CONTRIBUCIONES CIENTÍFICAS
EN HONOR DE MIRIAN ANDRÉS GÓMEZ
(Laureano Lambán, Ana Romero y Julio Rubio, editores),
Servicio de Publicaciones, Universidad de La Rioja,
Logroño, Spain, 2010.

TRIANGULATIONS OF COMPLEX PROJECTIVE SPACES

FRANCIS SERGERAERT

In memory of Mirian Andrés

RESUMEN. Mirian trabajaba en un tema que, en el tercer milenio, podría parecer agotado: el teorema de Eilenberg-Zilber. Sin embargo, más de sesenta años después de su descubrimiento se han realizado varios experimentos en ordenador que muestran que todavía estamos lejos de entender la naturaleza profunda de este resultado. Por ejemplo los informes sobre la ejecución del programa Kenzo nos indican que la mayor parte de su tiempo de ejecución se dedica a la utilización del teorema de Eilenberg-Zilber, más concretamente, la versión fuerte que describe una *reducción* $C_*(X \times Y) \Rightarrow\!\!\!\!\!\Rightarrow C_*(X) \otimes C_*(Y)$: es simplemente el inevitable puente entre Topología y Álgebra. La implementación actual, que combina varios trucos de codificación de una forma bastante complicada, aunque es mejor que las primeras, no puede ser la más adecuada. Teniendo en cuenta el objetivo inicial de una versión *certificada* del programa Kenzo, cualquier trabajo que reconsidere varios aspectos del teorema de Eilenberg-Zilber será bienvenido; y obligarse a obtener una prueba *certificada* de ese teorema es el mejor modo de descubrir las preciosas propiedades ocultas de esta reducción. Esperamos que otros colegas del equipo de Mirian en Logroño continúen su espléndido trabajo, detenido tan trágicamente.

ABSTRACT. Mirian worked on a subject which, in the third millenium, could seem exhausted, namely the Eilenberg-Zilber theorem. More than sixty years after its discovery, various computer experiments show on the contrary we are far from having understood the deep nature of this result. For example the profiler accounts of the Kenzo program instruct us most of its runtime is devoted to using the Eilenberg-Zilber theorem, more precisely, the strong form describing a *reduction* $C_*(X \times Y) \Rightarrow\!\!\!\!\!\Rightarrow C_*(X) \otimes C_*(Y)$: this is nothing but the initial inevitable bridge between Topology and Algebra. The current implementation, combining several dirty tricks in a rather weird way, though better than the first ones, cannot be the right one. Bearing in mind the ideal goal of a *proved* version of the Kenzo program, any work reconsidering the various aspects of the Eilenberg-Zilber theorem is welcome; and forcing oneself to obtain a *certified* proof of such a fundamental theorem is one of the best ways to discover precious hidden properties in this reduction. We hope other people of Mirian's team at Logroño will continue her beautiful work, so tragically stopped.

Key words and phrases. Simplicial sets, Complex projective spaces, Triangulations, Effective Homology.

Totally supported by Ministerio de Educación y Ciencia, Project MTM2006-06513.

1. Introduction

The simple Eilenberg-Zilber theorem is nothing but a preferred description of a triangulation of the product of two simplices $\Delta^p \times \Delta^q$. In Combinatorial Topology, simplicial *sets* are more flexible than simplicial *complexes*, with this amusing terminological paradox: the definition of a simplicial set is more complex than the definition of a simplicial… complex.

The nice paper [5] obtains and describes the unique *minimal* triangulation of $P^2(\mathbb{C})$ as a simplicial *complex* with $(9, 36, 84, 90, 36)$ simplices, that is, 9 vertices, 36 edges, 84 triangles, 90 tetrahedrons and 36 4-simplices. The Kenzo program obtains here a triangulation of $P^2(\mathbb{C})$ with only $(1, 0, 2, 3, 3)$ simplices; this does not contradict the previous claim, for the last model is a simplicial *set*, not a simplicial *complex*: for example it is legal in a simplicial set to attach the boundary of a triangle to a point to obtain a $(1, 0, 1)$ "triangulation" of the 2-sphere S^2 as a simplicial *set*, while, as a simplicial *complex*, the minimal triangulation of S^2 needs $(4, 6, 4)$ simplices. The main interest of the Kühnel triangulation of $P^2(\mathbb{C})$ is not really in the triangulation itself but in the remarkable symmetry properties that are used and described in it, a subject not at all considered in our triangulation as a simplicial set.

Another work around this subject must be quoted. In [1, Exemple 1.19], Clemens Berger obtains as a consequence of his *effective* version of the Hurewicz theorem a triangulation of the Hopf map $S^3 \to S^2$. The mapping cone of this map again is our $P^2(\mathbb{C})$, which produces with this method a $(1, 0, 5, 9, 6)$-triangulation.

We do not know any *use* of our triangulation. The matter is just to highlight how *Effective Homology* [7] is a tool which can be used in some unexpected situations. The common advertisement about effective homology underlines it is so possible to process objects *not of finite type* such as huge chain complexes or simplicial sets, and to compute the corresponding homology or homotopy groups, when they are guaranteed being of finite type by Jean-Pierre Serre [10]. This short paper is devoted to an amusing side effect: effective homology can also be used to obtain *finite geometrical* objects by a process going through *infinite geometrical* objects. It seems this method can be used for arbitrary complex projective sets. For example the Kenzo program obtains in a few seconds a triangulation of $P^5(\mathbb{C})$ with $(1, 0, 5, 40, 271, 1197, 3381, 5985, 6405, 3780, 945)$ simplices. More precisely, the object so obtained has the *homotopy type* of $P^5(\mathbb{C})$ and it is an open – and interesting – question to determine whether it is *homeomorphic* to $P^5(\mathbb{C})$.

This could recall Thomas Chapman's result about simple homotopy types, see [3, 11]: Chapman proved Whitehead's conjecture about the simple homotopy type of homeomorphisms between *finite* CW-complexes through an essential use of Hilbert cube manifolds, some exotic manifolds of *infinite* dimension. The similarity is clear but there is also a difference: Chapman's result has a very general scope, valid for every finite CW-complex while which is explained here on the contrary is rather limited: only the first complex projective spaces are currently covered.

The main ingredients of our construction:

- A triangulation of $P^n\mathbb{C}$ defines also a $(2n)$-cycle, the homology class of which is the canonical generator of $H_{2n}(P^n\mathbb{C}, \mathbb{Z})$.
- The inclusion $P^n\mathbb{C} \hookrightarrow P^\infty\mathbb{C}$ induces an isomorphism between the respective H_{2n} groups.
- The infinite projective space $P^\infty\mathbb{C}$ and the Eilenberg-MacLane space $K(\mathbb{Z}, 2)$, in particular its canonical minimal Kan model, have the same homotopy type.
- The Kenzo program can compute the *effective* homology of $K(\mathbb{Z}, 2)$, in particular a generator of $H_{2n}K(\mathbb{Z}, 2)$ as a simplicial cycle.

Combining these facts gives easily the desired triangulations.

2. The Eilenberg-MacLane space $K(\mathbb{Z}, 2)$

The projective spaces $P^n\mathbb{C}$ can be organized as an inductive system

$$* = P^0\mathbb{C} \hookrightarrow P^1\mathbb{C} \hookrightarrow \cdots \hookrightarrow P^n\mathbb{C} \hookrightarrow P^{n+1}\mathbb{C} \hookrightarrow \cdots \hookrightarrow P^\infty\mathbb{C}$$

In particular, the limit of this system $P^\infty\mathbb{C}$ is the most common model for the Eilenberg-MacLane space $K(\mathbb{Z}, 2)$. The Kan simplicial model for this space is obtained in the Kenzo program by a process totally independent from the projective spaces, and appropriately using this *simplicial* model, we obtain triangulations for the projective spaces by a rather strange and lucky process.

We recommend the small book [6] as an ideal reference for the simplicial techniques which are used below, and also for the notions of principal fibration and classifying space. The introductory text [9] could also be useful.

2.1. Complex projective spaces. The complex n-vector spaces can be considered as defining an inductive system:

$$\mathbb{C} \hookrightarrow \mathbb{C}^2 \hookrightarrow \cdots \hookrightarrow \mathbb{C}^n \hookrightarrow \mathbb{C}^{n+1} \hookrightarrow \cdots \hookrightarrow \mathbb{C}^{(\mathbb{N})}$$

where the last space is made of the infinite sequence of complex numbers, all null except a finite number of them. The inclusion $\mathbb{C}^n \hookrightarrow \mathbb{C}^{n+1}$ adds a null component at the end of a vector.

This gives an inductive system of unit spheres:

$$S^1 \hookrightarrow S^3 \hookrightarrow \cdots \hookrightarrow S^{2n-1} \hookrightarrow S^{2n+1} \hookrightarrow \cdots \hookrightarrow S^\infty$$

The diagonal action $S^1 \times S^{2n-1} \to S^{2n-1} : (s, (z_1, \ldots, z_n)) \mapsto (sz_1, \ldots, sz_n)$ of the unit circle over these spheres is compatible with the inductive structure, so that the corresponding homogeneous spaces $S^{2n-1}/S^1 =: P^{n-1}\mathbb{C}$ are also organized as an inductive system:

$$* \hookrightarrow P^1\mathbb{C} \hookrightarrow P^2\mathbb{C} \hookrightarrow \cdots \hookrightarrow \cdots \hookrightarrow P^{n-1}\mathbb{C} \hookrightarrow P^n\mathbb{C} \hookrightarrow \cdots \hookrightarrow P^\infty\mathbb{C}$$

2.2. $P^\infty\mathbb{C}$ as a classifying space. The action of the topological multiplicative group S^1 over the infinite sphere S^∞ is free, that is, $sz = z$ implies $s = 1$, and defines a *principal fibration*:

$$S^1 \hookrightarrow S^\infty \longrightarrow P^\infty\mathbb{C}$$

The infinite sphere S^∞ is contractible. Let us define a contracting homotopy $h : I \times S^\infty \to S^\infty$. The shift operator $\sigma : S^\infty \to S^\infty$ is defined as $\sigma(z_0, z_1, \ldots) =$

$(0, z_0, z_1, \ldots)$. If two points z and z' of S^∞ are not opposite, a geodesic $\gamma_{z,z'} : I \to S^\infty$ is defined connecting both points; it is the radial projection of the segment joining z and z'. Furthermore the value $\gamma_{z,z'}(t)$ depends continuously on z, z' and t. Finally let $P = (1, 0, 0, \ldots)$ be the "north pole" of S^∞.

In particular a point z is never opposite to its shift $\sigma(z)$, and a shift $\sigma(z)$, being on the "equator" $z_0 = 0$, cannot be opposite to the north pole P.

Then we can define $h(t, z)$ as follows:

$$\begin{aligned} h(t,z) &= \gamma_{z,\sigma(z)}(2t) && \text{if } 0 \le t \le 1/2 \\ &= \gamma_{\sigma(z),P}(2t-1) && \text{if } 1/2 \le t \le 1 \end{aligned}$$

It so happens the shift σ is a homeomorphism between the whole sphere S^∞ and the equator $z_0 = 0$, a strange world.

The total space of our principal fibration is contractible, so that this fibration is *universal* and the base space $P^\infty\mathbb{C}$ can be qualified as the *classifying space* of the group S^1. Most often, this is denoted by $P^\infty\mathbb{C} = BS^1$.

The topological group S^1 is not discrete, but it is also a classifying space, namely the classifying space of the *discrete* group $\mathbb{Z}$. This comes from the canonical action of $\mathbb{Z} \times \mathbb{R} \to \mathbb{R} : (n, x) \mapsto n + x$. It is again a free action, the total space $\mathbb{R}$ is again contractible and the quotient $\mathbb{R}/\mathbb{Z}$ is nothing but the circle S^1. So that $S^1 = B\mathbb{Z}$ and $P^\infty\mathbb{C} = B^2\mathbb{Z}$.

2.3. $P^\infty\mathbb{C}$ as an Eilenberg-MacLane space. Iterating the classifying space construction is possible for *commutative* groups. In particular, if G is a *discrete* commutative group, the Eilenberg-MacLane space $K(G, n)$ is defined as the iterated classifying space $K(G, n) := B^nG$.

If G is a topological group, the homotopy groups of BG are the same as those of G, shifted: $\pi_nG = \pi_{n+1}BG$; furthermore, $\pi_0BG = 0$, that is, the classifying space BG is connected.

For a *discrete* group G, all the homotopy groups are null except $\pi_0G = G$, using here the standard convention that π_0X is the *set* of the (arc-) connected components of X, which is also a group when G is a topological group. So that, if G is a discrete group, all the homotopy groups of B^nG are null except $\pi_nB^nG = G$. In fact this defines unambiguously the homotopy type of B^nG, then often denoted by $K(G, n)$.

In particular $\mathbb{Z}$ is a commutative discrete group, so that $\mathbb{Z} = K(\mathbb{Z}, 0)$, $S^1 = K(\mathbb{Z}, 1)$ and $P^\infty\mathbb{C} = K(\mathbb{Z}, 2)$.

2.4. $K(\mathbb{Z}, 2)$ in the Kenzo environment. The Kenzo program has a predifined function `k-z` constructing the Kan minimal model of $K(\mathbb{Z}, n)$ for $n > 0$.

```
> (setf kz2 (k-z 2)) ✠
[K13 Abelian-Simplicial-Group]
```

The Lisp prompt is the greater character '`>`' and the user then *enters* a Lisp *statement* to be *evaluated*, here the statement `(setf kz2 (k-z 2))`. On this display, the end of the Lisp statement is marked by the maltese character '✠', in fact not visible on the user's screen; the end of the Lisp statement is automatically detected by the Lisp interpreter, which then *evaluates* the given statement and *returns* the result of the evaluation, here the Kenzo object #13, which happens to be an abelian simplicial group. Only a simple external reference to this object is displayed, the internal object, a package of rather sophisticated algorithms, cannot be properly displayed.

The evaluated statement here also assigns the returned object to the symbol `kz2`, arbitrarily chosen by the user; this symbol can be used later to refer to our model of $K(\mathbb{Z},2)$.

What about the *origin* of `kz2`?

```
> (orgn kz2) ✠
(CLASSIFYING-SPACE [K1 Abelian-Simplicial-Group])
```

It is the classifying-space of the Kenzo object #1, which is also an abelian simplicial group, but what about the origin of the latter?

```
> (orgn (k 1)) ✠
(K-Z-1)
```

2.4.1. $K(\mathbb{Z},1)$. As explained in the previous section, $K(\mathbb{Z},2)$ can be obtained from a general constructor, the classifying-space constructor $G \mapsto BG$, valid in the Kenzo environment if G is a connected simplicial group, not necessarily abelian; but if G is abelian, BG is also an abelian simplicial group, so that the construction can be iterated. This recursive process must therefore start from a *connected* simplicial group. The starting point is $K(\mathbb{Z},1)$ constructed by Kenzo "from scratch", because of the specific well known properties of the minimal Kan model of $K(\mathbb{Z},1)$.

For convenient further references, let us assign $K(\mathbb{Z},1)$, that is, the Kenzo object #1, to the sympol `kz1`.

```
> (setf kz1 (k 1)) ✠
[K1 Abelian-Simplicial-Group]
```

The most important property of `kz1` is its *effective homology.*

```
> (efhm kz1) ✠
[K34 Homotopy-Equivalence K1 <= K1 => K28]
```

Let us examine the mysterious object `K28` and its origin:

```
> (k 28) ✠
[K28 Chain-Complex]
> (orgn (k 28)) ✠
(CIRCLE)
```

It is the chain complex deduced from the *ordinary* model of the circle, one vertex and one (loop) edge starting from and ending at the unique vertex. Let us compare the *basis* for example in dimension 1 of `kz1` and the circle `k28`.

```
> (basis kz1 1) ✠
Error: The object [K1 Abelian-Simplicial-Group] is locally-effective.
> (basis (k 28) 1) ✠
(S1)
```

In the Kenzo environment, the notion of *basis* has different meanings depending on the context. For a simplicial set, the basis in dimension 1 is the set of the non-degenerate 1-simplices. It happens the 1-basis of `kz1` is $\mathbb{Z}_*^1 = \mathbb{Z}_*$, the non-null integers, it is an infinite object which cannot be displayed on a finite (!) machine; such an object in the Kenzo environment is called *locally effective*, which explains the error which is obtained and its descriptor; see [7, 8] for the meaning and the reason of the qualifiers *effective* and *locally effective*. While the (algebraic) basis of the chain group of dimension 1 of the chain complex `k28` is made of a *unique* object, the symbol `S1`, corresponding to the unique 1-simplex of the ordinary simplicial model of a circle.

The simplicial model of $K(\mathbb{Z}, 1)$ here located through the symbol `kz1` is a simplicial set where the n-basis $K(\mathbb{Z}, 1)_n$ is $\mathbb{Z}_*^n$, the sequences of length n made of non-null integers. The associated chain complex is not of finite type, so that its homology groups cannot be elementarily computed. But the homotopy type is well defined by the characteristic property: all the homotopy groups are null except $\pi_1 = \mathbb{Z}$, a property satisfied as well by the circle, so that the homology groups of our $K(\mathbb{Z}, 1)$ are certainly isomorphic to those of the circle, namely $(\mathbb{Z}, \mathbb{Z}, 0, 0, \ldots)$.

Now the effective homology of $K(\mathbb{Z}, 1)$, obtained before through the operator `efhm`, is a reduction connecting $C_*(\texttt{kz1})$, the chain complex associated to the simplicial group $K(\mathbb{Z}, 1)$, and the chain complex `k28`.

As already explained, the homology groups of $C_*(\texttt{kz1})$ cannot be directly computed: this chain complex *is not* of finite type. But the Kenzo program has recorded the reduction over `k28`, so that if we ask for example for the first homology group of $K(\mathbb{Z}, 1)$:

```
> (homology kz1 1) ✠
Homology in dimension 1 :
Component Z
```

in fact Kenzo uses the chain complex `k28` to obtain the requested homology group, here $H_1K(\mathbb{Z}, 1) = \mathbb{Z}$.

2.4.2. $K(\mathbb{Z}, 2)$. The next Eilenberg-MacLane space $K(\mathbb{Z}, 2)$ is the classifying space of $K(\mathbb{Z}, 1)$.

```
> (classifying-space kz1) ✠
[K13 Abelian-Simplicial-Group]
```

In this case, Kenzo remembers this space has already been constructed, and returns it. It is also an object with effective homology.

```
> (efhm kz2) ✠
[K153 Homotopy-Equivalence K13 <= K143 => K139]
```

The situation is more complex. The effective homology is the chain equivalence `k153` connecting the chain complex of $K(\mathbb{Z}, 2)$, denoted *also*[1] by `k13` to the *effective*

[1] Such a situation is possible because of the rich class system of Common Lisp; here the class `abelian-simplicial-group` is a subclass of the class `chain-complex`.

chain complex k139 through an intermediate chain complex k143 not of finite type either.

```
> (basis kz2 4) ✠
Error: The object [K13 Abelian-Simplicial-Group] is locally-effective.
> (basis (k 143) 4) ✠
Error: The object [K143 Chain-Complex] is locally-effective.
> (basis (k 139) 4) ✠
(<<Abar[2 S1][2 S1]>>)
```

You see only the 4-basis of k139 is returned, made of a unique element, the bar generator which would be traditionally denoted by [S1 | S1]. It happens the chain complex k139 is the bar construction of k1, see [2] for this notion which allowed Henri Cartan to completely solve the problem of computing the (ordinary) homology of $K(G, n)$ for G an abelian group of finite type.

This *algebraic* chain equivalence k153 will be the main ingredient allowing us to *geometrically* triangulate the complex projective spaces.

3. Using the effective homology of $K(\mathbb{Z}, 2)$

At this time of our Kenzo environment, no projective space is "visible".

3.1. $H_*P^\infty\mathbb{C} = H_*K(\mathbb{Z}, 2)$**.** Let us examine a little the nature of $H_*K(\mathbb{Z}, 2)$. It depends only on the homotopy type, so that the structure of this homology can be also studied by an examination of the cellular presentation:

$$* = P^0\mathbb{C} \hookrightarrow P^1\mathbb{C} \hookrightarrow \cdots \hookrightarrow P^n\mathbb{C} \hookrightarrow P^{n+1}\mathbb{C} \hookrightarrow \cdots \hookrightarrow P^\infty\mathbb{C}$$

In fact $P^n\mathbb{C}$ is obtained from $P^{n-1}\mathbb{C}$ by attaching a disk D^{2n} by the projection map $S^{2n-1} \to P^{n-1}\mathbb{C}$. For example $D^4 = \{(z_0, z_1) \in \mathbb{C}^2 \ \underline{\mathbf{st}}\ |z_0|^2 + |z_1|^2 \leq 1\}$ and a surjective map $D^4 \to P^2\mathbb{C}$ is defined sending (z_0, z_1) to the projective class of $(z_0, z_1, \sqrt{1 - |z_0|^2 - |z_1|^2})$. The restriction of this map to the *open* disk $|z_0|^2+|z_1|^2 < 1$ is a homeomorphism $(D^4-S^3) \to (P^2\mathbb{C}-P^1\mathbb{C})$ while the restriction to the boundary S^3 is the projective projection $S^3 \to P^1\mathbb{C}$; so that $P^2\mathbb{C}$ is obtained by attaching a D^4 to $P^1\mathbb{C}$ through this projection.

The cellular complex allowing one to compute the homology groups of $P^\infty\mathbb{C}$ is therefore very simple: only one generator in the even positive degrees, and the $\mathbb{Z}$-homology is also made of one copy of $\mathbb{Z}$ for every even positive degree. The projective space $P^n\mathbb{C}$ is also an orient*ed* $2n$-manifold, and the canonical orientation defines also the fundamental $2n$-homology class as canonically associated to any triangulation.

3.2. Is some converse possible? In other words, would it be possible to obtain a triangulated projective space $P^n\mathbb{C}$ *from* some homology class produced by a different process? The answer is positive and rather amazing.

The simplicial model kz2 produced by the Kenzo program, the (essentially unique) minimal Kan simplicial model of $K(\mathbb{Z}, 2)$, has the same homotopy type as $P^\infty\mathbb{C}$. The Kenzo program also knows the homology groups of this space, certainly isomorphic to those described in the previous section: exactly one copy of $\mathbb{Z}$ for every even degree. But because the Kenzo program computes the *effective* homology of this space, it can also produce *explicit* cycles $z_{2n} \in Z_{2n}K(\mathbb{Z}, 2)$ whose corresponding homology classes are the generators of the homology.

Now we can try the following game: the cycle z_{2n} so obtained should have some "similarity" with $P^n\mathbb{C}$, the fundamental homology class of which also represents the canonical generator of $H_{2n}P^\infty\mathbb{C}$. This cycle z_{2n} is a $\mathbb{Z}$-linear combination of $2n$-simplices, and these simplices must fit to each other along their boundaries rather nicely, for this combination of simplices is a *cycle*. We can then consider the smallest simplicial subset $Z_{2n} \subset K(\mathbb{Z}, 2)$ containing the cycle z_{2n} and, who knows, with some luck, maybe Z_{2n} is the triangulation of an object which could be $P^n\mathbb{C}$? Yes it is, in fact it is the triangulation of a simplicial set having the *homotopy type* of $P^n\mathbb{C}$, and Kenzo *proves* it. Of course we would prefer the simplicial set so obtained is *homeomorphic* to $P^n\mathbb{C}$, but this is an open problem.

3.3. Triangulating the homotopy type of $P^2\mathbb{C}$. The fundamental homology class of $P^2\mathbb{C}$ is a generator of $H_4P^\infty\mathbb{C}$, and it is therefore interesting to consider the fourth homology group $H_4(K(\mathbb{Z}, 2), \mathbb{Z})$.

```
> (homology kz2 4) ✠
Homology in dimension 4 :
Component Z
```

As expected, we obtain $H_4(K(\mathbb{Z}, 2), \mathbb{Z}) = \mathbb{Z}$. Let us recall the *effective* homology of `kz2` and assign it to the symbol `efhm-kz2`:

```
> (setf efhm-kz2 (efhm kz2)) ✠
[K153 Homotopy-Equivalence K13 <= K143 => K139]
```

where the chain complex #139 is of finite type; then a variant of *homology* can compute the same homology *and* a list of generators for the homology group:

```
> (chcm-homology-gen (k 139) 4) ✠
Homology in dimension 4 :
Component Z
(
----------------------------------------------------------------------{CMBN 4}
<1 * <<Abar[2 S1][2 S1]>>>
------------------------------------------------------------------------------
)
```

Only one generator, already mentioned Section 2.4.2, in fact also the generator of the chain group $\texttt{k139}_4$. We extract it from this generator *list* in fact made of this unique generator, and assign it to the symbol `g`:

```
> (setf g (first *)) ✠
----------------------------------------------------------------------{CMBN 4}
<1 * <<Abar[2 S1][2 S1]>>>
------------------------------------------------------------------------------
```

Now we can use the equivalence `efhm-kz2`:

$$C_*K(\mathbb{Z}, 2) = C_*(\texttt{kz2}) \Lleftarrow \texttt{k143} \Rrightarrow \texttt{k139}$$

to obtain the corresponding cyle in $C_*K(\mathbb{Z}, 2)$:

```
> (setf z4 (lf efhm-kz2 (rg efhm-kz2 g))) ✠
-------------------------------------------------------------------{CMBN 4}
<-1 * <<GBar<1-0 (1)><1-0 NIL><- (1)><- NIL>>>>
<1 * <<GBar<2-0 (1)><1 (1)><0 NIL><- NIL>>>>
<-1 * <<GBar<2-1 (1)><0 (1)><0 NIL><- NIL>>>>
--------------------------------------------------------------------------
```

We obtain a $\mathbb{Z}$-linear combination of three 4-simplices of $K(\mathbb{Z}, 2)$. The partial statement `(rg efhm-kz2 g)` computes the image of `g` in the central chain complex `k143` and `(lf efhm-kz2 ...)` computes the image of the previous result in the left-hand chain complex $C_*(K(\mathbb{Z}, 2))$. The obtained cycle should have some similarity with $P^2\mathbb{C}$. Let us verify it is really a cycle!

```
> (? kz2 z4) ✠
-------------------------------------------------------------------{CMBN 3}
--------------------------------------------------------------------------
```

A statement `(?  kz2 xxx)` computes the boundary of `xxx` in the chain complex associated to `kz2`.

The components of the cycle z_4 can be used to construct a *simplicial set* Z_4, more precisely a simplicial *subset* of `kz2`. The last simplicial set is not of finite type, it is only *locally effective*, but nevertheless this allows a user to undertake any "local" work, for example to compute all the faces, faces of faces, and so on, of some simplices, to construct a finite simplicial set from the initial 4-simplices. The Kenzo function `gmsms-subsmst` (= geometrical-simplices-to-subsimplicial-set) does this work and returns *two* results, a high level Lisp technicality. Only the first one is displayed. But the technical Lisp function `multiple-value-setq` assigns both values respectively to two symbols, here `ssz4` and `incl`.

```
> (multiple-value-setq (ssz4 incl) (gmsms-subsmst kz2 z4)) ✠
[K154 Simplicial-Set]
```

The first value is simply the simplicial set constructed from the cycle, assigned to the symbol `ssz4` and displayed. This simplicial set Z_4 is a simplicial *subset* of $K(\mathbb{Z}, 2)$ and we will see later the canonical inclusion $Z_4 \hookrightarrow K(\mathbb{Z}, 2)$ will play an essential role in our study; this inclusion is also computed by the `gmsms-subsmst` function, returned as a second value, here assigned to the symbol `incl`. We can display the value of the symbol `incl` and verify it is really a simplicial morphism $Z_4 \hookrightarrow K(\mathbb{Z}, 2)$, that is, `k154` $\hookrightarrow$ `k13`.

```
> incl ✠
[K159 Morphism (degree 0): K154 -> K13]
```

3.4. Studying the obtained Z_4. It was explained above we "hope" the simplicial set Z_4 maybe is strongly connected to the standard $P^2\mathbb{C}$. We will prove here, using the Kenzo program, that it really has the homotopy type of $P^2\mathbb{C}$.

We can firstly examine the homology groups.

```
> (homology ssz4 0 5) ✠
Homology in dimension 0 :
Component Z
Homology in dimension 1 :

Homology in dimension 2 :
Component Z
Homology in dimension 3 :

Homology in dimension 4 :
Component Z
```

Z_4 is a finite simplicial set of dimension 4, so that it is enough to examine the homology groups $H_i Z_4$ for $0 \leq i < 5$. Good! we find the homology groups of $P^2\mathbb{C}$, namely $(\mathbb{Z}, 0, \mathbb{Z}, 0, \mathbb{Z})$.

But it is well known this does not guarantee the right homotopy type. For example the wedge $S^2 \vee S^4$ has the same homology groups and does not have the homotopy type of $P^2\mathbb{C}$.

```
> (setf s2vs4 (sphere-wedge 2 4)) ✠
[K164 Simplicial-Set]
> (homology s2vs4 0 5) ✠
Homology in dimension 0 :
Component Z
Homology in dimension 1 :

Homology in dimension 2 :
Component Z
Homology in dimension 3 :

Homology in dimension 4 :
Component Z
```

3.5. Using the Hurewicz-Whitehead theorem. The Hurewicz-Whitehead theorem states that if a continuous map $f : X \to Y$ between two simply connected CW-complexes X and Y induces an isomorphism between the respective homology groups H_*X and H_*Y, then the map f is a homotopy equivalence.

We know there exists a homotopy equivalence between $P^\infty\mathbb{C}$ and our Kan minimal model $K(\mathbb{Z}, 2)$; let $f : K(\mathbb{Z}, 2) \to P^\infty\mathbb{C}$ such a homotopy equivalence. Because of the cellular approximation theorem, we can assume the map f sends the $(2n)$- and $(2n+1)$-simplices of $K(\mathbb{Z}, 2)$ in $P^n\mathbb{C}$, this point will be essential. Note in these descriptions the objects $P^\infty\mathbb{C}$ and f are only "abstract", not available in our Kenzo environment: no possibility to install a "general" CW-complex X on a computer, because of the *arbitrary continuous* attaching maps between the added n-dimensional cells D^n's and the previous $(n-1)$-skeleton X_{n-1}.

Let us call $\alpha : Z_4 \hookrightarrow K(\mathbb{Z}, 2)$ the canonical inclusion, which was assigned to the symbol `incl` in our Kenzo environment. Now the composition again denoted by $\alpha := f\alpha : Z_4 \to P^\infty\mathbb{C}$ in fact has its image in $P^2\mathbb{C}$, which allows us to denote again as $\alpha : Z_4 \to P^2\mathbb{C}$ essentially the same map with a smaller target $P^2\mathbb{C}$ instead of $P^\infty\mathbb{C}$.

Theorem 1. — *The map $\alpha : Z_4 \to P^2\mathbb{C}$ is a homotopy equivalence.*

♣ The simplicial set Z_4 is a simplicial *subset* of $K(\mathbb{Z}, 2)$, the simplicial model of which has only one vertex and no non-degenerate 1-simplex. The same properties are satisfied by Z_4 which therefore is simply connected. The Appendix gives the detailed organisation of the simplices of Z_4 and their faces.

The spaces Z_4 and $P^2\mathbb{C}$ are simply connected CW-complexes and proving $\alpha : Z_4 \to P^2\mathbb{C}$ is a homotopy equivalence is equivalent to proving the maps induced between homology groups are isomorphisms.

The inclusion $P^2\mathbb{C} \hookrightarrow P^\infty\mathbb{C}$ induces isomorphisms between the homology groups for the degrees $i < 6$, so that, taking account of the homotopy equivalence f, it is enough to prove the inclusion $\alpha : Z_4 \hookrightarrow K(\mathbb{Z}, 2)$ induces isomorphisms between homology groups for the degrees ≤ 4.

If $\beta : C_* \to C'_*$ is a chain complex morphism, the *cone*of β, denoted by Cone^β, is a chain complex defined as follows: $\mathrm{Cone}^\beta_i = C'_i \oplus C_{i-1}$ and the boundary operator $d : \mathrm{Cone}^\beta_i \to \mathrm{Cone}^\beta_{i-1}$ is the matrix:

$$\begin{bmatrix} d_{C'_*} & \beta \\ 0 & -d_{C_*} \end{bmatrix}$$

The homology groups of C_*, C'_* and Cone^β are then connected by a long exact sequence:

$$\cdots \to H_{i+1}\mathrm{Cone}^\beta \to H_i C_* \xrightarrow{\beta} H_i C'_* \to H_i \mathrm{Cone}^\beta \to H_{i-1} C_* \to \cdots$$

which allows one to connect isomorphisms induced by β between homology groups to null homology groups of Cone^β.

In the case of our simplicial map $\alpha : Z_4 \hookrightarrow K(\mathbb{Z}, 2)$, because $H_5 K(\mathbb{Z}, 2) = 0$, proving α induces isomorphisms between homology groups for degrees ≤ 4 is equivalent to proving $H_i\mathrm{Cone}^\alpha = 0$ for $0 \leq i \leq 5$. And the Kenzo program knows how to compute these homology groups.

First we construct the cone of $\alpha =$ `incl`.

```
> (setf cone-alpha (cone incl)) ✠
[K174 Chain-Complex]
```

It is a chain complex *not at all of finite type*, for $C_*K(\mathbb{Z}, 2)$ is not, but the methods of *effective homology*, see [7, 8], easily compute these homology groups.

```
> (homology cone-alpha 0 6) ✠
Homology in dimension 0 :

Homology in dimension 1 :

Homology in dimension 2 :

Homology in dimension 3 :

Homology in dimension 4 :

Homology in dimension 5 :
```

The absence of indication "`Component xxx`" means in fact these homology groups are null. ♣

3.6. Higher dimensions. So far, only the case of $P^2\mathbb{C}$ has been considered in this article, just to be simpler. Analogous computations give analogous results for $P^n\mathbb{C}$ for $n \leq 6$, needing a few hours of runtime in the case $n = 6$, and it is sensible to conjecture in fact our method works for every n. But we do not have any hint for a proof!

So that our Kenzo program obtains simplicial models for the homotopy types of $P^n\mathbb{C}$ for $n \leq 6$. The numbers of simplices in dimensions $\leq 2n$ are as follows:

	0	1	2	3	4	5	6	7	8	9	10	11	12
$P^0\mathbb{C}$	1												
$P^1\mathbb{C}$	1	0	1										
$P^2\mathbb{C}$	1	0	2	3	3								
$P^3\mathbb{C}$	1	0	3	10	25	30	15						
$P^4\mathbb{C}$	1	0	4	22	97	255	390	315	105				
$P^5\mathbb{C}$	1	0	5	40	271	1197	3381	5985	6405	3780	945		
$P^6\mathbb{C}$	1	0	6	65	627	4162	18496	54789	107933	139230	112770	51975	10395

Some "regularity" is observed, for example $\#P^n\mathbb{C}_2 = n$, $\#P^n\mathbb{C}_{2n} = 1.3.5\ldots(2n-1)$, $\#P^n\mathbb{C}_{2n-1} = (n-1)\#P^n\mathbb{C}_{2n}$, and Peter Paule, using the *On-Line Encyclopedia of Integer Sequences* [4], discovered that $\#P^n\mathbb{C}_3 = (n-1)n(n+7)/6$, thanks Peter! So far no other closed formula is known for the number of simplices, but it is clear some formulas must exist!

References

[1] C. Berger. *Une version effective du théorème de Hurewicz.* Thèse Institut Fourier, Grenoble, 1991.
`http://tel.archives-ouvertes.fr/tel-00339314/en/`

[2] H. Cartan. *Algèbres d'Eilenberg-MacLane et homotopie.* Séminaire Henri Cartan, 1954-5. Or *Henri Cartan, Œuvres*, Springer, 1979.

[3] T. A. Chapman. *Lectures on Hilbert cubes manifolds.* AMS Bookstore, 1976.

[4] *The On-line Encyclopedia of Integer Sequences.*
`http://www.research.att.com/~njas/sequences/`

[5] W. Kühnel, T. F. Banchoff. The 9-vertex Complex Projective Plane. *The Mathematical Intelligencer* **5**, 11–22, 1983.

[6] J. P. May. *Simplicial objects in algebraic topology.* Van Nostrand, 1967.

[7] J. Rubio, F. Sergeraert. Constructive Algebraic Topology. *Bulletin des Sciences Mathématiques* **126**, 389–412, 2002.

[8] J. Rubio, F. Sergeraert. *Genova Lecture Notes.*
`http://www-fourier.ujf-grenoble.fr/~sergerar/Papers/Genova-Lecture-Notes.pdf`

[9] F. Sergeraert. *Introduction to Combinatorial Homotopy Theory.*
Lecture Notes 2008 Ictp Summer School.
`http://www-fourier.ujf-grenoble.fr/~sergerar/Papers/Trieste-Lecture-Notes.pdf`

[10] J. P. Serre. Groupes d'homotopie et classes de groupes abéliens. *Annals of Mathematics* **58**, 258–294, 1953.

[11] L. Siebenmann. *L'invariance topologique du type simple d'homotopie.* Séminaire Bourbaki, 1972-1973, Exposé 428.

Appendix

We give in this appendix the complete description of the triangulation of $P^2\mathbb{C}$ used as example in the paper. Analogous descriptions can be obtained for $P^n\mathbb{C}$ for $n \leq 6$, but they are of course a little more lengthy. The Kenzo program can produce the following listing.

```
Dimension = 0
   S00
Dimension = 1
Dimension = 2
   S20
      Face 0 = <AbSm 0 S00>
      Face 1 = <AbSm 0 S00>
      Face 2 = <AbSm 0 S00>
   S21
      Face 0 = <AbSm 0 S00>
      Face 1 = <AbSm 0 S00>
      Face 2 = <AbSm 0 S00>
Dimension = 3
   S30
      Face 0 = <AbSm - S21>
      Face 1 = <AbSm - S20>
      Face 2 = <AbSm - S21>
      Face 3 = <AbSm 1-0 S00>
   S31
      Face 0 = <AbSm 1-0 S00>
      Face 1 = <AbSm - S21>
      Face 2 = <AbSm - S20>
      Face 3 = <AbSm - S21>
   S32
      Face 0 = <AbSm - S21>
      Face 1 = <AbSm - S21>
      Face 2 = <AbSm - S21>
      Face 3 = <AbSm - S21>
Dimension = 4
   S40
      Face 0 = <AbSm 0 S21>
      Face 1 = <AbSm 0 S21>
      Face 2 = <AbSm - S32>
      Face 3 = <AbSm 2 S21>
      Face 4 = <AbSm 2 S21>
   S41
      Face 0 = <AbSm 1 S21>
      Face 1 = <AbSm - S31>
      Face 2 = <AbSm - S32>
      Face 3 = <AbSm - S30>
      Face 4 = <AbSm 1 S21>
   S42
      Face 0 = <AbSm 2 S21>
      Face 1 = <AbSm - S31>
      Face 2 = <AbSm 1 S21>
      Face 3 = <AbSm - S30>
      Face 4 = <AbSm 0 S21>
```

Every simplex is named `Sij`, the character `i` being its dimension and the character `j` just an identification number. Every face is an "abstract" simplex, an important data type in the Kenzo program, representing some *possible degeneracy* of a non-degenerate simplex.

A notation as `<AbSm - S30>` means a non-degenerate simplex, namely in this case the simplex `S30`. So you can read in the listing that the face #3 of the simplex `S42` is the (non-degenerate) simplex `S30`. In the same way, the face #0 of `S42` is the 2-degeneracy η_2`S21` if η_i denotes an elementary degeneracy operator. You see also $\partial_3\mathtt{S30} = \partial_0\mathtt{S31} = \eta_1\eta_0\mathtt{S00}$, that is, the only possible degeneracy of the base point in dimension 2.

The author is very interested by a direct proof that this relatively simple (?) finite simplicial set is a triangulation of the homotopy type of $P^2\mathbb{C}$. Or even maybe a triangulation of $P^2\mathbb{C}$ itself?

Institut Fourier, Université Joseph Fourier, Grenoble, France
E-mail address: `Francis.Sergeraert@ujf-grenoble.fr`