g., for the strongly prime ones) will need the establishment of elementary but fundamental properties which at present are not fully settled. The *Kostrikin radical* of L, $\mathcal{K}(L)$, is the analogue in the Lie setting of Baer radical in the associative setting and McCrimmon radical in the Jordan setting, and is thus defined as the smallest ideal of L such that $L/\mathcal{K}(L)$ is nondegenerate. It was studied first in [Filippov'81] and developed in [Zel'manov'83],[Zel'manov'84]. At the moment it is not known whether the Kostrikin radical is the intersection of all strongly prime ideals, although some advances have been made in [García&Gómez'11]. This result would imply the deep fact that any nondegenerate Lie algebra is a subdirect product of strongly prime Lie algebras. This fact could then be used to prove many results by standard subdirect product arguments; for example, it could be used to show that if L is a nondegenerate Lie algebra, $a, b \in L$ and $I(a)$ denotes the ideal generated by a, then $[I(a), I(b)] = 0$ if and only if $[a, [b, L]] = 0$, the analogue to the well-known associative property (if R is semiprime and $a, b \in R$ then $I(a)I(b) = 0$ if and only if $aRb = 0$). For Lie algebras over arbitrary rings of scalars even less is known. So, for example, if L is a Lie algebra over a field of characteristic 0 and I is an ideal of L then $\mathcal{K}(I) = \mathcal{K}(L) \cap I$, but it is not known whether this result keeps being true for more general rings of scalars. This property of the Kostrikin radical was used by Zel'manov to set affirmatively a conjecture of Filippov: given a nondegenerate Lie algebra L, an ideal I of L and $a \in L$ such that $ad_a^2 I = 0$, it is always true that $a \in \mathrm{Ann}(I)$. In Chapter 2 we include a proof of this fact for Lie algebras over general rings of scalars (due to Fernández López and Gómez Lozano). Then we build on it to show that, indeed, if L is nondegenerate and free of 6-torsion then $[a, [b, L]] = 0$ implies $[I(a), I(b)] = 0$. The approach is combinatorial, based on identities of Jordan elements and absolute zero divisors, and is developed in a series of steps which involve identities of increasing complexity.

Resumen de la tesis (in Spanish)

0.1 Introducción

Las estructuras Lie y Jordan que provienen de álgebras asociativas (con o sin involución) juegan un papel esencial en la clasificación de las álgebras simples en ambos contextos no asociativos. En los años 1950, Herstein dio comienzo a un programa de análisis de estas estructuras, basado en un estudio cuidadoso de identidades elementales que se construyen paso a paso (en lo que llamaremos, de manera difusa, 'un enfoque combinatorio'). Comenzó mostrando ([Herstein'55]) que si R es un anillo simple entonces R^+ es simple, mientras que los ideales del anillo Lie R^- o bien caen dentro de $Z(R)$ o bien contienen $[R, R]$. También mostró ([Herstein'55(2)]) que todo ideal Lie de $[R, R]$ cae en $Z(R)$, de manera que $[R, R]/([R, R] \cap Z(R))$ es simple siempre que R es simple y $\mathrm{char}(R) \neq 2, 3$, lo cual prueba que este hecho, ya observado en los teoremas de clasificación, no se debe a ninguna condición de finitud. Con la misma motivación, demostró en [Herstein'56] que si R tiene involución y $\mathrm{char}(R) \neq 2$, entonces el subanillo Lie K de los elementos antisimétricos y el subanillo Jordan H de los elementos simétricos son simples excepto cuando R tiene dimensión 4 o menos sobre su centro. Este programa ha sido continuado, entre otros, por Baxter (que estudió los casos de característica 2 y 3 en [Baxter'56]), Erickson (que generalizó los resultados a anillos primos con involución), Benkart (quien determinó la estructura interna Lie de R y K en anillos simples en [Benkart'76]), Martindale (que introdujo en el programa las potentes herramientas de la teoría GPI, como ejemplifica [Martindale&Miers'86]) y Fernández

López (quien determinó la estructura Lie interna de los anillos primos centralmente cerrados en [Fernández'14]), y fue también desarrollada de manera extensa en los libros de Herstein [TopicsRingTheory] y [RingsInvolution]. La teoría de Herstein produce una bella interacción entre las propiedades asociativas de un anillo y sus estructuras no asociativas. Así por ejemplo, cuando R es simple con involución, $\operatorname{char}(R) \neq 2$ y R tiene dimensión mayor que 4 sobre su centro, entonces los subanillos asociativos generados por H y por K coinciden con R.

Buena parte de esta tesis puede ser adscrita al programa de Herstein. Uno de sus objetivos principales es la determinación de la estructura Lie interna de K cuando R es un anillo primo centralmente cerrado con involución, es decir, la clasificación de sus ideales internos (abelianos), que se consigue en el capítulo 3. Estos ideales internos aparecen en clases análogas a las de R^- (llamadas isotrópica, estándar y especial), más un tipo de ideal interno exclusivo de K, pero que ya aparece cuando R es simple con zócalo, el ideal interno *Clifford*. Los ideales internos Clifford fueron descritos por Benkart en función de bases del álgebra, y también lo fueron geométricamente en [Fernández,García&Gómez'06(2)] mediante planos hiperbólicos. Aún no existía en la literatura una descripción de estos ideales internos en términos de teoría de anillos; mostramos una en el mismo capítulo, en función de idempotentes minimales $*$-ortogonales. Para determinar la clasificación de los ideales internos de K cuando R es simple, Benkart utilizó la estructura Lie de R y K tal como estaba desarrollada en la teoría de Herstein. Lo mismo podría hacerse en el contexto primo utilizando resultados de Lanski y Martindale pero, aunque manteniendo el espíritu combinatorio, hemos preferido reducir el caso de K al caso de R^-. Para conseguir esto, demostramos en el capítulo 1 (dedicado por entero a la estructura de K), principalmente mediante la recolección e interrelación de resultados ya conocidos, que $\langle K \rangle$, el subanillo generado por K dentro de R, es primo y contiene un ideal de R excepto si $[K, K] = 0$.

Los ideales internos Clifford de K, al ser aquellos que no cuentan con contrapartida en R^-, son los ideales internos con un comportamiento más diferente al de los demás.

La razón principal es que son los únicos que contienen elementos Jordan c tales que $c^2 \neq 0$ pero $c^3 = 0$, llamados *elementos Clifford* por nosotros. Esto garantiza que R tiene zócalo e involución de tipo ortogonal. En el capítulo 4 llevamos a cabo un estudio de los elementos Clifford, pues poseen un interés independiente. Los elementos Jordan son llamados de esta manera porque asociada a cada elemento Jordan $a \in L$ existe un álgebra de Jordan L_a que se comporta como un álgebra local de L. Si L es un álgebra de Lie no degenerada sobre un cuerpo F con $\mathrm{char}(F) > 5$ que posee un elemento Jordan $c \in L$ tal que L_c es un álgebra de Jordan de Clifford, entonces puede demostrarse vía una graduación de L y la TKK del par de Jordan de sus extremos, que es un álgebra finitaria ortogonal, que c en realidad vive en los elementos antisimétricos de un anillo simple con zócalo e involución de tipo ortogonal, que además verifica $c^2 \neq 0$ y $c^3 = 0$, es decir, que c es un elemento Clifford. Recíprocamente, demostramos en el capítulo 4 que si R es un anillo primo centralmente cerrado con involución tal que $\mathrm{char}(R) > 5$ y con un elemento Clifford c, entonces K_c es un álgebra de Jordan de Clifford. Lo logramos basándonos en resultados del capítulo 1 y desarrollando un enfoque combinatorio al estilo de la teoría de Herstein.

Por otro lado, para conseguir una teoría general para álgebras de Lie sin condiciones de finitud (por ejemplo, para las fuertemente primas) se necesitaría establecer propiedades elementales pero fundamentales que a día de hoy no están completamente determinadas. El *radical de Kostrikin* de L, $\mathcal{K}(L)$ (estudiado originalmente en [Filippov'81] y desarrollado en [Zel'manov'83],[Zel'manov'84]), es el análogo en el contexto Lie del radical de Baer en el contexto asociativo y del radical de McCrimmon en el contexto Jordan. Por el momento se desconoce si el radical de Kostrikin es la intersección de todos los ideales fuertemente primos del álgebra, aunque se han producido algunos avances en este sentido en [García&Gómez'11]. Si esta conjetura fuera cierta, entonces cualquier álgebra de Lie no degenerada sería un producto subdirecto de álgebras de Lie fuertemente primas, un resultado profundo que a su vez podría ser utilizado para demostrar muchos otros resultados mediante argumentos estándar sobre productos subdirectos; por ejemplo, podría ser usado para mostrar que si L es un álgebra de Lie no degenerada, $a, b \in L$ y $I(a)$ es el ideal generado por a, entonces $[I(a), I(b)] = 0$ si y solamente si $[a, [b, L]] = 0$, el análogo a la bien conocida propiedad asociativa (si R es semiprima y $a, b \in R$ entonces $I(a)I(b) = 0$ si y sólo si $aRb = 0$). Se sabe aún menos sobre álgebras de Lie sobre anillos de escalares arbitrarios. Por ejemplo, si L es un álgebra de Lie sobre un cuerpo de característica 0 e I es un ideal de L, entonces $\mathcal{K}(I) = \mathcal{K}(L) \cap I$, pero no se sabe si este resultado sigue siendo cierto para anillos de escalares más generales. Esta propiedad del radical de Kostrikin fue utilizada por Zel'manov para responder afirmativamente una conjetura de Filippov: dada un álgebra de Lie L no degenerada, un ideal I de L y $a \in L$ tal que $ad_a^2 I = 0$, es siempre cierto que $a \in \mathrm{Ann}(I)$. En el capítulo 2 incluimos una demostración de este hecho para álgebras de Lie sobre anillos de escalares generales (debida a Fernández López y Gómez Lozano). Después nos basamos en ella para demostrar que, de hecho, si L es no degenerada y libre de torsión 6 entonces $[a, [b, L]] = 0$ implica que $[I(a), I(b)] = 0$.

0.2 Preliminares

Anillos y álgebras. Por regla general, los anillos considerados en esta tesis son no necesariamente unitarios, y las álgebras lo son sobre anillos de escalares Φ (que son conmutativos y unitarios). Toda álgebra asociativa R da lugar a un álgebra de Lie R^- (o simplemente R, por abuso de notación), cuyo grupo aditivo subyacente es el mismo, cuando se equipa con el producto *corchete* $[x, y] := xy - yx$. De manera similar, si $\frac{1}{2} \in \Phi$ entonces R da lugar a un álgebra de Jordan lineal R^+ con mismo grupo aditivo subyacente cuando se equipa con el producto $\frac{1}{2}(x \circ y)$, denotando $x \circ y := xy + yx$. Esto además dota a R de estructura de sistema triple de Jordan con producto cuadrático $P_x y := xyx$ y producto triple $\{x, y, z\} := xyz + zyx$. Si además R es un álgebra con involución $*$, entonces el conjunto $H := \mathrm{Sym}(R, *) := \{x \in R \mid x^* = x\}$ de los elementos simétricos de R es un álgebra de Jordan (y un sistema triple) sobre $\mathrm{Sym}(\Phi, *)$ con los productos heredados de R^+, y el conjunto $K := \mathrm{Skew}(R,*) := \{x \in R \mid x^* = -x\}$ de los elementos antisimétricos de R es un álgebra de Lie y un sistema triple de Jordan sobre $\mathrm{Sym}(\Phi,*)$ cuando se equipa con el producto corchete y el producto cuadrático.

Estructuras relacionadas. Dada un álgebra A, denotamos por $TF(A)$ el conjunto de enteros para los que A es libre de torsión. El *álgebra de multiplicación* $\mathrm{M}(A)$ es la subálgebra unitaria (asociativa) de $\mathrm{End}_\Phi(A)$ generada por todos los operadores de multiplicación a izquierda y derecha. El *centroide* $\Gamma_\Phi(A)$ (o simplemente Γ si A es un anillo) es el centralizador de $\mathrm{M}(A)$ dentro de $\mathrm{End}_\Phi(A)$. Si $a \in A$ denotamos por $I(a)$ el ideal generado por a en A.

Álgebras asociativas primas y semiprimas. Un álgebra A es *prima* (*semiprima*) cuando $IJ = 0$ ($I^2 = 0$) implica $I = 0$ o $J = 0$ ($I = 0$), con I, J ideales de A. Si R es un álgebra asociativa entonces R es prima (semiprima) si y solamente si $aRb = 0$ ($aRa = 0$) implica $a = 0$ o $b = 0$ ($a = 0$). Un anillo es primo (semiprimo) si lo es como $\mathbb{Z}$-álgebra. Si R es primo con zócalo, diremos que un elemento $a \in R$ es *minimal* si $I(a)$ es minimal. Un *elemento reducido* es un elemento $a \in R$ minimal tal que $aRa = Fa$ con F un cuerpo. Los anillos primos con zócalo pueden ser caracterizados como anillos de operadores de pares duales de espacios vectoriales, lo que permite anexarles un modelo geométrico que transporta ideas y métodos del contexto de la geometría lineal al algebraico (este modelo puede consultarse en el apéndice A).

Sea R un anillo semiprimo. Consideramos tanto el anillo de cocientes de Martindale bilátero (derecha) de R, $Q(R)$, como el simétrico $Q_s(R)$ (véase [RingsGIs, Section 2.2]). El centro de $Q_s(R)$ coincide con el de $Q(R)$ y es denominado el *centroide extendido* de R, denotado por $\mathcal{C}(R)$ (simplemente como $\mathcal{C}$ si R es un anillo). El centroide extendido contiene el centroide y el centro. La *clausura central* de R es el subanillo $\mathcal{C}R$ de $Q_s(R)$, y su *clausura central unitaria* es $\widehat{R} := \mathcal{C}R + \mathcal{C}$. R se dice *centralmente cerrado* cuando $\mathcal{C}R = R$. Tanto $\mathcal{C}R$ como $\widehat{R}$ son centralmente cerrados, como lo es cualquier anillo simple. Si R es primo entonces $\mathcal{C}$ es un cuerpo. Si $\overline{\mathcal{C}}$ denota la clausura algebraica de $\mathcal{C}$, entonces la extensión de escalares $\overline{R} := \widehat{R} \otimes_{\mathcal{C}} \overline{\mathcal{C}}$ es centralmente cerrada.

Representación adjunta. Sea L un álgebra de Lie, y denotemos por $\mathrm{Der}(L)$ el conjunto de sus derivaciones. Debido a la identidad de Jacobi, la *aplicación adjunta* que envía $x \in L$ a $[x, \cdot] \in \mathrm{Der}(L)$ es un homomorfismo de álgebras de Lie, cuyo núcleo es $Z(L)$. Las derivaciones del tipo $[a, \cdot]$ con $a \in L$ se denominan *derivaciones internas.* El conjunto de las derivaciones internas se denota por $\mathrm{Inn}(L)$. La aplicación adjunta se suele denotar por $ad : L \to \mathrm{Der}(L)$ con $ad_x(y) := [x, y]$, aunque en esta tesis también adoptamos de manera sistemática una notación más clara que denota mediante una letra mayúscula la adjunta del elemento representado por la misma letra en minúscula. Así, $A \equiv ad_a$ en $\mathrm{Inn}(L)$ con $a \in L$. Debido a la identidad de Jacobi, la aplicación adjunta transforma identidades del álgebra de Lie en identidades de sus endomorfismos (una técnica usada originalmente por Kostrikin, véanse [Kostrikin'59] y [AroundBurnside]). Por ejemplo, si $a \in L$ es tal que $A^2(x) = 0$ para todo $x \in L$, entonces $ad_{A^2(x)}$ también es 0 y por tanto $ad_{A^2(x)} = ad_{[a,[a,x]]} = [A, [A, X]] = 0$ para todo $X \in \mathrm{Inn}(L)$. Pero $[A, [A, X]] = A^2X - 2AXA + XA^2 = -2AXA$ ya que $A^2 = 0$, y en consecuencia $AXA = 0$ si $2 \in \mathrm{TF}(L)$.

Elementos Lie destacables. Un elemento $a \in L$ es un *divisor absoluto de cero* si $ad_a^2 L = 0$. L se dice *no degenerada* cuando no posee divisores absolutos de cero no nulos, *fuertemente prima* cuando es prima y no degenerada. El *radical de Kostrikin* de L, $\mathcal{K}(L)$, es el menor ideal de L tal que $L/\mathcal{K}(L)$ es no degenerada. Un elemento $a \in L$ es un *elemento Jordan* si $ad_a^3 L = 0$. Un *ideal interno* de L es un submódulo B tal que $[B, [L, B]] \subseteq B$, y es *abeliano* si además $[B, B] = 0$. Por ejemplo, si $L = \bigoplus_{-n \le i \le n} L_i$ es una $\mathbb{Z}$-graduación finita, entonces L_{-n} y L_n son ideales internos abelianos de L. Todos los elementos de un ideal interno abeliano son Jordan y recíprocamente, si $3 \in \mathrm{TF}(L)$ y $a \in L$ es Jordan, entonces $\mathrm{ad}_a^2 L$ es un ideal interno abeliano de L (véase 4.1.3).

0.3 Capítulo 1: K, elementos antisimétricos de un anillo con involución

El capítulo 1 está dedicado por entero a la estructura de K cuando R es un anillo con involución tal que $\frac{1}{2} \in \Gamma$ (condición asumida implícitamente a partir de ahora). Recoge resultados útiles e importantes que son necesarios posteriormente en los capítulos 3 y 4. La primera sección es recordatoria: incluye las definiciones relevantes y los resultados estándar sobre involuciones, el modelo geométrico para anillos primos con zócalo e involución y los tipos de involución asociados, cuyas propiedades resume la siguiente tabla (Δ es un anillo de división, F un cuerpo):

Tipo de involución	**Forma bilineal**	**Anillo de división**	**Elementos**
Traspuesta: ortogonal	Simétrica	$F = \mathrm{Sym}(F,^{-})$	$\exists a = a^*$ minimal
Traspuesta: unitaria	Hermítica o skew	$\Delta \neq \mathrm{Sym}(\Delta,^{-})$	$\exists a = a^*$ minimal
Simpléctica	Alternante	$F = \mathrm{Sym}(F,^{-})$	$a^*a = 0\ \forall a$ minimal

Además, si R es un anillo primo con involución $*$, ésta puede extenderse a $Q_s(R)$. Entonces se dice que $*$ es *de primera clase* si es trivial en $\mathcal{C}$, *de segunda clase* si existe un elemento no nulo en $\mathrm{Skew}(\mathcal{C}, *)$.

La segunda sección introduce propiedades elementales de K. Exponemos las dos más relevantes para esta tesis:

- Es bien conocido que si R es semiprimo entonces K, como sistema triple de Jordan, es no degenerado. Este resultado admite variantes, muy útiles para llevar a cabo cálculos con identidades:

  Lema 1.2.4 (Lemas de reducción).
  Sea R un anillo semiprimo con involución. Sean $k \in K$ y $0 \neq h \in H$.

  1. $kKk = 0$ implica $k = 0$.
  2. $hKh = 0$ implica $0 \neq hRh \subseteq \mathcal{C}h$ en $\widehat{R}$.
  3. Si $I(h)$ es esencial, entonces $hKh = 0$ y $hKk = 0$ implican $k = 0$.

Además, si R es primo y $hKh = 0$, entonces $h\widehat{R}h = \mathcal{C}h$ y $\mathcal{C}R$ y $\widehat{R}$ tienen zócalo no nulo e involución de tipo ortogonal.

- Supongamos que $a \in K$ es un elemento regular von Neumann de R. Entonces existe $b \in R$ tal que $aba = a$, elemento al que llamamos una *pareja de a*. Sean $b_h := \frac{1}{2}(b+b^*)$ y $b_k := \frac{1}{2}(b-b^*)$. Si deseamos una pareja de a que sea también antisimétrica, podemos tomar $b' := b_k$, puesto que $ab_ha + ab_ka = aba = a = -a^* = -(aba)^* = -ab_ha + ab_ka$, así que $0 = ab_ha$ y $a = aba = ab_ka$. Si además queremos una pareja c de a tal que a sea a su vez una pareja de c, podemos tomar $c := b'ab'$. Obsérvese que $c^* = (b'ab')^* = -b'ab' = -c$, así que c también es antisimétrico. Si además de todo eso a es un elemento de cuadrado cero, entonces el elemento $d := c - c^2a$ es una pareja de a tal que $dad = d$ y $d^2 = 0$. Pero d no es antisimétrico. Con algo más de esfuerzo se puede encontrar una pareja de a antisimétrica y de cuadrado cero, lo que denominamos una *bella pareja* (este resultado aparecerá en [Brox,Fernández&Gómez(2)]):

  Lema 1.2.5 (Lema de la bella pareja).
  Sea R un anillo con involución y sea $a \in K$ un elemento regular von Neumann tal que $a^2 = 0$. Sea c como en el párrafo anterior. Entonces el elemento $d := c - \frac{1}{2}(ac^2 + c^2a) + \frac{1}{4}ac^3a$ es tal que $d \in K$, $ada = a$, $dad = d$ y $d^2 = 0$.

  Existe un resultado análogo cuando a es simétrico.

La tercera sección desarrolla una técnica que permite transportar resultados de anillos primos a K. La mayoría de los hechos relevantes para esta sección ya eran conocidos por separado, pero pensamos que se gana algo de conocimiento al tenerlos en cuenta de manera simultánea. Denotemos por $\langle K\rangle$ el subanillo asociativo generado por K en R. Su estructura es conocida ([RingsGIs, Lemma 9.1.5]) y cercana a K: $\langle K\rangle = K \oplus (K \circ K)$, donde además $K \circ K$ coincide con el subgrupo generado por $\{k^2 \mid k \in K\}$. Por tanto, hablando de manera laxa, $\langle K\rangle$ es K junto con sus cuadrados. La clave de la técnica mencionada es el siguiente teorema:

Teorema 1.3.2 (**Buen comportamiento de $\langle K\rangle$**).

Sea R un anillo primo con involución. Si $[K,K]\neq 0$ entonces el ideal generado por $[K,K]^2$ en R es no nulo y está contenido en $\langle K\rangle$. En particular $\langle K\rangle$ es un anillo primo cuyo centroide extendido coincide con el de R.

Por tanto, si queremos probar un resultado P para K que sabemos cierto para anillos primos, siempre que $[K,K]\neq 0$ podemos usar P para $\langle K\rangle$ y demostrar que K hereda alguna versión de P (quizás deformada) debido a que $\langle K\rangle = K\oplus(K\circ K)$. Si necesitamos una relación específica entre P para K y P para R, entonces usaremos también la conexión entre $\langle K\rangle$ y R a través de su ideal común no nulo. La única restricción insalvable a este proceso es que ocurra $[K,K]=0$ (en cuyo caso decimos que K es *excepcional*). Afortunadamente, el siguiente teorema demuestra que esto sucede sólo en casos concretos de dimensión pequeña:

Teorema 1.3.9 (**Equivalencias de excepcionalidad**).

Sea R un anillo primo con involución. Las siguientes condiciones son equivalentes.

i) *R es conmutativo o $Z(K)\not\subseteq Z(R)$.*
ii) $[K,K]=0$.
iii) *R es conmutativo o $\mathcal{C}R=\widehat{R}$ es un álgebra central simple de dimensión 4 sobre $\mathcal{C}$ (es decir, un álgebra de cuaternios generalizada) con involución de primera clase y tipo traspuesto, y $\overline{R}\cong \mathbb{M}_2(\overline{\mathcal{C}})$ con la involución traspuesta.*

La demostración que realizamos para **ii)** $\Rightarrow$ **iii)** del teorema previo utiliza los teoremas fundamentales de la teoría PI aplicados a anillos primos. Dedicamos la última sección de este capítulo a demostrar (proposición 1.4.2), mediante un enfoque combinatorio más elemental, que si R es cualquier anillo con involución (no necesariamente semiprimo) tal que $[K,K]=0$, entonces R satisface la *identidad de Hall*, $[[x,y]^2,z]=0$, que es satisfecha por las álgebras de cuaternios.

0.4 Capítulo 2: Elementos ortogonales en álgebras de Lie

El objetivo de este capítulo es demostrar, siguiendo nuestro artículo [Brox,García&Gómez'14], que si L es un álgebra de Lie no degenerada tal que $6 \in \mathrm{TF}(L)$ y $a, b \in L$, entonces $[I(a), I(b)] = 0$ si y solamente si $AB = 0$. La implicación directa es obvia, y puesto que $[I(a), I(b)] = 0$ si y solamente si $a \in \mathrm{Ann}(I(b))$, es suficiente con demostrar que $AB = 0$ implica $a \in \mathrm{Ann}(I(b))$. El capítulo se subdivide en varias secciones con hipótesis de partida de complejidad creciente ($AXYB = 0$, $AXB = 0$ y finalmente $AB = 0$, con $X, Y \in \mathrm{Inn}(L)$ arbitrarios), cuyos resultados son aplicados sucesivamente para demostrar el paso siguiente. Los casos con más variables entre A y B son más sencillos de tratar, entre otras cosas, debido a la proposición Hacia Abajo ([García&Gómez'07, Proposition 1.3]):

Proposición 2.1.7 (Hacia Abajo).
Sea L un álgebra de Lie no degenerada y sean $a, b \in L$ tales que $AX_1 \dots X_n B = 0$ para todos los $x_1, \dots, x_n \in L$. Entonces, si $0 \leq m \leq n$, se tiene que $AX_1 \dots X_m B = 0$ para todos los $x_1, \dots, x_m \in L$. Además $[a, b] = 0$.

La demostración de que si L es no degenerada entonces $AXYB = 0$ implica $a \in \mathrm{Ann}(I(b))$ fue llevada a cabo en [García&Gómez'07, Proposition 1.5] a partir de la proposición Hacia Abajo.

Caso $AXB = 0$

Para demostrar que $AXB = 0$ implica $a \in \mathrm{Ann}(I(b))$ necesitamos contar con ciertas propiedades básicas de los anuladores de ideales, que a su vez necesitan de varias identidades sobre elementos Jordan y divisores absolutos de cero, que se demuestran mediante la técnica de Kostrikin:

Lema 2.1.1 (**Fórmula fundamental para elementos Jordan**).
Sea L un álgebra de Lie tal que $3 \in \mathrm{TF}(L)$ y sea $a \in L$ un elemento Jordan.
Sea $x \in L$ arbitrario. Entonces $ad^2_{A^2(x)} = A^2X^2A^2$.

Esta identidad ([Benkart'77, Lemma 1.7(i),(iii)]), $ad^2_{ad^2_a(x)} = ad^2_a ad^2_x ad^2_a$, toma su nombre de la fórmula fundamental para álgebras de Jordan $U_{U_x(y)} = U_xU_yU_x$ (véase [TasteJordanAlgebras, páginas 5 a 9]).

Lema 2.1.2 (**Identidades para divisores absolutos de cero**).
Sea L un álgebra de Lie tal que $2 \in \mathrm{TF}(L)$ y sea $a \in L$ un divisor absoluto de cero.
Sean $x, y \in L$ arbitrarios. Entonces:
1. $AXA = 0$. *2.* $AXYA = AYXA$. *3.* $ad^2_{A(x)} = -AX^2A$.
4. $AXYA(z) = AXZA(y) = AYZA(x)$. *5.* $AX^2AX^2A = 0$ *si además* $3 \in \mathrm{TF}(L)$.

También utilizamos que si $a \in L$ es un divisor absoluto de cero y $2 \in \mathrm{TF}(L)$, entonces $A(L)$ es un ideal interno abeliano, un caso particular de [García&Gómez'09, Theorem 2.3]. Con estas herramientas se prueba el siguiente resultado, útil para reducir identidades:

Teorema 2.1.5 (**Ideal no degenerado como álgebra**).
Sea L un álgebra de Lie tal que $6 \in \mathrm{TF}(L)$ y sea I un ideal de L que, como álgebra, es no degenerado. Entonces $\mathrm{Ann}(I) = \{x \in L \mid X^2(I) = 0\}$ *y además* $\mathrm{Ann}(I)$ *es un ideal no degenerado.*

En la demostración de la proposición principal de esta sección son necesarias identidades específicas, que se demuestran mediante la técnica de Kostrikin:

Proposición 2.3.1 (**Identidades del caso $AXB = 0$**).
Sea L un álgebra de Lie no degenerada y sean $a, b \in L$ tales que $AXB = 0$ para todo $x \in L$. Sean $x, y, z, w \in L$ arbitrarios. Entonces:
1. $AB = BA = BXA = 0$ *y* $[a, b] = 0$. *2.* $AXYB = AYXB$.
3. $AXYB = BYXA$. *4.* $A^2XYB = 0 = AXYB^2$. *5.* $AXAYZB = 0 = BXBYZA$.
6. $A^2XYZB = 0 = AXYZB^2$. *7.* $A^2XYZWB^2 = 0$.

La proposición principal del caso $AXB = 0$ se demuestra ahora usando las identidades previas para probar que $ad_{A^2(x)}ZWad_{B^2(y)} = 0$ para todos los $x, y, z, w \in L$, lo que lleva a $a \in \text{Ann}(I(b))$ a través de un lema técnico y del caso $AXYB = 0$.

Caso $AB = 0$

La demostración de que $AB = 0$ implica $a \in \text{Ann}(I(b))$ es similar a la del caso $AXB = 0$ en cuanto a estructura, pero más compleja en su ejecución. Las identidades utilizadas en este paso son las siguientes:

Proposición 2.4.2 (**Identidades del caso** $AB = 0$)**.**
Sea L un álgebra de Lie no degenerada y sean $a, b \in L$ tales que $AB = 0$.
Sean $x, y, z \in L$ arbitrarios. Entonces:

1. $BA = 0$ *y* $[a, b] = 0$. *2.* $AXB = -BXA$. *3.* $AXB^2 = A^2XB = A^2XYB^2 = 0$.

4. $AXAYB = BXAYA$. *5.* $AXYB^2 = AYXB^2$.

6. $A^2XYB = 2BXAYA + 2BYAXA - BXYA^2$ *y*

$AXYB^2 = 2AYBXB + 2AXBYB - B^2YXA$.

7. $A^2XYZB^2 = 2AXAYBZB + 2AXAZBYB + 2AYAXBZB+$

$+2AYAZBXB + 2AZAXBYB + 2AZAYBXB$.

El teorema principal se demuestra probando, gracias al caso $AXB = 0$, que a es un elemento Jordan de $L/\text{Ann}(I(b))$ y usando la fórmula fundamental de los elementos Jordan junto con las identidades previas para mostrar que $ad_{A^2(X^2A^2(y))}Vad_{B^2(Z^2B^2(w))} = 0$ para todos los $x, y, z, w, v \in L$. A partir de aquí, el caso $AXB = 0$ y un lema técnico implican el resultado deseado.

0.5 Capítulo 3: Ideales internos

Este capítulo comienza con un resumen de la historia de la clasificación de los ideales internos en distintos contextos y de las técnicas usadas en los artículos previos. Entre ellos cabe destacar [McCrimmon'71] (que los clasifica en álgebras de Jordan[1] de capacidad finita, y en particular en R^+ y H, gracias al segundo teorema de estructura), [Fernández&García'99] (que extiende la clasificación a álgebras de Jordan no degeneradas de capacidad infinita mediante el modelo geométrico), [Benkart'76] (que los estudia en las álgebras de Lie $[R,R]/Z([R,R])$ y $[K,K]/([K,K] \cap Z(R))$ cuando R es simple artiniano (con involución)), [Benkart&Fernández'09] (que extiende la clasificación previa a anillos simples con zócalo (e involución) mediante el modelo geométrico, y corrige una omisión en el artículo anterior) y [Fernández'14] (que extiende la clasificación a R primo centralmente cerrado mediante el lema de Herstein, véase más abajo, para ideales internos Lie abelianos). Nuestro artículo [Brox,Fernández&Gómez(1)] puede verse como el siguiente paso natural, pues lleva la clasificación a K cuando R es primo centralmente cerrado con involución. Para lograrlo, por un lado nos basamos en el modelo geométrico y por otro trasladamos el lema de Herstein de R a K mediante la técnica basada en $\langle K \rangle$ desarrollada en el capítulo 1. La siguiente sección del capítulo revisa brevemente el desarrollo histórico del resultado que en esta tesis denominamos *lema de Herstein*, que establece que (en ciertos contextos y con condiciones de torsión suficientemente buenas) cualquier elemento adnilpotente[2] es la suma de uno nilpotente y uno central. La primera versión de este lema apareció en [Herstein'63] para anillos simples, y fue posteriormente extendido a anillos primos centralmente cerrados ([Martindale&Miers'83]) y a semiprimos centralmente cerrados ([Grzeszczuk'92]). Además existe en la literatura una versión para K cuando R es primo centralmente cerrado y K no es excepcional ([Martindale&Miers'91]), que como ya hemos mencionado demostramos de manera sencilla en esta tesis mediante la técnica basada en $\langle K \rangle$ (incluimos sólo la demostración

[1]Un ideal interno de un álgebra de Jordan J es un submódulo B tal que $U_B J \subseteq B$.

[2]Un elemento a de un álgebra de Lie L se dice *adnilpotente* cuando existe $n \in \mathbb{N}$ tal que $A^n(L) = 0$.

para elementos Jordan):

Proposición 3.2.1 (**Lema de Herstein para elementos Jordan**).
Sea R un anillo centralmente cerrado con involución $*$ tal que $\mathrm{char}(R) \neq 2,3,5$ *y* $[K,K] \neq 0$*, y sea $a \in K$ un elemento Jordan de K. Entonces:*

1. *Si la involución es de segunda clase entonces $a = v + z$, donde $z \in \mathrm{Skew}(\mathcal{C}, *)$ y $v^2 = 0$.*
2. *Si la involución es de primera clase entonces $a^3 = 0$. Más aún, si $a^2 \neq 0$ entonces a^2 es un elemento reducido y R tiene zócalo no nulo e involución de tipo ortogonal.*

La demostración consiste esencialmente en probar que $ad_a^3 K = 0$ implica $ad_a^5\langle K\rangle = 0$ gracias a que $\langle K\rangle = K + K \circ K$ y a la regla de Leibniz y en usar entonces el lema de Herstein con $n = 5$ en $\langle K\rangle$, que es primo y con mismo centroide extendido que R porque K no es excepcional. La afirmación sobre la estructura de R cuando la involución es de primera clase y $a^2 \neq 0$ se deduce de los lemas de reducción.

El lema de Herstein es el resultado que abre las puertas de la clasificación de los ideales internos Lie abelianos de K (como ya había sido reconocido por Benkart en el contexto simple) debido a que todo elemento de un ideal interno abeliano es un elemento Jordan. Nuestro estudio se fundamenta en el estudio previo para R llevado a cabo en [Fernández'14], razón por la que revisamos los resultados de dicho artículo en la tercera sección. Nuestra clasificación comienza en la cuarta sección. Dada un álgebra semiprima R con involución, definimos varias clases de ideales internos Lie abelianos de K:

- Un ideal interno *isotrópico* de K es un submódulo V tal que $V^2 = 0$.
- Supongamos que $\mathrm{Skew}(Z(R), *) \neq 0$. Un ideal interno es *estándar* si es de la forma $V \oplus \Omega$ con V un ideal interno isotrópico y $0 \neq \Omega$ un submódulo de $\mathrm{Skew}(Z(R), *)$.
- Supongamos de nuevo que $\mathrm{Skew}(Z(R), *) \neq 0$. Un ideal interno *especial* de K es de la forma $\mathrm{inn}(V, f) := \{v + f(v) \mid v \in V\}$, donde V es un ideal interno isotrópico y $f : V \to \mathrm{Skew}(Z(R), *)$ es una aplicación lineal tal que $[V,[V,K]] \subseteq \ker f$.
- Supongamos ahora R primo con zócalo no nulo, de manera que K es una subálgebra de $\mathbf{o}(X)$ que contiene $\mathbf{fo}(X)$ (véase 3.4.3). Un ideal interno de K es *Clifford* si es

de la forma $[x, H^\perp] := \{[x, z] \mid z \in H^\perp\}$, con x un vector isotrópico no nulo y H un plano hiperbólico asociado (véase 3.4.14).

A lo largo de varias subsecciones demostramos las propiedades elementales de cada uno de los distintos tipos de ideal interno. Las más relevantes y complejas son las asociadas a los Clifford, de entre las que destacamos sus diferentes caracterizaciones (desde el punto de vista de la teoría de anillos y mediante elementos distinguidos):

Proposición 3.4.18 (Estructura de los ideales internos Clifford).
Sea L un álgebra de Lie tal que $\mathbf{fo}(X) \leq L \leq \mathbf{o}(X)$ y sea B un subconjunto de L.
B es un ideal interno Clifford de L si y solamente si $B = \kappa((1-e)\mathbf{fo}(X)e)$, donde $\kappa(x) := x - x^*$ es la antitraza *y $e \in \mathcal{F}(X)$ es un idempotente minimal $*$-ortogonal, en cuyo caso $B = \kappa((1-e)Se)$ para cualquier subconjunto $\mathbf{fo}(X) \subseteq S \subseteq \mathcal{L}(X)$.*

Proposición 3.4.19 (Caracterización de los ideales internos Clifford).
Sea R un álgebra prima centralmente cerrada con $\mathrm{char}(R) \neq 2, 3, 5$ e involución tal que $[K, K] \neq 0$. Si B es un ideal interno Lie abeliano de K tal que $b^2 \neq 0$ para algún $b \in B$, entonces B es un ideal interno Clifford de K.

Gracias a esta última caracterización y a la clasificación del caso R se puede demostrar que todos los ideales internos Lie abelianos de K en el caso primo centralmente cerrado son de una de las cuatro clases definidas previamente:

Teorema 3.4.20 (Clasificación de los ideales internos Lie abelianos de K).
Sea R un álgebra prima centralmente cerrada de $\mathrm{char}(R) \neq 2, 3, 5$ e involución $*$ tal que $[K, K] \neq 0$. Si B es un ideal interno Lie abeliano de K, entonces o bien
1. *$B = V$ es isotrópico,*
2. *$B = V \oplus \mathrm{Skew}(\mathcal{C}, *)$ es estándar,*
3. *$B = \mathrm{inn}(V, f)$ es especial, o*
4. *$B = \kappa((1-e)Re)$ es Clifford.*

Además, en los casos (2) y (3) R es unitaria y $*$ es de segunda clase, mientras que en el caso (4) R tiene zócalo no nulo y $*$ es de tipo ortogonal.

0.7 Capítulo 4: Elementos Clifford

Sea R un anillo primo centralmente cerrado con involución $*$ de primera clase tal que $\mathrm{char}(\mathcal{C}) \notin \{2,3,5\}$. Por el lema de Herstein sabemos que si $[K,K] \neq 0$ entonces cualquier elemento Jordan $a \in K$ cumple o bien $a^2 = 0$ o bien $a^2 \neq 0$ y $a^3 = 0$. Consecuentemente llamaremos *elemento Clifford* de R a cualquier elemento Jordan $c \in K$ tal que $c^2 \neq 0$ y $c^3 = 0$. Los cuadrados de los elementos Clifford poseen propiedades sencillas que son útiles para realizar cálculos. En concreto son reducidos, lo que determina en parte la estructura de $(R,*)$.

Proposición 4.2.2 (**Propiedades de los cuadrados de los elementos Clifford**).
Sea $c \in K$ un elemento Clifford de R. Entonces:

1. $c^2Kc^2 = 0$. *2. $c^2Rc^2 = \mathcal{C}c^2$.* *3. $c^2k_1k_2c^2 = c^2k_2k_1c^2$ para todos los $k_1, k_2 \in K$.*
4. R tiene zócalo no nulo e involución de tipo ortogonal.

Observemos que c^2 es regular von Neumann porque es reducido. Además c^2 es simétrico y de cuadrado cero (puesto que $c^3 = 0$). Por el lema de la bella pareja existe $d \in R$ tal que $d^* = d$, $d^2 = 0$, $c^2dc^2 = c^2$ y $d = dc^2d$. Las bellas parejas de c^2, y sus idempotentes asociados, satisfacen más propiedades interesantes.

Proposición 4.2.3 (**Propiedades de la bella pareja**).
Sean c un elemento Clifford y d una bella pareja de c^2.

1. *$dKd = 0$ y $dRd = \mathcal{C}d$.*
2. *Existe un idempotente $*$-ortogonal $e \in R$ tal que $eRe = \mathcal{C}e$, $e^*Re = \mathcal{C}c^2$, $eRe^* = \mathcal{C}d$ y $e^*Ke = 0 = eKe^*$.*
3. *$ec = ce^* = 0$, $e^*c^2e = e^*c^2 = c^2e = c^2$ y $ede^* = ed = de^* = d$.*
4. *$[K,K] \neq 0$. En particular R no es un álgebra de matrices 2×2 sobre $\mathcal{C}$.*
5. *$e^* \neq 1 - e$ en $\widehat{R}$.*

La existencia de elementos Clifford en R está ligada a la existencia de idempotentes del tipo de la proposición anterior.

Teorema 4.2.4 (**Existencia de elementos Clifford**).
R posee un elemento Clifford si y solamente si $[K,K] \neq 0$ y existe un idempotente $*$-ortogonal $e \in R$ tal que $eRe = \mathcal{C}e$ y $e^*Ke = 0$.

Observemos que si e es un idempotente asociado a c, puesto que $e^* \neq 1-e$, tenemos que el idempotente simétrico $g := 1-e-e^*$ es no nulo. El conjunto $\{e^*, g, e\}$ es un sistema completo de idempotentes ortogonales y por lo tanto genera la 5-graduación $R = gRe \oplus (gRe^* \oplus e^*Re) \oplus (gRg \oplus e^*Re^* \oplus eRe) \oplus (eRe^* \oplus e^*Rg) \oplus eRg$. Debido a las propiedades mostradas más arriba, este conjunto resulta generar también una 3-graduación de K:

Teorema 4.2.5 (**3-graduación**).
Sean $c \in K$ un elemento Clifford, d una bella pareja de c^2 y $e := dc^2$. Sea $g := 1-e-e^*$. Entonces $K = K_{-1} \oplus K_0 \oplus K_1$ con $K_{-1} := \kappa(gRe), K_0 := \kappa(eRe) \oplus gKg, K_1 := \kappa(eRg)$ es una 3-graduación de K en la que la componente homogénea i-ésima k_i de cualquier $k \in K$ coincide con $\bigoplus_{m-n=i} \kappa(e_m k e_n)$, con $e_0 := e^*, e_1 := g, e_2 := e$.

El teorema 4.2.6 demuestra que el elemento Clifford cae en la componente K_{-1} de cualquiera de estas 3-graduaciones (componente que de hecho es invariante para todas las bellas parejas de c^2), lo que implica que $cKc = \mathcal{C}c$, uno de los hechos más relevantes para el desarrollo de los resultados de este capítulo.

Álgebra de Jordan en un elemento Clifford

Existe un álgebra de Jordan asociada a cualquier elemento Jordan de un álgebra de Lie:

Teorema 4.1.2 (**Álgebra de Jordan en un elemento Jordan**).
Sea L un álgebra de Lie tal que $3 \in \mathrm{TF}(L)$ y sea $a \in L$ un elemento Jordan. Entonces L equipada con el producto $x \bullet y := [[x,a],y]$ es un álgebra, denotada por $L^{(a)}$, tal que:
1. $\ker(a) := \{x \in L \mid A^2(x) = 0\}$ *es un ideal de $L^{(a)}$.*
2. $L_a := L^{(a)}/\ker(a)$ *es un álgebra de Jordan tal que $U_{\bar{x}}(\bar{y}) = \overline{X^2A^2(y)}$.*

Cuando a es regular von Neumann con pareja b, L_a es isomorfa a $(ad_a^2 L, +, \bullet)$, con $x \bullet y := [x, [b, y]]$ (([Fernández,García&Gómez'06, Proposition 2.11])).

Si F es un cuerpo con $\mathrm{char}(F) \neq 2$ y X es un espacio F-vectorial equipado con una forma bilineal simétrica $\langle \cdot, \cdot \rangle$, el espacio vectorial $F \oplus X$ puede ser dotado de estructura de álgebra de Jordan con el producto $(\alpha + x) \bullet (\beta + y) := \alpha\beta + \langle x, y\rangle + \beta x + \alpha y$ para $\alpha, \beta \in F$ y $x, y \in X$. Esta álgebra de Jordan es unitaria y especial, pues es isomorfa a la subálgebra de Jordan del álgebra de Clifford asociativa definida por $\langle \cdot, \cdot \rangle$. Por esta razón $F \oplus X$ es llamada en ocasiones un *álgebra de Clifford de Jordan*, nomenclatura que seguimos en esta tesis. El resto del capítulo demuestra que el álgebra de Jordan K_c asociada a un elemento Clifford c es un álgebra de Jordan de Clifford. Para probarlo se necesitan más herramientas básicas. Por la razón obvia introducimos la notación

$$\sqrt{d} := cd + dc.$$

Proposición 4.3.2 (Propiedades de la raíz cuadrada de d).

1. $\sqrt{d} \in K_1$ *en la 3-graduación del teorema 4.2.5. En particular* $\sqrt{d}$ *es Jordan.*

2. $(\sqrt{d})^2 = d$. *3.* $(\sqrt{d})^3 = 0$. *4.* $\sqrt{d}K\sqrt{d} = \mathfrak{C}\sqrt{d}$. *5.* $\sqrt{d}c\sqrt{d} = \sqrt{d}$. *6.* $c\sqrt{d}c = c$.

7. $c^2 \circ \sqrt{d} = c$. *8.* $d \circ c = \sqrt{d}$. *9.* $ad_c^2(-\sqrt{d}) = c$. *10.* $ad^2_{-\sqrt{d}}c = -\sqrt{d}$.

11. $[[c, \sqrt{d}], b] = b$ *para todo* $b \in K_{-1}$.

La imagen de $\sqrt{d}$ juega el papel de elemento unidad en K_c. La estructura Clifford de K_c se construye sobre dos formas:

Formas

- El teorema 4.2.6(4) muestra que $ckc = \mu_k c$ para todo $k \in K$. Denotamos $\mathrm{tr}(k) := \mu_k$ y la denominamos *traza* de k.
- La proposición 4.2.2(2) muestra que $c^2xc^2 = \lambda_x c^2$ para todo $x \in R$. Denotamos $\langle k, k'\rangle := \lambda_{kk'}$ para todos los $k, k' \in K$. Entonces $\langle \cdot, \cdot \rangle$ es una forma bilineal simétrica sobre $\mathfrak{C}$ (por la proposición 4.2.2(3)).

La traza ayuda a identificar la estructura de suma directa del álgebra de Jordan de Clifford: puesto que c y $-\sqrt{d}$ son pareja regular (por la proposición 4.3.2(9),(10)), tenemos que K_c es isomorfa a $(ad_c^2K, +, \bullet)$. Vemos que ad_c^2K puede escribirse como $\mathcal{C}c \oplus B_0$, con B_0 definido a partir de los elementos de traza cero:

Proposición 4.3.4 (**Estructura de** ad_c^2K)**.**
Sean c un elemento Clifford, d una bella pareja de c^2 y $e := dc^2$. Entonces:

1. $K_{-1} = c^2 \circ K$. *En particular* $B := K_{-1}$ *es invariante.*
2. $B = B_0 \oplus \mathcal{C}c$, *donde* $B_0 := \{c^2 \circ k \mid k \in \ker(\mathrm{tr})\}$.
3. $B = \mathrm{ad}_c^2 K$.

Finalmente, la forma bilineal ayuda a construir el producto Clifford:

Teorema 4.4.2 (K_c **es un álgebra de Jordan de Clifford**)**.**
El álgebra de Jordan $(ad_c^2K, +, \bullet) \cong K_c$ es un álgebra de Jordan de Clifford en la que $\mathcal{C}c$ hace de parte escalar, B_0 hace de parte vectorial y la forma bilineal asociada de B_0 a $\mathcal{C}c$ es $\langle c^2 \circ k_1, c^2 \circ k_2 \rangle_0 := -\langle k_1, k_2 \rangle c$.

List of used symbols

a, b, c, d	Fixed elements of an algebra
x, y, z, w, v	Arbitrary elements of an algebra
v, z	A nilpotent element and a central element
e, f, g	Idempotents
$\bar{x}$	The class of equivalence of x modulo some ideal
$\vec{x}, \vec{u}, \vec{v}$	Vectors of some vector space, to distinguish them from operators
u, v, w	Vectors of some vector space, when ubiquitous
I_n	Identity matrix of size $n \times n$
h, h_1, k, k_1	h and h_1 symmetric elements, k and k_1 skew elements
$\lambda, \mu, \alpha, \beta$	Scalars, usually in the extended centroid
n, m, k	Natural numbers (coefficients, exponents and indices)
i, j, k	Indices
δ_{ij}	Kronecker delta, $\delta_{ij} := 1$ if $j = i$, $\delta_{ij} := 0$ if $j \neq i$
$\lfloor \frac{n}{m} \rfloor$	Nearest integer to $\frac{n}{m}$ from below
$\mathrm{char}(A)$	Characteristic of the algebra A. In this text, $\mathrm{char}(A) > n$ includes also the possibility $\mathrm{char}(A) = 0$.
$TF(A)$	Integer numbers for which A is free of torsion
∞	Infinity
A, B, C, K	The adjoint representations of the elements a, b, c, k, $A(x) := [a, x]$
I, J	One-sided or two-sided ideals
B, C	Abelian Lie inner ideals or Jordan inner ideals
V	An isotropic inner ideal
T, T_a	In an associative algebra, the linear Jordan operator $T_a(x) := ax + xa$

l_a, r_a	The left and right multiplication operators, $l_a(x) := ax, r_a(x) := xa$
l_A, r_A	The left and right multiplication operators of the adjoint representation of a
$\{f, I\}$	The equivalence class of the essential ideal I and the R-module homomorphism $f : I_R \to R_R$ inside $Q(R)$
$\exp_d$	The exponential automorphism associated to the nilpotent derivation d
$[x, y]$	The commutator or Lie product, in an associative algebra $[x, y] := xy - yx$
$[x, y, z]$	In an associative algebra, $[x, y, z] := xyz - zyx$
$x \bullet y$	The Jordan product of a Jordan algebra
$x \circ y$	Twice the Jordan product in an associative algebra, $x \circ y := xy + yx$
p_a	The quadratic Jordan operator, in an associative algebra $p_a(x) := axa$
$\{x, y, z\}$	The Jordan triple product, in an associative algebra $\{x, y, z\} := xyz + zyx$
$U_x y$	The Jordan U-operator, $U_x y := 2x \bullet (x \bullet y) - (x \bullet x) \bullet y$
$\tau(a), \kappa(a)$	The symmetric trace and the skew trace, $\tau(a) := a + a^*$, $\kappa(a) := a - a^*$
$\langle v, w\rangle$	A bilinear or sesquilinear form applied to the vectors v, w
$\mathrm{span}(S)$	The submodule of A generated by the set S
$\langle S\rangle$	The subalgebra of A generated by the set S
$I(S)$	The ideal of A generated by the set S
$I(a)$	The ideal of A generated by the set $\{a\}$
$\mathrm{Ann}_l(S), \mathrm{Ann}_r(S)$	The left and right annihilators of the set S
$\mathrm{Ann}(I)$	In a semiprime algebra, the annihilator of the two-sided ideal I
$\mathbb{N}$	The natural numbers, including 0
$\mathbb{Z}$	The integers
$\mathbb{Z}_n$	The integers modulo n, $\mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$
$\mathbb{C}$	The complex numbers
$\mathbb{H}(\alpha, \beta)$	The quaternion algebra such that $i^2 = \alpha$, $j^2 = \beta$
Φ	A ring of scalars, i.e., a commutative unital ring
Δ	A division algebra
F	A field
$\overline{F}$	The algebraic closure of the field F
X, V, W	Vector spaces over some field or division ring
(V, W)	A pair of dual vector spaces over some division ring

$\dim_F V$	The dimension of the vector space V over the field F
R	An associative algebra
$(A, +, \star)$	Any kind of algebra, with bilinear product $\star$
L	A Lie algebra
$\mathbb{M}_n(R)$	Ring of square $n \times n$ matrices over the ring R
$\mathbf{fo}(V, \langle\cdot\,,\cdot\rangle)$	The finitary orthogonal (Lie) algebra of the vector space V with bilinear form $\langle\cdot\,,\cdot\rangle$
$\mathbf{o}(V, \langle\cdot\,,\cdot\rangle)$	The orthogonal (Lie) algebra of the vector space V with bilinear form $\langle\cdot\,,\cdot\rangle$
$\mathbb{A}_1$	Weyl algebra
$*, {}^{-}$	Involutions
a^T	The transpose of the matrix a
K	The subgroup of skew elements of a ring with involution, as a Lie algebra or Jordan triple system
H	The subgroup of symmetric elements of a ring with involution
$\langle K\rangle$	The subring of R generated by K
$\mathcal{K}(L)$	The Kostrikin radical of the Lie algebra L
$Z(A)$	The center of the associative or Lie algebra A
Γ	The centroid of A as a ring
$\mathcal{C}$	The extended centroid of R as a ring
$\mathcal{C}R$	The central closure of the semiprime ring R
$\widehat{R}$	The unital central closure of R, $\widehat{R} := \mathcal{C}R + \mathcal{C}$
$\widehat{K}$	The skew elements of the unital central closure of R, $\widehat{K} := \mathrm{Skew}(\widehat{R}, *)$
$\widehat{H}$	The symmetric elements of the unital central closure of R, $\widehat{H} := \mathrm{Sym}(\widehat{R}, *)$
$\overline{R}$	The extension of scalars of $\widehat{R}$ to $\overline{\mathcal{C}}$, $\overline{R} := \widehat{R} \otimes_{\mathcal{C}} \overline{\mathcal{C}}$
$Q(R)$	The two-sided right ring of quotients of the semiprime ring R
$Q_s(R)$	The symmetric Martindale ring of quotients of the semiprime ring R
$\mathrm{Hom}_\Delta(V, W)$	The ring of homomorphisms of the Δ-vector spaces V and W
$\mathrm{End}_\Phi(A)$	The ring of endomorphisms of the Φ-module A
$\mathrm{M}(A)$	The multiplication algebra of the algebra A
$\mathrm{Der}(A)$	The Lie algebra of derivations of the algebra A
$\mathrm{Inn}(A)$	The Lie ideal of inner derivations of the algebra A

δ	The differential operator, $\delta(f) = f'$
A/I	Factor ring of the algebra A over its ideal I
L_a	Jordan algebra of L at the Jordan element $a \in L$
$\mathrm{sub}(B)$	The subquotient $\mathrm{sub}(B) := (B, L/\ker(B))$ of an abelian inner ideal $B \subseteq L$
$A \leq B$	A is a substructure of B
$A \cong B$	A and B have isomorphic structures
$A \equiv B$	A and B are different notations for the same object
φ	A morphism between two structures
$\ker \varphi$	The kernel of the morphism φ
$\mathrm{im}(a)$	The image of the linear operator a
tr	A trace, a linear map $\mathrm{tr} : V \to F$ from a vector space to its base field
$\langle \cdot , \cdot \rangle$	A bilinear form $\langle \cdot , \cdot \rangle : V \times W \to \Delta$ from two vector spaces to their base division ring
i),ii),iii)	Equivalent conditions

Preliminaries

In this preamble we set the definitions, notation and conventions that will get used throughout this dissertation, and quickly review some of the elementary properties of the relevant structures that get used the most, in a rather schematic and aseptic fashion.

Rings and algebras. By a *ring* we will understand an associative ring, not necessarily unital. By a *ring of scalars* we mean a commutative unital ring. The notation Φ will always be reserved for an underlying ring of scalars. By a Φ-*algebra* we understand a Φ-module equipped with a bilinear product. We are mostly interested in associative and Lie algebras, although (linear and quadratic) Jordan algebras and another Jordan systems (Jordan triple systems, Jordan pairs) do appear occasionally; we refer the reader to [TasteJordanAlgebras] and [JordanPairs] for the suitable definitions and conventions. We usually reserve the letter R for an associative ring or algebra and the letter L for a Lie algebra[3]. We choose to elide the algebras' underlying ring of scalars whenever possible. When very occasionally an algebra of undetermined kind is needed, we denote it by A, its product by $\star$. If $a \in A$ we denote the ideal generated by a inside A as $I_A(a)$, $I(a)$ when A is clear from context.

Every associative algebra R gives rise to a Lie algebra R^-, with same underlying additive group, when endowed with the *bracket* product $[x, y] := xy - yx$. Similarly, if $\frac{1}{2} \in \Phi$ then R gives rise to a linear Jordan R^+ with same underlying additive group when

[3]Except on one occasion, in which we use R and L to denote right and left ideals, and reserve the letter A for the relevant associative algebra.

endowed with the product $\frac{1}{2}(x \circ y)$, where we denote $x \circ y := xy + yx$. This in addition endows R with an structure of Jordan triple system with quadratic product $P_x y := xyx$ and triple product $\{x, y, z\} := xyz + zyx$. We will usually abuse the notation and simply talk about R as either an associative, Lie or Jordan algebra when clear from context. Similarly, if R is an algebra with involution, then the set H of symmetric elements of R is a Jordan algebra (and a triple system) with the products inherited from R, and the set K of skew elements of R is a Lie algebra and a triple system when endowed with the bracket and the quadratic products of R (see Chapter 1 for more information on this and other related subjects). Every (associative, Lie, Jordan) algebra can be seen as an (associative, Lie, Jordan) ring (by "forgetting" the ring of scalars and "peeling" it to $\mathbb{Z}$), and every ring is in particular a $\mathbb{Z}$-algebra.

Related structures. Given any algebra A, we denote by $TF(A)$ the set of integers for which A is torsion free. The *multiplication algebra* $\mathrm{M}(A)$ of a linear algebra A is the unital (associative) subalgebra of $\mathrm{End}_\Phi(A)$ generated by all left and right multiplication operators, which in associative algebras we denote respectively by l_a and r_a, for every $a \in R$. The *centroid* $\Gamma_\Phi(A)$ is the centralizer of $\mathrm{M}(A)$ inside $\mathrm{End}_\Phi(A)$. The notation Γ will always denote the centroid of A **as a ring**. Under mild conditions (for example, $\mathrm{Ann}_l(A) = 0$ or $A^2 = A$) the centroid is commutative. In an associative algebra the centroid always contains an homomorphic image of the center of the algebra and, if the algebra is unital, then[4] $Z(R) = \Gamma$. If R is simple then $\Gamma_\Phi(R)$ is a field.

A *derivation* of an algebra A is a map $d \in \mathrm{End}_\Phi(A)$ such that $d(a \star b) = d(a) \star b + a \star d(b)$. The set of all derivations of A, denoted by $\mathrm{Der}(A)$, is a Lie subalgebra of $\mathrm{End}_\Phi(A)^-$. The expansion of the power of a derivation applied to a product is calculated by the well-known Leibniz Rule.

[4]Here and in the remaining of this dissertation, when we embed a ring inside another via a monomorphism we see the second one as a superring of the first, and substitute the corresponding isomorphisms by equalities, by an abuse of notation. So, for example, $Z(R) = \Gamma$ here actually means that the monomorphic image of $Z(R)$ inside Γ actually fills Γ.

Lemma 0.4.3 (Leibniz Rule).

Let A be an algebra and let $d \in Der(A)$. Then for every $x, y \in A$ and $n \in \mathbb{N}$ we have

$$d^n(x \star y) = \sum_{i=0}^{n} \binom{n}{i} d^i(x) \star d^{n-i}(y),$$

where we understand that d^0 is the identity map of $\mathrm{End}_\Phi(A)$.

Prime and semiprime associative algebras. An algebra A is said *prime* (*semiprime*) when $IJ = 0$ ($I^2 = 0$) implies $I = 0$ or $J = 0$ ($I = 0$), where I, J are ideals of A. For an associative algebra R there exist useful characterizations by elements of these properties: R is prime (semiprime) if and only if $aRb = 0$ ($aRa = 0$) implies $a = 0$ or $b = 0$ ($a = 0$). A ring is prime (semiprime) if it is so as a $\mathbb{Z}$-algebra. The center of a prime associative algebra is an integral domain, while the center of a semiprime one is reduced, i.e., has no nonzero nilpotent elements.

Let R be a semiprime associative algebra for this and the following paragraph. The left socle and right socle[5] of R coincide ([StructureRings, Theorem 1 on page 65]). For brevity, when R has socle[6] we will call an element of R *minimal* whenever it generates a minimal right ideal (equivalently, a minimal left ideal[7]). In particular a minimal idempotent is an idempotent which is minimal as an element; equivalently, an idempotent $e \in R$ is minimal if and only eRe is a division ring ([RingsGIs, Proposition 4.3.3]). If R is prime with socle, a *reduced element* is a minimal element $a \in R$ such that $aRa = Fa$ with F a field[8]. In addition, prime rings with socle can be characterized as rings of

[5]The (left, right) *socle* is defined as the sum of all the minimal (left, right) ideals of R.

[6]Anytime we talk about a set 'with X' or state that 'a set has property P', we mean 'a set with a nonzero X' or 'a ring for which P is not trivial'. Thus, an algebra with socle is an algebra with nonzero socle.

[7]If $a \in R$ is a minimal element, then by Brauer Lemma ([Lam1, 10.22]) there exists a minimal idempotent $e \in aR$ such that $aR = eR$ and $a = ea$ (because $eax = ax$ for every $x \in R$ implies $(ea - a)R = 0$, and R is semiprime). Now by Schur Lemma $Ra = Rea \cong Re$, which is a minimal left ideal.

[8]In this case $F \cong eRe$ with e any minimal idempotent of R; equivalently, $F \cong \mathrm{End}_R(X)$ with X any faithful irreducible R-module.

operators of dual pairs of vector spaces and then a geometric model can be attached to them, which transports ideas and tools from the geometric setting to the algebraic one. Since throughout this dissertation we will make some explicit computations within this model, we have included our notation and the necessary results in Appendix A.

We will consider the two-sided right Martindale ring of quotients and the symmetric Martindale ring of quotients associated to R (refer to [RingsGIs, Section 2.2]), which we denote, respectively, by $Q(R)$ and $Q_s(R)$. If $I \subseteq R$ is an essential ideal[9] and $f : I_R \to R_R$ is a homomorphism of right R-modules, we denote the corresponding equivalence class inside $Q(R)$ as $\{f, I\}$. The center of $Q_s(R)$ coincides with the center of $Q(R)$ and is called the *extended centroid* of R, which is denoted by $\mathcal{C}(R)$. By $\mathcal{C}$ we will always denote the extended centroid of R **as a ring**. The extended centroid contains the centroid (hence its name), and therefore it also contains the center. The *central closure* of R is the subring $\mathcal{C}R$ of $Q_s(R)$, and hence the *unital central closure* of R is the subring $\mathcal{C}R + \mathcal{C}$, which we always denote by $\widehat{R} := \mathcal{C}R + \mathcal{C}$. R is said to be *centrally closed* whenever $\mathcal{C}R = R$ inside $Q_s(R)$, equivalently, whenever $\Gamma = \mathcal{C}$ (with R seen as a ring). The rings $\mathcal{C}R$ and $\widehat{R}$ are centrally closed, and every simple ring is centrally closed too[10]. If R is prime then $\mathcal{C}$ is a field. In this case we denote the algebraic closure of $\mathcal{C}$ by $\overline{\mathcal{C}}$, and denote the corresponding extension of scalars by $\overline{R} := \widehat{R} \otimes_{\mathcal{C}} \overline{\mathcal{C}}$, which is also centrally closed. If R is centrally closed prime, then[11] either $Z(R) = 0$ or $Z(R) = \mathcal{C}$, depending on whether R is unital or not; this is true in particular for simple rings.

Martindale Lemma (the original version is [Martindale'69, Theorem 1]) is a very powerful tool of prime rings that guarantees that if a certain type of identity is satisfied, then the involved elements must be linearly dependent over $\mathcal{C}$. The proof of [RingsGIs, Theorem 2.3.4] can be adapted in a straightforward manner to prove the following (the first item is well known and appears for example in [FunctionalIdentities,

[9] An ideal I of A is *essential* if $I \cap J \neq 0$ for every nonzero ideal J of A.

[10] If R is simple then $\Gamma = \mathcal{C}$ because the only essential ideal of R is R itself.

[11] Since R is prime and $Z(R) \neq 0$, there exists $0 \neq z \in Z(R) \subseteq \mathcal{C}$ and so $1 = z^{-1}z \in R$ since R is centrally closed. This implies that $Z(R) = \Gamma = \mathcal{C}$.

Theorem A.7]), the second one is a direct generalization to semiprime rings used in [Brox,García&Gómez(2)]).

Theorem 0.4.4 (**Martindale Lemma**).

1. *Let R be a prime ring and let $a_i, b_i \in Q(R)$ with $b_1 \neq 0$ be such that* $\sum_{i=1}^{n} a_i x b_i = 0$ *for every $x \in R$. Then* $a_1 \in \sum_{i=2}^{n} \mathcal{C} a_i$.
2. *Let R be a semiprime ring and let $a_i, b_i \in Q(R)$ with $b_1 \neq 0$ be such that* $\sum_{i=1}^{n} a_i x b_i = 0$ *for every $x \in R$. If in addition every nonzero ideal contained in $I(a_1)$ has nonzero intersection with $I(b_1)$, then* $a_1 \in \sum_{i=2}^{n} \mathcal{C} a_i$.

Adjoint representation. Let L be a Lie algebra. Due to Jacobi Identity, the *adjoint map* sending $x \in L$ to $[x, \cdot\,] \in \mathrm{Der}(L)$ is a homomorphism of Lie algebras, whose kernel is $Z(L)$. Any derivation of the kind $[a, \cdot\,]$ with $a \in L$ is called an *inner derivation*. The set of all inner derivations is usually denoted by $\mathrm{Inn}(L)$. The adjoint map is usually denoted as $ad : L \to \mathrm{Der}(L)$ with $ad_x(y) := [x, y]$. We will also adopt systematically a cleaner notation which denotes by a capital letter the adjoint of the element represented by the same lowercase letter. So, for example $A \equiv ad_a$ in $\mathrm{Inn}(L)$, and AXY is an associative product inside $\mathrm{M}(L)$. We will not limit our use of this notation to operate in $\mathrm{M}(L)$, but we will also usually take advantage of it to operate in L and make the computations less messy to the eye (because of the disappearance of brackets). With this notation, Jacobi Identity translates to

$$XY(z) = ZY(x) + YX(z).$$

It is of some importance for us that in associative algebras the inner derivations are not only Lie derivations, but also associative derivations: $[a, x]y + x[a, y] = axy - xay + xay - xya = axy - xya = [a, xy]$. Also, the powers of the adjoint operator have a nice expansion by Newton Binomial inside the multiplication algebra. If R is an associative algebra and $a \in R$, then since $A(x) = ax - xa$ we have $A = l_a - r_a$ in $\mathrm{M}(R)$, and thus, since l_a and r_a commute,

$$A^n = ad_a^n = (l_a - r_a)^n = \sum_{i=0}^{n}(-1)^i \binom{n}{i} l_a^{n-i} r_a^i = \sum_{i=0}^{n}(-1)^i \binom{n}{i} l_{a^{n-i}} r_{a^i}.$$

Due to Jacobi Identity, the adjoint map transforms identities of the Lie algebra into identities of its endomorphisms, which allow to find new identities by means of simplifications. This is a useful trick exploited by Kostrikin ([Kostrikin'59]; see also [AroundBurnside]). So, for example, if $a \in L$ is such that $A^2(x) = 0$ for every $x \in L$, then $ad_{A^2(x)}$ is also 0 and therefore $ad_{A^2(x)} = ad_{[a,[a,x]]} = [A,[A,X]] = 0$ for every $X \in \mathrm{Inn}(L)$. But $[A,[A,X]] = A^2X - 2AXA + XA^2 = -2AXA$ because $A^2 = 0$, and therefore $2AXA = 0$. We use this Kostrikin Trick without further mentioning from now on. We may sometimes go one level further and write down the adjoint of the element $A \in \mathrm{Inn}(L)$. To easily distinguish visually one adjoint from another, we will keep using $ad : L \to \mathrm{Inn}(L)$ and will use $\mathbf{ad} : \mathrm{Inn}(L) \to \mathrm{End}_\Phi(\mathrm{Inn}(L))$. Thus, we may say that $A^n(x) = 0$ implies $ad_{A^n(x)} = 0$, but we may equally say that it implies $\mathbf{ad}_A^n = 0$. Then we are working with the endomorphisms of an associative algebra (which is $\mathrm{End}_\Phi(L)$) and hence we have access to its multiplication algebra, so that we may say that $A^n = 0$ implies

$$0 = \mathbf{ad}_A^n = (l_A - r_A)^n = \sum_{i=0}^{n}(-1)^i \binom{n}{i} l_A^{n-i} r_A^i = \sum_{i=1}^{n-1}(-1)^i \binom{n}{i} l_{A^{n-i}} r_{A^i},$$

a notation which simplifies computations and is worth this little effort. Whenever we go up this level of abstraction, to ease the reasonings we will warn that we are working in $\mathrm{End}_\Phi(\mathrm{Inn}(L))$. Note that $\mathrm{Inn}(L)$ is *not* an associative subalgebra of $\mathrm{End}_\Phi(L)$ and therefore $\mathbf{ad}_A^n = 0$ in $\mathrm{End}_\Phi(\mathrm{Inn}(L))$ does not mean that $\mathbf{ad}_A^n$ is identically 0 in $\mathrm{End}_\Phi(\mathrm{End}_\Phi(L))$; for example it does not necessarily mean that $\mathbf{ad}_A^n(XY) = 0$. But by Leibniz Rule we have that $\mathbf{ad}_A^n(XY) = \sum_{i=1}^{n-1} \binom{n}{i} \mathbf{ad}_A^{n-i}(X)\mathbf{ad}_A^i(Y)$, since $\mathbf{ad}_A$ is also an associative derivation.

Lie highlighted elements. An element $a \in L$ is called an *absolute zero divisor*[12] if $ad_a^2 L = 0$. L is said to be *nondegenerate* when it does not have absolute zero divisors, *strongly prime* when it is prime and nondegenerate. The *Kostrikin radical* of L, $\mathcal{K}(L)$, is the smallest ideal of L such that $L/\mathcal{K}(L)$ is nondegenerate.

An element $a \in L$ is called a *Jordan element*[13] if $ad_a^3 L = 0$. An *inner ideal* of L is a submodule B such that $[B, [L, B]] \subseteq B$, which is called *abelian* if it is also an abelian subalgebra, i.e., if $[B, B] = 0$. For example, if $L = \bigoplus_{-n \leq i \leq n} L_i$ is a finite $\mathbb{Z}$-grading, then L_{-n} and L_n are easily checked to be abelian inner ideals of L. Every element in an abelian inner ideal is easily shown to be a Jordan element, and conversely, if L is 3-torsion free and $a \in L$ is Jordan, then $\mathrm{ad}_a^2 L$ is an abelian inner ideal of L (see 4.1.3).

[12]We remark that, in the literature, absolute zero divisors have received several different names, including *sandwhich elements* and *crusts of thin sandwiches* (e.g. [Benkart&Fernández'09, page 3833] and [Zel'manov'83, page 538]).

[13]The denomination of Jordan elements is adequately selected, as can be verified in Theorem 4.1.2, for associated to any Jordan element $a \in L$ there exists a Jordan algebra L_a which behaves as a local algebra for L in the sense of inheritance of important properties, as for example happens with nondegeneracy.

Chapter 1

K, skew elements of a ring with involution

Let R be a ring with involution $*$. In this chapter we collect useful and important facts concerning the skew elements of rings with involution, $K := \mathrm{Skew}(R, *)$, which will be needed in Chapters 3 and 4. The first section includes the relevant definitions and standard facts about involutions, their classifications, and the geometric model for prime rings with socle and involution. The second section introduces elementary properties of K, including the important well-known result we call here the Reduction Lemma (if R is semiprime then K, as a Jordan triple system, is nondegenerate), which we lightly generalize, and also our Beautiful Partner Lemma, which associates a regular skew element of zero square with another one of the same characteristics. The third section develops a technique which allows to carry results of prime rings to K, building on the fact that if R is prime then the subring generated by K is prime and contains an ideal of R, except if $[K, K] = 0$ or, equivalently, if R is commutative or $\widehat{R}$ is a quaternion algebra over $\mathcal{C}$ with an involution of the first kind and transpose type (for whose proof we will need to introduce some concepts from PI theory). Most of the facts relevant for this section were already known separately, but we feel that some knowledge has been gained by drawing them altogether. We devote the last section to show that if

$[K, K] = 0$ then we do not really need to resort to the fundamental theorems of PI theory to elucidate the structure of R, even for an arbitrary ring: in that case R always satisfies Hall Identity, a polynomial identity of degree 5 which is satisfied by quaternion algebras. We prove this result by elementary combinatorial means.

1.1 Involutions

Given a ring R, an *involution* $*$ on R is an additive map $* : R \to R$ such that $(a^*)^* = a$ and $(ab)^* = b^*a^*$, i.e., $*$ is antiautomorphism of R of order (one or) two. Observe that the identity map will be an involution if and only if R is commutative. If R is an algebra over a ring of scalars with involution $(\Phi, ^-)$, then an involution $*$ on R is an involution on the underlying ring of R which in addition satisfies $(\lambda a)^* = \bar{\lambda} a^*$ for every $\lambda \in \Phi$ and $a \in R$. If $^-$ is trivial (i.e., if it is the identity map) then $*$ is just an involution of R as a ring which is also a linear map. By abuse of notation, when appropriate we will also denote by $*$ the involution on the ring of scalars. Assume for the remaining of this section that R is a ring with an involution $*$.

An element $a \in R$ is said to be *symmetric* if $a^* = a$, *skew*[1] if $a^* = -a$. Clearly, if $2 \in TF(R)$ then the only element which is simultaneously symmetric and skew is 0. The addition of symmetric elements is symmetric, while the addition of skew elements is skew. Moreover, opposites also respect the symmetric and skew behaviors. This shows that the set of symmetric elements, $\mathrm{Sym}(R, *)$, and the set of skew elements, $\mathrm{Skew}(R, *)$, are subgroups of R which have trivial intersection (if $2 \in TF(R)$). Observe that $x + x^*, xx^*, x^*x \in \mathrm{Sym}(R, *)$ for every $x \in R$, while $x - x^* \in \mathrm{Skew}(R, *)$ for every $x \in R$. In addition, when k is skew we have that $(k^2)^* = (k^*)^2 = (-k)^2 = k^2$ and thus k^2 is symmetric. For unital rings 1 is always symmetric.

If Γ is commutative then it is a direct exercise to see that $*$ can be extended to Γ, defining $\lambda^* \in \Gamma$ by $\lambda^* x := (\lambda x^*)^*$ for every $x \in R$. Since Γ is a unital ring, this extension

[1] These elements are usually called *skew-symmetric*, we have chosen to shorten the name.

of $*$ is an antiautomorphism of unital rings and thus the homomorphic image Z of $\mathbb{Z}$ inside Γ is symmetric; moreover, if some element of Z is invertible in Γ, then its inverse is also symmetric. If R is a semiprime ring, then $*$ can be extended to $\mathcal{C}$ in a similar way, and hence to $\widehat{R} = \mathcal{C}R + \mathcal{C}$. In fact, as we show now, the involution can be extended up to the symmetric Martindale ring of quotients ([RingsGIs, Proposition 2.5.4])[2].

Proposition 1.1.1 (Involutions extend to the symmetric ring of quotients). *Let R be a semiprime ring with involution $*$. Then $*$ extends to $Q_s(R)$.*

Proof. Pick $q \in Q_s(R)$ and an essential ideal I such that $qI + Iq \subseteq R$. Since $*$ is an antiautomorphism, I^* is also an essential ideal. We define $f : I^* \to R$ such that $f(x) := (x^*q)^*$. If $y \in R$ and $x \in I^*$, then $f(xy) = ((xy)^*q)^* = ((y^*x^*)q)^* = (y^*(x^*q))^* = (x^*q)^*y = f(x)y$, so f is a right-module homomorphism. Hence $q^* := \{f, I^*\}$ is well defined and lies in $Q(R)$. Let us show that for every $x \in I$ we have

$$q^*x^* = (xq)^*. \qquad \textbf{(1)}$$

In $Q(R)$, q^*x^* means $\{f, I^*\}\{l_{x^*}, R\} = \{fl_{x^*}, I^*\}$. Now for every $y \in I^*$ we have $fl_{x^*}(y) = f(x^*y) = f(x^*)y = ((x^*)^*q)^*y = (xq)^*y = l_{(xq)^*}(y)$. Hence $q^*x^* \equiv \{fl_{x^*}, I^*\} = \{l_{(xq)^*}, I^*\} \equiv (xq)^*$. Let us see now that for every $x \in I$ we have

$$x^*q^* = (qx)^*. \qquad \textbf{(2)}$$

Fix $x \in I$. By **(1)** we get $x^*q^*y^* = x^*(yq)^* = (yqx)^* = (qx)^*y^*$ for every $y \in I$. Therefore $x^*q^* - (qx)^* \in \mathrm{Ann}_{Q(R)}(I^*) = 0$ and $x^*q^* = (qx)^*$.
Since for every $x \in I^*$ we have $q^*x = (x^*q)^* \in R$ and $xq^* = (qx^*)^* \in R$, I^* is an essential ideal of R such that $q^*I^* + I^*q^* \subseteq R$ and therefore $q^* \in Q_s(R)$ by definition. This shows that $*$ extends to $Q_s(R)$. Since the map f is a composition of three additive maps, $*$ is itself an additive map. Let us see that for every $p \in Q_s(R)$ it satisfies

$$(p^*)^* = p.$$

[2]Actually, the proof can be adapted to show that any antiautomorphism of R can be extended to $Q_s(R)$.

Consider the ideal $J := I \cap I^*$, which is essential because is the intersection of two essential ideals[3]. In addition if $x \in J$ then $x^* \in J$. Then for every $x \in J$ we have, by **(1)** and **(2)**, $(q^*)^* x^* = (xq^*)^* = ((qx^*)^*)^* = qx^*$ since $qx^* \in R$ as $x^* \in J \subseteq I$. This implies that $(q^*)^* = q$ since $\mathrm{Ann}_{Q_s(R)} J = 0$.
Last, we prove that for every $p, q \in Q_s(R)$ it is

$$(pq)^* = q^* p^*.$$

Choose $p, q \in Q_s(R)$ and take an essential ideal J such that $pJ, qJ, Jp, Jq, pqJ, Jpq \in R$ (which exists because the intersection of essential ideals is essential). Consider J^2, which is also essential[4]. Observe that for all $x \in J^2$ we have $xpq \in J^2 pq = J(Jpq) \subseteq JR \subseteq J$ and similarly $xp \in J$. Then, by **(1)**,

$$(pq)^* x^* = (xpq)^* = q^*(xp)^* = q^* p^* x^*.$$

This implies that $(pq)^* = q^* p^*$ since $\mathrm{Ann}_{Q_s(R)}((J^2)^*) = 0$. $\square$

We will always denote with the same symbol the involution on R and its extension to $Q_s(R)$.

Let R be a prime ring. The involution is said to be *of the first kind* if its extension is trivial on $\mathcal{C}$ (i.e., if $\mathrm{Sym}(\mathcal{C}, *) = \mathcal{C}$), *of the second kind* if it is not trivial on $\mathcal{C}$, equivalently, if there exists $0 \neq \lambda \in \mathrm{Skew}(\mathcal{C}, *)$, for if $\mu \in \mathcal{C}$ is such that $\mu^* \neq \mu$, then $0 \neq \mu - \mu^* \in \mathrm{Skew}(\mathcal{C}, *)$. If $*$ is of the first kind, then it can also be extended to $\overline{R} = \widehat{R} \otimes_{\mathcal{C}} \overline{\mathcal{C}}$ by the rule $(x \otimes \lambda)^* := x^* \otimes \lambda$.

If $e \in R$ is an idempotent then e^* is also an idempotent, since the involution is an antiautomorphism. An idempotent will be said to be left (resp. right) *isotropic* if $e^* e = 0$ (resp. $ee^* = 0$). An idempotent is $*$-*orthogonal* if it is left and right isotropic.

[3]If I, J are essential ideals and K is any nonzero ideal, then $K \cap (I \cap J) = (K \cap I) \cap J$, which is not zero because J is essential and $K \cap I$, being I essential, is a nonzero ideal.

[4]Let I be an essential ideal and J be any nonzero ideal. Then $I \cap J \neq 0$ implies $(I \cap J)^2 \neq 0$ since R is semiprime and, since $(I \cap J)^2 \subseteq I^2 \cap J$, we conclude that I^2 is essential.

If R is prime with socle, then the involution is said to be of *symplectic type* whenever every minimal idempotent is $*$-orthogonal. If there exists a minimal symmetric idempotent then the involution is said to be *of transpose type*. These two cases are mutually exclusive ([RingsGIs, Theorem 4.6.2]).

It is possible a characterization of involutions of transpose and symplectic type which explains their names (here we are following [Martindale&Miers'91, page 1047]). If R is a prime ring with socle which is not commutative and which has an involution of the first kind, then (by the geometric model of Appendix A) it turns out that $\overline{R}$ always contains a symmetric idempotent e such that $e\overline{R}e \cong \mathbb{M}_2(\overline{\mathcal{C}})$ as $\overline{\mathcal{C}}$-algebras. Since $e\overline{R}e$ inherits the involution, by choosing matrix units properly it can be shown (see [RingsGIs, pages 163 to 168]) that $*$ in $e\overline{R}e$ is either the usual transpose for matrices (denoted by ^{T} henceforth) or the symplectic involution for matrices, which we define below. Then the involution on R is termed of *transpose* or *symplectic type* accordingly.

Definition 1.1.2 (**Symplectic involution for** 2×2 **matrices**).
Let F be a field, consider $R := \mathbb{M}_2(F)$ and denote $s := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then the map $* : R \to R$ such that $a^* := sa^Ts^{-1}$ is an involution called the *symplectic* involution.

Hence $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ for the symplectic involution[5]. A direct computation reveals the patterns for skew and symmetric elements for 2×2 matrices with either involution.

Lemma 1.1.3 (**Structure of skew and symmetric matrix elements**).
Let F be a field with $\mathrm{char}(F) \neq 2$ *and* $R := \mathbb{M}_2(F)$.

[5]A similar definition can be given for $\mathbb{M}_{2n}(F)$ for every $n \geq 1$: just replace the 1's in the definition of s by the identity matrix I_n. Then the application of the involution produces a similar pattern, being a, b, c, d blocks of $n \times n$ matrices in this case. In this dissertation only the 4-dimensional case concerns us in an explicit manner.

1. For the transpose involution,

$$\mathrm{Sym}(R,{}^T) := \left\{ \begin{pmatrix} x & y \\ y & z \end{pmatrix} \mid x, y, z \in F \right\}, \mathrm{Skew}(R,{}^T) := \left\{ \begin{pmatrix} 0 & x \\ -x & 0 \end{pmatrix} \mid x \in F \right\}.$$

2. For the symplectic involution,

$$\mathrm{Sym}(R,*) := \left\{ \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \mid x \in F \right\}, \mathrm{Skew}(R,*) := \left\{ \begin{pmatrix} x & y \\ z & -x \end{pmatrix} \mid x, y, z \in F \right\}.$$

We can see that the transpose case presents symmetric elements of rank 1 (those nonzero matrices with $xz = y^2$), in particular minimal symmetric idempotents (e.g. $x = 1$, $y = 0 = z$), while all the nonzero symmetric elements are of rank 2 in the symplectic case (since their determinant is x^2). We will show in Proposition 1.1.8 that the involution is of transpose type if and only if R has a minimal symmetric element[6]. There is a similar skew counterpart: in the transpose case all the nonzero skew elements are of rank 2 (with determinant x^2), while the symplectic case presents skew elements of rank 1 (those with $yz = -x^2 \neq 0$) although obviously no skew idempotents (the square of a skew element is symmetric)[7]. Other useful observations are the following ones about commutativity, which are also proved by a direct check.

Lemma 1.1.4 (Commutativity results in $\mathbb{M}_2(F)$ with an involution).

1. In the symplectic case:

- a) *All the symmetric elements are in the center.*
- b) *No nonzero skew element is in the center.*
- c) *The squares of all the skew elements are in the center.*
- d) *No nonzero skew element commutes with all the skew elements.*

2. In the transpose case:

- a) *Some symmetric elements are in the center (those with $y = 0$, $z = x$).*
- b) *No nonzero skew element is in the center.*

[6]Recall that we call an element *minimal* whenever it generates a minimal right ideal, equivalently in semiprime rings, a minimal left ideal.

[7]It does present minimal nilpotent elements, for example the matrix with $x = 0 = z, y = 1$.

c) *The squares of all the nonzero skew elements are nonzero and in the center.*

d) *All the skew elements commute with each other, with products lying in the center.*

1.1.1 Geometric model for prime rings with socle and involution

As is well known (refer to Appendix A), prime rings with socle possess a geometric model in terms of dual pairs of vector spaces that allows to pose complicated calculations with the aid of the powerful tools of linear algebra. Informally speaking, if a prime ring with socle has in addition an involution, then in its geometric model (V, W) we may take $W = V$ (in what is called a selfdual vector space) and then the involution can be realized as the adjoint involution of $\mathcal{L}_V(V)$. Moreover the kind of the inner product (which, more generally in this case, is a sesquilinear form) is determined by the type of the involution. This is the important Kaplansky Theorem. Before presenting it officially, we need to consider some different kinds of sesquilinear form.

Definitions 1.1.5 (**Sesquilinear forms**).
Let $(\Delta, ^-)$ be a division ring with involution and let V be a left vector space over Δ. Then a *sesquilinear form* is a biadditive map $\langle \cdot\,, \cdot\rangle : V \times V \to \Delta$ such that $\langle \alpha v, \beta w\rangle = \alpha\langle v, w\rangle\bar{\beta}$. The form is called:

- *Nondegenerate* if $\langle v, V\rangle = 0$ forces $v = 0$ and $\langle V, v\rangle = 0$ forces $v = 0$.
- *Symmetric* if $^-$ is the trivial involution and $\langle w, v\rangle = \langle v, w\rangle$. This forces Δ to be a field.
- *Alternate* if $\mathrm{char}(\Delta) \neq 2$, $^-$ is the trivial involution and $\langle w, v\rangle = -\langle v, w\rangle$.

  This forces Δ to be a field and $\langle v, v\rangle = 0$.
- *Hermitian* if $\langle w, v\rangle = \overline{\langle w, v\rangle}$.

  Note that symmetric forms are a special kind of hermitian forms.
- *Skew*[8] if $\langle w, v\rangle = -\overline{\langle w, v\rangle}$.

  Note that alternate forms are a special kind of skew forms.

[8] These sesquilinear forms are usually called *skew-hermitian*, we have chosen to shorten the name.

For symmetric and alternate forms, Δ is a field by the following argument: denote $\epsilon := 1$ if the form is symmetric and $\epsilon := -1$ if it is alternate, and pick $\alpha, \beta \in \Delta$ and $v, w \in V$ such that $\langle v, w\rangle \neq 0$. Then $\alpha\beta\langle v, w\rangle = \alpha\langle \beta v, w\rangle = \epsilon\alpha\langle w, \beta v\rangle = \epsilon\langle \alpha w, \beta v\rangle = \epsilon^2\langle \beta v, \alpha w\rangle = \beta\langle v, \alpha w\rangle = \epsilon\beta\langle \alpha w, v\rangle = \epsilon\beta\alpha\langle w, v\rangle = \epsilon^2\beta\alpha\langle v, w\rangle = \beta\alpha\langle v, w\rangle$. Multiplying by the inverse of $\langle v, w\rangle$ at the right we get $\alpha\beta = \beta\alpha$.

Definition 1.1.6 (Selfdual vector space).
Let $(\Delta, ^-)$ be a division ring with involution. A *selfdual vector space* V is a left vector space over Δ equipped with a nondegenerate hermitian or skew form.

With this, more general definition (in the sesquilinear form used) than that of dual pair, it seems that the previous theory is lost. But this is not the case; if V is a selfdual vector space we can consider the dual pair (V, V) with V defined as a right vector space over Δ by the action $v\alpha := \bar{\alpha}v$. In this way $\langle \alpha v, w\beta\rangle = \langle \alpha v, \bar{\beta}v\rangle = \alpha\langle v, w\rangle\beta$ and $\langle \cdot\, , \cdot\rangle$ is a nondegenerate inner product for (V, V), which is indeed a dual pair. We denote $\mathcal{L}(V) := \mathcal{L}_V(V)$ and $\mathcal{F}(V) := \mathcal{F}_V(V)$. Observe that the adjoint map $^\# : \mathcal{L}(V) \to \mathcal{L}(V)$ is an involution.

Although selfdual vector spaces are defined with either a hermitian or a skew form, this is not a real dichotomy for our purposes. Suppose that V is a selfdual space over $(\Delta, ^-)$ equipped with a hermitian form $\langle \cdot\, , \cdot\rangle$. If the form is not symmetric, then there exists $\alpha \in \Delta$ such that $\bar{\alpha} \neq \alpha$. If we denote $\beta := \alpha - \bar{\alpha} \neq 0$, we get that the map $^\beta : \Delta \to \Delta$ such that $\lambda^\beta := \beta^{-1}\bar{\lambda}\beta$ is an involution with $\beta^\beta = -\beta$. Then the form $\langle \cdot\, , \cdot\rangle_\beta := \langle \cdot\, , \cdot\rangle\beta$ has the following properties:

1. It is skew with respect to the involution $^\beta$.

   This is because
   $(\langle v, w\rangle_\beta)^\beta = (\langle v, w\rangle\beta)^\beta = \beta^\beta\langle v, w\rangle^\beta = -\beta\beta^{-1}\overline{\langle v, w\rangle}\beta = -\langle w, v\rangle\beta = -\langle w, v\rangle_\beta$.
2. Clearly, the adjoint map is kept invariant in the pass from $\langle \cdot\, , \cdot\rangle$ to $\langle \cdot\, , \cdot\rangle_\beta$.

Observe that $\langle \cdot\, , \cdot\rangle_\beta$ can never be symmetric with respect to the involution $^\beta$, since $\beta^\beta = -\beta \neq 0$ and thus $^\beta$ is not trivial on Δ.

Analogously, if V is equipped with a skew form which is not alternate, then after a change of involution we can equip V with a hermitian form which does not change the adjoint map.

Hence, since we are mainly interested in modeling, inside the geometric model, the involution of a ring by means of the adjoint map, and we do not really care about the specific shape of the associated sesquilinear form, we may say that

If V is a selfdual space then it is either equipped with an alternate form or a hermitian form. Alternatively, V is either equipped with a symmetric form or a skew form.

We may also say that V is equipped either with a symmetric form, an alternate form, or a form which can be hermitian or skew, but which has an associated involution in Δ which is not trivial.

We are now prepared to present the celebrated Kaplansky Theorem (see [RingsGIs, Theorem 4.6.8]).

Theorem 1.1.7 (Kaplansky Theorem).
Let R be a prime ring with socle and involution $*$ such that $\mathrm{char}(R) \neq 2$. Then there exists a selfdual vector space V over Δ such that $\mathcal{F}(V) \subseteq R \subseteq \mathcal{L}(V)$ and $*$ is the adjoint map restricted to R. Moreover, if $*$ is of transpose type then V can be equipped with a hermitian form, while if $*$ is of symplectic type then V can be equipped with an alternate form and Δ is a field.

Observe that if $*$ is of transpose type then it may be the case that V can be equipped with a symmetric form. In that case Δ would also be a field. Such an involution $*$ will be said to be *of orthogonal type*[9]. If $*$ is of transpose type and e is a minimal symmetric

[9]Our definitions of involution of symplectic and orthogonal type are more restrictive than the ones usually given for finite-dimensional central simple algebras. For them it is said (see [BookInvolutions, Definition 2.5]) that an involution of the first kind is *orthogonal* (*symplectic*) if, when extended by scalars to a splitting field, the corresponding sesquilinear form is symmetric (alternate). With our definition, a division ring cannot be endowed with an involution of symplectic type, since its only

idempotent, then $*$ is of orthogonal type if the restriction of $*$ to eRe is trivial[10].

If the involution is of transpose type but not of orthogonal type, we follow loosely [BookInvolutions, Page 2] and say that it is *of unitary type*. Note that the division ring of an involution of unitary type may be a field, while $^-$ is not trivial on it. For example, the usual conjugation $^-$ on $\mathbb{C} \cong \mathrm{End}_{\mathbb{C}}(\mathbb{C}) = \mathfrak{F}(\mathbb{C})$ is a unitary involution where the underlying division ring $\mathbb{C}$ is a field such that when equipped also with $^-$ gives the associated hermitian form $\langle x, y\rangle := x\bar{y}$.

Lastly, it should not be misunderstood that an involution of the first kind must be either orthogonal or symplectic[11]: under our conventions, an involution is of the first kind if its extension to the extended centroid is trivial, but the extended centroid is not necessarily Δ, but is isomorphic to its center (by Theorem A.0.4(5)).

As promised after Lemma 1.1.3, we are going to see that in the characterization of involutions (in the transpose/symplectic classification) the elements do not have to be idempotent.

Proposition 1.1.8 (**Characterization of involutions by elements**).

Let R be a prime ring with socle, $\mathrm{char}(R) \neq 2$ and involution $*$.

The following conditions are equivalent:

i) *The involution is of transpose type.*

ii) *R has a minimal symmetric element.*

---

nonzero idempotent is 1 and thus it does not contain nonzero $*$-orthogonal idempotents, while if it is equipped with an involution of orthogonal type then it must be a field and the involution must be trivial. In contrast, with the definition coming from central simple algebras, any involution on any division ring which acts as the identity on its center is either orthogonal or symplectic. An specific example of a ring with involution of symplectic type in this second sense which is not of symplectic type in our sense is the quaternions with their usual conjugation. Those more relaxed notions of symplectic and orthogonal type correspond in our convention, respectively, to involutions of the first kind and symplectic or transpose type.

[10] This can be checked by analyzing the proof of [RingsGIs, Theorem 4.6.5] and observing that the involution on eRe remains unchanged.

[11] This assertion is indeed true for the usual definition of involution types in central simple algebras.

iii) *There exists a minimal element $a \in R$ such that $a^*a \neq 0$.*

Proof. Let $*$ denote the involution of R. By the dichotomy of involutions $*$ is either of transpose type or of symplectic type. Suppose first that the involution is of transpose type. By definition there exists a minimal symmetric idempotent $e \in R$. This proves **i)** $\Rightarrow$ **ii)**. Moreover, $ee^* = ee = e^2 = e \neq 0$. This proves **i)** $\Rightarrow$ **iii)**.

Now suppose that the involution is of symplectic type. By Kaplansky Theorem (1.1.7) we can see R as a ring of operators of a selfdual space V over a field F equipped with an alternate bilinear form $\langle \cdot, \cdot \rangle$. A minimal element is then an operator of rank one, and every rank-one continuous operator can be written as $u \otimes v$ for some $u, v \in V$, since $\mathcal{F}(V) = V \otimes V$ by Lemma A.1.2. We claim that $(u \otimes v)^* = -v \otimes u$. To see this we have to prove that $v \otimes u$ is the adjoint operator to $u \otimes v$. Pick $x, y \in V$; then

$$\langle u \otimes v(x), y \rangle = \langle \langle x, u \rangle v, y \rangle = \langle x, u \rangle \langle v, y \rangle = -\langle x, u \rangle \langle y, v \rangle = -\langle x, \langle y, v \rangle u \rangle = \langle x, -v \otimes u(y) \rangle.$$

So, a symmetric minimal element must satisfy $u \otimes v = -v \otimes u$; when applied to $x \in V$ this identity produces $\langle x, u \rangle v = -\langle x, v \rangle u$, and since $\langle \cdot, \cdot \rangle$ is nondegenerate this means that $v = \alpha u$ for some $\alpha \in F$. Then it must be $u \otimes \alpha u = u \otimes v = -v \otimes u = -(\alpha u) \otimes u = -u\alpha \otimes u = -u \otimes \alpha u$ and, since $\mathrm{char}(R) \neq 2$, this implies $u \otimes \alpha u = 0$. Therefore in this case R has no symmetric minimal elements, which shows **ii)** $\Rightarrow$ **i)**. In addition, since $(u \otimes v)^* = -v \otimes u$ we get that $(u \otimes v)^*(u \otimes v) = (-v \otimes u)(u \otimes v) = -u \otimes (v \otimes u(v))$ by the Absorption Law 1 (A.1.2(4)), and this gives $-u \otimes (v \otimes u(v)) = -u \otimes \langle v, v \rangle u = 0$ due to $\langle v, v \rangle = 0$ because $\langle \cdot, \cdot \rangle$ is alternate. This proves **iii)** $\Rightarrow$ **i)** and finalizes the proof. $\square$

For convenience, we include a look-it-up chart with the relationship between sesquilinear forms and involutions.

Involution type	**Bilinear form**	**Division ring**	**Elements** (*)
Transpose: orthogonal	Symmetric	$F = \mathrm{Sym}(F, ^-)$	$\exists a = a^*$ minimal
Transpose: unitary	Hermitian or skew	$\Delta \neq \mathrm{Sym}(\Delta, ^-)$	$\exists a = a^*$ minimal
Symplectic	Alternate	$F = \mathrm{Sym}(F, ^-)$	$a^*a = 0 \ \forall a$ minimal

(*) The elements a can be taken idempotent.

1.2 $K := \mathrm{Skew}(R, *)$

Let R be a ring with involution $*$. We denote the subgroups of skew and symmetric elements of R as

$$K := \mathrm{Skew}(R, *) = \{x \in R \mid x = -x^*\} \text{ and } H := \mathrm{Sym}(R, *) = \{x \in R \mid x = x^*\}.$$

From this chapter on we will always assume that if R is a ring with involution then K and H are as above and $\frac{1}{2} \in \Gamma$. Since $(\frac{1}{2})^* = \frac{1}{2}$ in the extension of the involution to Γ (see Section 1.1), we get that $\frac{1}{2}K = K$ and $\frac{1}{2}H = H$. In this way we may decompose any element $x \in R$ as $x = x_h + x_k$, with $x_h := \frac{1}{2}(x + x^*) \in H$ and $x_k := \frac{1}{2}(x - x^*) \in K$, so that

$$R = H \oplus K.$$

Hence, for any $x \in R$, we will also assume that x_h and x_k mean the same as above.

Sometimes in later sections we will talk, generically and simultaneously, about R and K. In those occasions it will be understood that K are the skew elements of some ring with involution. So we will make informal statements like 'It is true for R centrally closed prime and for K with R as before', meaning 'It is true for every centrally closed prime ring and every Lie algebra of the skew elements of a centrally closed prime ring with involution'.

When working in the central closure we will denote $\widehat{K} := \mathrm{Skew}(\widehat{R}, *)$ and $\widehat{H} := \mathrm{Sym}(\widehat{R}, *)$. Recall that the involution of a prime ring may be of the first or of the second kind. The following well-known lemma shows that $\widehat{K}$ is well behaved with respect to K for the first kind, and that dealing with the second kind in $\widehat{R}$ is usually easy.

Lemma 1.2.1 (Results on $\widehat{K}$).

Let R be a prime ring with involution.

1. *If the involution is of the first kind, then $\widehat{K} = \mathcal{C}K$.*
2. *If $0 \neq \lambda \in \mathrm{Skew}(\mathcal{C}, *)$ then $\widehat{R} = \widehat{K} \oplus \lambda\widehat{K}$.*

Proof. Recall that $\widehat{R} = \mathcal{C}R + \mathcal{C}$.

1. Suppose $a \in \widehat{K}$ is such that $a = \lambda b + \mu$, with $b \in R$ and $\lambda, \mu \in \mathcal{C}$. Then $a = -a^*$ implies that $\lambda b + \mu = -\lambda b^* - \mu$, and therefore $2\lambda b_h = \lambda(b + b^*) = -2\mu$. If $\lambda = 0$ then $\mu = 0$ and $a = 0$; otherwise $b_h = -\lambda^{-1}\mu$. In this case we have $b = -\lambda^{-1}\mu + b_k$. Thus $a = \lambda b + \mu = \lambda(-\lambda^{-1}\mu + b_k) + \mu = \lambda b_k$. In consequence $\widehat{K} \subseteq \mathcal{C}K$. But clearly $\mathcal{C}K \subseteq \widehat{K}$.
2. As $\widehat{R} = \widehat{K} \oplus \widehat{H}$, all we need to show is that $\widehat{H} = \lambda\widehat{K}$. Pick $h \in \widehat{H}$, $k \in \widehat{K}$. Since $(\lambda k)^* = k^*\lambda^* = \lambda^* k^* = \lambda k$ we have $\lambda\widehat{K} \subseteq \widehat{H}$. Since $h = \lambda(\lambda^{-1}h)$ and $(\lambda^{-1}h)^* = -\lambda^{-1}h$, we get $\widehat{H} \subseteq \lambda\widehat{K}$. □

From elements $x \in R$ we can easily get elements in K and H, since $x = x_h + x_k$. We denote $\tau(x) := x + x^* \in H$ and $\kappa(x) := x - x^* \in K$, and call them, respectively, the *symmetric trace* and the *skew trace*. Then $x_h = \frac{1}{2}\tau(x)$ and $x_k = \frac{1}{2}\kappa(x)$. Note that κ and τ are additive as maps and that $\kappa^*(x) := (x - x^*)^* = x^* - x = \kappa(x^*) = -\kappa(x)$. We say that a subgroup M of R is *selfadjoint* if and only if $M^* = M$. We have under our belt at least two ways of constructing a selfadjoint subgroup from any subgroup M, namely $\mathrm{Skew}(M,*) := M \cap K$ and $\kappa(M) := \{\kappa(m) \mid m \in M\}$. The first one reduces the set to those elements which are already skew, and is therefore contained in M, while the second one essentially takes the skew part of every element, and therefore is not contained in M if M is not selfadjoint. Suppose $2M = M$. We always have $\mathrm{Skew}(M,*) \subseteq \kappa(M)$, for if $k \in \mathrm{Skew}(M,*)$ then $2k = \kappa(k) \in \kappa(M)$. If in addition M is selfadjoint, then we also have $\kappa(M) \subseteq \mathrm{Skew}(M,*)$ and the two notions coincide, because $\kappa(x) = x - x^* \in M \cap K = \mathrm{Skew}(M,*)$ for every $x \in M$ (in particular $\kappa(R) = K$). If M is not selfadjoint we may ask how far apart are the two constructions from one another. The fact that $\kappa(-x) = \kappa(x^*)$ produces the observation $\kappa(M) = \kappa(M^*)$, which implies that $\kappa(M) = 2\kappa(M) = \kappa(M) + \kappa(M) = \kappa(M) + \kappa(M^*) = \kappa(M + M^*)$. Since $2(M + M^*) = M + M^*$ and $M + M^*$ is selfadjoint, we get $\kappa(M) = \mathrm{Skew}(M + M^*, *)$.

We can also construct elements in K and H taking into account that some common operations of skew and symmetric elements lie always inside K, while some others lie always inside H. The following ones, which are useful in this section, are confirmed by a simple check. We denote $[a, b, c] := abc - cba$. Recall that $a \circ b := ab + ba$.

Lemma 1.2.2 (Operations in K and in H).
Let $k, k_1, k_2 \in K$ and $h, h_1, h_2 \in H$.

1. *In K we have:*
   $k \circ h$, $[k_1, k_2]$, $[h_1, h_2]$, $P_k h$, $\{k, k_1, k_2\}$, $\{k, h_1, h_2\}$, $\{h_1, k, h_2\}$, $[h, k_1, k_2]$, $[k_1, h, k_2]$.
2. *In H we have:*
   k^2, $h_1 \circ h_2$, $k_1 \circ k_2$, $[h, k]$, $P_h k$, $\{h, h_1, h_2\}$, $\{h, k_1, k_2\}$, $\{k_1, h, k_2\}$, $[k_1, k_2, k_3]$, $[k, h_1, h_2]$, $[h_1, k, h_2]$.

A practical hint (actually a nonrigorous simultaneous proof) is the following:
These operators are either symmetric or antisymmetric. To find if one of them lies in K or in H it is enough to do a simple computation. Say that $s := 0$ if the operator is symmetric (i.e., either $(\cdot)^2$, $\circ$, p, $\{\cdot, \cdot\}$ or $\{\cdot, \cdot, \cdot\}$) and say that $s := 1$ if the operator is skew (i.e., either $[\cdot, \cdot]$ or $[\cdot, \cdot, \cdot]$); count the number N_k of skew elements inside the operator, and compute

$$S := (-1)^{N_k + s}.$$

If $S = +1$ then the resulting element lies in H, while if $S = -1$ the resulting element lies in K. For example $[h, k_1, k_2]$ has $S = (-1)^{2+1} = -1$, so the element is in K. We ought to take care with $(\cdot)^2$ and p, since they carry respectively *two* and *three* elements inside.

Note that we may take combinations of the operators, e.g., as in $[k^2, h] \in K$ (because $N_k = 2, s_1 = 1, s_2 = 0$, $S = (-1)^3 = -1$). Observe also that we have

$$h \circ k = hk + kh = hk - (hk)^* = \kappa(hk) = \kappa(kh),$$

a simple identity that will show up often in Chapter 4.

We should make the observation, since K is closed for $[\cdot, \cdot]$ and $\{\cdot, \cdot, \cdot\}$, that it inherits from R the structure of a Lie ring, and the structure of a Lie algebra and of a

Jordan triple system over $\mathrm{Sym}(\Phi, *)$ if R is an algebra over $(\Phi,{}^*)$. In particular, if R is a centrally closed prime ring then K is a Lie algebra over $\mathrm{Sym}(\mathcal{C}, *)$, which equals $\mathcal{C}$ if the involution is of the first kind.

If R is a graded algebra with a selfadjoint grading, then K, as a Lie algebra, is graded too.

Lemma 1.2.3 (**Inheritance of gradings**).
Let $R := \bigoplus_{g\in G} R_g$ *be an associative or Lie grading with* R_g *selfadjoint for every* $g \in G$.
Then $K := \bigoplus_{g\in G} K_g$ *is a Lie grading, with* $K_g := R_g \cap K$.

Proof. By construction it is obvious that the K_g are $\mathrm{Sym}(\Phi, *)$-modules, that $\sum_{g\in G} K_g \subseteq K$ and that the sum is direct. Let $k \in K$ be decomposed in homogeneous components as $k = \sum_{g\in G} k_g$. Then $-k = \sum_{g\in G} k_g^*$ and, since $K_g^* = K_g$ for every $g \in G$, $-k_g^*$ is the homogeneous component of degree g of k, i.e., $k_g^* = -k_g$ and $k_g \in R_g \cap K = K_g$. Therefore $K \subseteq \bigoplus_{g\in G} K_g$.
Now observe that $[K_g, K_{g'}] \subseteq [R_g, R_{g'}] \cap [K, K] \subseteq R_{g+g'} \cap K = K_{g+g'}$. □

As we know, in prime rings it is satisfied the useful condition that $aRb = 0$ implies $a = 0$ or $b = 0$, which may be used to reduce larger identities to shorter ones. Similarly, in semiprime rings we have got that $aRa = 0$ implies $a = 0$. For K we have analogous results at our disposal. In particular, if R is centrally closed prime and the involution is of the second kind, then $aKa = 0$ implies $a = 0$: by Lemma 1.2.1(2) we have $R = K \oplus \lambda K$ for every $0 \neq \lambda \in \mathrm{Skew}(\mathcal{C}, *)$, so that $aRa = aKa + \lambda aKa = 0$ and $a = 0$ because R is prime.
For involutions of the first kind, and for semiprime rings, we can resort to the following lemmas (item (1) is [TopicsRingTheory, Remark on page 43][12], item (2) is a generalization of [Martindale&Miers'91, Lemma 5] and item (3) will appear in [Brox,García&Gómez(2)]).

[12]There it is proved by invoking Levitzki Lemma, [TopicsRingTheory, Lemma 1.1].

Lemma 1.2.4 (**Reduction Lemmas**).

Let R be a semiprime ring with involution. Let $k \in K$ and $0 \neq h \in H$.

1. *$kKk = 0$ implies $k = 0$.*
2. *$hKh = 0$ implies $0 \neq hRh \subseteq \mathcal{C}h$ in $\widehat{R}$.*
3. *If $I(h)$ is essential, then $hKh = 0$ and $hKk = 0$ imply $k = 0$.*

In addition, if R is prime and $hKh = 0$, then $h\widehat{R}h = \mathcal{C}h$ and $\mathcal{C}R$ and $\widehat{R}$ have socle and involution of orthogonal type.

Proof.

1. Pick $x \in R$. Note that $k\kappa(x)k = 0$, so that $kxk = kx^*k$. Then

$$k(xkx)k = k(xkx)^*k = -kx^*kx^*k = -(kx^*k)x^*k = -kxkx^*k = -kx(kx^*k) = -kxkxk$$

and since $\mathrm{char}(R) \neq 2$ we have $kxkxk = 0$. Then it is also true that $kxkxkyk = 0$ for every $y \in R$. Hence

$$0 = -kxk(xky)k = -kxk(xky)^*k = kxky^*kx^*k = kxkykxk,$$

so $(kxk)R(kxk) = 0$ and $kxk = 0$ since R is semiprime. Now $kRk = 0$ implies, again by semiprimeness, that $k = 0$.

2. Pick $x, y \in R$. Note that $h\kappa(x)h = 0$ and therefore $hxh = hx^*h$. Then

$$0 = h\kappa(xhy)h = h(xhy - (xhy)^*)h = hxhyh - hy^*hx^*h =$$

$$= hxhyh - (hy^*h)x^*h = hxhyh - hy(hx^*h) = hxhyh - hyhxh = (hxh)yh - hy(hxh).$$

By Martindale Lemma for semiprime rings (Theorem 0.4.4), since $h \neq 0$ and $I(hxh) \subseteq I(h)$, we get $hxh = \lambda_x h$ for every $x \in R$. Hence $hRh \subseteq \mathcal{C}h$ and, being $h \neq 0$, it cannot be $hRh = 0$ since R is semiprime.

3. Pick $x, y \in R$. Note that we have $h\kappa(x)h = 0 = h\kappa(x)k$ and therefore $hx^*k = hxk$ and $kx^*h = -(hxk)^* = -(hx^*k)^* = kxh$. Then

$$0 = h(xky - (xky)^*)h = hxkyh + hy^*kx^*h = hxkyh + hykxh = (hxk)y(h) + (h)y(kxh).$$

By Martindale Lemma for semiprime rings, since $h \neq 0$ and $I(hxk) \subseteq I(h)$, we get $hxk = \lambda_x h$ for every $x \in R$. Pick now $x \in H$. Then $xkx \in K$ and $h(xkx)k = 0$, hence $0 = (hxk)xk = \lambda_x hxk = \lambda_x^2 h$; but since h is essential, $\lambda_x^2 = 0$, and since $\mathcal{C}$ is a reduced ring, $\lambda_x = 0$. This means that $hxk = \lambda_x h = 0$ for every $x \in H$. Now we have $hHk = 0$ and $hKk = 0$, so that $hRk = h(H+K)k = 0$. Since $I(h)$ is essential and R is semiprime we finally get $k = 0$.

4. Suppose that R is prime and $hKh = 0$. We reason for $\widehat{R}$; the $\mathcal{C}R$ case is analogous. By the observation previous to this lemma we know that the involution must be of the first kind. Then by Lemma 1.2.1(1) we know that $\widehat{K} = \mathcal{C}K$ and hence $h\widehat{K}h = \mathcal{C}hKh = 0$. By item (2) we get that $0 \neq h\widehat{R}h \subseteq \mathcal{C}h$. Pick $a \in \widehat{R}$ such that $hah = \lambda h \neq 0$ with $0 \neq \lambda \in \mathcal{C}$; then $h\mathcal{C}ah = \mathcal{C}h$ since $\mathcal{C}$ is a field. Therefore $h\widehat{R}h = \mathcal{C}h$, i.e., h is a reduced element of $\widehat{R}$. Since $\mathcal{C}$ is a field there exists $a \in \widehat{R}$ such that $hah = h$ and hence $h\widehat{R} = e\widehat{R}$, where $e := ha$ is an idempotent of $\widehat{R}$. Then $e\widehat{R}e = h\widehat{R}ha = \mathcal{C}ha = \mathcal{C}e$, which being a field proves that $e\widehat{R}$ is a minimal right ideal of $\widehat{R}$ ([RingsGIs, Proposition 4.3.3]). This has two direct consequences: $\widehat{R}$ has socle and, by Proposition 1.1.8, the involution is of transpose type since h is a minimal symmetric element. In addition, by the definition of involution of transpose type there exists a symmetric minimal idempotent $f \in \widehat{R}$, with $f\widehat{R}f \cong e\widehat{R}e$ since $\widehat{R}$ is prime ([RingsGIs, Theorem 4.3.7(i)]). This implies that $f\widehat{R}f$ is a subspace of dimension 1 of the $\mathcal{C}$-vector space $\widehat{R}$ and therefore $f\widehat{R}f = \mathcal{C}x$ for any nonzero $x \in f\widehat{R}f$; in particular $f\widehat{R}f = \mathcal{C}f$, which shows that, for every $x \in \widehat{R}$, $(fxf)^* = (\lambda_x f)^* = \lambda_x f = fxf$ since $f^* = f$ and the involution is of the first kind. This implies that the involution is of orthogonal type (see the paragraph after Kaplansky Theorem 1.1.7). □

Note that in a prime ring any nonzero element generates an essential ideal, so item (3) is always valid for prime rings.

Suppose $a \in K$ is a von Neumann regular element of R. Then there exists $b \in R$ such that $aba = a$, which we call a *partner of* a. If we want a partner of a which is also skew, then we may take $b' := b_k$, since $ab_ha + ab_ka = aba = a = -a^* = -(aba)^* = -ab_ha + ab_ka$, so that $0 = ab_h a$ and $a = aba = ab_k a$. If in addition we want a partner c of a such that a is a partner of c, then we may take $c := b'ab'$; it is checked that $ab'a = a$ and $b'ab' = b'$. Note that $c^* = (b'ab')^* = -b'ab' = -c$, so c is still skew. If, in top of all that, a is an element of zero square, then the element $d := c - c^2a$ is a partner of a such that $dad = d$ and $d^2 = 0$. But d is not skew. To find a partner of a which is skew and of zero square we need to work a little more (this result will appear in [Brox,Fernández&Gómez(2)]).

Lemma 1.2.5 (**Beautiful Partner Lemma**).
Let R be a ring with involution and let $a \in K$ (resp. $a \in H$) be a von Neumann regular element such that $a^2 = 0$.
Then there exists $b \in K$ (resp. $b \in H$) such that $aba = a$, $bab = b$ and $b^2 = 0$.

Proof. Denote by $*$ the involution on R. The symmetric and skew proofs are analogous and we develop both at the same time.
Since a is von Neumann regular it is $a = ac'a$ for some $c' \in R$. Denote $c := c'_k a c'_k$ (respectively $c := c'_h a c'_h$). Then $aca = a$, $cac = c$ and $c^* = -c$ (respectively $c^* = c$).
Take $b := c - \frac{1}{2}(ac^2 + c^2a) + \frac{1}{4}ac^3a$. Then $b^* = -b$ (resp. $b^* = b$), $aba = aca = a$,

$$bab = \left(c - \tfrac{1}{2}(ac^2 + c^2a) + \tfrac{1}{4}ac^3a\right) a \left(c - \tfrac{1}{2}(ac^2 + c^2a) + \tfrac{1}{4}ac^3a\right) =$$
$$= (c - \tfrac{1}{2}ac^2)a(c - \tfrac{1}{2}c^2a) = cac - \tfrac{1}{2}cac^2a - \tfrac{1}{2}ac^2ac + \tfrac{1}{4}ac^2ac^2a =$$
$$= c - \tfrac{1}{2}(ac^2 + c^2a) + \tfrac{1}{4}ac^3a = b \text{ and}$$

$$b^2 = \left(c - \tfrac{1}{2}(ac^2 + c^2a) + \tfrac{1}{4}ac^3a\right)\left(c - \tfrac{1}{2}(ac^2 + c^2a) + \tfrac{1}{4}ac^3a\right) =$$
$$= c^2 - \tfrac{1}{2}(cac^2 + c^3a) + \tfrac{1}{4}cac^3a - \tfrac{1}{2}(ac^3 + c^2ac) + \tfrac{1}{4}(ac^2ac^2 + ac^4a + c^2ac^2a) - \tfrac{1}{8}ac^2ac^3a +$$
$$\tfrac{1}{4}ac^3ac - \tfrac{1}{8}ac^3ac^2a =$$
$$= c^2 - \tfrac{1}{2}(c^2 + c^3a) + \tfrac{1}{4}c^3a - \tfrac{1}{2}(ac^3 + c^2) + \tfrac{1}{4}(ac^3 + ac^4a + c^3a) - \tfrac{1}{8}ac^4a + \tfrac{1}{4}ac^3 - \tfrac{1}{8}ac^4a =$$
$$= c^2 - \tfrac{1}{2}c^2 - \tfrac{1}{2}c^2 - \tfrac{1}{2}(c^3a + ac^3) + \tfrac{1}{4}(c^3a + ac^3) + \tfrac{1}{4}(c^3a + ac^3) + \tfrac{1}{4}ac^4a - \tfrac{1}{8}ac^4a - \tfrac{1}{8}ac^4a =$$
$$= 0.$$ □

Whenever we have two elements in the conditions of the statement of the theorem, we will say that they are *beautiful partners*.

We will put the Beautiful Partner Lemma to work in Chapter 4, **Clifford Elements**. There we will study, in a centrally closed prime ring, elements $c \in K$ such that $C^3(K) = 0$, $c^2 \neq 0$ and $c^3 = 0$. Denote $b := c^2 \in H$. The conditions on c imply that $b^2 = 0$ and also that $bKb = 0$, so that b is von Neumann regular by the Reduction Lemma (Proposition 1.2.4(2)) and hence possess a beautiful partner, fact that unleashes a bunch of interesting results, among them that c is also von Neumann regular.

1.3 $\langle K\rangle$, the subring generated by K

In this section and in Chapter 3, **Inner Ideals**, we will use $\langle K\rangle$, the subring of R generated by K, to bring to K results valid for prime rings. The following lemma ([RingsGIs, Lemma 9.1.5]) reveals that the algebraic structure of $\langle K\rangle$ is near to the structure of K, and is the keystone to some of our results.

Lemma 1.3.1 (**Structure of $\langle K\rangle$**).

$$\langle K\rangle = K \oplus (K \circ K),$$

where $K \circ K$ coincides with the subgroup generated by $\{k^2 \mid k \in K\}$.

Proof. First of all note that $a \circ b = (a+b)^2 - a^2 - b^2$ implies that $K \circ K$ coincides with the additive subgroup generated by $\{k^2 \mid k \in K\}$. Observe that $K \circ K \subseteq H$, so that $(K \circ K) \cap K = 0$. Since $\langle K\rangle = \sum K^n$, it is now clear that $K \oplus (K \circ K) \subseteq \langle K\rangle$. Pick $k_1, k_2 \in K$. Observe that $2k_1k_2 = k_1 \circ k_2 + [k_1, k_2]$, so we have $K^2 \subseteq K \oplus (K \circ K)$. Now, since $K^3 = K^2K \subseteq K^2 + (K \circ K)K$ and so on with K^n, it suffices to show that $(K \circ K)K \subseteq K \oplus (K \circ K)$. Note that $2k_1^2k_2 = k_1^2 \circ k_2 + [k_1^2, k_2]$. The first monomial lies in K ($S = (-1)^3$), while the second one equals, using that the adjoint is an associative derivation, $k_1[k_1, k_2] + [k_1, k_2]k_1 = k_1 \circ [k_1, k_2] \in K \circ K$. □

Apart from its good structure, and related to it, $\langle K\rangle$ has another very desirable property: if R is prime then $\langle K\rangle$ is almost always prime. We achieve this by finding an ideal of R inside $\langle K\rangle$. As we show, this happens when $[K, K] \neq 0$, and a bit later we prove that $[K, K] = 0$ if and only if R is commutative or an order in a 4-dimensional central simple algebra (with involution of transpose type). In these two special cases, which are of less interest due to the triviality of K as a Lie algebra, $\langle K\rangle$ is sometimes prime and sometimes not (see the paragraph after Definition 1.3.10). The existence of an ideal of R inside $\langle K\rangle$ when this dimension condition is asked for was known to Erickson ([Erikcson'72, paragraph after Theorem 2, remark on page 529]), but unfortunately he stated his theorems ([Erikcson'72, Theorems 2,3,4]) with a less strict condition, pointing out as an exception R as an order in a central simple algebra 'of dimension at most 9' (other times 'at most 16') because these conditions were enough to state his main theorem about the Lie ideal structure of K, and it seems that the best condition of dimension 4 has not been fully assumed in the literature (see for example [Benkart'76, Theorem 3.8] and [RingsGIs, Lemma 9.1.14]).

Theorem 1.3.2 (Good behavior of $\langle K\rangle$).
Let R be a prime ring with involution.

1. *If $[K, K] \neq 0$ then the ideal generated by $[K, K]^2$ in R is nonzero and contained in $\langle K\rangle$. In particular $\langle K\rangle$ is a prime ring such that $\mathcal{C}(\langle K\rangle) = \mathcal{C}$.*
2. *If K is abelian then $\langle K\rangle$ is commutative.*

Proof. Suppose first that $[K, K] = 0$ and let us show that $[\langle K\rangle, \langle K\rangle] = 0$. By Lemma 1.3.1 we have $\langle K\rangle = K + K \circ K$, with $K \circ K$ coinciding with the subgroup generated by $\{k^2 \mid k \in K\}$. Therefore it is enough to show that $[k_1^2, k_2^2] = 0$. Observe that $[k_1, k_2^2] = k_2[k_1, k_2] + [k_1, k_2]k_2 = 0$ because ad_{k_1} is an associative derivation. Then in the same vein $[k_1^2, k_2^2] = k_1[k_1, k_2^2] + [k_1, k_2^2]k_1 = 0$.

Now suppose $[K, K] \neq 0$. The existence of an ideal $0 \neq I$ of R contained in $\langle K\rangle$ is enough to show that $\langle K\rangle$ is prime with extended centroid $\mathcal{C}$, for then, by [Lam1, Theorem 14.14 and subsequent Remark], $\langle K\rangle$ is prime with $Q_s(\langle K\rangle) = Q_s(R)$ (recall that the extended centroid is the center of the symmetric ring of quotients). Let I be the ideal of R generated by $[K, K]^2$. By [RingsGIs, Proof of Lemma 9.1.4], $I \subseteq \langle K\rangle$: the

proof for this is just and beautifully $[k_1,k_2]h = k_1(k_2\circ h)-(k_1\circ h)k_2+[h,k_1,k_2]\in\langle K\rangle$. Therefore it is sufficient to prove that if $[K,K]\neq 0$, then $[K,K]^2\neq 0$.[13]
Suppose that $[K,K]\neq 0$ but $[K,K]^2=0$ and consider $k:=[k_1,k_2]$ with $k_1,k_2\in K$. Pick $k_3\in K$; then $kk_3k=[k_1,k_2](k_3[k_1,k_2])=[k_1,k_2]([k_3,[k_1,k_2]]+[k_1,k_2]k_3)=[k_1,k_2][k_3,[k_1,k_2]]+[k_1,k_2][k_1,k_2]k_3=0$ because $[K,K]^2=0$, which implies $kKk=0$. By the Reduction Lemma 1.2.4(1), $k=0$. Therefore $[K,K]=0$, a contradiction. □

The main idea of our $\langle K\rangle$ technique is then the following: suppose we are in a situation in which, given K of a prime ring R such that $[K,K]\neq 0$, we want to prove for K a result P we know true for prime rings; we then use P for $\langle K\rangle$ and see that K inherits some (maybe twisted) form of P because of $\langle K\rangle = K\oplus(K\circ K)$. If a relation between P for K and P for R is needed, then we use also the $\langle K\rangle$–R connection through their common nonzero ideal.

1.3.1 Exceptionality

There is one handicap to the technique schemed above, if we want the result on K to follow the result on R. How restrictive is the condition $[K,K]\neq 0$? As we claimed before, the only exceptions are commutative or of low dimension (over $\mathcal{C}$); we are going to show in addition another characterization based on $Z(K)$. But first we need a bit of PIs theory (note that, informally speaking, $[K,K]=0$ is a multilinear PI for K over $\mathcal{C}$). If Φ is a ring of scalars, by $\Phi\langle X\rangle$ we denote the nonunital associative free algebra on a countable number of generators $X\equiv\{X_1,X_2,\ldots\}$. We call the X_i *variables* and call *polynomials* to the elements of $\Phi\langle X\rangle$. We also maintain the rest of denomination conventions coming from commutative polynomials (monomials, coefficients, degree, etc), with the obvious changes.

Definitions 1.3.3 (Polynomial identities).

[13] This could be shown by Herstein's theory of the Lie ideals of K as in [RingsGIs, Theorem 9.1.13(d), taking U:=K], but here we have preferred a more elementary approach.

Let R be a prime ring and let G be an abelian subgroup of $(R, +)$. We say that $p(X_1, \ldots, X_n) \in \mathcal{C}\langle X \rangle$ is a *polynomial identity* for G (or PI for short) if $p \neq 0$ and $p(x_1, \ldots, x_n) = 0$ in $\mathcal{C}R$ for all n-uples $(x_1, \ldots, x_n) \in G^n$. If p is a PI which is multilinear as a polynomial we say that p is a *multilinear PI.* If $p \in \mathcal{C}\langle X \rangle$ is a PI for G we also say that G *is PI* over $\mathcal{C}$, or that G *satisfies a PI* over $\mathcal{C}$. Finally, if R is PI over $\mathcal{C}$, the *degree* of R will be defined by us as the number $d \in \mathbb{N}$ such that R satisfies some PI of degree d but not any PI of degree $d-1$ (over $\mathcal{C}$).

By a process of successive linearization we can reduce the degree of any variable in a PI until it becomes linear, at the cost of introducing new variables. This is an elementary and fundamental tool of the PIs theory which comes from [Kaplansky'48, Lemma 2].

Lemma 1.3.4 (PIs can be taken multilinear).
Let R be a prime ring and let G be a subgroup of $(R, +)$ which is PI of degree d over $\mathcal{C}$. Then G satisfies a multilinear PI of degree at most d.

Now we state two of the fundamental structure results of the theory. The following version of Kaplansky Theorem is a combination of [FurtherAlgebra, Theorem 8.3.6] and [RingsGIs, Theorem 6.1.10].

Theorem 1.3.5 (Kaplansky-for-PIs Theorem).
Let R be a primitive ring which is PI over $\mathcal{C}$ of degree d.
Then R is central simple over $\mathcal{C}$, of dimension not greater than $(d/2)^2$.
More precisely, $R \cong \mathrm{End}_{eRe}(eR)$, where $e \in R$ is a minimal idempotent.

GPIs theory generalizes and is more complex than PIs theory. For this dissertation we just need to know that if R is PI, then R is GPI, to unlock the doors of Martindale Theorem ([RingsGIs, Theorem 6.1.6]).

Theorem 1.3.6 (Martindale Theorem).
Let R be a prime ring. If R is GPI over $\mathcal{C}$ then $\mathcal{C}R$ has nonzero socle and $\dim_C(eRe) < \infty$ *for any minimal idempotent $e \in R$.*

With Kaplansky and Martindale Theorems in hand we can completely characterize the central closure of prime PI rings.

Corollary 1.3.7 (**Characterization of prime PI rings**).
Let R be a prime PI ring over $\mathcal{C}$ of degree d.
Then $\mathcal{C}R = \widehat{R}$ is a central simple algebra over $\mathcal{C}$ of dimension not greater than $(d/2)^2$.

Proof. By Martindale Theorem (1.3.6) $\mathcal{C}R$ has nonzero socle and hence is prime with socle, equivalently, primitive with socle. Let p be a multilinear PI for R of degree at most d, which exists by Lemma 1.3.4. Let us see that $\mathcal{C}R$ also satisfies p. Pick $x_1, \dots, x_n \in R$ and $\lambda_1, \dots, \lambda_n \in \mathcal{C}$. Then since p is multilinear we get $p(\lambda_1 x_1, \dots, \lambda_n x_n) = \lambda_1 \dots \lambda_n p(x_1, \dots, x_n) = 0$. So $\mathcal{C}R$ is a primitive PI ring. By Kaplansky-for-PIs Theorem (1.3.5) we find that $\mathcal{C}R$ is a central simple algebra over $\mathcal{C}$ of dimension not greater than $(d/2)^2$. In addition $\widehat{R} = \mathcal{C}R + \mathcal{C} = \mathcal{C}R$, since $\mathcal{C}R$ is already unital. □

For our purpose we need to relate a PI for K with a PI for R. The following version of Amitsur Theorem is a combination of [Amitsur'68, Theorem 5] and [RingsGIs, proof of Theorem 9.1.10].

Theorem 1.3.8 (**Amitsur Theorem: PI condition lifts from K to R**).
Let R be a prime ring with involution $*$. All the PIs are over $\mathcal{C}$.
1. If $*$ is of the second kind and p is a multilinear PI for K then p is a PI for R.
2. If $*$ is of the first kind and K is PI of degree d then R is PI of degree at most $2d$.

We are finally in position to exhibit the structure of prime rings such that $[K, K] = 0$. The result **ii)** $\Rightarrow$ **iii)** is a particular case of [RingsGIs, Theorem 9.1.13(a)] (taking $U := K$), whose proof builds on Herstein's theory of the Lie structure of K and on applying PIs theory to $\overline{R}$; instead we apply PIs theory directly to $\widehat{R}$, to get also information about this ring (a similar idea appears already in [Erikcson'72, proof of Theorem 2]).

Theorem 1.3.9 (**Equivalences of exceptionality**).
Let R be a prime ring with involution. The following conditions are equivalent.

i) *R is commutative or $Z(K) \not\subseteq Z(R)$.*

ii) $[K, K] = 0$.

iii) *R is commutative or $\mathcal{C}R = \widehat{R}$ is a 4-dimensional central simple algebra over $\mathcal{C}$ with involution of the first kind and transpose type and $\overline{R} \cong \mathbb{M}_2(\overline{\mathcal{C}})$ with the transpose involution.*

Proof.

i) $\Rightarrow$ ii). If R is commutative then trivially $[K, K] = 0$. Pick $z \in Z(K) \setminus Z(R)$ and suppose $[K, K] \neq 0$. By Theorem 1.3.2 we have that $\langle K\rangle$ is a prime ring such that $\mathcal{C}(\langle K\rangle) = \mathcal{C}$. Since $K = K + K \circ K$ and $K \circ K$ is the subgroup generated by $\{k^2 \mid k \in K\}$, we see from $[z, k^2] = [z, k]k + k[z, k] = 0$ that $z \in Z(\langle K\rangle)$. Hence $z \in \mathcal{C}$, but also $z \in R$. Therefore $z \in Z(R)$, a contradiction.

ii) $\Rightarrow$ iii). Observe that $[K, K] = 0$ implies that $p(X, Y) := XY - YX$ is a multilinear PI for K of degree 2. If the involution is of the second kind then by Amitsur Theorem 1.3.8(1) p is also a PI for R, i.e., R is commutative. Suppose then that the involution is of the first kind. Since p is of degree 2, by Amitsur Theorem 1.3.8(2) we get that R is PI of degree at most 4. If R is not commutative, by the characterization of prime PI rings (1.3.7) we get that $\mathcal{C}R = \widehat{R}$ is a central simple algebra of dimension 4 over $\mathcal{C}$ (and R is an order in $\widehat{R}$). Consider $\overline{R} := \widehat{R} \otimes_{\mathcal{C}} \overline{\mathcal{C}}$; since $\widehat{R}$ has dimension 4 over $\mathcal{C}$, $\overline{R}$ has dimension 4 over $\overline{\mathcal{C}}$ and is central simple over $\overline{\mathcal{C}}$, hence by Wedderburn Theorem it is a matrix algebra over some finite-dimensional division algebra over $\overline{\mathcal{C}}$. But $\overline{\mathcal{C}}$ is algebraically closed and thus it is the only finite-dimensional division algebra over itself[14]. Therefore $\overline{R} \cong \mathbb{M}_2(\overline{\mathcal{C}})$. $\overline{R}$ can be equipped with the extended involution $(x \otimes \lambda)^* = x^* \otimes \lambda$ and, by choosing matrix units properly, we can assume that $*$ is either the transpose or the symplectic involution on $\mathbb{M}_2(\overline{\mathcal{C}})$ (see Section 1.1). As the tensor product respects direct sums we know that $\widehat{R} \otimes_{\mathcal{C}} \overline{\mathcal{C}} = (\widehat{H} \oplus \widehat{K}) \otimes_{\mathcal{C}} \overline{\mathcal{C}} = (\widehat{H} \otimes_{\mathcal{C}} \overline{\mathcal{C}}) \oplus (\widehat{K} \otimes_{\mathcal{C}} \overline{\mathcal{C}})$, which shows that $\overline{K} := \mathrm{Skew}(\overline{R}, *) = \widehat{K} \otimes_{\mathcal{C}} \overline{\mathcal{C}}$. Since the involution is of the first kind, by

[14]Suppose Δ is a division algebra over $\overline{\mathcal{C}}$ of dimension n. Then for every $a \in \Delta$ the set $\{1, a, \ldots, a^n\}$ is linearly dependent and hence a is algebraic of degree at most n. Now the minimal polynomial of a factorizes in linear factors in $\overline{\mathcal{C}}[X]$ and Δ has no zero divisors. Therefore $a \in \overline{\mathcal{C}}$.

Lemma 1.2.1(1) it is $[\widehat{K},\widehat{K}] = \mathcal{C}[K,K] = 0$. Therefore $[\overline{K},\overline{K}] = [\widehat{K},\widehat{K}] \otimes \mathcal{C} = 0$ (since $[k\otimes\lambda, k'\otimes\mu] = kk'\otimes\lambda\mu - k'k\otimes\mu\lambda = [k,k']\otimes\lambda\mu$). This implies, by Lemma 1.1.4(1d,2d), that the involution on $\mathbb{M}_2(\overline{\mathcal{C}})$ is the transpose involution.

iii) $\Rightarrow$ i). If R is commutative there is nothing to prove. Suppose that $\overline{R} \cong \mathbb{M}_2(\overline{\mathcal{C}})$ endowed with the transpose involution. Denote $\overline{K} := \mathrm{Skew}(\overline{R},*) = \widehat{K}\otimes_{\mathcal{C}}\overline{\mathcal{C}}$. By Lemma 1.1.4(2b,2d) we get that $\overline{K}\cap\overline{C} = 0$ and $[\overline{K},\overline{K}] = 0$. This implies in particular that $\widehat{K}\cap\mathcal{C} = 0$ and $[\widehat{K},\widehat{K}] = 0$, since $\widehat{K} \cong \widehat{K}\otimes 1 \subseteq \overline{K}$ and $\mathcal{C} \subseteq \overline{\mathcal{C}}$. Pick $0 \neq a \in \widehat{K}$. Since the involution is of the first kind, by Lemma 1.2.1(1) there exist $b \in R$ and $\lambda \in \mathcal{C}$ such that $a = \lambda b$, with $\lambda \neq 0$ because $a \neq 0$. It is clear that $b \notin Z(R)$, because otherwise we would have $a = \lambda b \in \mathcal{C}$, a contradiction. Now $[b,K] = [\lambda^{-1}a, K] = \lambda^{-1}[a,K] = 0$, since $K \subseteq \widehat{K}$ and $[\widehat{K},\widehat{K}] = 0$. This proves that $b \in Z(K)\setminus Z(R)$. $\square$

Observe that when $[K,K] = 0$ and R is not commutative we can say more about the structure of $\widehat{R}$. Given a field F such that $\mathrm{char}(F) \neq 2$ and $0 \neq \alpha,\beta \in F$, the *quaternion algebra* $\mathbb{H}(\alpha,\beta)$ is the 4-dimensional algebra over F with basis $\{1,i,j,ij\}$ and multiplication relations $i^2 = \alpha$, $j^2 = \beta$, $ij = -ji$ (refer to [CSAs, Chapter 1] or [BasicAlgebra1, Section 7.6]). A quaternion algebra is either a division ring or isomorphic to $\mathbb{M}_2(F)$ ([CSAs, Proposition 1.1.7]), in which case it is termed *split*. The proof above shows that if $[K,K] = 0$ then $\widehat{R}$ is a 4-dimensional central simple algebra over $\mathcal{C}$, that is, $\widehat{R} \cong \mathbb{M}_n(\Delta)$ with Δ a division ring with center $\mathcal{C}$. Since $4 = \dim_{\mathcal{C}}(\widehat{R}) = n^2\dim_{\mathcal{C}}(\Delta)$, either $n = 2$, $\dim_{\mathcal{C}}(\Delta) = 1$ and hence $\Delta = \mathcal{C}$ and $\widehat{R} \cong \mathbb{M}_2(\mathcal{C})$ with the transpose involution, or $n = 1$ and $\widehat{R} \cong \Delta$ with Δ a 4-dimensional division algebra over $\mathcal{C}$ which, by [CSAs, Proposition 1.2.1], is isomorphic to a division quaternion algebra, equipped with an involution of the first kind and transpose type[15]. In any case $\widehat{R}$ is a quaternion algebra (either split or division).

[15] By [BookInvolutions, Proposition 2.21], any involution of the first kind and transpose type on $\mathbb{H}(\alpha,\beta)$ (there called of orthogonal type) is of the form $a^* := u\bar{a}u^{-1}$, where $^-$ is the usual conjugation involution and u is a noncentral unit, skew with respect to conjugation.

Definition 1.3.10 (K **exceptional).**
We will say that K is *exceptional* when R satisfies any of the equivalent conditions of the previous theorem.

So, if K is not exceptional we have at our disposal the $\langle K\rangle$ technique in its entirety. A concrete example of its use is our proof of Herstein Lemma carried out in Section 3.2.1.

It is known that $\langle K\rangle$ is well behaved for semiprimeness: if R is semiprime, then $\langle K\rangle$ is semiprime ([RingsInvolution, Theorem 6.5.7]). In contrast, $\langle K\rangle$ is not that well behaved for primeness. Theorem 1.3.9 allows us to identify sufficient conditions for $\langle K\rangle$ to be prime (or not) when R is prime.

- If $[K,K]\neq 0$, then $\langle K\rangle$ is prime by Theorem 1.3.2(1).
- If $[K,K]=0$ then by Theorem 1.3.9 either R is commutative or $\widehat{R}$ is a 4-dimensional central simple algebra (i.e., a quaternion algebra) over $\mathcal{C}$ with involution of the first kind and transpose type. Moreover $\langle K\rangle$ is commutative by Theorem 1.3.2(2), so it is prime if and only if it does not contain zero divisors.
  - If R is commutative then it is an integral domain, so the subring $\langle K\rangle$ is also an integral domain.
  - Suppose $\widehat{R}$ is a quaternion algebra. We know that $\widehat{K}=\mathcal{C}K$ because the involution is of the first kind (Lemma 1.2.1(1)). This and $\langle K\rangle = K+K\circ K$ imply that $\langle K\rangle\subseteq\langle\widehat{K}\rangle=\mathcal{C}K+\mathcal{C}(K\circ K)$, which proves that $\langle\widehat{K}\rangle$ is commutative and that $\langle K\rangle$ has a zero divisor if and only if it inherits it from $\langle\widehat{K}\rangle$.
    * If $\widehat{R}$ is a division algebra, then it is an integral domain and $\langle\widehat{K}\rangle\subseteq\widehat{R}$ has no zero divisors.
    * If $\widehat{R}$ is split, then $\widehat{R}\cong\mathbb{M}_2(\mathcal{C})$ with the transpose involution. Then, as we know from Lemma 1.1.3,
$$\widehat{K}\cong\left\{\begin{pmatrix}0 & x\\ -x & 0\end{pmatrix}\mid x\in\mathcal{C}\right\}$$
with $k^2\in\mathcal{C}$ for every $k\in\widehat{K}$. Therefore

$$\langle\widehat{K}\rangle \cong \left\{ \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \mid x, y \in \mathcal{C} \right\},$$

with x being a sum of squares, which amounts to saying nothing[16] since $x = \frac{(x+1)^2}{4} - \frac{(x-1)^2}{4}$ for every $x \in \mathcal{C}$ (recall that we always ask for $\frac{1}{2} \in \Gamma$ in our rings with involution). If the product of two nonzero matrices of $\langle\widehat{K}\rangle$ is zero, at least one of them must have left rank less than 2 (see for example [GeometricAlgebra, page 151]). Then $(x, y) = \lambda(-y, x)$ for some $\lambda \in \mathcal{C}$, what implies $y = \lambda x = -\lambda^2 y$ and hence $\lambda^2 = -1$, since $y = 0$ implies $x = 0$ and the matrix would be zero. Therefore, if $\mathcal{C}$ does not contain a square root of -1, $\langle K\rangle$ is prime.

* On the other hand, if F is a field with a $\lambda \in F$ such that $\lambda^2 = -1$ and we consider the simple ring $R := \mathbb{M}_2(F)$, then $a := \begin{pmatrix} \lambda x & x \\ -x & \lambda x \end{pmatrix}$, $b := \begin{pmatrix} y & \lambda y \\ -\lambda y & y \end{pmatrix}$ with $0 \neq x, y \in F$ are (nonzero) zero divisors lying in $\langle K\rangle$, since

$$a = \begin{pmatrix} 0 & x \\ -x & 0 \end{pmatrix} - \lambda \left(\begin{pmatrix} 0 & \frac{x+1}{2} \\ -\frac{x+1}{2} & 0 \end{pmatrix}^2 - \begin{pmatrix} 0 & \frac{x-1}{2} \\ -\frac{x-1}{2} & 0 \end{pmatrix}^2 \right), \text{ and similarly for } b.$$

Therefore $\langle K\rangle$ is not always prime.

1.4 Hall Identity

In Theorem 1.3.9, to show that when R is prime $[K, K] = 0$ implies that $\widehat{R}$ is a central simple algebra of dimension 4 over $\mathcal{C}$ we have used Amitsur Theorem (1.3.8), which for prime (and semiprime) rings guarantees that the degree of the PI for R is at most $2d$ whenever the degree of the PI for K is d. For arbitrary rings, Amitsur Theorem still guarantees that R is PI (over $\mathbb{Z}$) whenever K is PI (over $\mathbb{Z}$), but does not bound the degree of R. In this section we will prove that if R is an arbitrary ring such that $[K, K] = 0$ then R always satisfies a certain PI called Hall Identity, which has degree 5:

[16] This trick is borrowed from [Joly'70, Exemple (7.10)].

$$[[X_1, X_2]^2, X_3] \quad (\textbf{Hall Identity}).$$

As a starting point we show that the two exceptional cases of Theorem 1.3.9 do indeed satisfy Hall Identity, which is the fact that motivates this endeavour.

Consider the matrix algebra $R := \mathbb{M}_2(F)$ over a field F. By the Cayley-Hamilton Theorem every $x \in R$ satisfies $x^2 - \operatorname{tr}(x)x + |x|1 = 0$, where 1 is the identity matrix and $\operatorname{tr}(x)$, $|x|$ denote the trace and the determinant of x, respectively. If $x, y \in R$ then, as is well known, $\operatorname{tr}(xy) = \operatorname{tr}(yx)$ and therefore $\operatorname{tr}([x, y]) = 0$; hence by Cayley-Hamilton we get $[x, y]^2 = -|[x, y]|1$, and in particular $[x, y]^2 \in Z(R)$. This argument (which comes from [PIRings, Examples 1.15(v)]) shows that Hall Identity is a PI for $\mathbb{M}_2(F)$.

Consider now the quaternion algebra $\mathbb{H} := \mathbb{H}(\alpha, \beta)$ with $0 \neq \alpha, \beta \in F$ for a given field F. The linear map $^-$ such that $\bar{1} = 1$, $\bar{i} = -i, \bar{j} = -j, \overline{ij} = -ij$ is an involution called *conjugation* such that $\operatorname{Sym}(\mathbb{H}, ^-) = F1$ and $\operatorname{Skew}(\mathbb{H}, ^-) = Fi \oplus Fj \oplus Fij$. The center is also $Z(\mathbb{H}) = F1$. Hence the image of the symmetric trace $\tau(x) = x + \bar{x}$ lies in the center for every $x \in \mathbb{H}$. In addition, if $x := \lambda_1 1 + \lambda_i i + \lambda_j j + \lambda_k ij$ then the map $q(x) := \bar{x}x = (\lambda_1^2 - \alpha\lambda_i^2 - \beta\lambda_j^2 + \alpha\beta\lambda_k^2)1 \in F1$ defines a *quadratic norm* $q : \mathbb{H} \to F$. Then it is clear that $x^2 - \tau(x)x + q(x)1 = x^2 - (x + \bar{x})x + \bar{x}x = 0$, which parallels the matrix case above (in fact, it generalizes it[17]). Following this parallelism we can see that since the only symmetric elements are in the center, which is the kernel of the adjoint map, we have that $[x, y]$ is skew for every $x, y \in \mathbb{H}$ and therefore $\tau([x, y]) = 0$, since τ adds an element with its conjugate. Therefore $[x, y]^2 = -q([x, y])1 \in Z(\mathbb{H})$, and Hall Identity is a PI for $\mathbb{H}$.

Reciprocally, as proved by Hall in his memoir about projective planes ([Hall'43, Theorem 6.2]), every division ring in which Hall Identity holds is either a field or a quaternion

[17]It can be proved ([BookInvolutions, paragraph previous to Proposition 2.21]) that when a split quaternion algebra is represented by 2×2 matrices, the conjugation involution is represented by the symplectic involution $*$ (see Definition 1.1.2). It is then easily checked that for every $x \in \mathbb{M}_2(F)$ we have $x + x^* = \operatorname{tr}(x)1$ and $x^*x = |x|1$, where 1 is the identity matrix.

algebra over its center. According to [PIRings, Examples 1.15(v)], Hall Identity was actually published in [Wagner'37] for 2×2 matrices over a field, but the proof of Hall surprised Kaplansky because it implied the finite dimensionality of a division ring satisfying Hall Identity, and motivated him to prove that any PI for a division ring would oblige it to be finite-dimensional, giving birth to one of the founding papers of PIs theory ([Kaplansky'48]).

Our result that $[K, K] = 0$ implies that R satisfies Hall Identity generalizes the matrix case since, as observed in Lemma 1.1.4(2d), if we equip $\mathbb{M}_2(F)$ with the transpose involution then all skew elements commute with each other; more in general it generalizes the quaternion case, as we explain now. If we denote $\mathbb{H} := \mathbb{H}(\alpha, \beta)$, then we have that $i \in \mathbb{H}$ is invertible with $i^{-1} = \alpha^{-1} i$. The map $*$ such that $x^* := i\bar{x}i^{-1}$ for every $x \in \mathbb{H}$ can be checked to be an involution such that $1^* = 1$, $i^* = -i$, $j^* = j$ and $(ij)^* = ij$. Then $K = Fi$ is 1-dimensional and therefore $[K, K] = 0$.

Now we prove, by combinatorial means, that if $[K, K] = 0$ then R satisfies Hall Identity. First we need to note a consequence of $[K, K] = 0$.

Lemma 1.4.1 (K^2 **in the center).**
Let R be a ring with involution such that $[K, K] = 0$. Then $K^2 \subseteq Z(R)$.

Proof. Since $2ab = [a, b] + a \circ b = [a, b] + (a + b)^2 - a^2 - b^2$ and $[K, K] = 0$ we have that $K^2 = \{\sum_{i=1}^{n} k_i^2 \mid k_i \in K, n \in \mathbb{N}\}$, so it suffices to show that $[k^2, R] = 0$ for every $k \in K$. Pick $a \in K$ and $x \in R$ with $x = x_h + x_k$. Work inside $M(R)$. Denote $T := l_a + r_a$, which commutes with A. Note that $A(K) = 0$ and that $AT(x_h) = 0$ because $T(x_h) = a \circ x_h \in K$. Then in $M(R)$ we have $ad_{a^2} = l_{a^2} - r_{a^2} = l_a^2 - r_a^2 = (l_a + r_a)(l_a - r_a) = TA$. Therefore

$$[a^2, x] = ad_{a^2}(x) = TA(x_h + x_k) = AT(x_h) + TA(x_k) = 0.$$

Hence $K^2 \subseteq Z(R)$. □

Proposition 1.4.2 ($[K,K]=0$ **implies Hall Identity).**

Let R be a ring with involution such that $[K,K]=0$.

Then $[[x,y]^2,z]=0$ for every $x,y,z\in R$.

Proof. We can interpret Hall Identity as saying that $[x,y]^2\in Z(R)$ for every $x,y\in R$. To show this we will prove that $[x,y]^2\in K^2$, since $K^2\subseteq Z(R)$ by Lemma 1.4.1.

So, pick $x,y\in R$ and decompose them as $x=k_1+h_1, y=k_2+h_2$, with $k_1,k_2\in K$ and $h_1,h_2\in H$. Then

$$[x,y]^2=[k_1+h_1,k_2+h_2]^2=([k_1,h_2]+[h_1,k_2]+[h_1,h_2])^2=$$

$$=[k_1,h_2]^2+[h_1,k_2]^2+[h_1,h_2]^2+[k_1,h_2]\circ[h_1,k_2]+[h_1,h_2]\circ([k_1,h_2]+[h_1,k_2]).$$

We will study every term separately, showing that all of them lie in K^2. Any element with a letter h is meant to belong to H, and any with a letter k is meant to belong to K. We use several times that $[k,k']=0$, that $kk'=k'k$ and that $kk'x=xkk'$ (because $K^2\in Z(R)$). We give hints by the use of parentheses.

- $[h_1,h_2]^2$**:**

  $[h_1,h_2]\in K$ implies that $[h_1,h_2]^2\in K^2$.

- $[k,h]^2$**:**

  Note that $hkh\in K$. Then

  $$[k,h]^2=khkh+(hkh)k-kh^2k-h(k^2)h=2khkh-kh^2k-k^2h^2=$$

  $$=2khkh-(kh^2k+k^2h^2)=2k(hkh)-k(h^2\circ k)\in K^2.$$

- $[k_1,h_2]\circ[h_1,k_2]$**:**

  We further decompose in

  $$[k_1,h_2]\circ[h_1,k_2]=(k_1h_2)\circ(h_1k_2)-(h_2k_1)\circ(h_1k_2)-(k_1h_2)\circ(k_2h_1)+(h_2k_1)\circ(k_2h_1).$$

  - $(k_1h_2)\circ(h_1k_2)$ **and** $(k_2h_1)\circ(h_2k_1)$**:**

    Note that $h_1(k_2k_1)h_2=h_1h_2(k_2k_1)=h_1h_2k_1k_2$ and hence we have

    $$(k_1h_2)\circ(h_1k_2)=k_1h_2h_1k_2+h_1(k_2k_1)h_2=(k_1h_2h_1)k_2+(h_1h_2k_1)k_2=\{k_1,h_2,h_1\}k_2\in K^2.$$

Similarly, $(k_2h_1) \circ (h_2k_1) = \{k_2, h_1, h_2\}k_1 \in K^2$.

– $(h_1k_2) \circ (h_2k_1)$ **and** $(k_1h_2) \circ (k_2h_1)$**:**

First we show that $k_1hk_2 = k_2hk_1$. We use that K_1 is an associative derivation.

We have $0 = [k_1, h \circ k_2] = [k_1, h]k_2 + h[k_1, k_2] + [k_1, k_2]h + k_2[k_1, h] = [k_1, h]k_2 + k_2[k_1, h] = k_1hk_2 - hk_1k_2 + (k_2k_1)h - k_2hk_1 = k_1hk_2 - hk_1k_2 + hk_1k_2 - k_2hk_1 = k_1hk_2 - k_2hk_1$, so that $k_1hk_2 = k_2hk_1$.

This implies that

$$(h_1k_2)\circ(h_2k_1) = h_1(k_2h_2k_1)+h_2k_1h_1k_2 = h_1k_1h_2k_2+h_2k_1h_1k_2 = \{h_1, k_1, h_2\}k_2 \in K^2.$$

In a similar fashion, $(k_1h_2) \circ (k_2h_1) = k_1\{h_1, k_2, h_2\} \in K^2$.

- $[h_1, h_2] \circ [k, h_2]$**:**

We further decompose in

$$[h_1, h_2] \circ [k, h_2] = [h_1, h_2] \circ (kh_2) - [h_1, h_2] \circ (h_2k).$$

– $[h_1, h_2] \circ (kh_2)$ **and** $[h_1, h_2] \circ (h_2k)$**:**

First we show that $\{h_1, h_2, k\} = \{h_2, h_1, k\}$.

We have $0 = [[h_1, h_2], k] = h_1h_2k - kh_1h_2 - h_2h_1k + kh_2h_1 = \{h_1, h_2, k\} - \{h_2, h_1, k\}$.

This implies that

$$[h_1, h_2] \circ (kh_2) = h_1h_2kh_2 + kh_2h_1h_2 - h_2h_1kh_2 - kh_2^2h_1 =$$

$$= (\{h_1, h_2, k\} - \{h_2, h_1, k\})h_2 + kh_1h_2^2 - kh_2^2h_1 = k[h_1, h_2^2] \in K^2.$$

Similarly, $[h_1, h_2] \circ (h_2k) = [h_1, h_2^2]k \in K^2$.

In conclusion, $[x, y]^2 \in K^2$ and the result is proven. □

Chapter 2

Orthogonal elements in Lie algebras

For semiprime associative algebras, the orthogonality of the ideals generated by two elements has a simple characterization in terms of the elements: if R is semiprime and $a, b \in R$, then $I(a)I(b) = 0$ if and only if $aRb = 0$ (equivalently, $bRa = 0$). May this characterization be exported to Lie algebras? In this chapter we answer this question in the affirmative, exposing the results of our paper [Brox,García&Gómez'14]. As we know, in the associative context semiprimeness and nondegeneracy are equivalent properties, but in the nonassociative setting this is no longer true, and it turns out that stronger properties and characterizations are usually found when nondegeneracy is assumed. This happens even in the prime case, where the stronger notion is that of strong primeness, i.e., primeness plus nondegeneracy (there exist even simple finite dimensional Lie algebras which are degenerate). So, for example, a Jordan algebra J satisfies that $\{a, J, b\} = 0$ implies either $a = 0$ or $b = 0$ if and only if J is strongly prime ([Beĭdar,Mikhalëv&Slin'ko'87]), while a Lie algebra satisfies that $[a, [b, L]] = 0$ implies either $a = 0$ or $b = 0$ if and only if L is strongly prime ([García&Gómez'07])[1].
Hence, the natural translation to Lie algebras of the characterization by elements would read:

[1]The list of examples extends to alternative algebras ([Beĭdar,Mikhalëv&Slin'ko'87]), to Jordan pairs, and to quadratic Jordan algebras J with the condition $U_aU_JU_b = 0$ ([Anquela,Cortés&McCrimmon'96]).

If L is a nondegenerate Lie algebra, then $[I(a), I(b)] = 0$ if and only if $[a, [b, L]] = 0$ (equivalently, $[b, [a, L]] = 0$).

There exists a nice approach to this question in a general setting, which works for nondegenerate alternative algebras and for nondegenerate Jordan systems, but that, for the moment, cannot be applied to Lie algebras in general: in those cases, the nondegenerate radical (the smallest ideal of the system such that the factor system is nondegenerate) has been proved to be the intersection of all strongly prime ideals[2], what implies that those nondegenerate systems are a subdirect product of strongly prime systems. We do a generic proof with a generic system $(T, +, \star)$ (all the proofs are similar). If $a, b \in T$ are such that $(a \star T) \star b = 0$, then the same condition is satisfied inside every strongly prime factor of the product, so that $\bar{a} = 0$ or $\bar{b} = 0$ in each of them because of the characterization by elements of strong primeness, and therefore $\overline{I(a) \star I(b)} = \bar{0}$ in each factor system. This shows that $I(a) \star I(b)$ is contained in every strongly prime ideal of T, but their intersection is 0, and hence $I(a) \star I(b) = 0$. The other implication is trivial.

As mentioned earlier, the characterization of strong primeness by elements is available in Lie algebras, but unfortunately it is not known whether the Kostrikin radical, which is the nondegenerate radical in the Lie setting, is always the intersection of all strongly prime ideals. It is known that the answer is yes for Lie algebras over fields of characteristic 0, for Lie algebras with a finite grading of length n and every number $1, 2, \ldots, 4n$ invertible, for Lie algebras arising from associative algebras free of 2-torsion, and for nondegenerate Lie algebras with chain condition on annihilator ideals, among others (see [García&Gómez'11] for all these results).

Nevertheless, we are able to show by combinatorial means, building heavily on the results of [García&Gómez'07], that the characterization of orthogonality by elements is true for any nondegenerate Lie algebra free of 6-torsion. Since $[I(a), I(b)] = 0$ already implies $[a, [b, L]] = 0$, and since we have $[I(a), I(b)] = 0$ if and only if $a \in \mathrm{Ann}(I(b))$ (equiv. $b \in \mathrm{Ann}(I(a))$) because $\mathrm{Ann}(I(b))$ is an ideal, we settle to the task of showing

[2]Proved in [Beĭdar&Mikhalëv'87] and in [Thedy'85].

that $AB = 0$ in $\mathrm{End}(L)$ implies $a \in \mathrm{Ann}(I(b))$. To this end we will work in three different cases with hypotheses of increasing difficulty ($AXYB = 0$, $AXB = 0$ and, finally, $AB = 0$): in each case we will establish valid identities (more diffuse in every step) and then will apply the previous case to prove the studied one. In several steps we will witness cameos of Jordan elements, which are responsible, jointly with absolute zero divisors, for the appearance of the condition $6 \in TF(L)$.

2.1 Preliminaries

We prove an interesting identity of Jordan elements that we will need in what follows ([Benkart'77, Lemma 1.7(i),(iii)]).

Lemma 2.1.1 (**Fundamental Formula for Jordan elements**).
Let L be a Lie algebra such that $3 \in \mathrm{TF}(L)$ and let $a \in L$ be a Jordan element.
Let $x \in L$ be arbitrary. Then $ad^2_{A^2(x)} = A^2X^2A^2$ (the **Fundamental Formula**).

Proof. We develop the proof in three steps.

1. $A^3 = 0$ implies $0 = \mathbf{ad}_A^3 = (l_A - r_A)^3$ in $\mathrm{End}(\mathrm{Inn}(L))$. So $0 = -3l_A^2 r_A + 3l_A r_A^2$ and $l_A^2 r_A = l_A r_A^2$ **(1)** because $3 \in \mathrm{TF}(L)$.
2. Multiplying **(1)** by l_A we get $0 = l_A^2 r_A^2$ **(2)**.
3. $ad^2_{A^2(x)} = (\mathbf{ad}_A^2(X))^2 = (A^2X - 2AXA + XA^2)(A^2X - 2AXA + XA^2) =$ $(A^2XA^2)X - 2(A^2XA)XA + A^2X^2A^2 + 4AXA^2XA - 2AX(AXA^2) + X(A^2XA^2) =$ $A^2X^2A^2$ by **(1)** and **(2)**. □

The Fundamental Formula for Jordan elements, $ad^2_{ad^2_a(x)} = ad^2_a ad^2_x ad^2_a$, gets its name due to its resemblance with the Fundamental Formula of Jordan algebras, $U_{U_x(y)} = U_x U_y U_x$ (see [TasteJordanAlgebras, pages 5 to 9]).

We will also need the following properties of absolute zero divisors.

Lemma 2.1.2 (**Identities for absolute zero divisors**).

Let L be a Lie algebra such that $2 \in \mathrm{TF}(L)$ and let $a \in L$ be an absolute zero divisor. Let $x, y \in L$ be arbitrary. Then:

1. $AXA = 0$.
2. $AXYA = AYXA$.
3. $ad^2_{A(x)} = -AX^2A$.
4. $AXYA(z) = AXZA(y) = AYZA(x)$.
5. $AX^2AX^2A = 0$ *if in addition* $3 \in \mathrm{TF}(L)$.

Proof.

1. $A^2 = 0$ implies $0 = \mathbf{ad}^2_A = (l_A - r_A)^2$ in $\mathrm{End}(\mathrm{Inn}(L))$. Hence $-2l_Ar_A = 0$ and therefore $l_Ar_A = 0$ because $2 \in \mathrm{TF}(L)$.
2. By item (1), $0 = Aad_{[x,y]}A = A[X,Y]A = AXYA - AYXA$, so that $AXYA = AYXA$.
3. By item (1), $ad^2_{A(x)} = (\mathbf{ad}_A(X))^2 = (AX - XA)^2 = (AXA)X - AX^2A - X(A^2)X + X(AXA) = -AX^2A$.
4. By Jacobi Identity, $AX(YA(z)) = AX(AY(z)) + AX(ZA(y)) =$ $= (AXA)Y(z) + AXZA(y) = AXZA(y)$ by item (1). Now use item (2) to get $AXYA(z) = AYXA(z)$ and repeat the same reasoning to get $AYZA(x)$.
5. By item (1), $0 = Aad_{X^4(a)}A = A\mathbf{ad}^4_X(A)A = A[X[X[X[X,A]]]]A = 6AX^2AX^2A$, since the remaining terms have either AXA or A^2 as a factor. Since $6 \in \mathrm{TF}(L)$, $AX^2AX^2A = 0$. □

The following result is a particular case of [García&Gómez'09, Theorem 2.3].

Lemma 2.1.3 (**Little Kostrikin Lemma (for absolute zero divisors)**).

Let L be a Lie algebra such that $2 \in \mathrm{TF}(L)$ and let $a \in L$ be an absolute zero divisor. Then $A(L)$ is an abelian inner ideal.

Proof. Pick $x, y \in R$ and denote $b := A(x)$, $c := A(y)$. Then by Lemma 2.1.2(1)

$$BC = ad_{A(x)}ad_{A(y)} = [A,X][A,Y] =$$

$$= (AXA)Y - X(A^2)Y - AXYA + X(AYA) = -AXYA.$$

Therefore $BC(L) \subseteq A(L)$. In addition $A(L)$ is abelian because
$B(c) = [A, X]A(y) = AXA(y) - XA^2(y) = 0$, again by Lemma 2.1.2(1). □

Now we state several important facts about the ideals of a nondegenerate Lie algebra and their annihilators. The Fundamental Formula for Jordan elements shows that nondegeneracy is inherited by ideals.

Lemma 2.1.4 (**Ideals inherit nondegeneracy**).
Let L be nondegenerate Lie algebra such that $3 \in \mathrm{TF}(L)$.
Then every ideal I of L is nondegenerate as an algebra.

Proof. Suppose that $a \in I$ is such that $A^2(I) = 0$. Then $A^3(L) = A^2([a, L]) = 0$ since $[a, L] \subseteq I$, i.e., a is a Jordan element of L. Then the Fundamental Formula (2.1.1) shows that $ad^2_{A^2(x)} = A^2(X^2A^2) = 0$ for every $x \in L$, since $X^2A^2(y) \in I$ for every $y \in L$ and $A^2(I) = 0$. But L is nondegenerate, so $ad^2_{A^2(L)} = 0$ implies $A^2(L) = 0$ and this, again, implies that $a = 0$. □

Filippov asked ([Filippov'81]), given a nondegenerate Lie algebra L, an ideal I of L and $a \in L$ such that $A^2(I) = 0$, if it is always true that $a \in \mathrm{Ann}(I)$. Zel'manov ([Zel'manov'83, Corollary 2]) gave the following positive answer, over a field of characteristic 0, using the properties of the Kostrikin radical: define $J := I + \mathbb{Z}a$, which is a subalgebra of L such that $A^2(J) = A^2(I + \mathbb{Z}a) = A^2(I) + A^2(\mathbb{Z}a) = 0$. Then $a \in \mathcal{K}(J)$ and therefore $A(I) \subseteq \mathcal{K}(J)$ (because $\mathcal{K}(J)$ is an ideal). We also have $A(I) \subseteq I$ and thus $A(I) \subseteq \mathcal{K}(J) \cap I = \mathcal{K}(I)$ (by [Zel'manov'83, Corollary 1]), but $\mathcal{K}(I) = 0$ because I is nondegenerate as an algebra. Therefore $A(I) = 0$ and $a \in \mathrm{Ann}(I)$.
It is not known if this property of the Kostrikin radical is true in general for nondegenerate Lie algebras over rings of scalars, but we are able to show, by combinatorial means, that Filippov question also has a positive answer when L is free of 6-torsion[3].

[3] This comes from a private communication by Férnandez López and Gómez Lozano.

Theorem 2.1.5 (**Ideal nondegenerate as an algebra**).
Let L be a Lie algebra such that $6 \in \mathrm{TF}(L)$ and let I be an ideal of L which is nondegenerate as an algebra. Then:

1. $\mathrm{Ann}(I) = \{x \in L \mid X^2(I) = 0\}$.

2. $\mathrm{Ann}(I)$ *is a nondegenerate ideal.*

Proof.

1. Pick $a \in L$. Trivially, $A(I) = 0$ implies $A^2(I) = 0$. Suppose then that $A^2(I) = 0$. Since I is an ideal, the uppercase notation for adjoint representations is well defined while we restrict their arguments to I. Also, the identities of Lemma 2.1.2 keep being valid for this restriction. So, assume that all implicit arguments are in I. Pick $x \in L$ and denote $b := A(x) \in I$. By Lemma 2.1.2(3) we know that $B^2 = -AX^2A$. By the Little Kostrikin Lemma (2.1.3) b is a Jordan element of I and therefore it satisfies the Fundamental Formula (2.1.1), which when used for $y = x$ gives in particular

$$ad^2_{B^2(x)} = B^2X^2B^2 = AX^2AX^2AX^2A = 0,$$

because $AX^2AX^2A = 0$ by Lemma 2.1.2(5). Since I is a nondegenerate algebra, $ad^2_{B^2(x)} = 0$ implies that $B^2(x) = 0$. Note that x here is fixed and related to $b = A(x)$, so we are not finished yet. $B^2(x) = 0$ translates to $AX^2A(x) = 0$. Choose $n \in \{1, 2\}$ and $y \in I$. Recall that by Lemma 2.1.2(2) we have $AXYA = AYXA$. Linearize by

$$0 = Aad^2_{x+ny}A(x + ny) = A(X + nY)^2A(x + ny) =$$

$$= AX^2A(x + ny) + n^2AY^2A(x + ny) + 2nAXYA(x + ny) =$$

$$= AX^2A(x) + n^3AY^2A(y) + nAX^2A(y) + 2nAXYA(x) + n^2AY^2A(x) + 2n^2AXYA(y).$$

Now, $AX^2A(x) = 0 = AY^2A(y)$, and by Lemma 2.1.2(4) we get that $AXYA(x) = AX^2A(y)$ and $AXYA(y) = AY^2A(x)$. Therefore

$$\left\{\begin{matrix} 3AX^2A(y) + 3AY^2A(x) = 0 \\ 6AX^2A(y) + 12AY^2A(x) = 0 \end{matrix}\right\}.$$

This system has determinant $9(1\cdot4-(1\cdot3)) = 9$. Since $3 \in \mathrm{TF}(L)$ we get[4] $AX^2A(y) = 0$ for every $x, y \in I$, i.e., $B^2(I) = ad_b^2(I) = 0$ and b is an absolute zero divisor of I. Since I is a nondegenerate algebra we get $A(x) = 0$, as we wanted to show.

2. Suppose $a \in L$ is such that $A^2(L) \subseteq \mathrm{Ann}(I)$, i.e., such that $[A^2(L), I] = 0$. Then in particular $[A^2(I), I] = 0$ with $A^2(I) \subseteq I$, what implies, since I is nondegenerate as an algebra and so in particular it has trivial center, that $A^2(I) = 0$. Now by the previous item we get that $a \in \mathrm{Ann}(I)$. Therefore $\mathrm{Ann}(I)$ is a nondegenerate ideal. $\square$

Thus, the theorem above applies to any ideal when L is nondegenerate and $6 \in \mathrm{TF}(L)$, by Lemma 2.1.4. This property is our principal tool to pass from one step of our proof to the next. Loosely speaking, we will find later that if $AX_1 \ldots X_k B = 0$ with $k \in \{0, 1\}$ then there exist $0 \neq c \in I(a)$, $0 \neq d \in I(b)$ such that $ad_{C^2(y)} X_1 \ldots X_{k+1} ad_{D^2(z)} = 0$, what will allow us to use the previous case by the technical lemma we show now.

Lemma 2.1.6 (**Technical Lemma**).
Let L be a nondegenerate Lie algebra such that $6 \in \mathrm{TF}(L)$ and let $n \in \mathbb{N}$ be fixed. Suppose that L has the following property: if $a, b \in L$ are such that $AX_1 \ldots X_n B = 0$ for every $x_1, \ldots, x_n \in L$, then $a \in \mathrm{Ann}(I(b))$. In that case L also satisfies the following property: if $a, b \in L$ are such that $ad_{A^2(y)} X_1 \ldots X_n ad_{B^2(z)} = 0$ for every $y, z, x_1, \ldots, x_n \in L$, then $a \in \mathrm{Ann}(I(b))$.

Proof. Since $ad_{A^2(y)} X_1 \ldots X_n ad_{B^2(z)} = 0$, by hypothesis we have that $A^2(y) \in \mathrm{Ann}(I(B^2(z)))$ for every $y, z \in L$. For a fixed z, denote $I_z := \mathrm{Ann}(I(B^2(z)))$. Then we know that $\overline{A^2(y)} = \bar{0}$ in the factor ring L/I_z. Since L is nondegenerate, $I(B^2(z))$ is nondegenerate as an algebra by Lemma 2.1.4 and then, by Theorem 2.1.5(2), I_z is nondegenerate and L/I_z is a nondegenerate Lie ring. Therefore $\overline{A^2(y)} = \bar{0}$ for every $y \in L$ implies $\bar{a} = \bar{0}$, that is, that $a \in I_z$ and a annihilates $I(B^2(z))$. Then it is true that $AX_1 \ldots X_n ad_{B^2(z)} = 0$

[4]Let $\vec{a} := (a_1, \ldots, a_n)^T \in L^n$ and $M \in \mathbb{M}_n(\mathbb{Z})$ be such that $M\vec{a} = 0$. M possess an adjoint matrix M^{adj}, which satisfies $M^{\mathrm{adj}}M = |M|I_n$. Then $M\vec{a} = 0$ implies $M^{\mathrm{adj}}M\vec{a} = |M|I_n\vec{a} = |M|\vec{a} = 0$. It is then enough that $|M| \in \mathrm{TF}(L)$ to assure $\vec{a} = 0$.

for every $z, x_1, \ldots, x_n \in L$. Repeating the argument in the other side we get that $b \in \mathrm{Ann}(I(a))$ (and therefore that $a \in \mathrm{Ann}(I(b))$). $\square$

Thus our strategy is more or less the following: given that $AX_1 \ldots X_k B = 0$, we will first find suitable c, d such that $ad_{C^2(y)} X_1 \ldots X_{k+1} ad_{D^2(z)} = 0$ and then, by the lemma above and the case $k+1$, we will able to conclude that $c \in \mathrm{Ann}(I(d))$, which will imply that $a \in \mathrm{Ann}(I(b))$.

The fact that cases with more variables between A and B are easier to tackle is caused by the Going Down Proposition ([García&Gómez'07, Proposition 1.3]).

Proposition 2.1.7 (Going Down).
Let L be a nondegenerate Lie algebra and let $a, b \in L$ be such that

$$AX_1 \ldots X_n B = 0$$

for every $x_1, \ldots, x_n \in L$. Then, if $0 \leq m \leq n$,

$$AX_1 \ldots X_m B = 0$$

for every $x_1, \ldots, x_m \in L$. In addition $[a, b] = 0$.

From this point of view, our endeavor can also be understood as an effort to find a converse Going Up Proposition, which would assure that if $AX_1 \ldots X_n B = 0$ for every $x_1, \ldots, x_n \in L$, then also $AX_1 \ldots X_m B = 0$ for $m \geq n$ and for every $x_1, \ldots, x_m \in L$: we would go, by the Going Down Proposition, from $AX_1 \ldots X_n B = 0$ up to $AB = 0$ and then, by $a \in \mathrm{Ann}(I(b))$, we would find $AX_1 \ldots X_m B = 0$ for every $m \in \mathbb{N}$.

2.2 $AXYB = 0$

The case with $AXYB = 0$ is the first we are able to prove directly, without resorting to a previous, better case. The following is [García&Gómez'07, Proposition 1.5]. We include its proof for completeness of our exposition.

Proposition 2.2.1 (Case $AXYB = 0$).

Let L be a nondegenerate Lie algebra and let $a, b \in L$ be such that $AXYB = 0$ for every $x, y \in L$. Then $a \in \mathrm{Ann}(I(b))$.

Proof. Consider the set $S := \{XYB(z) \mid x, y, z \in L\}$. Denote $I := \mathrm{Ann}(S)$. Note that $A(S) = 0$, so that $a \in I$. If we are able to show that I is an ideal of L, then we will have $I(a) \subseteq I$ and $\mathrm{Ann}(I) \subseteq \mathrm{Ann}(I(a))$. Moreover, by the Going Down Proposition, if $c \in I$ then $CXYB = 0$ implies $[c, b] = 0$, so that $b \in \mathrm{Ann}(I)$, and therefore $b \in \mathrm{Ann}(I(a))$.
It is clear that I is a submodule. Consider $c \in I$ and let us see that $[c, x] \in I$ for every $x \in L$, i.e., that $[C(L), S] = 0$. Let $x, y, z, w \in L$ be arbitrary. By the Going Down Proposition we know that $CB = BC = CXB = BXC = 0$. Moreover, since $CXYB(z) = 0$, it is $0 = ad_{CXYB(z)} = [C, [X, [Y, [B, Z]]]] = -CXYZB - BZYXC$ because all the other terms have no more than two variables between a C and a B. Hence $CXYZB = -BZYXC$. Now

$$ad_{ad_{C(x)}YZB(w)} = [[C, X], [Y, [Z, [B, W]]]] =$$

$$= CXYZBW - CXYZWB - (CXYB)WZ + CXYWBZ - (CXZB)WY +$$
$$+CXZWBY + (CXB)WZY - (CXWB)ZY - X(CYZB)W + XCYZWB +$$
$$+X(CYB)WZ - X(CYWB)Z + X(CZB)WY - X(CZWB)Y - X(CB)WZY +$$
$$+X(CWB)ZY - YZ(BWC)X + YZW(BC)X + Y(BWZC)X - YW(BZC)X +$$
$$+Z(BWYC)X - ZW(BYC)X - BWZYCX + W(BZYC)X + YZ(BWX)C -$$
$$-YZW(BXC) - YBWZXC + YW(BZXC) - ZBWYXC + ZW(BYXC) +$$
$$+BWZYXC - WBZYXC =$$

$$= \boldsymbol{C}XYZ\boldsymbol{B}W - \boldsymbol{C}XYZW\boldsymbol{B} + \boldsymbol{C}XYW\boldsymbol{B}Z + \boldsymbol{C}XZW\boldsymbol{B}Y + X\boldsymbol{C}YZW\boldsymbol{B} -$$
$$-\boldsymbol{B}WZY\boldsymbol{C}X - Y\boldsymbol{B}WZX\boldsymbol{C} - Z\boldsymbol{B}WYX\boldsymbol{C} + \boldsymbol{B}WZYX\boldsymbol{C} - W\boldsymbol{B}ZYX\boldsymbol{C}.$$

Observe that in $ad^2_{ad_{C(x)}YZB(w)}$ all the terms will have an internal factor with no more than two variables between B and C, and hence it equals 0; since L is nondegenerate, we conclude that $ad_{C(x)}YZB(w) = 0$ and $[C(L), S] = 0$, as we wanted to prove. □

2.3 $AXB = 0$

In order to find, from $AXB = 0$, the appropriate $c, d \in L$ such that $ad_{C^2(z)} XY ad_{D^2(w)} = 0$, we will use the identities stated below.

Proposition 2.3.1 (Identities).
Let L be a nondegenerate Lie algebra and let $a, b \in L$ be such that $AXB = 0$ for every $x \in L$. Let $x, y, z, w \in L$ be arbitrary. Then:

1. $AB = BA = BXA = 0$ *and* $[a, b] = 0$.
2. $AXYB = AYXB$.
3. $AXYB = BYXA$.
4. $A^2XYB = 0 = AXYB^2$.
5. $AXAYZB = 0 = BXBYZA$.
6. $A^2XYZB = 0 = AXYZB^2$.
7. $A^2XYZWB^2 = 0$.

Proof. We use constantly that $X[Y, Z]W = XYZW - XZYW$ implies
$XYZW = X[Y, Z]W + XZYW$.

1. $AB = 0 = BA$ and $[a, b] = 0$ are proved by a direct application of the Going Down Proposition.

   To show $BXA = 0$ note that

   $0 = ad_{AB(x)} = [A, [B, X]] = (AB)X - (AXB) - BXA + X(BA) = -BXA$.
2. By item (1), $AXYB = A([X, Y] + YX)B = A[X, Y]B + AYXB = AYXB$, since $[X, Y] = ad_{[x,y]}$.
3. By item (1),

$$AXYB = AX[Y, B] + (AXB)Y = AX[Y, B] = A[X, [Y, B]] + A[Y, B]X =$$

$$= A[X, [Y, B]] + (AYB)X - (AB)YX = A[X, [Y, B]] = [A, [X, [Y, B]]] + [X, [Y, B]]A =$$

$$= -ad_{AXB(y)} + XY(BA) - X(BYA) - Y(BXA) + BYXA = BYXA,$$

   since $AXB(y) = 0$ by item (1).

4. Multiplying item (3) by A on the left, $A^2XYB = (AB)YXA = 0$ by item (1). The case $AXYB^2 = 0$ is analogous.
5. By item (3), $AX(AYZB) = AX(BZYA) = (AXB)ZYA = 0$ due to item (1).
6. By items (5) and (1),

$$A^2XYZB = A[A,X]YZB + (AXAYZB) = A[A,X]YZB =$$

$$= A[[A,X],Y]ZB + AY[A,X]ZB =$$

$$= A[[A,X],Y]ZB + (AYAXZB) - AYX(AZB) = A[[A,X],Y]ZB =$$

$$= (A[[A,X],Y],Z]B) + AZ[[A,X],Y]B = AZ[[A,X],Y]B =$$

$$= (AZAXYB) - AZX(AYB) - AZY(AXB) + AZYX(AB) = 0.$$

7. By items (6) and (4),

$$A^2XYZWB^2 = A[A,X]YZWB^2 + AX(AYZWB^2) = A[A,X]YZWB^2 =$$

$$= (A[[A,X],Y]ZWB^2) + AY[A,X]ZWB^2 = AY[A,X]ZWB^2 =$$

$$= AY(AXZWB^2) - AYX(AZWB^2) = 0. \qquad \square$$

Proposition 2.3.2 (Case $AXB = 0$).
Let L be a nondegenerate Lie algebra such that $6 \in \mathrm{TF}(L)$ and let $a, b \in L$ be such that $AXB = 0$ for every $x \in L$. Then $a \in \mathrm{Ann}(I(b))$.

Proof. For every $x, y, z, w \in L$ we have:

$$ad_{A^2(x)}ZWad_{B^2(y)} = (A^2X + XA^2 - 2AXA)ZW(B^2Y + YB^2 - 2BYB) =$$

$$= A^2XZWB^2Y + XA^2ZWB^2Y - 2AXAZWB^2Y + A^2XZWYB^2 + XA^2ZWYB^2 -$$

$$-2AXAZWYB^2 - 2A^2XZWBYB - 2XA^2ZWBYB + 4AXAZWBYB = 0,$$

due to Proposition 2.3.1:

- $(A^2XZWB^2)Y = X(A^2ZWYB^2) = AX(AZWYB^2) = (A^2XZWB)YB = 0$ by item (6).

- $X(A^2ZWB^2)Y = 0 = X(A^2ZWB)YB$ by item (4).
- $(AXAZWB^2)Y = 0 = (AXAZWB)YB$ by item (5).
- $A^2XZWYB^2 = 0$ by item (7).

Then, since $ad_{A^2(x)}ZWad_{B^2(y)} = 0$ for every $x, y, z, w \in L$, by the case $AXYB$ (Proposition 2.2.1) and the Technical Lemma (2.1.6) we get that $a \in \mathrm{Ann}(I(b))$. □

2.4 $AB = 0$

By the Going Down Proposition, $AXB = 0$ implies $AB = 0$, but we have no guarantee yet that $AXB = 0$ can be recovered from $AB = 0$. That is the reason why the identities of Proposition 2.4.2 below are less strong that those of Proposition 2.3.1. This fact notwithstanding, when $AB = 0$ we can guarantee that terms with enough factors of A and B are zero in any nondegenerate Lie algebra. This is [García&Gómez'07, Proposition 1.2]; we state it without proof.

Proposition 2.4.1 (Mixing).
Let L be a nondegenerate Lie algebra and let $a, b \in L$ be such that $AB = 0$. Let $x_1, \ldots, x_k$ be a list of elements of L such that some of them are a or b, with at least one of each. Denote $n := |\{x_i \mid x_i = a, 1 \leq i \leq k\}| \geq 1$ and $m := |\{x_i \mid x_i = b, 1 \leq i \leq k\}| \geq 1$. If $k + 1 < 2(n + m)$ then

$$X_1X_2 \ldots X_n = 0.$$

The fact that L is nondegenerate helps to prove identities more restrictive than those of the Mixing Proposition. The first two items where proved in [García&Gómez'07, Lemma 1.1].

Proposition 2.4.2 (Identities).
Let L be a nondegenerate Lie algebra and let $a, b \in L$ be such that $AB = 0$. Let $x, y, z \in L$ be arbitrary.

1. *$BA = 0$ and $[a, b] = 0$.*
2. *$AXB = -BXA$.*

3. $AXB^2 = A^2XB = A^2XYB^2 = 0$.
4. $AXAYB = BXAYA$.
5. $AXYB^2 = AYXB^2$.
6. $A^2XYB = 2BXAYA + 2BYAXA - BXYA^2$ *and*

   $AXYB^2 = 2AYBXB + 2AXBYB - B^2YXA$.
7. $A^2XYZB^2 = 2AXAYBZB + 2AXAZBYB + 2AYAXBZB+$

   $+2AYAZBXB + 2AZAXBYB + 2AZAYBXB$.

Proof.

1. Since $AB = 0$, $ad^2_{[a,b]} = [A, B]^2 = (AB - BA)^2 = (BA^2) = B(AB)A = 0$. Since L is nondegenerate, this implies $[a, b] = 0$. Then $0 = ad_{[a,b]} = AB - BA$ implies $BA = 0$.
2. By item (1),

   $0 = ad_{AB(x)} = [A, [B, X]] = (AB)X - AXB - BXA + X(BA) = -AXB - BXA$, so that $AXB = -BXA$.
3. These identities are due to the Mixing Proposition.
4. By item (2), $AX(AYB) = -(AXB)YA = BXAYA$.
5. By items (2), (4) and (1),

$$A^2XYB = (A[A, X]YB) + AXAYB = A[[A, X], Y]B + AY[A, X]B + (AXAYB) =$$

$$= -B[[A, X], Y]A + (AYAXB) - AYX(AB) + BXAYA =$$

$$= -(BA)XYA + BXAYA + BYAXA - BXYA^2 + BYAXA + BXAYA =$$

$$= 2BXAYA + 2BYAXA - BXYA^2.$$

   The other case is analogous.
6. By items (5) and (3),

$$A^2XYZB^2 = A[A, X]YZB^2 + AXAYZB^2 =$$

$$= (A[[A, X], Y]ZB^2) + AY[A, X]ZB^2 + AXAYZB^2 =$$

$$= AZ[[A, X], Y]B^2 + AYAXZB^2 - AYX(AZB^2) + AXAYZB^2 =$$

$$= AZAXYB^2 - AZX(AYB^2) - AZY(AXB^2) + AYAXZB^2 + AXAYZB^2 =$$

$$AZ(AXYB^2) + AY(AXZB^2) + AX(AYZB^2).$$

Now we use item (6) and (4), taking into account that $AXB^2YZA = 0$, to find that the computation above equals

$$2AXAYBZB + 2AXAZBYB + 2AYAXBZB+$$

$$+2AYAZBXB + 2AZAXBYB + 2AZAYBXB. \qquad \square$$

Theorem 2.4.3 (**Case** $AB = 0$)**.**
Let L be a nondegenerate Lie algebra such that $6 \in \mathrm{TF}(L)$ and let $a, b \in L$ be such that $AB = 0$. Then $a \in \mathrm{Ann}(I(b))$.

Proof. Denote $I_x := \mathrm{Ann}(I(x))$. By the Mixing Proposition we have that $\mathrm{ad}_{A^3(x)}YB = [A,[A,[A,X]]]YB = 0$ for every $x, y \in L$ and therefore by the AXB case (Proposition 2.3.2) we get $A^3(x) \in I_b$ for every $x \in L$. This means that a is a Jordan element of L/I_b. Analogously, b is a Jordan element of L/I_a. We are going to show, thanks to the Fundamental Formula for Jordan elements, that $ad_{ad^2_{A^2(x)}(y)}Vad_{ad^2_{B^2(z)}(w)} = 0$ for every $x, y, z, w, v \in L$. We start by analyzing $ad_{A^2(x)}Yad_{B^2(z)}$. By items (3), (6), (7) and (4) of Proposition 2.4.2 we get:

$$ad_{A^2(x)}Yad_{B^2(z)} = (A^2X + XA^2 - 2AXA)Y(B^2Z + ZB^2 - 2BZB) =$$

$$= (A^2XYB^2)Z + X(A^2YB^2)Z - 2AX(AYB^2)Z + A^2XYZB^2 + X(A^2YZB^2)-$$

$$-2AXAYZB^2 - 2A^2XYBZB - 2X(A^2YB)ZB + 4AXAYBZB =$$

$$= A^2XYZB^2 - 2AX(AYZB^2) - 2(A^2XYB = ZB + 4AXAYBZB =$$

$$= 2AXAYBZB + 2AXAZBYB + 2AYAXBZB + 2AYAZBXB+$$

$$+2AZAXBYB + 2AZAYBXB - 4AXAYBZB - 4AXAZBYB-$$

$$-4AXAYBZB - 4AYAXBZB + 4AXAYBZB =$$

$$= 2A\boldsymbol{Z}AYB\boldsymbol{X}B - 2A\boldsymbol{X}AYB\boldsymbol{Z}B+$$

$$+2A\boldsymbol{Z}A\boldsymbol{X}BYB - 2A\boldsymbol{X}A\boldsymbol{Z}BYB+$$

$$+2AYA\boldsymbol{Z}B\boldsymbol{X}B - 2AYA\boldsymbol{X}B\boldsymbol{Z}B.$$

Note that in this expression the roles of X and Z are skew symmetric. So if we swap x and z we obtain

$$ad_{A^2(x)}Yad_{B^2(z)} = -ad_{A^2(z)}Yad_{B^2(x)}.$$

Therefore, if we take as arguments $X^2A^2(y)$ and $Z^2B^2(w)$ for $x, y, z, w \in L$, then for every $v \in L$ we get

$$ad_{A^2(X^2A^2(y))}Vad_{B^2(Z^2B^2(w))} = -ad_{A^2(Z^2B^2(w))}Vad_{B^2(X^2A^2(y))} = 0,$$

since $A^2Z^2B^2 = 0$ by the Mixing Proposition.

Now recall that a is a Jordan element of the factor algebra L/I_b. By the Fundamental Formula for Jordan elements (Lemma 2.1.1) there exists $c \in I_b$ such that $ad^2_{A^2(x)}(y) = A^2X^2A^2(y) + c$. Note that $c \in I(a)$. Analogously, there exists $d \in I_a \cap I(b)$ such that $ad^2_{B^2(z)}(w) = A^2Z^2A^2(w) + d$. Therefore

$$ad_{ad^2_{A^2(x)}(y)}Vad_{ad^2_{A^2(z)}(w)} = ad_{A^2(X^2A^2(y))}Vad_{B^2(Z^2B^2(w))} = 0,$$

since $CVD = 0$.

Now, by the AXB case (Proposition 2.3.2) and the Technical Lemma (2.1.6) we get that $A^2(x) \in I_{B^2(z)}$ for every $x, z \in L$. This implies that $ad_{A^2(x)}Yad_{B^2(z)} = 0$ for every $x, y, z \in L$ and then, by the same reasoning, we get $AXB = 0$ for every $x \in L$ and finally $a \in I_b$ by the AXB case. □

As a corollary we get the Going Up Proposition, which we may mix with the Going Down one.

Corollary 2.4.4 (Going Up and Down).
Let L be a nondegenerate Lie algebra such that $6 \in TF(L)$ and let $a, b \in L$ be such that there exists $n \in \mathbb{N}$ such that $AX_1 \dots X_n B = 0$. Then $AX_1 \dots X_m B = 0$ for every $m \in \mathbb{N}$.

To finish, another corollary, proven by the fact that the Kostrikin radical is the smallest ideal whose factor ring is nondegenerate.

Corollary 2.4.5 (Product inside the Kostrikin radical).
Let L be a nondegenerate Lie algebra such that $6 \in TF(L)$ and let $a, b \in L$ be such that $[a, [b, L]] \subseteq \mathcal{K}(L)$. Then $[I(a), I(b)] \subseteq \mathcal{K}(L)$.

Chapter 3

Inner ideals

Loosely speaking, an inner ideal of an structure endowed with some product is a substructure that absorbs quadratically the entire structure. In more precise terms:

Definitions 3.0.1 (Inner ideal).

- Let J be a Jordan triple system. A submodule B of J is an *inner ideal* if $\{B, J, B\} \subseteq B$, equivalently, if $P_B J \subseteq B$.
- If J is a Jordan algebra then its inner ideals are the inner ideals of J considered as a Jordan triple system.
- Let L be a Lie algebra. A submodule B of L is an *inner ideal* if $[B, [L, B]] \subseteq B$. In addition B is called *abelian* when $[B, B] = 0$.
- Let R be an associative algebra. A submodule B of R is an *inner ideal* if $BRB \subseteq B$. An inner ideal of R^+ will be called a *Jordan inner ideal* of R, and similarly an inner ideal of R^- will be called a *Lie inner ideal* of R.
- Let R have in addition a ring involution. By a *Jordan inner ideal* of K we will mean an inner ideal of K seen as a Jordan triple system, while by a *Lie inner ideal* of K we will refer to an inner ideal of K seen as a Lie algebra.

If R is associative, any inner ideal B is a Jordan inner ideal (since $BRB \subseteq B$ implies $bRb \subseteq B$ for every $b \in B$) but the converse is not true (this can be seen considering, for example, the Jordan inner ideal generated by two elements in the free associative algebra). An associative inner ideal is not necessarily a Lie inner ideal, and reciprocally. If J is a Jordan algebra and $b \in J$, then by the Fundamental Formula $U_{U_bJ}J \subseteq U_bU_JU_bJ \subseteq U_bJ$ and therefore U_bJ is an inner ideal, called a *principal inner ideal* of J. If L is a Lie algebra and $B \subseteq L$ is an abelian inner ideal, then every $a \in B$ is a Jordan element, since $A^3(L) = A(A^2(L)) \subseteq [a, B] \subseteq [B, B] = 0$. Conversely, if $a \in L$ is a Jordan element then, due to identities close to the Fundamental Formula for the A^2 operator (see 2.1.1), the submodule $A^2(L)$ can be proved to be an inner ideal as in the Jordan case, which is hence called a *principal inner ideal* of L and which in addition is abelian ([Fernández,García&Gómez'06, Lemma 2.7(i)], see Proposition 4.1.3 for a proof). Thus, a Lie algebra has abelian inner ideals if and only if it has Jordan elements.

The notion of inner ideal is important to classify and determine the structure of nonassociative algebras. Inner ideals appeared first in the Jordan setting (under the denomination of 'quadratic ideals', for a time). According to [StructureJordan, page 153], the concept was introduced in [Topping'65] for Jordan algebras of operators in Hilbert spaces. Jacobson then used it to develop an structure theory for Jordan algebras analogue to Artin's theory for associative algebras, substituting one-sided ideals by inner ideals. In [Jacobson'66] he showed that a simple nondegenerate unital Jordan algebra J which satisfies

a) the descending chain condition on inner ideals of the form U_eJ with e idempotent and

b) that every such U_eJ contains a minimal inner ideal,

is either a division algebra, H of a $*$-simple artinian ring, Clifford[1] or Albert[2]. Shortly after that, McCrimmon generalized these results to the quadratic setting in [McCrimmon'66], [McCrimmon'69], in which is known as the Second Structure Theorem: a simple non-

[1]The Jordan algebra $F \oplus V$ whose product comes from a symmetric bilinear form in a vector space V over F.

[2]The symmetric elements of the 3×3 matrices over the octonions endowed with the conjugate transpose involution.

degenerate unital quadratic Jordan algebra which satisfies a) and b) follows the same classification as above, throwing several 'quadratic' and 'ample subalgebra of' qualifiers in the proper places.

The inner ideal concept was exported to the Lie setting by Faulkner ([Faulkner'73]). After that Benkart, in her celebrated paper [Benkart'77], gave a characterization of classical Lie algebras by means of inner ideals: a simple, finite-dimensional Lie algebra over an algebraically closed field F of characteristic $p > 5$ is classical if and only if it is nondegenerate and has a nonzero abelian inner ideal. This generalized a previous result of Kostrikin ([Kostrikin'67]); Strade had given another, different generalization in [Strade'73]. Benkart's result was improved by Premet in [Premet'86], where he removed the existence of an inner ideal from the hypotheses (actually, by showing that any finite-dimensional algebra over F already has a one-dimensional inner ideal).

In her paper, Benkart also expressed her hope that an artinian theory for Lie algebras could be established building on inner ideals, as it had been done by Jacobson and McCrimmon for Jordan algebras. Fernández López, García and Gómez Lozano followed her proposal in [Fernández,García&Gómez'08]. They called a Lie algebra *artinian* if it satisfies the descending chain condition on inner ideals, and then showed that if L is a simple Lie algebra over a field of characteristic $p > 7$ then L is artinian and nondegenerate if and only if L is either a division Lie algebra, a simple exceptional Lie algebra, $[R, R]/Z([R, R])$ with[3] R a simple artinian associative algebra, or $[K, K]/([K, K] \cap Z(R))$ with R simple with socle and either $Z(R) = 0$ or $\dim_{Z(R)} R > 16$.

In the same vein, inner ideals serve also to construct a socle theory in the Lie setting. In [Draper,Fernández,García&Gómez'08] the socle of a nondegenerate Lie algebra is defined as the sum of all its minimal inner ideals, and it is shown to be an ideal which is a direct sum of simple ideals and which satisfies the descending chain condition on principal inner ideals. Moreover, every finite-dimensional classical Lie algebra is shown

[3] This kind of result is usually stated taking the factor ring modulo $Z(R) \cap [R, R]$, but for semiprime rings this ideal coincides with $Z([R, R])$ by [RingsInvolution, Lemma 1.1.8], which shows that if $a \in R$ centralizes $[R, R]$ then $a \in Z(R)$.

to coincide with its socle.

Recently, Baranov and Rowley generalized the Kostrikin-Strade-Benkart Theorem in [Baranov&Rowley'13]. It turns out that a simple, infinite-dimensional locally-finite Lie algebra over an algebraically closed field of characteristic 0 has a nonzero abelian inner ideal[4] if and only if it is of diagonal type (equivalently, a Lie subalgebra of a locally finite associative algebra). Shortly afterwards, but by quite different techniques, Hennig proved a similar result in positive characteristic ([Hennig'14]): a simple, infinite-dimensional locally-finite Lie algebra over an algebraically closed field of characteristic $p > 7$ is locally nondegenerate and has a nonzero abelian inner ideal if and only if it is of the form $[R, R]/Z([R, R])$ with R a simple locally finite associative algebra, or of the form $[K, K]$ with R as before. Observe that these generalizations of the Kostrikin-Strade-Benkart Theorem cannot suffer an improvement like the one Premet made for the finite-dimensional case, since by the result of Baranov and Rowley there exist simple locally-finite Lie algebras of characteristic 0 that do not contain minimal abelian inner ideals (namely, those which are not of diagonal type).

3.1 Classification results

Since inner ideals are important in the determination of the structure of nonassociative algebras, it is sensible to try to characterize and classify the inner ideals of those algebras. This has been done in several contexts, as we will briefly review below. Some of the tools in which those results are based are the relevant structure theorems, the determination of minimal and maximal inner ideals, the geometric model for prime rings with socle (detailed in Appendix A and Section 1.1.1), Herstein's Lie theory of associative structures as epitomized by [Herstein'61], and the combinatorial properties of Jordan elements, with special emphasis in Herstein Lemma (to which we devote the following

[4]The statement of their theorem does not claim that the inner ideal is abelian, but this can be checked following the proof.

section, 3.2). In addition, is also relevant that if L is a Lie algebra and B is an abelian inner ideal of L of finite length, then L possess a finite $\mathbb{Z}$-grading with B at one extreme (this was proved with the aid of grid theory in [Fernández,García,Gómez&Neher'07] and later by considerations of classical Lie theory in [Draper,Fernández,García&Gómez'12, Appendix], for classical Lie algebras). Another important tool is the subquotient of an abelian Lie inner ideal[5]:

Let B be an abelian inner ideal of a Lie algebra L. The *kernel* of B is defined as the submodule $\ker B := \{x \in L \mid [B,[B,x]] = 0\}$. The pair of submodules $Sub(B) := (B, L/\ker B)$, when equipped with the triple products

$$\{x, \bar{y}, z\} := [[x,y],z] \text{ for every } x,z \in B \text{ and } y \in L$$
$$\{\bar{x}, y, \bar{z}\} = \overline{[[x,y],z]} \text{ for every } x,z \in L \text{ and } y \in B,$$

become a Jordan pair called the *subquotient* of B (see [Fernández,García,Gómez&Neher'07, Lemma 3.2]). Due to this notion we can define a relation between Lie abelian inner ideals of different Lie algebras: if B and B' are abelian inner ideals of L and L' respectively, then B and B' are said to be *Jordan-isomorphic* if $\mathrm{Sub}_L B$ and $\mathrm{Sub}_{L'} B'$ are isomorphic as Jordan pairs.

The classification of inner ideals was started in the Jordan setting by McCrimmon, who in [McCrimmon'71] characterized them in quadratic Jordan algebras of finite capacity[6] by a case-by-case analysis based on the Second Structure Theorem. In particular he proved that if A is a regular artinian associative algebra then any inner ideal of A^+ is of the form eAf with e, f idempotents[7], while the inner ideals of H are either of the form eAe^*, or point spaces (which can only appear with involutions of symplectic type). Later, Neher ([Neher'91]) also characterized the inner ideals of these algebras, by the use

[5]This notion is modeled upon the similar notion for Jordan pairs, see [Fernández,García,Gómez&Neher'07, Lemma 3.1].

[6]A Jordan unital ring J is said to be *of finite capacity* when the identity element decomposes as a finite sum of orthogonal idempotents e_i such that every $U_{e_i}J$ is a division Jordan ring ([TasteJordanAlgebras, page 96]).

[7]Note that this implies that every Jordan inner ideal of A is an associative inner ideal.

of grid theory. Fernández López and García Rus ([Fernández&García'99]) extended the classification to nondegenerate quadratic Jordan algebras of infinite capacity by means of the geometric model. Their results can be interpreted as saying that if A is a simple associative algebra with socle then the Jordan inner ideals of A are of the form RL, with R and L a right and a left ideal of A, respectively[8], while the Jordan inner ideals of H are either of the form[9] $\tau(RR^*)$, or point spaces (which can only appear with involutions of symplectic type).

Benkart carried the classification of inner ideals to the Lie setting in [Benkart'76]. She proved that if A is a simple artinian ring with $\mathrm{char}(A) \neq 2, 3$ then the inner ideals of $[A, A]/Z([A, A])$ are of the form eAf, with e, f idempotents such that $fe = 0$.[10] She built on Herstein's Lie theory and on the properties of Jordan elements, translated part of the problem to the Jordan context and used the McCrimmon's classification previously mentioned. She also classified the inner ideals of $[K, K]/([K, K] \cap Z(A))$ when A is as before and in addition $\dim_{Z(A)} A > 16$, claiming that they are either of the form eKe^* or of a special type we nowadays call *Clifford*, which arises when $A = \mathbb{M}_n(Z(A))$ with the transpose involution, and which she described as the span of $\{e_{1i} - e_{i2}\}$ in some basis $\{e_{ij}\}$; however, a case was omitted from the classification: point spaces can also appear when the involution is of orthogonal type, as was recognized and mended in [Benkart&Fernández'09]. The next steps were the classifications in finitary simple Lie algebras of characteristic 0 ([Fernández,García&Gómez'06(2)]) and in Lie algebras arising from simple algebras with socle (carried as part of [Fernández,García&Gómez'08]),

[8] They write these inner ideals as $W_1 \otimes V_1$ with V_1, W_1 subspaces of a dual pair (V, W). Observe that $R := W \otimes V_1$ and $L := W_1 \otimes V$ are, respectively, a right and a left ideal of $\mathcal{F}_W(V)$, by Theorem A.1.3. By the Product Law for subspaces (A.1.2(3)) we have then that $W_1 \otimes V_1 = (W \otimes V_1)(W_1 \otimes V) = RL$, since $\langle V, W\rangle \neq 0$ by nondegeneracy.

[9] They write these inner ideals as $(V_1 \otimes V_1) \cap H$ with V_1 a subspace of a selfdual space V. By the Product Law for subspaces we can write $V \otimes V = (V \otimes V_1)(V_1 \otimes V) = RR^*$, since the involution is the adjoint and so $(u \otimes v)^* = \pm v \otimes u$ implies $(V \otimes V_1)^* = V_1 \otimes V$.

[10] Hence all inner ideals of this Lie algebra are abelian and associative inner ideals of A (and thus also Jordan inner ideals of A).

which are similar both in results and techniques. They apply the geometric model and a direct limit argument over the artinian case; since the artinian classification was actually incomplete at that moment, the point spaces were also missing from those classifications, omission also mended in [Benkart&Fernández'09]. In conclusion, if A is a simple associative algebra with socle and $\mathrm{char}(A) \neq 2, 3$, then every inner ideal of $[A, A]$ is[11] of the form RL with $LR = 0$, where R and L are[12], respectively, a right and a left ideal of A, while every Lie inner ideal of $[K, K]$ is either of the form $\kappa(RR^*)$ with $R^*R = 0$, a point space, or Clifford, the two last cases only possible with involutions of orthogonal type. The authors describe Clifford inner ideals in geometric terms as sets of the form $[x, H^{\perp}]$, where x is an isotropic vector and H is an associated hyperbolic plane (refer to 3.4.14 below for the corresponding definitions). In [Benkart&Fernández'09], these results are completed and reproven, using not only the geometric model but also the notion of subquotient, which allows to reduce the problem to the classification of the Jordan inner ideals of the subquotients of the maximal abelian inner ideals.

In [Draper,Fernández,García&Gómez'12] the authors classified the inner ideals of classical Lie algebras, i.e., of the simple finite-dimensional Lie algebras over an algebraically closed field of characteristic 0, extending the previous results of Benkart and Fernández López to include the exceptional Lie algebras (G_2, F_4, E_6, E_7 and E_8), but adopting a rather different approach. They exploited the $\mathbb{Z}$-gradings which arise from abelian inner ideals[13] and related them with root systems, expressing the inner ideals as sums of root spaces.

The next generalization step was achieved in [Fernández'14], where the Lie abelian inner ideals of a centrally closed prime ring were characterized as being either isotropic, standard or special (see Section 3.3 below for the definitions), by elementary algebraic consid-

[11] Observe that A is unital if and only if it is artinian. Therefore, if A is not artinian then $Z(A) = 0$ because A is simple and thus $[A, A] = [A, A]/([A, A] \cap Z(A))$.

[12] Hence all inner ideals of $[A, A]$ are abelian and associative inner ideals of A.

[13] In a simple finite-dimensional Lie algebra L every proper inner ideal is abelian and of finite length. Therefore, associated to every proper inner ideal B of L there is a finite $\mathbb{Z}$-grading of L which has B as an extreme.

erations and the aid of Herstein Lemma. Recently, in our paper [Brox,Fernández&Gómez(1)] we have developed a similar approach (including in addition computations with the geometric model) to classify the Lie abelian inner ideals of K of centrally closed prime rings, showing that they are either isotropic, standard, special or Clifford, and we have described Clifford inner ideals from a ring-theoretic point of view as sets of the form $\kappa((1-e)Ke)$, where e is a minimal $*$-orthogonal idempotent. These results are the subject of Section 3.4. Since they build on the previous ideas and results of [Fernández'14], we include also a summary of these in Section 3.3.

3.2 Herstein Lemma

In this section we pause to present a result (called **Herstein Lemma** by us) which is a fundamental tool when working with Jordan elements in an associative context, and whose thesis practically determines the whole structure of Lie abelian inner ideals, both in R and in K.

Given a Lie algebra L, an element $a \in L$ is called *adnilpotent* if its adjoint representation is a nilpotent derivation, i.e., if $A^n = 0$ in $\mathrm{End}(L)$ for some $n \in \mathbb{N}$. If n is the index of nilpotency of A, this is, if n is such that $A^n = 0$ but there exists $b \in L$ such that $A^{n-1}(b) = 0$, then we call n the *index of adnilpotency* of a. So, Jordan elements are adnilpotent elements of index at most 3, absolute zero divisors are adnilpotent elements of index at most 2, and central elements are the only adnilpotent elements of index 1. Loosely speaking, Herstein Lemma guarantees that, in sufficiently good conditions, every adnilpotent element of R or K decomposes as the sum of a nilpotent part and a central part, and furthermore, the index of nilpotency of its nilpotent part is bounded by (a function of) its index of adnilpotency.

Historically, this result has suffered several generalizations. Herstein ([Herstein'63]) proved it for simple rings of characteristic greater than the index of the adnilpotent element. We call these facts collectively **Herstein Lemma** because, up to our knowledge, Herstein was the first person to prove a result of this kind. A bit later, but apparently unaware of Herstein's paper, Jacobson proved the same result for central simple algebras, as communicated by Benkart in [Benkart'76, Theorems 3.1 and 3.2]. Later on, taking advantage of the properties of the extended centroid, Martindale and Miers ([Martindale&Miers'83]) extended the result of Herstein, showing among other things that if R is a centrally closed prime ring, $a \in R$ is an adnilpotent element of index n and $\mathrm{char}(R) > n$, then $a = v + z$ with $z \in \mathcal{C}$ and $v^{\lfloor \frac{n+1}{2} \rfloor} = 0$. Coming from a different path, Grezeszczuk in [Grzeszczuk'92] extended to semiprime rings important results on nilpotent derivations (those of [Kharchenko'78], [Chung'85]), which in particular imply Herstein Lemma for centrally closed semiprime rings.

The validity of Herstein Lemma for K was also studied by Martindale and Miers ([Martindale&Miers'91]), mostly by combinatorial manipulations, when R is a centrally closed prime ring of characteristic zero. They concluded that if K is not exceptional (see 1.3.10) and $a \in K$ is an adnilpotent element of index n, then either $a = v + z$ with $z \in \mathcal{C}$ and v a nilpotent element of index of nilpotency at most $\lfloor \frac{n+1}{2} \rfloor$, or the involution is of the first kind and $a^{\lfloor \frac{n+1}{2} \rfloor + 1} = 0$.

The techniques we have developed in Chapter 2 allow us to give a simple proof of Herstein Lemma for K based on Herstein Lemma for R. We will do a proof just for Jordan elements, which will be published in [Brox,Fernández&Gómez(1)]. The main idea is to view K inside $\langle K \rangle$ and use the technique described in Section 1.3. A similar argument for Jordan elements in simple rings was given by Benkart ([Benkart'76, Lemma 4.22]).

Proposition 3.2.1 (Herstein Lemma for Jordan elements).

Let R be a centrally closed prime ring with involution $*$ such that $\mathrm{char}(R) \neq 2, 3, 5$ and $[K, K] \neq 0$, and let $a \in K$ be a Jordan element of K. Then:

1. *If the involution is of the second kind then $a = v + z$, where $z \in \mathrm{Skew}(\mathcal{C}, *)$ and $v^2 = 0$.*
2. *If the involution is of the first kind then $a^3 = 0$. Moreover, if $a^2 \neq 0$ then a^2 is a*

reduced element and R has socle and involution of orthogonal type.

Proof. Let $a \in K$ be such that $A^3(K) = 0$. We consider first an involution of the second kind, so there exists $0 \neq \lambda \in \operatorname{Skew}(\mathcal{C}, *)$. Then a is also a Jordan element of R, since $R = K \oplus \lambda K$ by Lemma 1.2.1(2) and thus $A^3(R) = A^3(K) + \lambda A^3(K) = 0$. By Herstein Lemma for prime rings ([Martindale&Miers'83, Corollary 1]) applied to R, it suffices $\operatorname{char}(R) > 3$ to get that $a = v + z$ with $v^2 = 0$ and $z \in \mathcal{C}$. Since $a^* = -a$ we have $v^* + z^* = -v - z$, so $v^* = -v - z - z^*$ and thus $[v^*, v] = [v, -v - z - z^*] = 0$, i.e., v and v^* are nilpotent elements which commute. Therefore $v^* + v = -z - z^* \in \mathcal{C}$ is a central nilpotent element and hence 0 because R is prime, which forces $v^* = -v$ and $z^* = -z$. Consider now an involution of the first kind. We work in $\langle K \rangle$, the subring generated by K, which is centrally closed prime with extended centroid $\mathcal{C}$ by Theorem 1.3.2 and which has the same characteristic as R. Recall that $\langle K \rangle = K + K \circ K$ and that $K \circ K$ equals the subgroup generated by $\{k^2 \mid k \in K\}$ (Lemma 1.3.1). By Leibniz Rule, $A^5(k^2) = \sum_{i=0}^{5} \binom{5}{i} A^i(k) A^{5-i}(k) = 0$ for every $k \in K$ since $A^3(K) = 0$ and in every summand either $i \geq 3$ or $5 - i \geq 3$. Hence $A^5(\langle K \rangle) = A^5(K) + A^5(K \circ K) = 0$. By Herstein Lemma for prime rings applied to $\langle K \rangle$, it suffices $\operatorname{char}(R) > 5$ to get that $a = v + z$ with $v \in \langle K \rangle$ such that $v^3 = 0$ and $z \in \mathcal{C}(\langle K \rangle) = \mathcal{C}$. Since $a = -a^*$, by the same reasoning as before we get $z \in \operatorname{Skew}(\mathcal{C}, *) = 0$, so that $a = v$ and hence $a^3 = 0$. Now suppose $a^2 \neq 0$. Then $a^2 \in H$ and $a^2 K a^2 = 0$ because $0 = A^4(k) = -6a^2 k a^2$ for every $k \in K$ and $\operatorname{char}(R) \neq 2, 3$. Then, by the Reduction Lemma 1.2.4(2), $hRh = \mathcal{C}h$ and R (which, being centrally closed, equals $\mathcal{C}R$) has socle and involution of orthogonal type. □

The hypothesis $[K, K] \neq 0$ of the previous theorem is not superfluous. Recall from Theorem 1.3.9 that if $[K, K] = 0$ then either R is commutative or $\widehat{R}$ is a central simple algebra of dimension 4 over $\mathcal{C}$ with involution of the first kind and transpose type. The case with R commutative is trivially uninteresting, for then every element is central. We show that Herstein Lemma is false for noncommutative centrally closed prime rings with $[K, K] = 0$, without using explicitly the matrix structure of $\overline{R}$.

Counterexample 3.2.2 (K exceptional).
Let R be a noncommutative centrally closed prime ring with involution such that $[K, K] = 0$. Let us show that no nonzero adnilpotent satisfies the conclusion of Herstein Lemma. Pick a nonzero adnilpotent $a \in K$ (i.e., any nonzero element), which is necessarily of index 1. By Theorem 1.3.9 the involution is of the first kind, and therefore $a \notin \mathcal{C}$. By Lemma 1.4.1, $K^2 \subseteq \mathcal{C}$ and therefore $a^2 \in \mathcal{C}$. Suppose we can decompose $a = v + \lambda$ with v nilpotent and $\lambda \in \mathcal{C}$. Then $v \neq 0$, for otherwise we would have $a = z \in \mathcal{C}$, a contradiction. On the other hand $a^2 = (v + \lambda)^2 = v^2 + 2\lambda v + \lambda^2$, so that $v^2 + 2\lambda v = a^2 - \lambda^2 \in \mathcal{C}$. But v^2 and $2\lambda v$ are two nilpotent elements which commute, and thus $v^2 + 2\lambda v$ is nilpotent and central, so $v^2 = -2\lambda v$ and $a^2 = \lambda^2$. Consider the index of nilpotency k of v. Note that $k \geq 2$. Then $0 = v^k = v^{k-2}v^2 = v^{k-2}(-2\lambda v) = -2\lambda v^{k-1}$, contradicting the minimality of k unless $\lambda = 0$ (recall that we always have $\mathrm{char}(R) \neq 2$). Then $a = v$ is an element of zero square. We already know that $[a, K] = 0$; hence $0 = [a, k]a = aka - ka^2 = aka$ for every $k \in K$. Therefore $aKa = 0$ and, by the Reduction Lemma 1.2.4(1), $a = 0$. □

3.3 Lie abelian inner ideals of R centrally closed prime

For simplicity **we call an abelian Lie inner ideal just a Lie inner ideal**. In this section we present the classification of Lie inner ideals of a centrally closed prime ring, which appeared in [Fernández'14]. We are not going to elaborate in the details and reasonings which lead to these definitions and results, since in our next section we will present an study of the Lie inner ideals of K, study which parallels in a good amount the one of that paper (although with the bit more of casuistic with which K always treats us). So most of the commentaries of the next section would apply here.

Definitions 3.3.1 (**Types of Lie inner ideal**).

Let R be a semiprime algebra.

- A Lie inner ideal V of R is said *isotropic*[14] if $V^2 = 0$.
- Suppose that $Z(R) \neq 0$. Let V be an isotropic inner ideal and $0 \neq \Omega$ be a submodule of $Z(R)$. Then the submodule $V \oplus \Omega$ is a nonisotropic Lie inner ideal said to be an *standard inner ideal*.
- Suppose in addition that R is unital. Let V be an isotropic inner ideal and let $f : V \to Z(R)$ be a functional such that $[V, [V, R]] \subseteq \ker f$. Then the set $\mathrm{inn}(V, f) := \{v + f(v) \mid v \in V\}$ is a nonisotropic and nonstandard Lie inner ideal called *special*[15].

As it happens, those types of Lie inner ideals are the only ones that can appear in a centrally closed prime ring ([Fernández'14, Theorem 5.4]).

Theorem 3.3.2 (**Classification of Lie inner ideals of** R **centrally closed prime**).
Let R be a centrally closed prime algebra such that $\mathrm{char}(R) \neq 2, 3$ and let B be a Lie inner ideal of R. Then either

1. $B = V$ *is isotropic,*
2. $B = V \oplus \mathcal{C}$ *with V isotropic, or*
3. $B = \mathrm{inn}(V, f)$ *is special.*

Since R is centrally closed and $\Gamma = \mathcal{C}$ is a field, either $Z(R) = 0$ or $Z(R) = \mathcal{C}$. Hence in the previous theorem the standard and special cases can only occur if R is unital (i.e., if R has no identity element then all Lie inner ideals are isotropic).

In the same paper it was proved that if a semiprime algebra R over a field has elements of zero square which are not von Neumann regular then it has special inner ideals

[14] In [Fernández'14] these inner ideals were originally called *Jordan-Lie*, but we have abandoned this denomination because in the classification of K do appear other inner ideals that are Jordan and Lie at the same time besides the isotropic ones, the Clifford inner ideals.

[15] In [Fernández'14] these inner ideals were originally called *non-standard*, but we have abandoned this denomination because in the classification of K do appear other inner ideals that are not standard besides the special ones, the Clifford inner ideals.

([Fernández'14, Corollary 4.2]). The converse is also true; see Proposition 3.4.9 below, which is straightly adapted to this context. As an aside, we provide here an specific example of special inner ideal based on Weyl algebra (refer to [Coutinho, Chapters 1&2] and [Lam1, Examples 1.3c)]), which was not present in that paper.

Example 3.3.3 (Special inner ideal of R).
Given a field F and a polynomial ring $F[Y]$, we call the *Weyl algebra* $\mathbb{A}_1(F)$ over F to the differential polynomial ring $(F[Y])[X;\delta]$ (see [Lam1, Example 1.9]), where δ denotes the derivative operator of R. $\mathbb{A}_1(F)$ is simple when $\mathrm{char}(F) = 0$ ([Coutinho, Theorem 2.1]) and is not von Neumann regular because a degree can be defined for Weyl algebra just like for usual polynomial rings (see [Beachy, Proposition 1.5.13(b)]).
Let F be a field with $\mathrm{char}(F) = 0$ and consider the F-algebra $R := \mathbb{M}_2(\mathbb{A}_1(F))$. R is simple since $\mathbb{A}_1$ is simple. Therefore R is centrally closed, i.e., $\mathcal{C} = F$. The element $a := \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}$ has zero square and is not von Neumann regular, since $aRa = \begin{pmatrix} 0 & X\mathbb{A}_1X \\ 0 & 0 \end{pmatrix}$ and X is not von Neumann regular in $\mathbb{A}_1$. Then $V := Fa \oplus aRa = \begin{pmatrix} 0 & FX \oplus X\mathbb{A}_1X \\ 0 & 0 \end{pmatrix}$ endowed with $f : V \to F$ linear such that $f(X\mathbb{A}_1X) := 0$ and $f(X) := 1$ generates the special inner ideal

$$\mathrm{inn}(V, f) = \left\{ \begin{pmatrix} \lambda & \lambda X + XpX \\ 0 & \lambda \end{pmatrix} \mid p \in \mathbb{A}_1, \lambda \in F \right\}.$$

3.4 Lie abelian inner ideals of K for R centrally closed prime

Throughout this section we present the results of our paper [Brox,Fernández&Gómez(1)], in which we classified the Lie abelian inner ideals of K for a centrally closed prime ring R with $\mathrm{char}(R) \neq 2, 3, 5$, following the main scheme crafted in [Fernández'14] and also building on its results. From now on, let R be an algebra with involution $*$ (and, as always, $\frac{1}{2} \in \Gamma$) over the ring of scalars with involution $(\Phi, *)$. **By a Lie inner ideal of K we will always understand an abelian inner ideal** of the Lie $\mathrm{Sym}(\Phi, *)$-algebra K.

As we know, any element of a Lie inner ideal is a Jordan element. If R is a centrally closed prime ring such that $\mathrm{char}(R) \neq 2, 3, 5$, then by Herstein Lemma (3.2.1) every Jordan element $a \in K$ is either of zero cube or decomposes as $v + \lambda$ with $v^2 = 0$ and $\lambda \in \mathrm{Skew}(\mathcal{C}, *)$ (which may be zero). It is therefore natural to try to carry this information to the Lie inner ideals that contain the Jordan elements, task that resolves with a positive balance, since there comes to exist a great similarity between the structure of Jordan elements and the classification of Lie inner ideals.

3.4.1 Standard inner ideals

In this section R will denote a semiprime algebra.

Since the starting point should be the easiest one, we begin by analyzing those Lie inner ideals full of (Jordan) elements of zero square.

Lemma 3.4.1 (Isotropy).
Let V be a submodule of K.

1. *$V^2 = 0$ in R if and only if V is commutative and $v^2 = 0$ for every $v \in V$.*
2. *If $V^2 = 0$ then $\{u, k, v\} = [[u, k], v]$ for every $u, v \in V$ and $k \in K$. In particular, V is a Jordan inner ideal if and only if it is a Lie inner ideal.*

Proof.

1. If $V^2 = 0$ then $uv = 0 = vu$ for every $u, v \in V$, so obviously $[u, v] = 0$ and $v^2 = 0$. On the other hand, if V is commutative then for every $u, v \in V$ we have $(u + v)^2 = u^2 + v^2 + 2uv$ and since $x^2 = 0$ for every $x \in V$ we get $2uv = 0$, which implies $uv = 0$ since $2 \in TF(R)$.
2. By item (1) we know that V is abelian. In addition, for every $u, v \in V$ and $k \in K$ we have $[[u, k], v] = ukv - k(uv) - (vu)k + vku = ukv + vku = \{u, k, v\}$. □

As the nilpotent elements make up the founding basis of Herstein Lemma, so these inner ideals which consist only of elements of zero square are the cornerstone of our classification. Let us give them a name.

Definition 3.4.2 (Isotropic inner ideal).
A Lie inner ideal V of K such that $V^2 = 0$ will be called an *isotropic inner ideal*.

Next we bring central elements into the equation. Since we are treating with algebras which are not necessarily unital, it may be the case that $Z(R) = 0$. Suppose on the contrary that $Z(R) \neq 0$ and denote $K(Z) := \mathrm{Skew}(Z(R), *)$. If V is an isotropic inner ideal and Ω is a $\mathrm{Sym}(\Phi, *)$-submodule of $K(Z)$, then $B := V + \Omega$ is clearly a Lie inner ideal of K, since $[B, [B, K]] = [V + \Omega, [V + \Omega, K]] = [V, [V, K]] \subseteq V$. In addition, the sum $V + \Omega$ is direct because V is full of nilpotent elements and $Z(R)$ does not contain any of them, by semiprimeness of R. Let us set another name.

Definition 3.4.3 (Standard inner ideal).
A Lie inner ideal B of K will be called *standard* if $B = V \oplus \Omega$, where V is an isotropic inner ideal of K and Ω is a $\mathrm{Sym}(\Phi, *)$-submodule of $K(Z)$.

Given an arbitrary Lie inner ideal we would like to attach an isotropic one to it. We could try and just take its subset of zero square elements, but in view of Herstein Lemma it is better to take also into account its skew central elements.

Definition 3.4.4 (V_B, isotropic set attached to B).
Given a Lie inner ideal B we denote $V_B := \{v \in B + K(Z) \mid v^2 = 0\}$.

As we see, in V_B we are essentially[16] considering the nilpotent parts of all Jordan elements of B of the kind $v+z$, $v^2=0$, $z \in \mathcal{C}$ which actually have their central parts inside $Z(R)$.

Let us see when can we assure that V_B is an inner ideal.

Lemma 3.4.5 (Properties of V_B).
Let B be a Lie inner ideal such that $B \subseteq V_B + K(Z)$.
Then V_B is an isotropic inner ideal such that $[B,[B,K]] \subseteq \{V_B, K, V_B\} \subseteq B$.

Proof. We will show that V_B is a submodule of K. It is clear by the definition of V_B that $\mathrm{Sym}(\Phi,*)V_B \subseteq V_B$. Let us see that V_B is also a subgroup of K. Since $B+K(Z)$ is a subgroup of K,

$$V_B+V_B \subseteq (B+K(Z))+(B+K(Z)) \subseteq B+K(Z) \subseteq (V_B \oplus K(Z))+K(Z) = V_B \oplus K(Z).$$

Thus for any $u,v \in V_B$ there exist $w \in V_B$ and $z \in K(Z)$ such that $u+v=w+z$, with $z=0$ since $u+v-w$ is a central nilpotent element and R is semiprime. This proves that $V_B+V_B \subseteq V_B$, hence V_B is a subgroup of K. Now let us see that V_B is an isotropic inner ideal. By Lemma 3.4.1 $uv=0$ for every $u,v \in V_B$ and $\{u,k,v\} = ukv+vku = [[u,k],v]$ for every $k \in K$. In addition $\{u,k,v\}^2=0$ since $uv=vu=0$. Hence $\{u,k,v\} = [[u,k],v] \in [[B+K(Z),K],B+K(Z)] = [[B,K],B] \subseteq B$ with $\{u,k,v\}^2=0$, so that $\{u,k,v\} \in V_B$ by definition. This proves that V_B is an isotropic inner ideal of K satisfying $\{V_B,K,V_B\} \subseteq B$. Note also that by hypothesis $[B,[B,K]] \subseteq [V_B+K(Z),[V_B+K(Z),K]] = \{V_B,K,V_B\}$. $\square$

V_B can be used to determine if B is standard.

Theorem 3.4.6 (Characterization of standard inner ideals).
A Lie inner ideal B is standard if and only if the following condition holds:

$$V_B \subseteq B \subseteq V_B + K(Z). \tag{ST}$$

[16] Note that as of today Herstein Lemma has not been proved for K of a semiprime ring.

Proof. Suppose first that $V_B \subseteq B \subseteq V_B + K(Z)$. Then by Lemma 3.4.5 V_B is an isotropic inner ideal. Since $V_B \subseteq B$, by the Modular Law we have

$$B = B \cap (V_B + K(Z)) = V_B \oplus (B \cap K(Z)),$$

so B is standard by definition.

Now suppose that B is standard, i.e., $B = V \oplus \Omega$ where V is an isotropic inner ideal and Ω a $\mathrm{Sym}(\Phi, *)$-submodule of $K(Z)$. Let us show that $V_B = V$. Since V_B contains in particular all the zero square elements of B we get $V \subseteq V_B$. On the other hand, $V_B \subseteq B + K(Z) = (V \oplus \Omega) + K(Z) = V \oplus K(Z)$. Thus for every $u \in V_B$ there exist $v \in V$, $z \in K(Z)$ such that $u = v + z$. Hence $u - v$ is a central nilpotent element and therefore $u = v$ since R is semiprime. This proves that $V_B \subseteq V$. Thus $B = V_B \oplus \Omega$ and therefore it satisfies **(ST)**. $\square$

3.4.2 Special inner ideals

In this section R will denote a unital semiprime algebra whose involution does not act as the identity on its center, i.e., such that $K(Z) \neq 0$.

In the definition of standard ideal we have allowed the elements of zero square and the central elements to dance with each other freely, but we can also have inner ideals B whose Jordan elements a are such that $a = v_a + z_a$ with $0 \neq z_a \in K(Z)$, $v_a^2 = 0$ and $v_a v_b = 0 = v_b v_a$ for every $a, b \in B$, but not necessarily $v_a, z_a \in B$. Since $[a,[b,K]] = [v_a + z_a, [v_b + z_b, K]] = [v_a, [v_b, K]]$ and we need $[a,[b,K]] \subseteq B$, a good way to guarantee this is to ask for $v_a \in B$ for some a's and ask for $[B,[B,K]] \subseteq \{v_a \mid v_a \in B\}$. This motivates the following definition.

Definition 3.4.7 (Special inner ideal).

Let V be a nonzero isotropic inner ideal of K and let $f : V \to K(Z)$ be a nonzero linear map such that $[V,[V,K]] \subseteq \ker f$. We define $\mathrm{inn}(V, f) := \{v + f(v) \mid v \in V\}$ and call it an *special inner ideal.*

Let us see that special inner ideals are in fact inner ideals.

Theorem 3.4.8 (Special are inner ideals).
$\mathrm{inn}(V, f)$ is a Lie inner ideal of K which is not standard and such that $V_B = V$.

Proof. Set $B := \mathrm{inn}(V, f)$. Then:

1. B is a Lie inner ideal of K.

   Observe that if $u \in V$ then $[u + f(u), k] = [u, k]$ for every $k \in K$ because $\mathrm{im} f \subseteq K(Z)$. Then, since for every $b \in B$ we have $b = u + f(u)$ for some $u \in V$, we get $[B, [B, K]] = [V, [V, K]] \subseteq \ker f \subseteq B$, the last inclusion due to $v = v + f(v) \in B$ for every $v \in \ker f$. In addition $[B, B] = [V, V] = 0$.

2. $V \cap B = \ker f$.

   Note that $\ker f \subseteq B$ is shown in (1) and that $\ker f \subseteq V$ by definition of f. Hence $\ker f \subseteq V \cap B$. Now let $v \in V \cap B$. Then $v = u + f(u)$ for some $u \in V$ and thus $v - u = f(u) \in V \cap Z(R) = 0$, so $v = u \in \ker f$.

3. $V_B = V$.

   By definition of B we have $V \subseteq B + K(Z)$, and since $V^2 = 0$ we get $V \subseteq V_B$ by the very definition of V_B. Conversely, pick $u = b + z \in V_B$ with $b = v + f(v)$ for some $v \in V$ and $z \in K(Z)$. Then $u = v + f(v) + z$ implies $[v, u] = [v, v + f(v) + z] = 0$, so $u - v = f(v) + z$ is a central nilpotent element and hence is zero because R is semiprime, which forces $u = v \in V$.

4. B is not standard.

   By Theorem 3.4.6 it is enough to show that V_B is not contained in B, and by (3) we have $V_B = V$. Suppose otherwise that $V \subseteq B$. Then by (2) we get that $V = V \cap B = \ker f$, which yields a contradiction with $f \neq 0$. □

Actually, how special are special inner ideals? The following proposition gives an answer.

Proposition 3.4.9 (Characterization of K with special inner ideals).
Let Φ be an integral domain. Then K contains a special inner ideal if and only if there exists an element $v \in K$ which is of zero square and such that $vKv \cap \Phi v = 0$.

Proof. Let $B := \mathrm{inn}(V, f)$ be a special inner ideal. By Lemma 3.4.1(2) and by the definition of special inner ideal we get that $P_V K = \{V, K, V\} = [V, [V, K]] \subseteq \ker f$. Pick $v \in V$ such that $f(v) \neq 0$ and let $k \in vKv \cap \Phi v$. Then $k = \lambda v$ for some $\lambda \in \Phi$, but $\lambda f(v) = f(\lambda v) = f(k) \in f(vKv) \subseteq f(P_V K) = 0$. This implies $\lambda = 0$, since $f(v) \neq 0$ and Φ is an integral domain. Therefore $k = 0$. Note also that any $v \in V$ is of zero square.

Now suppose conversely that there exists $v \in K$ such that $v^2 = 0$ and $vKv \cap \Phi v = 0$. Then the sum $\Phi v + vKv$ is direct and it is easily checked that $V := \mathrm{Sym}(\Phi, *)v \oplus vKv$ is an isotropic inner ideal of K. Given a nonzero $z \in K(Z)$ (which exists because $*$ does not act as the identity in $Z(R)$, by assumption), consider the additive map $f : V \to K(Z)$ defined by $f(vKv) = 0$ and $f(v) = z$. Then $\mathrm{inn}(V, f)$ is a special inner ideal of K, since $[V, [V, K]] = vKv = \ker f$ and $f \neq 0$. $\square$

In particular, if Φ is a field, then K contains an special inner ideal if and only if it contains a nonzero element of zero square which is not von Neumann regular, since in that case $vKv \cap \Phi v \neq 0$ implies v regular ($vkv = \lambda v$ with $0 \neq \lambda \in \mathrm{Sym}(\Phi, *)$, then $v\lambda^{-1}kv = v$).

Despite the fact that special inner ideals are not standard, isotropic inner ideals and special inner ideals are the same kind of thing from the Jordan point of view.

Proposition 3.4.10 (Special are Jordan-isomorphic to isotropic).
The Lie inner ideals V and $\mathrm{inn}(V, f)$ are Jordan-isomorphic for every suitable map f.

Proof. Set $B := \mathrm{inn}(V, f)$ for some arbitrary but fixed f. Since $b \in B$ implies $b = v + f(v)$ with $v \in V$ and $f(v) \in K(Z)$, and $\ker B = \{x \mid [B, [B, x]] = 0\}$, it is clear that $\ker B = \ker V$. Denote $\overline{K} := K/\ker B = K/\ker V$. Then $\mathrm{Sub}V = (V, \overline{K})$ and $\mathrm{Sub}B = (B, \overline{K})$. We claim that the pair of linear maps $(\varphi, \mathrm{id}_{\overline{K}}) : \mathrm{Sub}V \to \mathrm{Sub}B$ is an isomorphism of Jordan pairs, where $\varphi(v) := v + f(v)$ and $\mathrm{id}_{\overline{K}}$ is the identity on $\overline{K}$. Clearly $\varphi : V \to B$ is a linear isomorphism, and for $u, v \in V$ and $x, y \in K$, we have

$$\varphi(\{u, \overline{x}, v\}) = [[u, x], v] + f([[u, x], v]) = [[u, x], v] = [[u + f(u), x], v + f(v)] = \{\varphi(u), \overline{x}, \varphi(v)\}$$

since $[[V,K],V] \subseteq \ker f$ and $f(V) \subseteq K(Z)$, and

$$\{\overline{x}, v, \overline{y}\} = \overline{[[x,v],y]} = \overline{[[x,v+f(v)],y]} = \overline{[[x,\varphi(v)],y]} = \{\overline{x}, \varphi(v), \overline{y}\},$$

which completes the proof. $\square$

Now we provide an specific example of special inner ideal.

Example 3.4.11 (**Special inner ideal of K**).
Consider the field $F := \mathbb{Z}_3[i]$, where i is a root of X^2+1, and the F-algebra $R := \mathbb{M}_2(F[X])$, where $F[X]$ is the ring of polynomials in one variable over F. Since $F[X]$ is prime, R is prime, although R is not centrally closed: the center of R is isomorphic to $F[X]$, which is not a field and hence cannot be isomorphic to $\mathcal{C}$. The base field possess a *conjugation* involution $\overline{x+yi} := x - yi$ for $x, y \in \mathbb{Z}_3$. Observe that $\mathrm{Sym}(F,^{-}) = \mathbb{Z}_3$ and $\mathrm{Skew}(F,^{-}) = \mathbb{Z}_3 i$. We can extend the involution from F to $F[X]$ in a straight way, defining $\overline{\sum_{k=1}^{n} c_k X^k} := \sum_{k=1}^{n} \overline{c_k} X^k$, and then we can further extend it to R by taking the conjugate tranpose, so that

$$\text{if } a := \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \text{ with } p_{ij}(X) \in F[X], \text{ then } a^* := \begin{pmatrix} \overline{p_{11}} & \overline{p_{21}} \\ \overline{p_{12}} & \overline{p_{22}} \end{pmatrix}.$$

Then it is directly checked that $K := \left\{ \begin{pmatrix} p_1 & q \\ -\overline{q} & p_2 \end{pmatrix} \right\}$, where p_1, p_2 are polynomials whose coefficients are skew with respect to conjugation, i.e., purely imaginary. Since $Z(R) \cong F[X]$ we get that $K(Z) \neq 0$. For example, $z := \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$ lies in $K(Z)$.
Since F is a field, by the observation after Proposition 3.4.9, to find an special inner ideal it is enough to find an skew element of zero square not von Neumann regular. Consider the element $a := \begin{pmatrix} i & 1+i \\ -1+i & -i \end{pmatrix} X$, which lies in K. The modulo 3 restriction guarantees that $a^2 = 0$:

$$a^2 = \begin{pmatrix} i & 1+i \\ -1+i & -i \end{pmatrix} X \begin{pmatrix} i & 1+i \\ -1+i & -i \end{pmatrix} X =$$

$$= \begin{pmatrix} i^2+(i+1)(i-1) & (1+i)i-(1+i)i \\ (-1+i)i-(-1+i)i & (i+1)(i-1)+i^2 \end{pmatrix} X^2 = 0,$$

since $i^2+(i+1)(i-1) = i^2+i^2-1 = -3 = 0$. The polynomial degree guarantees that a is not von Neumann regular: since we have $aRa \subseteq RX^2$, if $a \in aRa$ then $a = bX^2$ for some $b \in R$ and so $iX = a_{11} = b_{11}X^2$ with $b_{11} \in F[X]$, an impossibility. Therefore the isotropic inner ideal $V := \mathbb{Z}_3 a + aKa$ endowed with $f : V \to K(Z)$ linear such that $f(a) := z$ and $f(aKa) := 0$ generates the special inner ideal

$$\operatorname{inn}(V, f) = \{M_{p_1,p_2,q}X^2 \mid p_1, p_2 \in \operatorname{Skew}(F[X], ^-), q \in F[X]\} \oplus \{\lambda M \mid \lambda \in \mathbb{Z}_3\}$$

with $M_{p_1,p_2,q} := \begin{pmatrix} p_1-2p_2-\kappa(q)+\tau(q)i & (p_1+p_2)(1-i)+q-2\overline{q}i \\ (p_1+p_2)(1+i)-\overline{q}-2qi & 2p_1-p_2+\kappa(q)+\tau(q)i \end{pmatrix}$ and

$M := \begin{pmatrix} i(1+X) & (1+i)X \\ (-1+i)X & i(1-X) \end{pmatrix}$.

3.4.3 Clifford inner ideals

In our approach to define inner ideals from the properties of their Jordan elements, as collected in Herstein Lemma, we finally arrive to those Jordan elements $a \in K$ such that $a^2 \neq 0$ is minimal in R and $a^3 = 0$. In a centrally closed prime ring those are only possible if R has socle and involution of orthogonal type, what implies by Kaplansky Theorem (1.1.7) that R is isomorphic to an algebra of endomorphisms of a selfdual[17] space X over a field F, equipped with a nondegenerate symmetric bilinear form $\langle \cdot , \cdot \rangle$, being the involution on R the adjoint involution inherited from $\mathcal{L}(X)$. In this section we will restrict to that kind of algebras, and moreover we will suppose that $\dim_F X \geq 3$, in order to get the so-called Clifford inner ideals, which are associated to Jordan elements such that $a^2 \neq 0$. Since Clifford inner ideals already appear in the simple with socle case (see Section 3.1), we may restrict ourselves mostly to an study inside $\mathcal{F}(X)$. Refer to Sections A.1 and 1.1.1 for the geometric model of continuous and finite-rank operators.

[17]In here we will use X to denote the vector space, since V is reserved for an isotropic inner ideal.

Geometric model of K in simple rings with socle

It can be shown that $\mathcal{F}(X)$ is a selfadjoint ideal of $\mathcal{L}(X)$ ([RingsGIs, page 156]). This implies that $\mathrm{Skew}(\mathcal{F}(X),*) = \kappa(\mathcal{F}(X))$. Since $\mathrm{Skew}(\mathcal{L}(X),*)$ is the *orthogonal algebra*, denoted by $\mathbf{o}(X)$, $\mathrm{Skew}(\mathcal{F}(X),*)$ is called the *finitary orthogonal algebra* and denoted by $\mathbf{fo}(X)$ ([Baranov'99, 6.Finitary simple Lie algebras]).
If $u, v \in X$ we define their *bracket* $[\cdot,\cdot] : X \times X \to \mathcal{F}(X)$ as $[u, v] := u \otimes v - v \otimes u$. The linear span of all brackets is denoted by $[X, X]$.

We state now some elementary computational facts about $K := \mathbf{fo}(X)$ which will see use in what follows, in occasions without further remark.

- The adjoint of a rank-one operator is $(u \otimes v)^* = v \otimes u$, since

$$\langle u \otimes v(x), y\rangle = \langle\langle x, u\rangle v, y\rangle = \langle x, u\rangle\langle v, y\rangle = \langle x, u\rangle\langle y, v\rangle = \langle x, \langle y, v\rangle u\rangle = \langle x, v \otimes u(y)\rangle.$$

- This implies that $\kappa(u \otimes v) = u \otimes v - v \otimes u = [u, v]$ and hence that $K = [X, X]$.
  In particular every skew linear operator has even rank.
- In addition all the symmetric operators of rank one are of the form $\alpha u \otimes u$ with $\alpha \in F$, since $(u \otimes v) = (u \otimes v)^* = v \otimes u$ implies, evaluating in $x \in X$, that $\langle x, u\rangle v = \langle x, v\rangle u$, so that $v = \alpha u$ for some $\alpha \in F$.
- If $\{v_i\}$ is a dual set to $\{u_i\}$, so that $\langle u_i, v_j\rangle = \delta_{ij}$, then the reverse product is

$$\langle v_j, u_i\rangle = \langle u_i, v_j\rangle = \delta_{ij}.$$

- If $a \in K$, then $\langle au, u\rangle = 0$ for every $u \in X$, i.e., a is *alternating*, because

$$\langle au, u\rangle = \langle u, a^* u\rangle = -\langle u, au\rangle = -\langle au, u\rangle.$$

  The converse is also true (this is the skew version of [McCrimmon'66, Lemma 6]).

Lemma 3.4.12 (Structure of the orthogonal complement of a vector). *Let $0 \neq u \in X$. Then $\{u\}^\perp = Ku$.*

Proof. $Ku \subseteq \{u\}^\perp$, since every $a \in K$ is alternating. If $\{u\}^\perp = 0$ then trivially $\{u\}^\perp \subseteq Ku$. Otherwise, take $0 \neq w \in \{u\}^\perp$ and choose $v \in X$ such that $\langle u, v\rangle = 1$,

which can be done thanks to the existence of dual sets. Consider $[v, w] \in K$; then $[v, w]u = v \otimes w(u) - w \otimes v(u) = \langle u, v\rangle w - \langle u, w\rangle v = 1 \cdot w - 0 \cdot v = w$, i.e.,
$w = [v, w]u \in Ku$. Hence $\{u\}^{\perp} \subseteq Ku$. $\square$

- Since $K = [X, X]$, the associative and Lie products of elements of K are determined by the product of 'pure' brackets of the form $[x, y]$.

Lemma 3.4.13 (Bracket products).
Let $a, b, c, d \in X$. Then:

1. $[a, b][c, d] = (\langle b, c\rangle d - \langle b, d\rangle c) \otimes a - (\langle a, c\rangle d - \langle a, d\rangle c) \otimes b$.
2. $ad_{[a,b]}([c, d]) = [a, \langle b, c\rangle d + \langle b, d\rangle c] + [b, \langle a, c\rangle d - \langle a, d\rangle c]$.

Proof.

1. A direct calculation using the Product Law (A.1.2(2)) shows that
$$[a, b][c, d] = (a \otimes b - b \otimes a)(c \otimes d - d \otimes a) =$$
$$= (a \otimes b)(c \otimes d) - (a \otimes b)(d \otimes c) - (b \otimes a)(c \otimes d) + (b \otimes a)(d \otimes a) =$$
$$= \langle d, a\rangle c \otimes b - \langle c, a\rangle d \otimes b - \langle d, b, \rangle c \otimes a + \langle a, b\rangle d \otimes a =$$
$$= (\langle b, c\rangle d - \langle b, d\rangle c) \otimes a - (\langle a, c\rangle d - \langle a, d\rangle c) \otimes b.$$
2. $[a, b][c, d] - [c, d][a, b] =$
$$(\langle b, c\rangle d - \langle b, d\rangle c) \otimes a - (\langle a, c\rangle d - \langle a, d\rangle c) \otimes b -$$
$$- (\langle d, a\rangle b - \langle d, b\rangle a) \otimes c + (\langle c, a\rangle b - \langle c, b\rangle a) \otimes d =$$
$$= \langle b, d\rangle [a, c] - \langle b, c\rangle [a, d] - \langle a, d\rangle [b, c] + \langle a, c\rangle [b, d] =$$
$$= [a, \langle b, d\rangle c - \langle b, c\rangle d] + [b, \langle a, c\rangle d - \langle a, d\rangle c].$$ $\square$

The notion of hyperbolic plane is fundamental to define Clifford inner ideals from a geometric perspective.

Definitions 3.4.14 (Hyperbolic pair and hyperbolic plane).

- A vector $u \in X$ is said to be *isotropic* if $\langle u, u\rangle = 0$. A vector which is not isotropic is called *anisotropic*.
- By a *hyperbolic pair* we mean a pair (x, y) of isotropic vectors of X such that $\langle x, y\rangle = 1$.

- A *hyperbolic plane* is a subspace $H := Fx \oplus Fy$ such that (x, y) is a hyperbolic pair.

The following are essential properties of the definitions above.

- Any nonzero isotropic vector can be extended to a hyperbolic pair, and hence to a hyperbolic plane, as follows. Pick $0 \neq x \in X$ isotropic and consider a dual vector $y' \in X$ to x. Then $y := -\frac{1}{2}\langle y', y'\rangle x + y'$ satisfies $\langle x, y\rangle = -\frac{1}{2}\langle y', y'\rangle\langle x, x\rangle + \langle x, y'\rangle = 1$ and $\langle y, y\rangle = \frac{1}{4}\langle y', y'\rangle^2\langle x, x\rangle - \langle y', y'\rangle\langle x, y'\rangle + \langle y', y'\rangle = 0$ since $\langle x, x\rangle = 0$ and $\langle x, y'\rangle = 1$.
- A minimal idempotent $e \in \mathcal{F}_W(X)$ is $*$-orthogonal if and only if $e = x \otimes y$, with (x, y) a hyperbolic pair. This is due to:
  - $e^2 = (x \otimes y)(x \otimes y) = x \otimes (x \otimes y(y)) = x \otimes \langle y, x\rangle y$ by the Absorption Law 1 (A.1.2(4)), which gives $e^2 = e$ if and only if $\langle x, y\rangle = 1$.
  - Since $(x \otimes y)^* = y \otimes x$, $e^*e = (y \otimes x)(y \otimes x) = x \otimes (y \otimes x(y)) = x \otimes \langle y, y\rangle x$, which gives $e^*e = 0$ if and only if $\langle y, y\rangle = 0$. Analogously $ee^* = 0$ if and only if $\langle x, x\rangle = 0$.
- If H is a hyperbolic plane, then by the Direct Summand Theorem (see [LinearGeometry, Theorem 2]) we have that

$$X = H \oplus H^\perp,$$

  where $H^\perp := \{u \in X \mid \langle u, H\rangle = 0\}$ is the orthogonal complement of H.

Enter Clifford inner ideals

Definition 3.4.15 (Clifford inner ideal).

Let L be a subalgebra of $\mathbf{o}(X)$ containing $\mathbf{fo}(X)$. A *Clifford inner ideal* is a subspace of L of the form $[x, H^\perp] := \{[x, z] \mid z \in H^\perp\}$, where x is a nonzero isotropic vector and H is an associated hyperbolic plane.

This terminology is motivated by the fact that the subquotient of B is the Clifford Jordan pair[18] $(H^\perp, H^\perp)$ (see [Benkart&Fernández'09, Proposition 4.4(i)]).

[18] If X is a vector space over a field and q is a quadratic form on X then (X, X) becomes a Jordan pair with product $Q_x y := q(x, y)x - q(x)y$, with $q(x, y)$ the bilinear form associated to q, see [Fernández,García&Gómez'08, 5.7].

Let us prove that Clifford inner ideals are indeed Lie inner ideals and state some of their properties.

Proposition 3.4.16 (**Properties of Clifford inner ideals**).
Let x be a nonzero isotropic vector with associated hyperbolic plane H and set $B := [x, H^\perp]$. Then:

1. *B is a Lie inner ideal of $\mathbf{o}(X)$ contained in $\mathbf{fo}(X)$.*
2. *$B = ad^2_{[x,z]}(\mathbf{fo}(X))$ for every anisotropic $z \in H^\perp$.*
3. *$b^3 = 0$ for every $b \in B$ and $b_0^2 \neq 0$ is of rank one for some $b_0 \in B$.*

   In particular B is neither standard nor special.
4. *B coincides with its centralizer in $\mathbf{o}(X)$ and hence is a maximal Lie inner ideal of $\mathbf{o}(X)$.*

Proof.

1. This was first proved in [Fernández,García&Gómez'06(2), Lemma 3.7(i)].

   To show that B is a Lie inner ideal is to show that $[B, B] = 0$ and that $ad^2_{[x,z]}(a) \in B$ for every $a \in \mathbf{o}(X)$ and $z \in H^\perp$. Take into account that $\langle x, x\rangle = 0 = \langle x, z\rangle$ because x is isotropic and $z \in H^\perp$, and that $\langle ax, z\rangle = -\langle x, az\rangle$ and $\langle ax, x\rangle = 0 = \langle az, z\rangle$ because $a^* = -a$. Pick $z, z_1, z_2 \in H^\perp$ and $a \in \mathbf{o}(X)$. Then:

   - By Lemma 3.4.13(1), $[x, z_1][x, z_2] = -\langle z_1, z_2\rangle x\otimes x$ **(1)**, which is symmetric in z_1, z_2. Hence $ad_{[x,z_1]}([x, z_2]) = 0$.
   - $ad_{[x,z]}(a) = (x\otimes z - z\otimes x)a - a(x\otimes z - z\otimes x) = (a^*x)\otimes z - (a^*z)\otimes x - x\otimes az + z\otimes ax$ by the Absorption Laws (A.1.2(4,5)), which in turns gives

     $$z \otimes ax - (ax) \otimes z + (az) \otimes x - x \otimes az = [z, ax] + [az, x].$$

   - $ad^2_{[x,z]}(a) = ad_{[x,z]}(ad_{[x,z]}(a)) = ad_{[x,z]}([z, ax]) + ad_{[x,z]}([az, x])$ which, by Lemma 3.4.13(2), gives

     $$[x, \langle z, ax\rangle z - \langle z, z\rangle ax] + [z, \langle x, z\rangle ax - \langle x, ax\rangle z] + [x, \langle z, x\rangle az - \langle z, az\rangle x] + [z, \langle x, az\rangle x - \langle x, x\rangle az] =$$

     $$= \langle z, ax\rangle[x, z] - \langle z, z\rangle[x, ax] + \langle x, az\rangle[z, x] = 2\langle ax, z\rangle[x, z] - \langle z, z\rangle[x, ax], \qquad (2)$$

since $\langle x, az\rangle[z, x] = -\langle x, az\rangle[x, z] = \langle ax, z\rangle[x, z]$.

Let us see that $[x, ax] \in [x, H^{\perp}]$. Let $y \in H$ denote the isotropic vector such that $\langle x, y\rangle = 1$. Then $V = H \oplus H^{\perp} = Fx \oplus Fy \oplus H^{\perp}$ and $ax = \alpha x + \beta y + w$ with $\alpha, \beta \in F$, $w \in H^{\perp}$. Since $a^* = -a$ we get $0 = \langle ax, x\rangle = \langle \alpha x + \beta y + w, x\rangle = \beta\langle x, y\rangle = \beta$. Hence $ax = \alpha x + w$ and $[x, ax] = [x, \alpha x + w] = [x, w] \in B$.

2. This was first proved in [Fernández,García&Gómez'06(2), Lemma 3.7(iii)].

Pick $w, z \in H^{\perp}$ with z anisotropic and denote $\lambda := \langle z, z\rangle^{-1}$. Let us see that there exists $a \in \mathbf{fo}(X)$ such that $ad^2_{[x,z]}(a) = [x, w]$. Suppose by reverse engineering that we already have a. By **(2)** we get $[x, w] = 2\langle ax, z\rangle[x, z] - \langle z, z\rangle[x, ax]$, so that $[x, w - 2\langle ax, z\rangle z + \langle z, z\rangle ax] = 0$ and $ax = \lambda^{-1}(2\langle ax, z\rangle z - w)$ since $x \neq 0$. Then $\langle ax, z\rangle = \lambda^{-1}(2\langle ax, z\rangle\langle z, z\rangle - \langle w, z\rangle) = 2\langle ax, z\rangle - \lambda^{-1}\langle w, z\rangle$ and hence $\langle ax, z\rangle = \lambda^{-1}\langle w, z\rangle$. Therefore

$$ax = \lambda^{-1}(2\lambda^{-1}\langle w, z\rangle - w) =: v.$$

Now, since $\langle x, y\rangle = 1$ we can take $a := y \otimes v$ to guarantee that $ax = v$. This element is checked to indeed satisfy $ad^2_{[x,z]}(a) = [x, w]$.

3. This was first proved in [Fernández,García&Gómez'06(2), Lemma 3.7(ii)].

By formula **(1)** for every $z \in H^{\perp}$ we have $[x, z]^2 = -\langle z, z\rangle x \otimes x$, which if not zero is of rank one. Then $[x, z]^3 = -\langle z, z\rangle x \otimes x(x \otimes z - z \otimes x) = 0$ since the two factors arising from the Product Law (A.1.2(2)) yield respectively $\langle z, x\rangle = 0$ and $\langle x, x\rangle = 0$. In addition it is $[x, z]^2 = 0$ if and only if $\langle z, z\rangle = 0$. Since $\dim_F X \geq 3$, $H^{\perp}$ must contain some anisotropic vector, so there exists $b \in B$ such that $b^2 \neq 0$. But the involution is of the first kind and thus we have that $K(Z) = 0$, so there are no special inner ideals in $\mathbf{o}(X)$ and all its standard inner ideals are isotropic. Therefore B is neither standard nor special.

4. Let $a \in \mathbf{o}(X)$ be such that

$$a(x \otimes z - z \otimes x) = (x \otimes z - z \otimes x)a \; z \in H^{\perp}. \tag{3}$$

We need to show that $a \in [x, H^{\perp}]$. The proof will be complete if we prove in particular that $ay \in H^{\perp}$ and $a = [x, ay]$ for the isotropic vector $y \in H$ such that $\langle x, y\rangle = 1$.

Since $a^* = -a$, by the Absorption Laws (A.1.2(4,5)) equation **(3)** can be written as

$$x \otimes az - z \otimes ax = (az) \otimes x - (ax) \otimes z \quad \text{for every } z \in H^{\perp}, \tag{4}$$

which evaluated in y, since $\langle x, y\rangle = 1$ and $\langle y, z\rangle = 0$ because $y \in H$, yields

$$az = \langle y, az\rangle x - \langle y, ax\rangle z, \;\; z \in H^{\perp}. \tag{5}$$

Consider $z \in H^{\perp}$ anisotropic, which is possible because $\dim_F X \geq 3$. Since $\langle z, az\rangle = 0$ because $a = -a^*$, by **(5)** we get

$$0 = \langle z, az\rangle = \langle z, \langle y, az\rangle x\rangle - \langle z, \langle y, ax\rangle z\rangle = \langle z, x\rangle\langle y, az\rangle - \langle z, z\rangle\langle y, ax\rangle = -\langle z, z\rangle\langle y, ax\rangle,$$

which implies $\langle y, ax\rangle = 0$ since $\langle z, z\rangle \neq 0$ by the choice of z. Thus by **(5)**

$$az = \langle y, az\rangle x, \;\; z \in H^{\perp}. \tag{6}$$

Evaluating **(4)** in z and applying **(6)** we get that for any $z \in H^{\perp}$,

$$-\langle z, z\rangle ax = -\langle z, ax\rangle z = \langle az, x\rangle z = \langle\langle y, az\rangle x, x\rangle z = \langle y, az\rangle\langle x, x\rangle z = 0.$$

Taking z anisotropic we get

$$ax = 0. \tag{7}$$

Then $\langle ay, x\rangle = -\langle y, ax\rangle = 0$ and, since $\langle ay, y\rangle = 0$, we get that $ay \in H^{\perp}$ since $H = Fx \oplus Fy$. Using the decomposition $X = H \oplus H^{\perp}$ we prove that $a = [x, ay]$ to complete the proof.

(a) $[x, ay]x = \langle x, x\rangle ay - \langle x, ay\rangle x = 0 = ax$ by **(7)**,

(b) $[x, ay]y = \langle y, x\rangle ay - \langle y, ay\rangle x = ay$ and, for $z \in H^{\perp}$,

(c) $[x, ay]z = \langle z, x\rangle ay - \langle z, ay\rangle x = -\langle z, ay\rangle x = \langle az, y\rangle x = az$, by **(6)**. □

The Clifford denomination for this kind of inner ideal is also justified by another reason apart from the subquotient one. In Chapter 4, **Clifford elements**, we will show how to attach a Jordan algebra L_a to any Jordan element a of a Lie algebra L (4.1.2), Jordan algebra that behaves much like local rings do in the associative setting. By item

(3) of the previous proposition, there exist Jordan elements $b \in B$ such that $b^2 \neq 0$. Then it happens that $\widehat{K}_b$ is a Clifford Jordan algebra over the extended centroid of R for any such $b \in B$.

We show now that there are no other kinds of inner ideal associated to Jordan elements with minimal square.

Proposition 3.4.17.
(Characterization of Clifford inner ideals: prime with socle case)
Let L be such that $\mathbf{fo}(X) \leq L \leq \mathbf{o}(X)$ and let B be an abelian inner ideal of L. If B contains an element b such that $b^3 = 0$ and b^2 has rank one then B is Clifford.

Proof. Since b^2 is symmetric and of rank one we have that $b^2 = \alpha x \otimes x$, where both $\alpha \in F$ and $x \in X$ are not zero. Now $b^3 = 0$ implies that $0 = b^2 b^2 = \alpha^2 \langle x, x\rangle x \otimes x$, so x is isotropic. Extend x to the hyperbolic pair (x, y) and set $H := Fx \oplus Fy$. We have the following identities:

- $b^2 y = (\alpha x \otimes x) y = \alpha\langle y, x\rangle x = \alpha x$.
- $\langle by, by\rangle = -\langle y, b^2 y\rangle = -\alpha$, so by is anisotropic.
- $by \in H^\perp$, since $\langle by, y\rangle = 0$ and $\langle by, x\rangle = \langle by, \alpha^{-1} b^2 y\rangle = \langle b^3 y, \alpha^{-1} y\rangle = 0$.

Let $z \in H^\perp$ and set $a := [y, z]$. Then:

- $ax = \langle x, y\rangle z - \langle x, z\rangle y = z$.
- $b^2 a = \alpha(x \otimes x)a = -\alpha(ax) \otimes x = -\alpha z \otimes x$.
- $ab^2 = \alpha a(x \otimes x) = \alpha x \otimes ax = \alpha x \otimes z$.
- $bab = b(y \otimes z - z \otimes y)b = (bz) \otimes by - (by) \otimes bz = [bz, by]$.
- $\mathrm{ad}_b^2 a = b^2 a + ab^2 - 2bab = \alpha[x, z] - 2[bz, by]$.

Taking $z := by$ in the last identity we get, since $b^2 y = \alpha x$, that

$$\mathrm{ad}_b^2 [y, by] = \alpha[x, by] - 2[b^2 y, by] = \alpha[x, by],$$

so $[x, by] \in B$. Since by is anisotropic, by Proposition 3.4.16(2) we have $[x, H^\perp] = \mathrm{ad}^2_{[x,by]} \mathbf{fo}(X) \subseteq B$, and hence $B = [x, H^\perp]$ since $[x, H^\perp]$ is maximal by Proposition 3.4.16(4). This proves that B is Clifford. □

Now we describe Clifford inner ideals in ring-theoretic terms.

Proposition 3.4.18 (**Ring-theoretic structure of Clifford inner ideals**). *Let L be such that $\mathbf{fo}(X) \leq L \leq \mathbf{o}(X)$ and let B be a subset of L.*
B is a Clifford inner ideal of L if and only if $B = \kappa((1-e)\mathbf{fo}(X)e)$, where $e \in \mathcal{F}(X)$ is a minimal $*$-orthogonal idempotent, in which case $B = \kappa((1-e)Se)$ for any set $\mathbf{fo}(X) \subseteq S \subseteq \mathcal{L}(X)$.

Proof. We will actually prove slightly more than what is claimed in the statement. We will prove that B is Clifford if and only if $B = \kappa((1-e)Se)$, where $e \in \mathcal{F}(X)$ is a minimal $*$-orthogonal idempotent and S is any set $S \subseteq \mathcal{L}(X)$ satisfying the technical condition $(1-e-e^*)Se = (1-e-e^*)\mathcal{F}(X)e$, all these sets giving rise to the same Clifford inner ideal; after that we will show that any S such that $\mathbf{fo}(X) \subseteq S$ satisfies the technical condition, proving the claim.

Let (x, y) be a hyperbolic pair. As commented after 3.4.14, (x, y) is a hyperbolic pair if and only if $e := x \otimes y$ is an $*$-orthogonal idempotent. Let $H := Fx \oplus Fy$ be the associated hyperbolic plane, set $f := e + e^*$ and suppose $S \subseteq \mathcal{L}(X)$ is such that $(1-f)Se = (1-f)\mathcal{F}(X)e$. Note that $\mathcal{F}(X)y = X$. By the Absorption Law 1 (A.1.2(4))

$$\mathcal{F}(X)e = \mathcal{F}(X)(x \otimes y) = x \otimes \mathcal{F}(X)y = x \otimes X.$$

Observe that $eX = (x \otimes y)X = Fy$ while $e^*X = (y \otimes x)X = Fx$. Hence $1-f$ is the orthogonal projection onto $H^\perp$ and we have that

$$(1-f)Se = (1-f)\mathcal{F}(X)e = (1-f)x \otimes X = x \otimes (1-f)X = x \otimes H^\perp. \qquad \mathbf{(1)}$$

Since $\langle by, y \rangle = 0$ for every $b \in \mathbf{o}(X)$,

$$e^*be = (x \otimes y)^*b(x \otimes y) = (y \otimes x)b(x \otimes y) = (y \otimes x)(x \otimes by) = \langle by, y \rangle x \otimes x = 0.$$

Hence, since for every $a \in S$ we have $\kappa(a) \in \mathbf{o}(X)$,

$$\kappa((1-f)ae) = \kappa((1-e)ae - e^*ae) = \kappa((1-e)ae) - e^*\kappa(a)e = \kappa((1-e)ae). \qquad \mathbf{(2)}$$

Then, by **(1)** and **(2)**, $[x, H^\perp] = \kappa(x \otimes H^\perp) = \kappa((1-f)Se) = \kappa((1-e)Se)$.

Now suppose S is such that $\mathbf{fo}(X) \subseteq S$. By Lemma 3.4.12 we know that $\mathbf{fo}(X)y = \{y\}^\perp = Fx \oplus H^\perp$. Hence Sy is either $Fx \oplus H^\perp$ or X. In any case $H^\perp \subseteq Sy$ and $(1-f)Sy = H^\perp$, since $1-f$ is the projection onto $H^\perp$. Therefore

$$(1-f)Se = (1-f)S(x \otimes y) = (1-f)x \otimes Sy = x \otimes (1-f)Sy = x \otimes H^\perp = (1-f)\mathcal{F}(X)e. \quad \square$$

3.4.4 Classification of Lie inner ideals

In this section R will be a centrally closed prime algebra. Recall that we deem K exceptional whenever $[K, K] = 0$ (see Theorem 1.3.9).

Proposition 3.4.19 (Characterization of Clifford inner ideals: prime case). *Let R be a centrally closed prime algebra with $\mathrm{char}(R) \neq 2, 3, 5$ and involution such that $[K, K] \neq 0$. If B is a Lie inner ideal of K such that $b^2 \neq 0$ for some $b \in B$, then B is a Clifford inner ideal of K.*

Proof. Since B is a Lie inner ideal of K, b is a Jordan element of K such that $b^2 \neq 0$. By Herstein Lemma (3.2.1) R has socle and involution of orthogonal type, and therefore $\mathbf{fo}(X) \subseteq K \subseteq \mathbf{o}(X)$ for some selfdual vector space X over $\mathcal{C}$. We claim that $\dim_{\mathcal{C}} X \geq 3$. Suppose on the contrary that $\dim_{\mathcal{C}} X < 3$. Then either $K = 0$ or there exist two linearly independent vectors $x, y \in X$ and we have $K = \mathcal{C}[x, y]$; in both cases $[K, K] = 0$, a contradiction. It follows from Proposition 3.4.17 that B is a Clifford inner ideal. $\square$

Theorem 3.4.20.
(Classification of Lie inner ideals of K for R centrally closed prime)
Let R be a centrally closed prime algebra with $\mathrm{char}(R) \neq 2, 3, 5$ and involution $*$ such that $[K, K] \neq 0$. If B is a Lie inner ideal of K, then either

1. $B = V$ is an isotropic inner ideal, *2.* $B = V \oplus \mathrm{Skew}(\mathcal{C}, *)$ *is a standard inner ideal,*

3. $B = \mathrm{inn}(V, f)$ *is special, or*

4. $B = \kappa((1-e)Re)$ *is Clifford.*

Moreover, in cases (2) and (3) R *is unital and* $*$ *is of the second kind, while in case (4)* R *has nonzero socle and* $*$ *is of orthogonal type.*

Proof. Suppose first that $*$ is of the second kind and let ξ be a nonzero skew element of $\mathcal{C}$. Then by Lemma 1.2.1(2) we know that $R = K \oplus \xi K$. Set $C := B \oplus \xi B$. It is straightforward to see that C is a Lie inner ideal of R, selfadjoint and with $B = \mathrm{Skew}(C, *) = C \cap K$. By the classification of the Lie inner ideals of R (Theorem 3.3.2), either

1. $C = V$, where V is an isotropic inner ideal, or
2. R is unital and $C = V \oplus \mathcal{C}$, where V is isotropic, or
3. R is unital and $C = \{v + g(v) \mid v \in V\}$, where V is isotropic and $g : V \to \mathcal{C}$ is a nonzero additive form such that $[V, [V, R]] \subseteq \ker g$.

If $C = V$ as in (1), then $B = \mathrm{Skew}(V, *)$ is an isotropic inner ideal of K. Suppose then that C is as in (2) or (3). In both cases V is selfadjoint:

$V^* \subseteq C^* = C \subseteq V \oplus \mathcal{C}$ and hence $[V^*, V] = 0$ since $V^2 = 0$. Thus for any $u \in V$, $u^* = v + z$ where $v \in V$ and $z \in \mathcal{C}$. Since $u^* - v$ is nilpotent, $u^* - v = 0$, so $u^* = v \in V$ as claimed.

If (2), then $B = (V \oplus \mathcal{C}) \cap K = \kappa(V \oplus \mathcal{C}) = \kappa(V) \oplus \kappa(\mathcal{C}) = \mathrm{Skew}(V, *) \oplus \mathrm{Skew}(\mathcal{C}, *)$ since V and $\mathcal{C}$ are selfadjoint (see the first paragraphs of Chapter 1), with $\mathrm{Skew}(V, *)$ being an isotropic inner ideal of K. If (3), then $\kappa(v + g(v)) = \kappa(v) + \kappa(g(v)) = \kappa(v) + g(\kappa(v))$ with $\kappa(g(v)) \in \kappa(\mathcal{C})$ implies that $B = \{u + f(u) \mid u \in U\}$, where $U = \mathrm{Skew}(V, *)$ is an isotropic inner ideal of K and $f : V \to \mathrm{Skew}(\mathcal{C}, *)$ is the restriction of g to V, which satisfies $[U, [U, K]] \subseteq \ker f$.

Suppose now that the involution is of the first kind. If $b^2 = 0$ for every $b \in B$ then B is an isotropic inner ideal by Lemma 3.4.1(1). Thus we may assume that $b_0^2 \neq 0$ for some $b_0 \in B$. By Herstein Lemma for K (3.2.1) we find that $b_0^3 = 0$ since the involution is of the first kind. Then we have by Proposition 3.4.19 that B is a Clifford inner ideal.

Since in this case $\mathcal{F}(X) \subseteq R \subseteq \mathcal{L}(X)$ for some selfdual space X and $\mathbf{fo}(X) \subseteq K \subseteq R$, by Proposition 3.4.18 there exists a minimal $*$-orthogonal idempotent $e \in R$ such that we can write $B = \kappa((1-e)Re)$. □

Isotropic inner ideals, in addition to being Lie inner ideals, are also Jordan inner ideals of K by Lemma 3.4.1(2). Therefore the isotropic, standard and special inner ideals of K arise from Jordan inner ideals and are very near to them. What about Clifford inner ideals? Actually, Clifford inner ideals are also Jordan! Consider $[x, H^\perp]$ for some isotropic vector x. Pick $z \in H^\perp$ and $a \in K$. Then, taking into account that $\langle x, x\rangle = 0 = \langle x, z\rangle$, by the Absorption Law 1 and the Product Law (A.1.2(2,4)) we get

$$[x, z]a[x, z] = [x, z](x \otimes az - z \otimes ax) = (x \otimes z - z \otimes x)(x \otimes az - z \otimes ax) =$$

$$= \langle az, x\rangle x \otimes z - \langle az, z\rangle x \otimes x - \langle ax, x\rangle z \otimes z + \langle z, ax\rangle z \otimes x = \langle az, x\rangle x \otimes z - \langle az, x\rangle z \otimes x = \langle az, x\rangle [x, z].$$

Therefore $[x, z]K[x, z] \subseteq [x, H^\perp]$. We can perfectly conclude that, in this context, the McCrimmon Motto[19] which goes

Nine times out of ten, when you open up a Lie algebra
you find a Jordan algebra inside which makes it tick

is wrong –it is too conservative!

To end this chapter we are going to show that the exception $[K, K] \neq 0$ in the statement of the previous theorem is not superfluous.
As we know from Theorem 1.3.9, if R is not commutative the condition $[K, K] = 0$ is satisfied if and only if $\widehat{R}$ is a quaternion algebra with an involution which is orthogonal on $\overline{R}$. If we take $R := \mathbb{M}_2(F)$ with the transpose involution we find that K is in itself a Lie inner ideal which does not lie in any of the four cases of the theorem above: since the involution is of the first kind, the standard and special cases are discarded and we are left with the isotropic and Clifford cases as candidates, in which every element is of

[19] This comes from [TasteJordanAlgebras, page 14].

zero cube. But the elements of K are not nilpotent because their squares are nonzero and lie in the center by Lemma 1.1.4(2c) (so every even power of the elements is not zero). The same happens if we take $R := \mathbb{H}(\alpha, \beta)$ with the involution $x^* := i\bar{x}i^{-1}$ (see Section 1.4): then $K = Fi$, $Z(R) = F1$, the involution is of the first kind and every even power of an element of K lies in the center and is not zero.

Chapter 4

Clifford elements

If F is a field with $\text{char}(F) \neq 2$ and X is an F-vector space with a symmetric bilinear form $\langle \cdot\,, \cdot \rangle$, the vector space $F \oplus X$ is endowed with an structure of Jordan algebra when equipped with the product

$$(\alpha + x) \bullet (\beta + y) := \alpha\beta + \langle x, y \rangle \; + \; \beta x + \alpha y$$

for $\alpha, \beta \in F$ and $x, y \in X$. This Jordan algebra is unital, with $1_F + 0$ as identity element, and special. In fact, it is isomorphic to a Jordan subalgebra of the Clifford (associative) algebra defined by $\langle \cdot\,, \cdot \rangle$ (refer to [StructureJordan, II.3]). For this reason, $F \oplus X$ is sometimes called a *Clifford Jordan algebra*, convention that we follow in this dissertation.

Let L be a Lie algebra over a field F with $\text{char}(F) \neq 2$ and let $c \in L$ be a Jordan element. A Jordan algebra L_c can be attached to c whenever $\text{char}(F) \neq 3$ (see the technical section 4.1 below for the definition and relevant proofs). In this context, we say that c is a *Clifford element* when L_c is a Clifford Jordan algebra. Suppose that L is nondegenerate, $\text{char}(F) > 5$, and c is a Clifford element of L. Since L_c is then unital, c is von Neumann regular ([Fernández,García&Gómez'06, 2.15(ii)]) and hence, by the Jacobson-Morozov Lemma (see [Draper,Fernández,García&Gómez'08, Proposition 1.18]), L has a 5-grading $L = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$ such that the Jordan pair

$V := (L_2, L_{-2})$ is isomorphic to the Clifford Jordan pair[1] defined by the Jordan algebra L_c, whose Tits-Kantor-Koecher algebra[2] $TKK(V)$ is a finitary orthogonal algebra by [Fernández,García&Gómez'08, 5.11], that is, $TKK(V) \cong \text{Skew}(R, *)$, where R is a simple ring coinciding with its socle and $*$ is an involution of orthogonal type, with R not being the algebra of 2×2 matrices over its center with the transpose involution. Thus every Clifford element c actually lives in a ring, and in that associative context verifies $c^3 = 0$ and $c^2 \neq 0$ (by Proposition 3.4.16(3)). In this chapter we prove, among other things, a strong converse of the above result: if R is a centrally closed prime ring with involution and $\text{char}(R) > 5$, and K has a Jordan element c such that $c^3 = 0$ and $c^2 \neq 0$, then R has socle, the involution is of orthogonal type, $R \not\cong \mathbb{M}_2(\mathcal{C})$ and c is a Clifford element of K.

In the first section, with the aid of the Beautiful Partner Lemma (1.2.5) we develop a set of important identities of (associative) Clifford elements c and their squares, which allow to compute every aspect of our problem. The key facts are that c^2 is reduced and that c, although not reduced in R, is reduced in K, in the sense that we have $cKc = \mathcal{C}c$. We pair c^2 (which being reduced is von Neumann regular) with an element d that behaves exactly like c^2. We also link the existence of Clifford elements to that of $*$-*orthogonal* reduced idempotents e such that $e^*Ke = 0$, and prove that Clifford elements grade K with 3-gradings.

Clifford elements of R are intimately associated with Clifford inner ideals (refer to Section 3.4.3). In the second section we see that the extremes of the aforementioned 3-gradings of K are Clifford inner ideals, we study their algebraic structure and use it to provide an specific algebraic construction for K_c. To achieve this we pair c (which being

[1] If X is a vector space over a field and q is a quadratic form on X then (X, X) becomes a Jordan pair with product $Q_x y := q(x, y)x - q(x)y$, with $q(x, y)$ the bilinear form associated to q (see [Fernández,García&Gómez'08, 5.7]).

[2] The TKK algebra of a Jordan pair V can be axiomatically defined as the unique Lie algebra with a 3-grading $TKK(V) = L_{-1} \oplus L_0 \oplus L_1$ such that the associated Jordan pair (L_1, L_{-1}) is isomorphic to V, $[L_1, L_{-1}] = L_0$ and $[x_0, L_1 \oplus L_{-1}] = 0$ implies $x_0 = 0$ for $x_0 \in L_0$ ([Draper,Fernández,García&Gómez'08, 1.8]).

reduced in K is von Neumann regular) with a (Lie and associative) partner $\sqrt{d}$ which behaves exactly like c and which squared gives the beautiful partner d of c^2. Then we attach a trace and a bilinear form to K, which are inherited by K_c, and show that it is in fact a Clifford Jordan algebra in the third section.

4.1 The Jordan algebra at a Jordan element

There exists a good reason for the denomination of Jordan elements: if L is a Lie algebra such that $3 \in \mathrm{TF}(L)$, then associated to any Jordan element $a \in L$ there exists a Jordan algebra L_a which behaves as a local ring for L in the sense of inheritance of important properties. For example, if L is nondegenerate then L_a is nondegenerate, while if L is nondegenerate and L_a has socle (as a Jordan algebra) then L has socle (as a Lie algebra). See [Fernández,García&Gómez'06] and [García&Gómez'07, Theorem 2.2] for these and more related results. Some evidence for the existence of such a Jordan algebra comes from the existence of the Fundamental Formula for Jordan elements.

Before we present L_a we need to establish several identities.

Proposition 4.1.1 (Identities for Jordan elements).
Let L be a Lie algebra such that $3 \in \mathrm{TF}(L)$ and let $a \in L$ be a Jordan element. Let $x, y \in L$ be arbitrary. Then:

1. $A^2XA = AXA^2$.
2. $A^2XA^2 = 0$.
3. $A^2X^2AXA = A^2XAX^2A$.
4. $[A^2(x), A(y)] = [A^2(y), A(x)]$.
5. $A^2XA(y) = A^2YA(x)$.

Proof. We use that $A^3 = 0$ implies $0 = \mathbf{ad}_A^3 = (l_A - r_A)^3$ in $\mathrm{End}(\mathrm{Inn}(L))$.

1. $0 = -3l_A^2 r_A + 3l_A r_A^2$. Hence $l_A^2 r_A = l_A r_A^2$ because $3 \in \mathrm{TF}(L)$.
2. Multiplying item (1) by l_A we get $0 = l_A^2 r_A^2$.

3. By item (2), $0 = A^2\mathbf{ad}_X^3(A)A^2 = A^2(X^3A - 3X^2AX + 3XAX^2 - AX^3)A^2 = -3A^2X^2AXA^2 + 3A^2XAX^2A^2$.
4. By Leibniz Rule, $A^3([x,y]) = -3[A^2(x), A(y)] + 3[A(x), A^2(y)]$.
5. By Leibniz Rule and using item (4) twice,
$A^2([x, A(y)]) = [A^2(x), A(y)] - 2[A(x), A^2(y)] = [A^2(y), A(x)] - 2[A(y), A^2(x)] = A^2([y, A(x)])$. □

Theorem 4.1.2 (**Jordan algebra at a Jordan element**).
Let L be a Lie algebra such that $3 \in \mathrm{TF}(L)$ and let $a \in L$ be a Jordan element.
Then L with product $x \bullet y := [[x,a],y]$ is a nonassociative algebra, denoted by $L^{(a)}$, such that:

1. $\ker(a) := \{x \in L \mid A^2(x) = 0\}$ is an ideal of $L^{(a)}$.
2. $L_a := L^{(a)}/\ker(a)$ is a Jordan algebra, with U-operator given by

$$U_{\bar{x}}(\bar{y}) = \overline{X^2A^2(y)}.$$

Proof. Observe that $[[x,a]y] = [y[a,x]] = YA(x)$.

1. Let us show that $\ker(a)$ is in fact an ideal. It is clear that $\ker(a)$ is a submodule, since A^2 is a linear endomorphism. Pick $b \in \ker(a)$ and $x \in L$. By Leibniz Rule we have $A^2([[b,a]x]) = -A^2([A(b),x]) = -[A^3(b),x] - 2[A^2(b), A(x)] - [A(b), A^2(x)] = 0$ since $A^2(b) = 0$ by hypothesis and $[A(b), A^2(x)] = [A(x), A^2(b)] = 0$ by Proposition 4.1.1(4). For the other product we have, by Proposition 4.1.1(5), that $A^2([[x,a]b]) = A^2BA(x) = A^2XA(b) = A^2([[b,a]x]) = 0$.
2. By Proposition 4.1.1(4), since $A^2(XA(y) - YA(x)) = 0$, $\overline{XA(y)} = \overline{YA(x)}$ and the product $\bullet$ is commutative in L_a. We need to verify the Jordan axiom. Denote, for the time being, $w := AXA(x)$.
On one hand we have, by the definition of $\bullet$ and by commutativity,

$$\overline{(x^2 \bullet y) \bullet x} = \overline{XA(x^2 \bullet y)} = \overline{XAYA(x^2)} = \overline{XAYAXA(x)} =$$

$$= \overline{XAY(w)} = \overline{ad_{Y(w)}(A(x))} = -\overline{ad_{A(x)}(Y(w))},$$

while on the other it is

$$\overline{x^2 \bullet (y \bullet x)} = \overline{ad_{y \bullet x} A(x^2)} = \overline{ad_{y \bullet x} AXA(x)} =$$

$$= \overline{ad_{y \bullet x}(w)} = -\overline{W(y \bullet x)} = -\overline{WYA(x)} = -\overline{YWA(x)} - \overline{\mathrm{ad}_{A(x)}(Y(w))}$$

by Jacobi Identity. Thus it is enough to show that $\overline{YWA(x)} = \bar{0}$. But note that $\overline{YWA(x)} = \overline{ad_{WA(x)}(y)}$, so in turn it is enough to show that $\overline{ad_{WA(x)}} = \bar{0}$, that is, that $A^2 ad_{WA(x)} = 0$ in $\mathrm{End}(L)$.
Observe that $A^2 ad_{WA(x)} = A^2[W[A,X]] = A^2WAX - A^2WXA + A^2XAW$. Taking into account that $w = AXA(x)$, so that $W = [A[X[A,X]] = -A^2X^2 + 2AXAX - 2XAXA + X^2A^2$, we find

$$A^2 ad_{WA(x)} = -2A^2XAXA^2X + 2A^2XAXAXA-$$

$$-A^2X^2A^2XA + 2A^2XA^2XAX - 2A^2XAXAXA + A^2XAX^2A^2.$$

Now we get $A^2XA^2XAX = 0$ by Proposition 4.1.1(2) and $A^2XAXA^2X = AXA^2XA^2X = 0$ by items (1) and (2) of the same proposition, while $A^2X^2A^2XA = A^2X^2AXA^2 = A^2XAX^2A^2$ by items (1) and (3). Therefore $\overline{YWA(x)} = \bar{0}$ and thus

$$\overline{(x^2 \bullet y) \bullet x} = \overline{x^2 \bullet (y \bullet x)}.$$

We will prove now that $U_{\bar{x}}(\bar{y}) = \overline{X^2A^2(y)}$. By definition and by commutativity, $U_{\bar{x}}(\bar{y}) = 2\bar{x} \bullet (\bar{x} \bullet \bar{y}) - (\bar{x} \bullet \bar{x}) \bullet \bar{y} = 2\overline{XAXA(y)} - \overline{ad_{XA(x)}A(y)}$. So it is enough to show that $\overline{2XAX} - \overline{ad_X A(x)} = \overline{X^2A}$. This is indeed true, because in $\mathrm{End}(L_a)$

$$\overline{ad_{XA(x)}} = \overline{[X[A,X]]} = \overline{XAX} - \overline{X^2A} - \overline{AX^2} + \overline{XAX} = 2\overline{XAX} - \overline{X^2A},$$

since $\overline{AX^2} = \bar{0}$ because $A^2(AX^2) = A^3X^2 = 0$. □

We call this Jordan algebra L_a simply *the Jordan algebra at the Jordan element a*.

When $a \in L$ is a Jordan element, $A^2(L)$ turns out to be an abelian inner ideal of L ([Benkart'77, Lemma 1.8]).

Proposition 4.1.3 (**Little Kostrikin Lemma for Jordan elements**).
Let L be a Lie algebra such that $3 \in \mathrm{TF}(L)$ and let $a \in L$ be a Jordan element. Then $A^2(L)$ is an abelian inner ideal such that, for every $x, y, z \in L$,

$$[A^2(x)[A^2(y), z]] = A^2XYA^2(z).$$

Proof. It is clear that $A^2(L)$ is a submodule. Pick $x, y \in L$ and denote $b := A^2(x)$, $c := A^2(y)$. To show that $A^2(L)$ is an inner ideal we have to show that $BC(L) \subseteq A^2(L)$, and for this it is enough to show that A^2 is a left divisor of BC. Since by Proposition 4.1.1(2) we have $A^2XA^2 = 0$, then

$$BC = ad_{A^2(x)}ad_{A^2(y)} = (A^2X - 2AXA + XA^2)(A^2Y - 2AYA + YA^2) =$$

$$= -2A^2XAYA + A^2XYA^2 + 4AXA^2YA - 2AXAYA^2 =$$

$$= -2(A^2XA)YA + 2(AXA^2)YA + 2AX(A^2YA) - 2AX(AYA^2) + A^2XYA^2 = A^2XYA^2$$

by Proposition 4.1.1(1). This proves that $A^2(L)$ is an inner ideal. To see that it is abelian, simply expand $B(c) = BA^2(y) = (A^2X - 2AXA + XA^2)A^2(y) = 0$. □

We say that the Jordan element a is *von Neumann regular* (in the Lie sense) if $a \in A^2(L)$. If the condition $\frac{1}{2}, \frac{1}{3}, \frac{1}{5} \in \Gamma$ is satisfied, then by [Draper,Fernández,García&Gómez'08, Proposition 1.18(i)] there exists another Jordan element $b' \in L$ such that $[[a, b'], a] = 2a$ and $[[a, b'], b'] = -2b'$; by choosing $b := -\frac{1}{2}b'$ we produce yet another Jordan element such that $A^2(b) = a$ and $B^2(a) = b$. We call b a (Lie) *regular partner* for a.
Adapted to this convention, [Benkart'77, Lemma 2.2] states that the inner ideal $A^2(L)$ is a Jordan algebra with product $-[x[b, y]]$. We see that this Jordan algebra is actually isomorphic to the Jordan algebra at a ([Fernández,García&Gómez'06, Proposition 2.11]).

Lemma 4.1.4 (**Realization of the Jordan algebra of a regular element**).
Let L be a Lie algebra such that $3 \in TF(L)$, let $a \in L$ be a von Neumann regular Jordan element and let b be a regular partner of a.
Then the Jordan algebra L_a is isomorphic to $(A^2(L), +, \hat{\bullet})$, with $x \hat{\bullet} y := [x, [b, y]]$.

Proof. Consider the map $\varphi : A^2(L) \to L_a$ defined by $\varphi(A^2(x)) := \bar{x}$. Let us confirm that φ is well defined. Suppose $x, y \in L$ are such that $A^2(x) = A^2(y)$. Then $A^2(x-y) = 0$, i.e., $x - y \in \ker(a)$ and hence $\varphi(A^2(x)) = \bar{x} = \bar{y} = \varphi(A^2(y))$. Since A^2 is linear, φ is linear too. To see that φ is multiplicative, note that, by the Little Kostrikin Lemma (4.1.3),

$$A^2(x)\hat{\bullet}A^2(y) = [A^2(x)[b, A^2(y)]] = -[A^2(x)[A^2(y), b]] = -A^2XYA^2(b) = -A^2XY(a).$$

Therefore $\varphi(A^2(x)\hat{\bullet}A^2(y)) = \varphi(-A^2XY(a)) = -\overline{XY(a)} = \overline{XA(y)} = \bar{x} \bullet \bar{y} = \varphi(A^2(x)) \bullet \varphi(A^2(y))$. It is clear that φ is onto, and it is also injective, since $\bar{x} = \phi(A^2(x)) = \bar{0}$ implies $x \in \ker(a)$, i.e., $A^2(x) = 0$. □

Thanks to this, if a Jordan element is von Neumann regular, then its associated Jordan ring possess a nice realization inside its associated inner ideal.

4.2 Clifford elements in prime rings

Throughout the rest of this chapter let R be a prime, centrally closed ring with involution such that $\mathrm{char}(R) \neq 2, 3, 5$.

By Herstein Lemma for K (3.2.1), if $[K, K] \neq 0$ and the involution is of the first kind then any Jordan element $a \in K$ satisfies either $a^2 = 0$ or $a^2 \neq 0$ and $a^3 = 0$. By the facts exposed in the introduction, the following definition is natural in the associative setting:

Definition 4.2.1 (Clifford element).
A *Clifford element* of R is a Jordan element c of K such that $c^2 \neq 0$ and $c^3 = 0$.

The squares of Clifford elements have simple properties associated that are useful to make computations. In particular they are reduced, which partly determines the structure of $(R, *)$.

Proposition 4.2.2 (Properties of the squares of Clifford elements).
Let $c \in K$ be a Clifford element of R. Then:

1. $c^2Kc^2 = 0$.
2. $c^2Rc^2 = \mathcal{C}c^2$.
3. $c^2k_1k_2c^2 = c^2k_2k_1c^2$ *for every* $k_1, k_2 \in K$.
4. *R has socle and involution of orthogonal type.*

Proof.

1. Since c is Jordan and $c^3 = 0$, for every $k \in K$ we have

$$0 = ad_c^4 k = c^4k - 4c^3kc + 6c^2kc^2 - 4ckc^3 + kc^4 = 6c^2kc^2 = c^2kc^2.$$

2. We prove items (2) and (4) at once. Note that $c^2 \in H$ and that R, being centrally closed, equals $\mathcal{C}R$. By item (1) we know that $c^2Kc^2 = 0$. Hence by the Reduction Lemma 1.2.4(2) for prime rings we get that $c^2Rc^2 = \mathcal{C}c^2$ and that R has socle and involution of orthogonal type.
3. Since $c^2Kc^2 = 0$ by item (1), $c^2\kappa(x)c^2 = 0$ for every $x \in R$, which implies $c^2xc^2 = c^2x^*c^2$. Hence for $k_1, k_2 \in K$ it is $c^2k_1k_2c^2 = c^2(k_1k_2)^*c^2 = c^2k_2k_1c^2$. □

Let c be a Clifford element of R. Observe that c^2 is von Neumann regular because it is reduced (by 4.2.2(2)). In addition c^2 is symmetric and of zero square (since $c^3 = 0$). By the Beautiful Partner Lemma (1.2.5) there exists a beautiful partner $d \in R$ of c^2 such that

$$d^* = d,\ d^2 = 0,\ c^2dc^2 = c^2 \text{ and } d = dc^2d.$$

Beautiful partners for c^2, and their associated idempotents, satisfy (even) more useful properties.

Proposition 4.2.3 (Beautiful partner properties).
Let c be a Clifford element and d a beautiful partner of c^2.

1. $dKd = 0$, $dRd = \mathcal{C}d$.

2. *There exists a $*$-orthogonal idempotent $e \in R$ with $eRe = \mathfrak{C}e$, $e^*Re = \mathfrak{C}c^2$, $eRe^* = \mathfrak{C}d$ and such that $e^*Ke = 0 = eKe^*$.*
3. *$ec = ce^* = 0$, $e^*c^2e = e^*c^2 = c^2e = c^2$ and $ede^* = ed = de^* = d$.*
4. *$[K, K] \neq 0$. In particular R is not an algebra of 2×2 matrices over $\mathfrak{C}$.*
5. *$e^* \neq 1 - e$ in $\widehat{R}$.*

Proof.

1. Note that $dRd \neq 0$ since $d \in dRd$. Then $dRd = (dc^2d)R(dc^2d) = dc^2(dRd)c^2d = d\mathfrak{C}c^2d = \mathfrak{C}dc^2d = \mathfrak{C}d$; also $dKd = dc^2(dKd)c^2d = 0$ since $c^2Kc^2 = 0$ and $d \in H$.
2. Denote $e := dc^2 = dc^2dc^2 = e^2$. Then $e^* = c^2d$, $ee^* = dc^4d = 0$ and $e^*e = c^2d^2c^2 = 0$. Moreover, taking into account that $Rd \neq 0$ because $0 \neq d = dc^2d$, we have $eRe = dc^2(Rd)c^2 = d\mathfrak{C}c^2 = \mathfrak{C}dc^2 = \mathfrak{C}e$, $e^*Re = (c^2d)R(dc^2) = c^2(dRd)c^2 = c^2\mathfrak{C}dc^2 = \mathfrak{C}c^2dc^2 = \mathfrak{C}c^2$ and $eRe^* = (dc^2)R(c^2d) = d(c^2Rc^2)d = d\mathfrak{C}c^2d = \mathfrak{C}dc^2d = \mathfrak{C}d$. Similar computations using $c^2Kc^2 = 0 = dKd$ show that $e^*Ke = 0 = eKe^*$.
3. $ec = (dc^2)c = dc^3 = 0$, $ce^* = c(c^2d) = c^3d = 0$, $e^*c^2e = (c^2d)c^2(dc^2) = (c^2dc^2)dc^2 = c^2dc^2 = c^2$, $ede^* = (dc^2)d(c^2d) = (dc^2d)c^2d = dc^2d = d$. Consequently $c^2e = (e^*c^2e)e = e^*c^2e = c^2$ (the computations for e^*c^2, ed and de are analogous).
4. $[c, e - e^*] = ce + e^*c = cdc^2 + c^2dc \neq 0$. Otherwise $cdc^2 = -c^2dc$ would lead, multiplying on the left by c, to the contradiction $c^2 = c^2dc^2 = -c^3dc = 0$. Since $[c, e - e^*] \in [K, K]$, $[K, K] \neq 0$. This means that K is not exceptional, so that by Theorem 1.3.9 $R = \mathfrak{C}R$ is not a central simple algebra of dimension 4 over $\mathfrak{C}$. In particular $R \ncong \mathbb{M}_2(\mathfrak{C})$.
5. Since $ec = 0 = ce^*$ and $eKe^* = 0 = e^*Ke$ we have $(e + e^*)c(e + e^*) = (ec)e + ece^* + e^*ce + e^*(ce^*) = 0$. Therefore $e + e^* \neq 1$. $\square$

Now we show that the existence of Clifford elements in R can be linked to the existence of idempotents of the kind of Proposition 4.2.3(2).

Theorem 4.2.4 (Existence of Clifford elements).
R has a Clifford element if and only if $[K, K] \neq 0$ and there exists a nonzero $*$-orthogonal idempotent $e \in R$ such that $eRe = \mathfrak{C}e$ and $e^*Ke = 0$.

Proof. The 'only if' part has been proved in Proposition 4.2.3(2,4). Let us show the 'if' part:

We will see first that $e^*Re = \mathcal{C}h$ for some $h \in H$. Note that $eRe = \mathcal{C}e$ implies $e^*Re^* = \mathcal{C}e^*$. Since R is prime, $e \neq 0$ and $e^*Ke = 0$ there exists $x_0 \in H$ such that $h := e^*x_0e \neq 0$. Note that $h^* = (e^*x_0e)^* = h \in H$. Pick $x \in R$; if $e^*xe = 0$ then $e^*xe \in \mathcal{C}h$; if $e^*xe \neq 0$ and being R prime there exists $y \in R$ such that $a := e^*xeyh \neq 0$. On one hand, $a = (e^*xeye^*)x_0e = \lambda_1 e^*x_0e = \lambda_1 h$ with $\lambda_1 \in \mathcal{C}$, while on the other hand $a = e^*x(eye^*x_0e) = \lambda_2 e^*xe$ with $0 \neq \lambda_2 \in \mathcal{C}$. Thus $\lambda_2 e^*xe = a = \lambda_1 h$ implies that $e^*xe = \lambda h$ with $\lambda \in \mathcal{C}$. Therefore

$$e^*Re = \mathcal{C}h.$$

Next we prove that $e^*Ke = 0$ implies $eKe^* = 0$. By a similar argument to that above we find that $eRe^* = \mathcal{C}h_2$ with $h_2 = ey_0e^* \in R$, where a priori we do not know whether h_2 is symmetric. But it does hold that either $h_2 \in H$ or $h_2 \in K$: if there exists $k \in K$ such that $k' := eke^* \neq 0$, then $k' = \lambda h_2$ for some $0 \neq \lambda \in \mathcal{C}$ and thus $-\lambda h_2 = -eke^* = ek^*e^* = (eke)^* = (k')^* = (\lambda h_2)^* = \lambda h_2^*$, which forces $h_2^* = -h_2$; if there exists $s \in H$ such that $s' := ese^* \neq 0$, then as above we find that it must be $h_2 = h_2^*$. The two results combined also imply that if $h_2 \in K$ then $eHe^* = 0$, while if $h_2 \in H$ then $eKe^* = 0$.

Now, since R is prime and $e \neq 0 \neq e^*$, we get that $0 \neq e^*ReRe^*Re = (e^*Re)(eRe^*)(e^*Re) = \mathcal{C}hh_2h$, so that $hh_2h \neq 0$. Next, note that $hh_2h = e^*x_0ey_0e^*x_0e \in e^*Re$ and thus $hh_2h = \lambda h$ for some $0 \neq \lambda \in \mathcal{C}$; therefore, since $h \in H$, $hh_2h = \lambda h = (\lambda h)^* = (hh_2h)^* = hh_2^*h$, which is incompatible with $h_2 \in K$. This gives us $h_2 \in H$ and

$$eKe^* = 0.$$

Now we will find a Clifford element c of R, that is, a Jordan element of K of zero cube but nonzero square:

It can be shown, exactly as in the proof of Proposition 4.2.3(4), that $e^* \neq 1 - e$, because $[K, K] = 0$ is forbidden by hypothesis. Denote $g := 1 - e - e^*$, which is then a nonzero symmetric idempotent orthogonal to e and e^* by both sides. Since R is prime and e, e^*, g are not zero, there exist $a_1, a_2 \in R$ such that $e^*a_1^*ga_2e \neq 0$. Consider $c_1, c_2 \in \kappa(gRe) \subseteq K$ such that $c_1 := \kappa(ga_1e), c_2 := \kappa(ga_2e)$. Then $c_1c_2 = (ga_1e - e^*a_1^*g)(ga_2e - e^*a_2^*g) = -e^*a_1^*ga_2e \neq 0$. Moreover, $c_1c_2 \in e^*Re$ implies $c_1c_2 = \lambda h$ for some $\lambda \in \mathcal{C}$, so that c_1 and c_2 commute: $c_2c_1 = (c_1c_2)^* = (\lambda h)^* = \lambda h = c_1c_2$. If $c_1^2 \neq 0$ (resp. $c_2^2 \neq 0$), take $c := c_1$ (resp. $c := c_2$); if $c_1^2 = 0 = c_2^2$, then $(c_1 + c_2)^2 = c_1^2 + c_2^2 + 2c_1c_2 = 2c_1c_2 \neq 0$ and we can take $c := c_1 + c_2 = \kappa(g(a_1 + a_2)e)$ in order to get $c^2 \neq 0$. Note that in any case $c = \kappa(gze)$ for some $z \in R$.

Finally we show that c is a Jordan element of K of zero cube. Note first that $c^2 = e^*c^2e$: $c^2 = (gze - e^*z^*g)(gze - e^*z^*g) = -e^*z^*gze$, so that $e^*c^2e = e^*(-e^*z^*gze)e = -e^*z^*gze = c^2$. Pick $k \in K$. Then we have $c^3 = c^2c = (e^*z^2e)(gze - e^*z^*g) = 0$ and $c^2kc = (e^*z^2e)k(gze - e^*z^*g) = c^2ekgze = \alpha c^2e = \alpha c^2$ for some $\alpha \in \mathcal{C}$ since $eKe^* = 0$ and $eRe = \mathcal{C}e$, which implies $ckc^2 = (c^2kc)^* = (\alpha c^2)^* = \alpha c^2 = c^2kc$. Thus

$$ad_c^3k = c^3k - 3c^2kc + 3ckc^2 - kc^3 = 0. \qquad \square$$

An interesting aside note is that, in the previous proof, $eKe^* = 0$ is implied by $eRe = \mathcal{C}e$ and $e^*Ke = 0$, being e an $*$-orthogonal idempotent.

As in the general Lie case, the presence of a Clifford element implies a finite grading of the algebra.

Theorem 4.2.5 (Short gradings).
Let $c \in K$ be a Clifford element, d be a beautiful partner of c^2 and $e := dc^2$. Set $g := 1 - e - e^*$. Then

$$K = K_{-1} \oplus K_0 \oplus K_1 \text{ with } K_{-1} := \kappa(gRe), K_0 := \kappa(eRe) \oplus gKg, K_1 := \kappa(eRg)$$

is a 3-grading of K in which the ith homogenous component k_i of any $k \in K$ coincides with $\bigoplus_{m-n=i} \kappa(e_mke_n)$ with $e_0 := e^*, e_1 := g, e_2 := e$.

Proof. Recall that $e^* \neq 1-e$, so $g = 1-e-e^*$ is a nonzero symmetric idempotent of R. Since $\{e_0, e_1, e_2\}$ is a complete system of orthogonal idempotents, by Smirnov's method ([Smirnov'97, p.174]) we have that the Peirce decomposition $R = \bigoplus_{-2\leq i\leq 2} R_i$ with $R_i := \bigoplus_{m-n=i} e_m R e_n$ is an (associative) 5-grading of R:

$$R = e^*Re \oplus (e^*Rg \oplus gRe) \oplus (e^*Re^* \oplus gRg \oplus eRe) \oplus (gRe^* \oplus eRg) \oplus eRe^*.$$

Now, since R_i is selfadjoint for every i, if we define $K_i := R_i \cap K = \mathrm{Skew}(R_i, *)$ we know by Lemma 1.2.3 that $K = \bigoplus_{-2\leq i\leq 2} K_i$ is a Lie (a priori) 5-grading of K which turns out to be a 3-grading: since each R_i is selfadjoint we have $K_i = \kappa(R_i)$ (see Chapter 1); thus $K_{-2} = \kappa(e^*Re) = e^*\kappa(R)e = e^*Ke = 0$ and similarly $K_2 = e^*Ke = 0$. Therefore

$$K = \kappa(gRe) \oplus (\kappa(eRe) \oplus gKg) \oplus \kappa(eRg).$$

Moreover, since the idempotents e_0, e_1, e_2 are orthogonal, the ith homogenous component k_i of any $k \in K$ coincides with $\bigoplus_{m-n=i} \kappa(e_m k e_n)$. As an example, $k_{-1} = \kappa(gke)$ because if $k = \kappa(gx_1e) + \kappa(ex_2e) + \kappa(gx_3g) + \kappa(ex_4g)$ with $x_i \in R$, then $gke = g\kappa(gx_1e)e + 0 = g(gx_1e - e^*x_1^*g)e = gx_1e$, so that $k_{-1} = \kappa(gx_1e) = \kappa(gke)$. □

Note that, for a given Clifford element c, we may get a different grading for each one of the beautiful partners of c^2. Despite of this the K_{-1} component is an invariant of all those gradings. This is true because (as we show below) c happens to lie in K_{-1} for every grading and, as we will prove in Proposition 4.3.1, every K_{-1} is a Clifford inner ideal. Then by Proposition 3.4.16(2) we get that $K_{-1} = ad_c^2 K$, which is independent of the chosen d. In Proposition 4.3.4 we will give an algebraic proof of this last fact (the proof of 3.4.16 is geometric) and will show another equivalent ways of describing K_{-1} independently of d.

We prove now that $c \in K_{-1}$. This implies that $cKc = \mathfrak{C}c$, one of the most important facts of this chapter.

Proposition 4.2.6 (Properties of Clifford elements).
Let c be a Clifford element, d be a beautiful partner of c^2 and $e := dc^2$. Then:

1. $c^2kc = ckc^2$ *for every* $k \in K$.
2. $c \in K_{-1}$ *in the* 3*-grading of* K *generated by* e.
3. $c = e^*c + ce = c^2dc + cdc^2$.
4. $cKc = \mathfrak{C}c$.

Proof.

1. Since c is a Jordan element of K, for every $k \in K$ we have $0 = ad_c^3 k = c^3k - 3c^2kc + 3ckc^2 - kc^3 = -3(c^2kc - ckc^2) = c^2kc - ckc^2$ and therefore $ckc^2 = c^2kc$.
2. Denote $g := 1-e-e^*$. By Theorem 4.2.5, $c = \kappa(gce)+(\kappa(ece)+gcg)+\kappa(ecg)$ with $c_{-1} = \kappa(gce)$. Since $ec = 0$, to show that $c \in K_{-1}$ is to show that $gcg = 0$. Denote $z := gcg$, which is a skew element. Recall that $ge = eg = e^*g = ge^* = 0$ and that $c^2 = c^2e$. As $c^2kc = ckc^2$ for every $k \in K$ by item (1), $c^2kcg = ckc^2g = ckc^2eg = 0$. Then, since in addition $ec = 0$ and $eKe^* = 0$, we have

   $$c^2kz = c^2kgcg = c^2k(1-e-e^*)cg = c^2kcg - c^2k(ec)g - c^2(eke^*)cg = c^2kcg = 0.$$

   Hence $c^2Kz = 0$. Now $c^2Kc^2 = 0$ and $c^2Kz = 0$ with $c^2 \in H$ and $z \in K$ imply, by the Reduction Lemma (1.2.4(3)), that $z = 0$.
3. Since $c = \kappa(gce)$, $(c^2d)c + c(dc^2) = e^*c + ce = e^*(gce + e^*cg) + (gce + e^*cg)e = e^*cg + gce = \kappa(gce) = c$.
4. For any $k \in K$ we have $ckc = (e^*c + ce)k(e^*c + ce) = e^*cke^*c + cekce$, since $eKe^* = 0$ by 4.2.3(2) and $ckc \in K$. Now, again by 4.2.3(2), $ekce = \lambda e$ for some $\lambda \in \mathfrak{C}$, and hence $e^*cke^* = (ekce)^* = (\lambda e)^* = \lambda e^*$, since the involution $*$ is of the first kind by 4.2.2(4). Then $ckc = \lambda e^*c + \lambda ce = \lambda c$. Hence $cKc \subseteq \mathfrak{C}c$. Now $cKc = 0$ would imply $c = 0$ by the Reduction Lemma (1.2.4(1)), so $cKc = \mathfrak{C}c$ since cKc is a $\mathfrak{C}$-subspace. □

We make a brief detour to study whether beautiful partners for the squares of Clifford elements are unique –and answer negatively. Since every adnilpotent element a of R carries with itself a nilpotent associative derivation, the exponential map for A is well defined and is an associative automorphism whenever the number $(2n-2)!$ is invertible in Γ, where n is the index of adnilpotency of a. We recall that

$$exp_A := \sum_{i=0}^{n-1} \frac{1}{i!} A^i,$$

understanding that $A^0 = 1$, the identity of $\mathrm{End}(L)$ (for a more detailed presentation refer to [Humphreys, Section 2.3]).

Proposition 4.2.7 (**Nonuniqueness of beautiful partners**).
Let R be a centrally closed prime ring with involution and $\mathrm{char}(R) > 7$. *Let c be a Clifford element and let d be a beautiful partner of c^2. Then the elements $d_\lambda := exp_{\lambda C}(d)$, where λ ranges in $\mathcal{C}$, are distinct beautiful partners for c^2.*

Proof. Since $c^3 = 0$, c is an adnilpotent element of R of index at most 5. Since $\mathrm{char}(R) > 7$, the numbers $1, \ldots, 8$ are invertible in $\mathcal{C}$ and therefore $exp_{\lambda C}$ is an associative automorphism of R. For every $x \in R$ we have $exp_{\lambda C}(x) = \sum_{i=0}^{4} \frac{\lambda^i}{i!} C^i(x)$ **(1)**. Then $exp_{\lambda C}(c) = c$, and $exp_{\lambda C}(H) \subseteq H$ because $c \in K$ and the involution is of the first kind by 4.2.2, so that $\lambda^* = \lambda$ for every $\lambda \in \mathcal{C}$. Hence $d_\lambda = exp_{\lambda C}(d) \in H$ because $d \in H$. Now clearly $d_\lambda^2 = 0$, $d_\lambda c^2 d_\lambda = d_\lambda$ and $c^2 d_\lambda c^2 = c^2$ because $d^2 = 0$, $dc^2 d = d$, $c^2 dc^2 = c^2$ and $exp_{\lambda C}$ is an automorphism which fixes c. Up to here we have proved that d_λ is a beautiful partner of c^2 for every $\lambda \in \mathcal{C}$. Now we show that all these elements are different. Recall that, by the proof of Theorem 4.2.5, R has a 5-grading which induces a 3-grading in K such that $K_{-1} = R_{-1} \cap K$, and observe that by Proposition 4.2.6 we have $c \in K_{-1} \subseteq R_{-1}$. Hence, when $x \in R$ is homogeneous, every term in the sum of **(1)** lies in a different homogeneous component of R. This implies that $d_\lambda \neq d_\mu$ if $\lambda \neq \mu$, since for example $d_\lambda = d_\mu$ implies $\lambda C(d) = \mu C(d)$, with $C(d) \neq 0$ because $c(cd - dc)c^2 = c^2dc^2 - dc^3 = c^2 \neq 0$. □

4.3 Clifford inner ideals

The extremes of a Lie grading are always abelian inner ideals. From Proposition 3.4.18 we infer that an abelian inner ideal of K is Clifford if it is of the form $\kappa((1-e)Ke) = \kappa((1-e)Re)$, since K and R contain all the skew elements of finite rank (in the geometric model for prime rings with socle and involution). The extremes of the 3-grading of Theorem 4.2.5 are in fact Clifford inner ideals. We prove this just for K_{-1}; the case for the K_1 component is analogous. Denote $B := \kappa(gRe)$ (as has been already noted, B is independent of d and e). We also give an algebraic proof of the fact that B is a Jordan inner ideal.

Proposition 4.3.1 (Properties of B).
Let $c \in K$ be a Clifford element, d be a beautiful partner of c^2, $e := dc^2$ and $g := 1-e-e^*$. Then:

1. *$B = \kappa(gKe) = \kappa((1-e)Ke) = \kappa((1-e)Re)$.*

   In particular B is a Clifford inner ideal.
2. *If $b \in B$ then $b = be + e^*b$ and $eb = 0 = be^*$.*
3. *B is a Jordan inner ideal of K (concretely, a point space).*
4. *$B^2 = \mathcal{C}c^2$.*
5. *$B^3 = 0$.*

Proof.

1. By Theorem 4.2.5 we know that all the homogeneous components of degree -1 of the elements of K lie in $\kappa(gKe)$. Therefore $\kappa(gKe) = \kappa(gRe) = B$. Now, for $k \in K$ it is $gke = (1-e-e^*)ke = (1-e)ke - e^*ke = (1-e)ke$ since $e^*Ke = 0$, so that $\kappa(gKe) = \kappa((1-e)Ke)$.
2. Pick $b \in B$. Then $b = \kappa(gbe)$ and $be = (gbe + e^*bg)e = gbe$, so that $b = \kappa(gbe) = \kappa(be) = be + e^*b$. Also $eb = e\kappa(gbe) = e(gbe + e^*bg) = 0$ since $eg = 0 = ee^*$, and $be^* = -(eb)^* = 0$.
3. Given $b \in B$ it can be shown that $P_bK = bKb = \mathcal{C}b \subseteq B$ by following exactly the same steps as in the proof of 4.2.6(4), but writing b instead of c. Therefore B is a

Jordan point space by definition.

4. Pick $b_1, b_2 \in B$. Then $b_1 b_2 = (b_1 e + e^* b_1)(b_2 e + e^* b_2) = e^* b_1 g b_2 e \in e^* R e = \mathcal{C}c^2$, since $eb_2 = ee^* = b_1 e^* = 0$, and $B^2 \neq 0$ since $c \in B$ and $c^2 \neq 0$.
5. $B = K_{-1} \subseteq R_{-1}$ in the 5-grading of R showed in the proof of Theorem 4.2.5. Therefore $B^3 \subseteq R_{-3} = 0$. □

In what follows we concentrate in proving that B, which will be seen to be the underlying set of the Jordan algebra K_c at the element c (see Theorem 4.1.2), has a direct sum structure with a scalar part. We also show that c endows K with a bilinear form, important later to define the Jordan product in K_c. To prove these facts some elementary tools more are needed. For the obvious reason, we introduce the notation

$$\sqrt{d} := cd + dc. \quad \textbf{(Square root of } d\textbf{)}$$

In addition to being a square root for d, $\sqrt{d}$ is a Clifford element which is an associative regular partner of c, and $-\sqrt{d}$ is a Lie regular partner of c.

Proposition 4.3.2 (**Properties of the square root of** d).
Let c be a Clifford element of R and d a beautiful partner for c^2. Then:

1. *$\sqrt{d} \in K_1$ in the 3-grading of Theorem 4.2.5. In particular $\sqrt{d}$ is a Jordan element.*
2. *$(\sqrt{d})^2 = d$.*
3. *$(\sqrt{d})^3 = 0$.*
4. *$\sqrt{d}K\sqrt{d} = \mathcal{C}\sqrt{d}$.*
5. *$\sqrt{d}c\sqrt{d} = \sqrt{d}$.*
6. *$c\sqrt{d}c = c$.*
7. *$c^2 \circ \sqrt{d} = c$.*
8. *$d \circ c = \sqrt{d}$.*
9. *$ad_c^2(-\sqrt{d}) = c$.*
10. *$ad^2_{-\sqrt{d}}c = -\sqrt{d}$.*
11. *$[[c, \sqrt{d}], b] = b$ for every $b \in B$.*

Proof.

1. Since $c \in K$ and $d \in H$, $\sqrt{d} = cd + dc \in K$. We have

$$\begin{aligned}\kappa(e\sqrt{d}(1-e)) &= e(cd+dc)(1-e) + (1-e^*)(dc+cd)e^* = \\ &= edc(1-e) + (1-e^*)cde^* = edc - edce + cde^* - e^*cde^* = \\ &= (dc^2d)c - e(dcd)c^2 + c(dc^2d) - c^2(dcd)e^* = dc + cd = \sqrt{d},\end{aligned}$$

since $ec = 0$, $e = dc^2$, $dc^2d = d$ and $dcd \in dKd = 0$. We have thus proved (see Theorem 4.2.5) that $\sqrt{d} \in \kappa(eK(1-e)) = K_1$. Now since K_1 is an abelian inner ideal, $\sqrt{d}$ is a Jordan element of K.

2. Recall that $d^2 = 0$ and $d = dc^2d$, and note that $dcd = 0$ since $dKd = 0$ and $c \in K$. Then $(\sqrt{d})^2 = (cd + dc)(cd + dc) = c(dcd) + cd^2c + dc^2d + (dcd)c = dc^2d = d$.
3. $(\sqrt{d})^3 = (\sqrt{d})^2 \cdot \sqrt{d} = d(cd + dc) = dcd + d^2c = 0$.
4. Since $\sqrt{d}$ is a Jordan element such that $(\sqrt{d})^2 = d \neq 0$, it is by definition a Clifford element. Therefore $\sqrt{d}K\sqrt{d} = \mathcal{C}\sqrt{d}$ as shown in Proposition 4.2.6(4).
5. $\sqrt{d}c\sqrt{d} = (cd+dc)c(cd+dc) = c(dc^2d)+c(dcd)c+dc^3d+(dc^2d)c = cd+dc = \sqrt{d}$, since $c^3 = 0$.
6. $c\sqrt{d}c = c(cd + dc)c = c^2dc + cdc^2 = c$, by Proposition 4.2.6(3).
7. $c^2 \circ \sqrt{d} = c^2(cd + dc) + (cd + dc)c^2 = c^2dc + cdc^2 = c$.
8. $d \circ c = dc + cd = \sqrt{d}$.
9. $ad_c^2(-\sqrt{d}) = c^2 \circ (-\sqrt{d}) + 2c\sqrt{d}c = -c + 2c = c$, by items (6) and (7).
10. $ad^2_{-\sqrt{d}}c = (-\sqrt{d})^2 \circ c - 2\sqrt{d}c\sqrt{d} = d \circ c - 2\sqrt{d} = \sqrt{d} - 2\sqrt{d} = -\sqrt{d}$, by items (2), (5) and (8).
11. In first place,
$[c, \sqrt{d}] = c\sqrt{d} - \sqrt{d}c = c(cd + dc) - (cd + dc)c = c^2d + cdc - cdc - dc^2 = e^* - e$.
Therefore $[[c, \sqrt{d}], b] = [e^* - e, b] = e^*b - eb - be^* + be = be + e^*b = b$, since $eb = 0 = be^*$ and $be + e^*b = b$ by Proposition 4.3.1. $\square$

The (image of) $\sqrt{d}$ plays the role of identity element in K_c. The Clifford structure of K_c is built on the two forms described below.

Definitions 4.3.3 (Forms).

Let $c \in K$ be a Clifford element of R.

- By 4.2.6(1) and the fact that $\mathcal{C}$ is a field, there exists a well-defined linear map $\mathrm{tr} : K \to \mathcal{C}$ such that, for every $k \in K$,

$$\mathrm{tr}(k)c = ckc.$$

Call $\mathrm{tr}(k)$ the *trace* of k. Note that

1. $\mathrm{tr}(\sqrt{d}) = 1$ since $c\sqrt{d}c = c$ by Proposition 4.3.2(6).
2. $K = \mathcal{C}\sqrt{d} \oplus \mathrm{Ker}(\mathrm{tr})$ by item (1).

- By Proposition 4.2.2(2,3) we have $c^2Rc^2 = \mathcal{C}c^2$ with $c^2k_1k_2c^2 = c^2k_2k_1c^2$ for all $k_1, k_2 \in K$. Therefore there exists a well-defined symmetric bilinear form $\langle\cdot\,,\cdot\rangle : K \times K \to \mathcal{C}$ defined for all $k_1, k_2 \in K$ by

$$\langle k_1, k_2\rangle c^2 := c^2k_1k_2c^2.$$

The trace can be realized from the bilinear form and vice versa. Let $k, k' \in K$.

1. $\langle\sqrt{d}, k\rangle c^2 = c^2\sqrt{d}kc^2 = c^2(cd+dc)kc^2 = c^3dkc^2 + c^2dckc^2 = c^2dckc^2 = c^2d(ckc)c = \mathrm{tr}(k)c^2dc^2 = \mathrm{tr}(k)c^2$, since $c^3 = 0$ and $c^2dc^2 = c^2$. Hence

$$\mathrm{tr}(k) = \langle k, \sqrt{d}\rangle.$$

2. $\mathrm{tr}(\kappa(ckk'))c^2 = (c\kappa(ckk')c)c = c^2kk'c^2 + ck'kc^3 = c^2kk'c^2 = \langle k, k'\rangle c^2$. Thus

$$\langle k, k'\rangle = \mathrm{tr}(\kappa(ckk')).$$

As an aside, observe that for the 3-grading of Theorem 4.2.5 we get $\mathrm{tr}(K_{-1}\oplus K_0) = 0$: By Proposition 4.3.1(5) we have $cBc \subseteq B^3 = 0$. In addition, for every $k \in K$ it is $c\kappa(eke)c = c(eke + e^*ke^*)c = 0$, since $ec = 0 = ce^*$, and $cgKgc = (ce + e^*c)gKg(ce + e^*c) = e^*(cgKgc)e = 0$, since $eg = 0 = ge^*$, $e^*Ke = 0$ and $cgKgc \subseteq K$.
Hence $K = \ker(\mathrm{tr}) \oplus \mathcal{C}\sqrt{(d)}$ with $\ker(\mathrm{tr}) = K_{-1} \oplus K_0 \oplus K_1^0$, where
$K_1^0 := \{k \in K_1 \mid \mathrm{tr}(k) = 0\}$.

The trace helps to identify the direct sum structure of the Clifford Jordan algebra of K_c. Since c and $-\sqrt{d}$ are Lie regular partners (Proposition 4.3.2(9),(10)), by Proposition 4.1.4 the Jordan algebra K_c is isomorphic to $(ad_c^2K, +, \bullet)$, with product $x \bullet y = [x, [\sqrt{d}, y]]$. We note that ad_c^2K may be written as $\mathcal{C}c \oplus B_0$, where B_0 is defined from elements of zero trace.

Proposition 4.3.4 (Structure of B).

Let $c \in K$ be a Clifford element. Then:

1. $B = c^2 \circ K$.
2. $B = B_0 \oplus \mathcal{C}c$, *where* $B_0 := \{c^2 \circ k \mid k \in \ker(\mathrm{tr})\}$.
3. $B = \mathrm{ad}_c^2 K$.

Proof.

Let d denote a beautiful partner of c^2 and $e := dc^2$.

1. Pick $k \in K$. Recall that $c^2 = e^*c^2 = e^*c^2e$ and that $eKe^* = 0$. Then

   $$c^2 \circ k = \kappa(kc^2) = \kappa(ke^*c^2 - (eke^*)c^2) = \kappa((1-e)ke^*c^2) = \kappa((1-e)kc^2e) \in \kappa((1-e)Ke).$$

   This shows that $c^2 \circ K \subseteq B$ by Proposition 4.3.1(1). Conversely, let $b \in B$. Then

   $$b = e^*b + be = (c^2d)b + b(dc^2) = c^2(d \circ b) + (d \circ b)c^2 = c^2 \circ (d \circ b) \in c^2 \circ K,$$

   since $e^* = c^2d$, $c^2 = c^2e$ and $c^2b = (c^2e)b = c^2(eb) = 0$.

2. By 4.3.3(2), $K = \mathrm{Ker}(\mathrm{tr}) \oplus \mathcal{C}\sqrt{d}$. Hence

   $$B = c^2 \circ K = c^2 \circ \ker(\mathrm{tr}) + \mathcal{C}c^2 \circ \sqrt{d} = c^2 \circ \ker(\mathrm{tr}) + \mathcal{C}c$$

   since $c^2 \circ \sqrt{d} = c$ by Proposition 4.3.2(7). Let us prove that the sum is direct. Suppose $\alpha c = c^2 \circ k$, with $\mathrm{tr}(k) = 0$ and $\alpha \in \mathcal{C}$. Then by multiplying on the left by c we get $\alpha c^2 = c^3k + ckc^2 = (ckc)c = \mathrm{tr}(k)c^2 = 0$, so that $\alpha = 0$ and $\alpha c = 0 = c^2 \circ k$.

3. For any $k \in K$ we have $ad_c^2 k = c^2k - 2ckc + kc^2 = c^2 \circ k - 2\mathrm{tr}(k)c \in B$. Conversely, let $c^2 \circ k_0 + \alpha c \in B$, with $k_0 \in \mathrm{Ker}(\mathrm{tr})$ and $\alpha \in \mathcal{C}$. Then

   $$c^2 \circ k_0 + \alpha c = \mathrm{ad}_c^2 k_0 - \alpha \mathrm{ad}_c^2 \sqrt{d} = \mathrm{ad}_c^2(k_0 - \alpha\sqrt{d}),$$

   since $ad_c^2 k_0 = c^2 \circ k_0 - 2ck_0c = c^2 \circ k_0 - 2\mathrm{tr}(k_0) = c^2 \circ k_0$ and $\mathrm{ad}_c^2\sqrt{d} = -c$ by Proposition 4.3.2(9). □

By analogy it can be proved that

$$K_1 = d \circ K = \mathcal{C}\sqrt{d} \oplus D_0 = ad_{\sqrt{d}}^2 K, \text{ with } D_0 := \{d \circ k \mid k \in K, \sqrt{d}k\sqrt{d} = 0\}.$$

4.4 Jordan algebra at a Clifford element

The bilinear form of K is involved in the construction of the Clifford product of K_c.

Lemma 4.4.1. *The symmetric bilinear form defined from B to $\mathcal{C}c$ by*

$$\langle c^2 \circ k_1, c^2 \circ k_2 \rangle_0 := -\langle k_1, k_2 \rangle$$

is well defined.

Proof. Pick $k_1, k_1', k_2 \in K$ and suppose that $c^2 \circ k_1 = c^2 \circ k_1'$. Since $\langle \cdot\, , \cdot \rangle_0$ is symmetric, all we have to show is that $\langle c^2 \circ k_1, c^2 \circ k_2 \rangle_0 = \langle c^2 \circ k_1', c^2 \circ k_2 \rangle_0$, that is, that $-\langle k_1, k_2 \rangle = -\langle k_1', k_2 \rangle$. By the definition of $\langle \cdot\, , \cdot \rangle$ this amounts to prove that $c^2 k_1 k_2 c^2 = c^2 k_1' k_2 c^2$. This identity is directly found from $c^2 \circ k_1 = c^2 \circ k_1'$, by multiplying on both sides by $k_2 c^2$ and taking into account that $c^2 K c^2 = 0$. □

Theorem 4.4.2 (K_c **is a Clifford Jordan algebra**).
The Jordan algebra K_c is isomorphic to the Clifford Jordan algebra

$$(\mathcal{C} \oplus B_0, \langle \cdot\, , \cdot \rangle_0).$$

Proof. Since $c = [[c, \sqrt{d}], c]$ (see 4.3.2(9)), we have by Lemma 4.1.4 that K_c is isomorphic to the Jordan algebra $J(c, \sqrt{d})$ defined on the $\mathcal{C}$-vector space $\mathrm{ad}_c^2 K = \mathcal{C}c \oplus B_0$ (see 4.3.4) by the product

$$(\alpha_1 c + c^2 \circ k_1) \bullet (\alpha_2 c + c^2 \circ k_2) = [[\alpha_1 c + c^2 \circ k_1, \sqrt{d}], \alpha_2 c + c^2 \circ k_2],$$

for all $\alpha_1, \alpha_2 \in \mathcal{C}$ and all $k_1, k_2 \in K$ such that $ck_1c = ck_2c = 0$. Endow the $\mathcal{C}$-vector space B_0 with the symmetric bilinear form $\langle \cdot\, , \cdot \rangle_0$ defined in 4.4.1 and consider the Clifford Jordan algebra $\mathcal{C} \oplus B_0$ defined by $\langle \cdot\, , \cdot \rangle_0$. We claim that the linear isomorphism $(\alpha c + c^2 \circ k) \mapsto (\alpha, c^2 \circ k)$ of $J(c, \sqrt{d})$ onto $\mathcal{C} \oplus X$ is actually an isomorphism of Jordan algebras. Since $\frac{1}{2} \in \mathcal{C}$, it suffices to check the identity

$$(\alpha c + c^2 \circ k)^2 = [[\alpha c + c^2 \circ k, \sqrt{d}], \alpha c + c^2 \circ k] = \alpha^2 c + \langle c^2 \circ k, c^2 \circ k \rangle_0 c + 2\alpha (c^2 \circ k).$$

The use of the bilinearity of the Lie product reduces this check to three products: (1) scalar by scalar, (2) scalar by vector and (3) vector by vector.

1. $[[\alpha c, \sqrt{d}], \alpha c] = \alpha^2[[c, \sqrt{d}], c] = \alpha^2 c$, by 4.3.2(9).
2. $[[\alpha c, \sqrt{d}], c^2 \circ k] = \alpha[[c, cd + dc], c^2k + kc^2] = \alpha[c^2d - dc^2, c^2k + kc^2] = \alpha(c^2 \circ k)$, where we have used $c^2dc^2 = c^2$, $c^4 = 0$ and $c^2kc^2 = c^2(dk + kd)c^2 = 0$, the latter because $c^2Kc^2 = 0$ and $(dk + kd)^* = -(kd + dk)$, since $d^* = d$ and $k^* = -k$.
3. $[[c^2 \circ k, \sqrt{d}], c^2 \circ k] = 2(c^2 \circ k)\sqrt{d}(c^2 \circ k) - (c^2 \circ k)^2 \circ \sqrt{d}$, with

$$(c^2 \circ k)\sqrt{d}(c^2 \circ k) = (c^2k + kc^2)(cd + dc)(c^2k + kc^2) = (c^2kdc + kc^2dc)(c^2k + kc^2) = 0,$$

since $c^3 = 0$ and $ckc = 0$ ($\mathrm{tr}(k) = 0$), and

$$(c^2 \circ k)^2 \circ \sqrt{d} = c^2k^2c^2(cd+dc)+(cd+dc)c^2k^2c^2 = c^2k^2c^2dc+cdc^2k^2c^2 = \langle k, k\rangle(c^2dc+cdc^2) = \langle k, k\rangle c$$

since $c = c^2dc + cdc^2$ by 4.2.6(1).

Therefore, $(c^2 \circ k) \bullet (c^2 \circ k) = -\langle k, k\rangle c = \langle c^2 \circ k, c^2 \circ k\rangle_0 c$, which completes the proof. □

Appendix A

Geometric model of prime rings with socle

As is well known, prime rings with socle possess a nice geometric model in terms of dual pairs of vector spaces that allows to pose complicated calculations with the aid of the powerful tools of linear algebra. In this appendix we introduce the needed concepts and notation to make computations with this model, without including proofs (good references for this material are [RingsGIs, Chapter 4] and [StructureRings, Chapter IV]). These tools are necessary in Sections 1.1.1 and 3.4.3.

Given a left and a right vector space V, W over a division ring Δ, we define a *bilinear form* to be a bilinear application $\langle \cdot\,,\cdot \rangle : V \times W \to \Delta$, where linearity for scalars in the second argument is understood as $\langle v, w\alpha \rangle = \langle v, w \rangle \alpha$ for every $v \in V$, $w \in W$ and $\alpha \in \Delta$. A bilinear form is said to be *nondegenerate* if $\langle v, W \rangle = 0$ implies $v = 0$ and if similarly $\langle V, w \rangle = 0$ implies $w = 0$.

Definition A.0.1 (Dual pair of vector spaces).
Let V, W be a left and a right vector space over the same division ring. Then $(V, W, \langle \cdot\,,\cdot \rangle)$ is a *dual pair of vector spaces* if $\langle \cdot\,,\cdot \rangle$ is a nondegenerate bilinear form on $V \times W$.

Usually we will not mention specifically the division ring nor the bilinear form, and will just talk about the dual pair (V, W).

We will realize the elements of prime rings with socle as particular endomorphisms of a dual pair. In this context we will call any element of $\mathrm{End}_\Delta(V)$ or $\mathrm{End}_\Delta(W)$ an *operator*. We always consider the operators as acting from the left. We need the notion of adjoint of an operator[1].

Definition A.0.2 (Adjoint).
Let (V, W) be a dual pair of vector spaces and let $a \in \mathrm{End}_\Delta(V)$. We say that a is *continuous* if there exists $a^\# \in \mathrm{End}_\Delta(W)$, the *adjoint* of a, such that $\langle av, w\rangle = \langle v, a^\# w\rangle$ for every $v \in V$, $w \in W$.

It is a direct consequence of the definition above and of the nondegeneracy of the bilinear form that the operator adjoint to a, if it exists, is unique.

Definitions A.0.3 (Continuous and finite-rank operators).
Let (V, W) be a dual pair of vector spaces.

- The subring of $\mathrm{End}_\Delta(V)$ of all continuous operators is denoted by $\mathcal{L}_W(V)$.
- The ideal of $\mathrm{End}_\Delta(V)$ of all continuous and finite-rank operators is denoted by $\mathcal{F}_W(V)$.

We are prepared to exhibit the geometric model of prime rings with socle ([RingsGIs, Theorems 4.3.7 and 4.3.8]).

Theorem A.0.4 (Geometric model for prime rings with socle).
Let R be a prime ring with socle and let $e \in R$ be a minimal idempotent. Then there exists a dual pair of vector spaces (V, W) over $\Delta := eRe$ such that:

1. $\mathcal{F}_W(V) \subseteq R \subseteq \mathcal{L}_W(V)$.
2. $\mathrm{Soc}(R) = \mathcal{F}_W(V)$, *which is the only minimal ideal of R.*
3. $Q_s(R) = \mathcal{L}_W(V)$.
4. $Q_m(R) = Q(R) = \mathrm{End}_\Delta(Re)$, *with $eQ(R)e = \Delta$.*[2]

[1] This notion should not be confused with the unrelated notion of adjoint operator for the adjoint representation of a Lie algebra described in the Preliminaries section.

[2] $Q_m(R)$ designates here the maximal right ring of quotients of Utumi (refer to [RingsGIs, Section 2.1]).

5. $\mathcal{C} \cong Z(\Delta)$.

The converse result also holds: if $\mathcal{F}_W(V) \subseteq R \subseteq \mathcal{L}_W(V)$ for some (V, W) then R is prime with socle. Therefore we get that the rings of quotients of a prime ring with socle are again prime with socle. On the other hand, if R is simple with socle, then R is prime with socle and such that $R = \mathrm{Soc}(R)$. Therefore we get the following result.

Corollary A.0.5 (Geometric model for simple rings with socle).
Let R be a simple ring with socle and let $e \in R$ be a minimal idempotent. Then there exists a dual pair of vector spaces (V, W) over eRe such that $R = \mathcal{F}_W(V)$.

It can be also seen that R simple with socle will be artinian if and only if V is finite dimensional, because that is a necessary and sufficient condition for the identity endomorphism (which is trivially continuous) to be of finite rank. In that case $Z(R) \cong \mathcal{C} \cong Z(eRe)$.

Since every element of a prime ring with socle can be seen as an operator of a dual pair, we will define the *rank* of an element as its rank as a linear operator. This notion is independent of the concrete dual pair chosen to represent the ring.

A.1 Linear algebra tools

By means of their geometric model, any useful result about dual pairs can be translated to the setting of prime rings with socle. We present now some of them.

A very important tool when working with a dual pair are dual linearly independent sets.

Definition A.1.1 (Dual sets).
Let (V, W) be a dual pair of vector spaces. If $S_1 := \{v_i\}_{i=1}^n \subseteq V$ is an linearly independent set, then there exists another linearly independent set $S_2 := \{w_i\}_{i=1}^n \subseteq W$ such that $\langle v_i, w_j \rangle = \delta_{ij}$ for every $i, j \in \{1, \ldots, n\}$, where δ_{ij} is Kronecker delta ([RingsGIs, Theorem 4.3.1]). Similarly, if we fix S_2 first we can find S_1 satisfying the same property. We will say that S_1 and S_2 are *dual sets*, or that S_i is a *dual set* to S_j ($i, j \in \{1, 2\}, i \neq j$).

The continuous and finite-rank operators can be described and operated in terms of (V, W). We follow the notation of [StructureRings, pages 74 and 75].

Lemma A.1.2 (**Model for $\mathcal{F}_W(V)$**).
Let (V, W) be a dual pair of vector spaces over Δ.
We denote $\otimes : W \times V \to \mathcal{F}_W(V)$ for the map such that, for every $x \in V$,

$$w \otimes v(x) := \langle x, w \rangle v.$$

For subspaces $V_1 \leq V$, $W_1 \leq W$ we denote $W_1 \otimes V_1 := \operatorname{span}(\{w \otimes v \mid v \in V_1, w \in W_1\})$. Then we get that

$$\mathcal{F}_W(V) = W \otimes V.$$

In addition the operation $\otimes$ satisfies the following properties:

1. $\otimes$ is additive and such that $w\alpha \otimes v = w \otimes \alpha v$ for every $\alpha \in \Delta$.
2. $(w_1 \otimes v_1)(w_2 \otimes v_2) = w_2 \otimes (\langle v_2, w_1 \rangle v_1)$ (**Product Law**).
   This is due to $w_1 \otimes v_1(w_2 \otimes v_2)(x) = w_1 \otimes v_1(\langle x, w_2 \rangle v_2) =$
   $\langle \langle x, w_2 \rangle v_2, w_1 \rangle v_1 = \langle x, w_2 \rangle \langle v_2, w_1 \rangle v_1 = w_2 \otimes (\langle v_2, w_1 \rangle v_1)(x)$.
3. We also have a **Product Law for subspaces** $V_1, V_2 \leq V$ and $W_1, W_2 \leq W$:

$$(W_1 \otimes V_1)(W_2 \otimes V_2) = 0 \text{ if } \langle V_2, W_1 \rangle = 0 \text{ and}$$

$$(W_1 \otimes V_1)(W_2 \otimes V_2) = W_2 \otimes V_1 \text{ otherwise.}$$

   This is due to the fact that if $\langle V_2, W_1 \rangle \neq 0$ then $\langle V_2, W_1 \rangle = \Delta$, because Δ is a division ring and $\langle \cdot, \cdot \rangle$ is (bi)linear.
4. For every $a \in \operatorname{End}_\Delta(V)$,

$$a(w \otimes v) = w \otimes (av) \text{ (\textbf{Absorption Law 1})}.$$

   This is due to $a(w \otimes v)(x) = a(\langle x, w \rangle v) = \langle x, w \rangle a(v) = (w \otimes a(v))(x)$.
5. For every $a \in \mathcal{L}_W(V)$,

$$(w \otimes v)a = (a^{\#} w) \otimes v \text{ (\textbf{Absorption Law 2})}.$$

   This is due to $(w \otimes v)a(x) = \langle a(x), w \rangle v = \langle x, a^{\#}(w) \rangle v = (a^{\#}(w) \otimes v)(x)$. □

Note that item (1) shows that in fact $\mathcal{F}_W(V) \overset{\Delta}{\cong} W \underset{\Delta}{\otimes} V$ with $\otimes$ the usual tensor product, so the nomenclature is well chosen.

This model allows to determine the one-sided ideals of simple rings with socle, which are relevant in our exposition of the classification results on inner ideals in Section 3.1. Observe that if $W_1 \leq W$ and $a \in \mathcal{F}_W(V)$, then by the Absorption Law 1 we get that $a(W_1 \otimes V) = W_1 \otimes aV \subseteq W_1 \otimes V$ and thus $W_1 \otimes V$ is a left ideal of $\mathcal{F}_W(V)$. The Absorption Law 2 gives us a similar result for right ideals. In fact all the one-sided ideals of $\mathcal{F}_W(V)$ follow these patterns (this is [StructureRings, Theorem 1, page 91]).

Theorem A.1.3 (One-sided ideals of a simple ring with socle).
Let (V, W) be a dual pair of vector spaces. Then every left (resp. right) ideal of $\mathcal{F}_W(V)$ is of the form $W_1 \otimes V$ (resp. $W \otimes V_1$), where $W_1 \leq W$ (resp. $V_1 \leq V$).

In particular, the minimal left ideals of $\mathcal{F}_W(V)$ are of the form $W \otimes \Delta v$, with $0 \neq v \in V$.

Index

Bibliography

[Amitsur'68] S.A. Amitsur. *Rings with involution.*
Israel J. Math. **6** (1968), 99–106.

[Anquela,Cortés&McCrimmon'96] J.A. Anquela, T. Cortés and K. McCrimmon.
An elemental characterization of strong primeness in Jordan systems.
J. Pure Appl. Algebra **109** (1996), 23–36.

[GeometricAlgebra] E. Artin. *Geometric Algebra.*
Interscience Tracts in Pure and Applied Mathematics **3**, Interscience Publishers, 1957.

[Baranov'99] A.A. Baranov. *Finitary simple Lie algebras.*
J. Algebra **219** (1999), 299–329.

[Baranov&Rowley'13] A.A. Baranov, J. Rowley.
Inner ideals of simple locally finite Lie algebras.
J. Algebra **379** (2013), 11–30.

[Baxter'56] W.E. Baxter.
Lie simplicity of a special class of associative rings.
Proc. Amer. Math. Soc. **7**(5) (1956), 855–863.

[Beachy] J.A. Beachy. *Introductory Lectures on Rings and Modules.*
London Mathematical Society Student Texts **47**, Cambridge University Press, 1999.

[RingsGIs] K.I. Beĭdar, W.S. Martindale 3rd and A.V. Mikhalëv.
Rings with Generalized Identities.
Pure and Applied Mathematics; Marcel Dekker Inc., 1995.

[Beĭdar&Mikhalëv'87] K.I. Beĭdar and A.V. Mikhalëv.
The structure of nondegenerate alternative algebras.
Trudy Sem. Petrovsk. **243**(12) (1987), 59–74 (in Russian).
English translation: J. Soviet Math. **47**(3) (1989), 2525–2536.

[Beĭdar,Mikhalëv&Slin'ko'87] K.I. Beĭdar, A.V. Mikhalëv and A.M. Slin'ko.
A criterion for primeness of nondegenerate alternative and Jordan algebras.
Tr. Mosk. Mat. Obs. **50** (1987), 130–137 (in Russian).
English translation: Trans. Moscow Math. Soc. **261**(50) (1988), 129–137.

[Benkart'76] G. Benkart. *The Lie inner ideal structure of associative rings.*
J. Algebra **43** (1976), 561–584.

[Benkart'77] G. Benkart. *On inner ideals and ad-nilpotent elements of Lie algebras.*
Trans. Amer. Math. Soc. **232** (1977), 61–81.

[Benkart&Fernández'09] G. Benkart and A. Fernández López.
The Lie inner ideal structure of associative rings revisited.
Comm. Algebra **37** (2009), 3833–3850.

[Benkart&Isaacs'77] G. Benkart and I.M. Isaacs.
On the existence of ad-nilpotent elements.
Proc. Amer. Math. Soc. **63**(1) (1977), 39–40.

[FunctionalIdentities] M. Brešar, M.A. Chebotar and W.S. Martindale 3rd.
Functional Identities. Frontiers in Mathematics, Birkhäuser Verlag, Basel-Berlin, 2007.

[Brox,Fernández&Gómez(1)] J. Brox, A. Fernández López and M. Gómez Lozano. *Lie inner ideals of skew elements of prime rings with involution.* In revision process by Proc. Amer. Math. Soc..

[Brox,Fernández&Gómez(2)] J. Brox, A. Fernández López and M. Gómez Lozano. *Clifford elements in prime rings*. Submitted to J. Algebra.

[Brox,García&Gómez'14] Jose Brox, E. García and M. Gómez Lozano. *An elemental characterization of orthogonal ideals in Lie algebras.* Mediterr. J. Math. **11**(1) (2014), 1061–1067.

[Brox,García&Gómez(2)] Jose Brox, E. García and M. Gómez Lozano. *A characterization of Jordan elements in semiprime rings*. Preprint.

[Chung'85] L.O. Chung. *Nil derivations.* J. Algebra **95** (1985), 20–30.

[FurtherAlgebra] P.M. Cohn. *Further Algebra and Applications.* Springer, London, 2003.

[Coutinho] S.C. Coutinho. *A Primer of Algebraic D-modules.* London Mathematical Society Student Texts **33**, Cambridge University Press, 1995.

[Dixmier'63] J. Dixmier. *Sur les algèbres de Weyl.* Bull. Soc. Math. France **96** (1963), 209–242 (in French).

[Draper,Fernández,García&Gómez'08] C. Draper, A. Fernández López, E. García and M. Gómez Lozano. *The socle of a nondegenerate Lie algebra.* J. Algebra **319** (2008), 2372–2394.

[Draper,Fernández,García&Gómez'12] C. Draper, A. Fernández López, E. García and M. Gómez Lozano. *The inner ideals of the simple finite dimensional Lie algebras.* J. Lie Theory **22** (2012), 907–929.

[PIRings] V. Drensky, E. Formanek. *Polynomial Identity Rings.*
Advanced Courses in Mathematics - CRM Barcelona, Springer Basel AG, Birkhäuser, 2004.

[Erikcson'72] T.S. Erickson. *The Lie structure in prime rings with involution.*
J. Algebra **2** (1972), 523–534.

[Faulkner'73] J.R. Faulkner. *On the geometry of inner ideals.*
J. Algebra **26** (1973), 1–9.

[Fernández'14] A. Fernández López. *Lie inner ideals are nearly Jordan inner ideals.*
Proc. Amer. Math. Soc. **142** (2014), 795–804.

[Fernández&García'99] A. Fernández López and E. García Rus.
Inner ideals in quadratic Jordan algebras of infinite capacity.
Int. J. Math. Game Theory Algebra **9**(1) (1999), 35–54.

[Fernández,García&Gómez'04] A. Fernández López, E. García and M. Gómez Lozano.
The Jordan socle and finitary Lie algebras. J. Algebra **280** (2004), 635–654.

[Fernández,García&Gómez'06] A. Fernández López, E. García and M. Gómez Lozano.
The Jordan algebras of a Lie algebra.
J. Algebra **308**(1) (2006), 164–177.

[Fernández,García&Gómez'06(2)] A. Fernández López, E. García and M. Gómez Lozano. *Inner ideals of finitary simple Lie algebras.*
J. Lie Theory **16**(1) (2006), 97–114.

[Fernández,García&Gómez'08] A. Fernández López, E. García and M. Gómez Lozano.
An Artinian theory for Lie algebras.
J. Algebra **319** (2008), 938–951.

[Fernández,García,Gómez&Neher'07] A. Fernández López, E. García, M. Gómez Lozano and E. Neher. *A construction of gradings of Lie algebras.* Int. Math. Res. Not. **16** (2007), 1–34.

[Filippov'81] V.T. Filippov. *On nil-elements of index* 2 *in Mal'tsev algebras.* Manuscript number 2499-80 deposited at VINITI 1980. Sibirsk. Mat. Zh. **22**(3) (1981), 232 (abstract, in Russian).

[Formanek] E. Formanek. *The Polynomial Identities and Invariants of* $n \times n$ *Matrices.* Regional Conference Series in Mathematics **78**, American Mathematical Society, 1991.

[García&Gómez'07] E. García and M. Gómez Lozano. *An elemental characterization of strong primeness in Lie algebras.* J. Algebra **312**(1) (2007), 132–141.

[García&Gómez'09] E. García and M. Gómez Lozano. *A note on a result of Kostrikin.* Comm. Algebra **37**(7) (2009), 2405–2409.

[García&Gómez'11] E. García and M. Gómez Lozano. *A characterization of the Kostrikin radical of a Lie algebra.* J. Algebra **346** (2011), 266–283.

[CSAs] P. Gille and T. Szamuely. *Central Simple Algebras and Galois Cohomology.* Cambridge Studies in Advanced Mathematics **101**, Cambridge University Press, 2006.

[Grzeszczuk'92] P. Grzeszczuk. *On nilpotent derivations of semiprime rings.* J. Algebra, **149**(2) (1992), 313–321.

[Hall'43] M. Hall. *Projective planes.* Trans. Amer. Math. Soc. **54** (1943), 229–277.

[Hennig'14] J. Hennig. *Simple, locally finite dimensional Lie algebras in positive characteristic.* J. Algebra **413** (2014), 270–288.

[Herstein'55] I.N. Herstein. *On the Lie and Jordan rings of a simple associative ring.* Amer. J. Math. **77** (1955), 393–403.

[Herstein'55(2)] I.N. Herstein. *On the Lie ring of a simple associative ring.* Duke Math. J. **22** (1955), 471–476.

[Herstein'56] I.N. Herstein. *Lie and Jordan systems in simple rings with involution.* Amer. J. Math. **78** (1956), 629–649.

[Herstein'61] I.N. Herstein. *Lie and Jordan structures in simple, associative rings.* Bull. Amer. Math. Soc. **67**(6) (1961), 517–531.

[Herstein'63] I.N. Herstein. *Sui commutatori degli anelli semplici.* Rend. Sem. Mat. Fis. Milano **33** (1963), 80–86.

[TopicsRingTheory] I.N. Herstein. *Topics in Ring Theory.* Chicago Lectures in Mathematics **2**, University of Chicago Press, Chicago-London, 1969.

[Herstein'70] I.N. Herstein. *On the Lie structure of an associative ring.* J. Algebra **14** (1970), 561–571.

[RingsInvolution] I.N. Herstein. *Rings with Involution.* Chicago Lectures in Mathematics **12**, University of Chicago Press, Chicago-London, 1976.

[Humphreys] J.E. Humphreys. *Introduction to Lie Algebras and Representation Theory.* Springer, New York, 1980 (third printing).

[StructureRings] N. Jacobson. *Structure of Rings.* Colloquium Publications *37*, American Mathematical Society, 1964 (revised edition).

[Jacobson'66] N. Jacobson. *Structure theory for a class of Jordan algebras.* Proc. Nat. Acad. Sci. U.S.A. **55** (1966), 243–251.

[StructureJordan] N. Jacobson. *Structure and Representations of Jordan Algebras.* Colloquium Publications *39*, American Mathematical Society, 1968.

[BasicAlgebra1] N. Jacobson. *Basic Algebra I.* W.H. Freeman and Company, New York, 1985 (second edition).

[Joly'70] J.R. Joly. *Sommes de puissances d-ièmes dans un anneau commutatif.* Acta Arith. **17** (1970), 37–114 (in French).

[Jordan,Neumann&Wigner'34] P. Jordan, J. Von Neumann and E. Wigner. *On an algebraic generalization of the quantum mechanical formalism.* Ann. of Math. **36** (1934), 29–64.

[LinearGeometry] I. Kaplansky. *Linear Algebra and Geometry, a Second Course.* University of Chicago, Chelsea Publishing, 1974 (second edition).

[Kaplansky'48] I. Kaplansky. *Rings with a polynomial identity.* Bull. Amer. Math. Soc. **54** (1948), 575–580.

[Kharchenko'78] V.K. Kharchenko. *Differential identities of prime rings.* Algebra and Logic **17** (1978), 155–167.

[Kharchenko'79] V.K. Kharchenko. *Differential identities of semiprime rings.* Algebra and Logic **18** (1979), 86–119.

[BookInvolutions] M.A. Knus, A. Merkurjev, M. Rost and J.P. Tignol. *The Book of Involutions.* Colloquium Publications **44**, American Mathematical Society, 1998.

[Kostrikin'59] A.I. Kostrikin. *The Burnside problem.* Izv. Akad. Nauk SSSR Ser. Mat. **23** (1959), 3–34 (in Russian). English translation: Izv. Amer. Math. Soc. Transl. **36**(2) (1964), 63–99.

[Kostrikin'67] A.I. Kostrikin.
Squares of adjoint endomorphisms in simple Lie p-algebras.
Izv. Akad. Nauk SSSR Ser. Mat. **31** (1967), 445–487 (in Russian).
English translation: Math. USSR Izv. **1** (1967).

[AroundBurnside] A.I. Kostrikin. *Around Burnside.*
Results in Mathematics and Related Areas (3), volume **20**, Springer-Verlag, Berlin-Heidelberg, 1990.

[Lam1] T.Y. Lam. *Lectures on Modules and Rings.*
Graduate Texts in Mathematics **189**, Springer-Verlag, New York, 1999.

[Littlewood'33] D.E. Littlewood. *On the classificaction of algebras.*
Proc. London Math. Soc.**35** (1933) 200–240.

[JordanPairs] O. Loos. *Jordan Pairs.*
Lecture Notes in Mathematics **460**; Springer-Verlag, Berlin-Heidelberg-New York, 1975.

[Martindale'69] W.S. Martindale 3$^{\text{rd}}$.
Prime rings satisfying a generalized polynomial identity.
J. Algebra **12** (1969), 576–584.

[Martindale'73] W. S. Martindale 3$^{\text{rd}}$. *On semiprime P.I. rings.*
Proc. Amer. Math. Soc. **40** (1973) 365–369.

[Martindale&Miers'83] W. S. Martindale 3$^{\text{rd}}$ and C.R. Miers.
On the iterates of derivations of prime rings.
Pacific J. Math. **104**(1) (1983) 179–190.

[Martindale&Miers'86] W. S. Martindale 3$^{\text{rd}}$ and C.R. Miers.
Herstein's Lie theory revisited.
J. Algebra **98** (1986) 14–37.

[Martindale&Miers'91] W.S. Martindale 3rd and C.R. Miers.
Nilpotent inner derivations of the skew elements of prime rings with involution.
Can. J. Math. **43** (1991), 1045–1954.

[TasteJordanAlgebras] K. McCrimmon. *A Taste of Jordan Algebras.*
Universitext, Springer-Verlag, New York, 2004.

[McCrimmon'66] K. McCrimmon. *A general theory of Jordan rings.*
Proc. Nat. Acad. Sci. U.S.A. *56* (1966), 1072–1079.

[McCrimmon'69] K. McCrimmon.
The Freudenthal-Springer-Tits constructions of exceptional Jordan algebras.
Trans. Amer. Mat. Soc. *139* (1969), 495–510.

[McCrimmon'71] K. McCrimmon. *Inner ideals in quadratic Jordan algebras.*
Trans. Amer. Math. Soc. **159** (1971), 445–468.

[Neher'91] E. Neher. *Jordan pairs with finite grids.*
Comm. Algebra **19** (1991), 455–478.

[Premet'86] A.A. Premet. *Lie algebras without strong degeneration.*
Mat. Sb. (N.S.) **129**(171)(1) (1986), 140–153 (in Russian).
English translation: Math. USSR Sbornik **57**(1) (1987), 151–164.

[PIRowen] L.H. Rowen. *Polynomial Identities in Ring Theory.*
Academic Press, Harcourt-Toronto, 1988.

[Rowen] L.H. Rowen. *Ring Theory (volume I).*
Academic Press, Harcourt-Toronto, 1988.

[Smirnov'97] O.N. Smirnov.
Simple associative algebras with finite Z-grading.
J. Algebra **196** (1997), 171–184.

[Smith'86] S.P. Smith.
Differential operators on the affine and projective lines in characteristic $p > 0$ in *Séminaire d'Algèbre Paul Dubreil et Marie-Paul Malliavin*,
Lectures Notes in Mathematics **1220**, Springer-Verlag, Berlin-New York, 1986, 157–177.

[Strade'73] H. Strade. *Nonclassical simple Lie algebras and strong degeneration.*
Arch. Math. (Basel) **24** (1973), 482–485.

[Thedy'85] A. Thedy. *Z-closed ideals of quadratic Jordan algebras.*
Comm. Algebra **13**(12) (1985), 2537–2565.

[Topping'65] D.M. Topping. *Jordan algebras of self-adjoint operators.*
Mem. Amer. Math. Soc. **53** (1965), 1–48.

[Wagner'37] W. Wagner.
Über die grundlagen der projektiven geometrie und allgemeine zahlsysteme.
Math. Z. **113** (1937), 528–567 (in German).

[Zel'manov'79] E. Zel'manov. *On prime Jordan algebras.*
Algebra i Logika, **18** (1979), 162–175 (in Russian).

[Zel'manov'83] E. Zel'manov. *Lie algebras with an algebraic adjoint representation.*
Mat. Sb. (N.S.) **121**(163)(4) (1983), 545–561 (in Russian).
English translation: Math. USSR Sbornik **49**(2) (1984), 537–552.

[Zel'manov'83(2)] E. Zel'manov. *On prime Jordan algebras II.*
Sib. Math. J. **24** (1983), 89–104.

[Zel'manov'84] E. I. Zelmanov. *Lie algebras with a finite grading.*
Mat. Sb. (N.S.), **124**(166)(3) (1984), 353–392.
English translation: Math. USSR Sbornik **52**(2) (1985).