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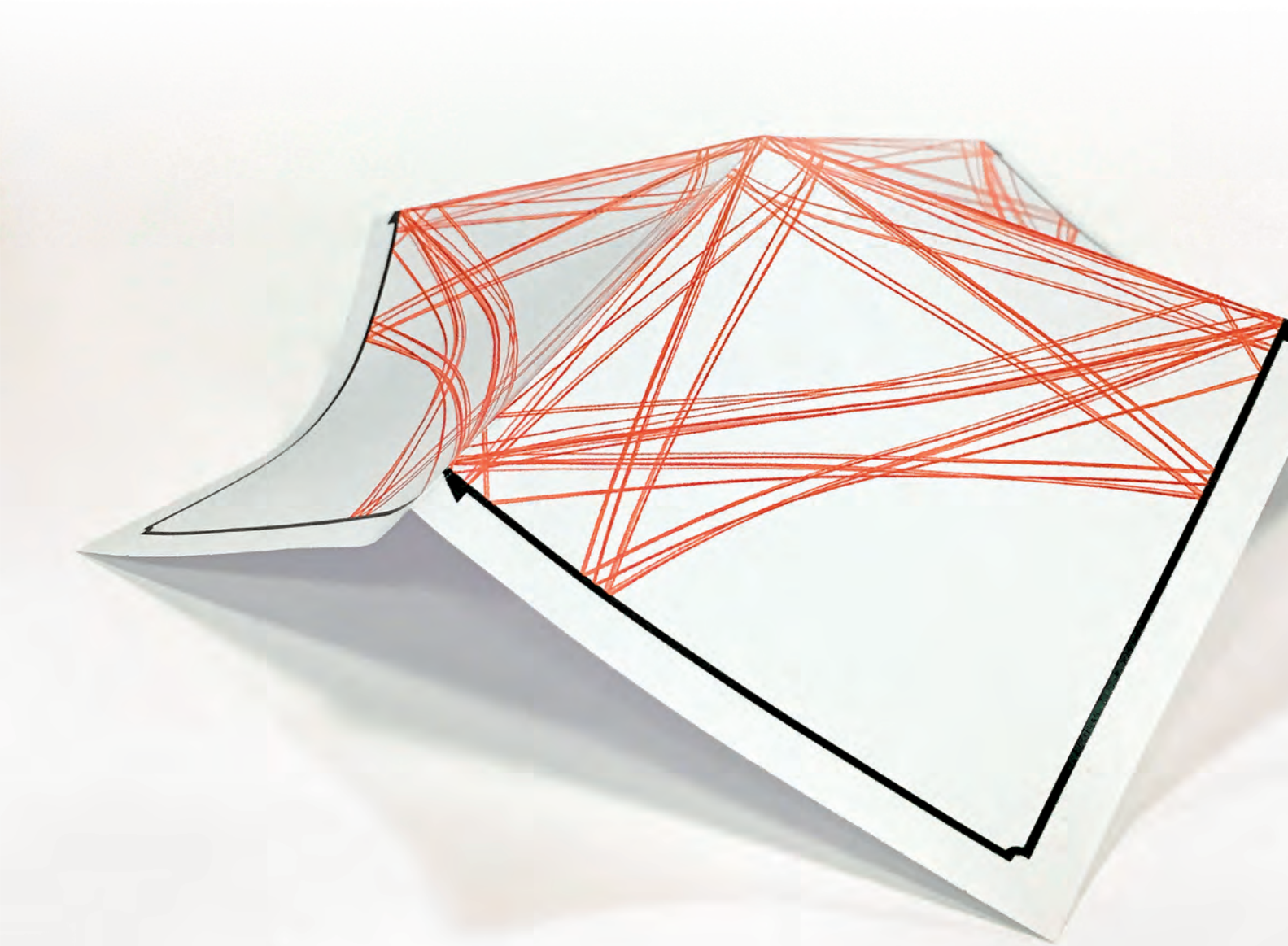

AMS American Mathematical Society
Advancing research. Creating connections.

The cover artwork is from Dan Margalit's feature article The Mathematics of Joan Birman, page 341.

Call for Nominations

LEROY P. STEELE PRIZES

The selection committee for these prizes requests nominations for consideration for the 2020 awards..

Three Leroy P. Steele Prizes are awarded each year in the following categories: (1) the Steele Prize for Lifetime Achievement: for the cumulative influence of the total mathematical work of the recipient, high level of research over a period of time, particular influence on the development of a field, and influence on mathematics through PhD students; (2) the Steele Prize for Mathematical Exposition: for a book or substantial survey or expository-research paper; and (3) the Steele Prize for Seminal Contribution to Research: for a paper, whether recent or not, that has proved to be of fundamental or lasting importance in its field, or a model of important research. In 2020 the prize for Seminal Contribution to Research will be awarded for a paper in Analysis/Probability.

Further information and instructions for submitting a nomination can be found at the Leroy P. Steele Prizes website: **www.ams.org/steele-prize**.

Nominations for the Steele Prizes for Lifetime Achievement and for Mathematical Exposition will remain active and receive consideration for three consecutive years.

For questions contact the AMS Secretary at secretary@ams.org.

The nomination period is February 1, 2019 through March 31, 2019.

AMS American Mathematical Society

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A WORD FROM...

Catherine A. Roberts, *AMS Executive Director*

Photo is courtesy of Maré Studios.

As Executive Director of the AMS, I am responsible for the general administration of the affairs of our Society in accordance with the policies set by the Board of Trustees and the Council. I applaud the transparency of our organization—throughout the year the *Notices of the AMS* publishes detailed reports about our budget and endowment, our membership and development efforts, and data on our profession.

At present, my attention focuses primarily on overseeing the implementation of the 2016–2020 Strategic Plan, and the following are a few recent advances:

- The AMS is increasing its support of early career mathematicians. Our successful Mathematics Research Communities are just one example. The new Next Generation Fund will maintain our graduate student travel grants and support many other important programs for those launching their careers. We are also learning more about how to educate graduate students about careers outside of academia and how to support those who choose this path. The new Early Career section in the *Notices* goes hand-in-hand with our increasing focus on early career mathematicians.
- With increased attention to diversity, inclusion, and education, the AMS can be a leader in serving all parts of the mathematics community. The inaugural Director of Education and Diversity, Helen G. Grundman, spent almost three years laying the groundwork for new relationships with stakeholders engaged in diversity and inclusion work. With her retirement, we are seeking new leadership to continue our efforts in both diversity and education. One focus in diversity and inclusion will be to work across all divisions of the AMS to effect change, in both our own hiring and in our efforts to diversify our meetings and conferences, prize and award nominee pools, committee membership, and governance. We will increase AMS attention to education matters, particularly at the graduate level, and ensure that we are a voice in conversations happening in DC and across the country.
- A strategic initiative to publish more mathematics content led to our acquisition of the MAA book program and to renewed efforts to publish a wider selection of high-quality math books, including those in applied areas and recreational math. If you are considering writing a book, I hope the AMS tops your list of potential publishers. Our books are beautiful and reasonably priced; revenue from their sales support the many programs and initiatives our Society offers that benefit our profession.
- We will be updating the MathSciNet® user interface after completing ambitious infrastructure upgrades designed to help integrate MathSciNet into the daily habits of mathematicians. This essential resource requires over 80 full-time staff and thousands of volunteer expert reviewers to remain current. MathSciNet curates the enormous body of mathematical literature and maintains accurate author attributions—if you are new to math research, please connect to MathSciNet through your library.
- Hiring a Director of Membership and creating a separate Membership Department has not only helped us implement strategies for recruiting and retaining AMS members (both individual and institutional), but it has also helped us more clearly identify the needs of AMS members and potential members. Through the support of our dedicated members, we are able to enrich the lives of countless mathematicians and students with our publications, meetings and conferences, MathSciNet, professional services, advocacy, and awareness programs. We thank our current members for their ongoing support of the global mathematical sciences community, and we encourage those who are not yet members to join and engage with us.

- The creation of new and consistent branding across the AMS ensures our publications, programs, and services are readily recognized as a resource in the mathematical sciences for our members and strategic partners, the media, and policy makers. Consistent branding also allows us to promote the AMS more effectively through all channels of communication and outreach. Keep an eye on our renewed website, https://www.ams.org, and please sign up to receive the member e-newsletter Headlines & Deadlines.

I am particularly proud of the AMS' willingness to have conversations with other mathematics advocates to explore new cooperative arrangements. For example, since January 2018, the Association for Women in Mathematics (AWM) engaged us to provide management services, which includes providing a space for their headquarters, a managing director, information technology support, and financial management. The AMS provides similar services (on a smaller scale) to the Friends of the International Mathematical Union and to the Mathematical Council of the Americas. The AMS is also active in several joint ventures, such as the Joint Policy Board for Mathematics and the Conference Board of the Mathematical Sciences. Working together, we can accomplish so much! With the recent announcement that the AMS will be the sole host of the Joint Mathematics Meetings starting in 2022, we have an opportunity to re-imagine, together, how this important conference can continue to serve the entire community through the participation of a constellation of mathematics organizations. Please submit your suggestions for how to ensure the JMM remains relevant to you at https://www.ams.org/jmm-reimagined.

As this column appears in March when we are celebrating women, I will take a moment to recognize and thank several women currently serving in some of the highest positions at the AMS: Jill C. Pipher (President), Abigail Thompson (Vice President), Bryna Kra (Chairperson of the Board of Trustees), Carla Savage (Secretary), Georgia Benkart (Associate Secretary), Jane M. Hawkins (Treasurer), Erica Flapan (Editor of *Notices*), Susan J. Friedlander (Editor of *Bulletin*), Emily D. Riley (Associate Executive Director of Administration and Chief Financial Officer), Karen Saxe (Associate Executive Director of Government Relations), and T. Christine Stevens (Associate Executive Director of Meetings and Professional Services).

Catherine A. Roberts
Executive Director

Notices
of the American Mathematical Society

March 2019

FEATURED

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ABOUT THE COVER

The cover image is an artist's interpretation of a figure from the feature article on Joan Birman. The original imagery was provided courtesy of David Dumas.

FROM THE AMS SECRETARY

IN EVERY ISSUE

CURRENT INVITATIONS FROM THE AMS

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For back issues see **www.ams.org/backvols**. Note: Single issues of the *Notices* are not available after one calendar year.

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LETTERS TO THE EDITOR

Reaction to John W. Dawson's Review in September 2018 *Notices*

To the Editor:

I was appalled by the last sentences in the otherwise excellent (in my humble opinion) John W. Dawson's review of Karl Sigmund's *Exact Thinking*…book, published in the September 2018 issue of the *Notices*.

I thought that the AMS in general, and the *Notices* as its publication in particular, is not a political entity and does not take partisan sides. Yes, we mathematicians are very much parts of the society. It's only normal that [an] individual mathematician, just like any individual, may or may not hold strong political views, and may or may not subscribe to different party affiliations. But as a community we are as diverse as the society we live in, with all kinds of political views represented among us. Moreover, we as a community are affected by what happens around us, including many controversial issues of the day, and in turn should participate in the life of our society and strive to change it for the better.

That being said, I always expect that *Notices* is a publication that doesn't go into politics per se, and when dealing with political issues out of necessity—when such issues are relevant to the mathematical community—it takes a balanced approach. A prime example would be the September 2017 issue, part of which was devoted to the immigration ban. One may or may not agree with the policy enforced by that ban—but everybody could benefit from the coverage offered in that issue, presented in a tasteful and respectful manner.

Unfortunately, that wasn't the case with this piece in the September 2018 issue. Whether one likes Trump or not (which shouldn't even matter, because that's what democracies are about—allowing people to have different views), it's impossible to ignore the fact that the constant anti-Trump bashing in the media has reached unprecedented levels, far exceeding any anti-Bush Jr. criticisms during his presidency, and especially alarming when compared with the media's love affair with Obama. And now the *Notices* has chosen to join the ranks of all anti-right liberally biased media in tasteless bashing of everything that can be attributed as right-wing. (I leave it as just a comment, but it's pretty clear that anti-Trumpism is lurking behind that piece too.)

Stating that "*the book's title … makes clear the relevance of the Circle's history to America today, where the scientific worldview and rational thinking in general is once again under assault by an extreme right-wing establishment*" is wrong on so many levels.

(*) First of all, it is a politically charged statement, which I find highly inappropriate for the pages of the *Notices*. In fact, over the many years that I have been reading the *Notices* I can't remember even a single other example of any statement like that.

(*) The so-called "extreme right-wing establishment" exists only in the biased minds of equally extreme … people with the opposite view. In fact, one of [the] reasons behind all these attacks is the fact that Trump is NOT part of the establishment, and the establishment, which is much more liberal than anything else, cannot accept him.

(*) Yes, there have been controversies with some opinions and actions of the current administration, e.g. related to the discussion on global warming, etc. But "scientific worldview under attack"??? Seriously??? Do we see academic institutions being closed? Scientists prohibited from publishing their views, their research, or their findings? Faculty losing their jobs because of their thinking? I'm afraid this "attack" exists only in the author's imagination.

(*) And the cherry on top is comparing the current state of affairs in the United States with the Nazi Germany and Austria. This is as tasteless as it gets, to say the least. That attack included physical terror, racial cleansing to resolve the "Jewish question," etc. Do I really need to make the point that the current life in the US, despite all the issues and controversies, is nothing like that, not even close? To be frank, I can't understand how a person who claims adherence to "scientific worldview" and "rational thinking" could possibly draw such a parallel. Sorry, but it's spitting at the memory of all those who perished during those truly terrible times.

Sincerely,

Dr. Mark S. Grinshpon
Principal Senior Lecturer
Math & Stat Club Faculty Advisor
Department of Mathematics and Statistics
Georgia State University

(Received November 27, 2018)

**We invite readers to submit letters to the editor at* `notices-letters@ams.org`.

Karen Uhlenbeck and the Calculus of Variations

Simon Donaldson

In this article we discuss the work of Karen Uhlenbeck, mainly from the 1980s, focused on variational problems in differential geometry.

The calculus of variations goes back to the 18th century. In the simplest setting we have a functional

$$F(u) = \int \Phi(u, u')dx,$$

defined on functions u of one variable x. Then the condition that $\mathcal{F}$ is stationary with respect to compactly supported variations of u is a second order differential equation—the *Euler–Lagrange equation* associated to the functional. One writes

$$\delta\mathcal{F} = \int \delta u\, \tau(u)\, dx,$$

where

$$\tau(u) = \frac{\partial\Phi}{\partial u} - \frac{d}{dx}\frac{\partial\Phi}{\partial u'}. \tag{1}$$

The Euler–Lagrange equation is $\tau(u) = 0$. Similarly for vector-valued functions of a variable $x \in \mathbf{R}^n$. Depending on the context, the functions would be required to satisfy suitable boundary conditions or, as in most of this article, might be defined on a compact manifold rather than a domain in $\mathbf{R}^n$, and u might not exactly be a function but

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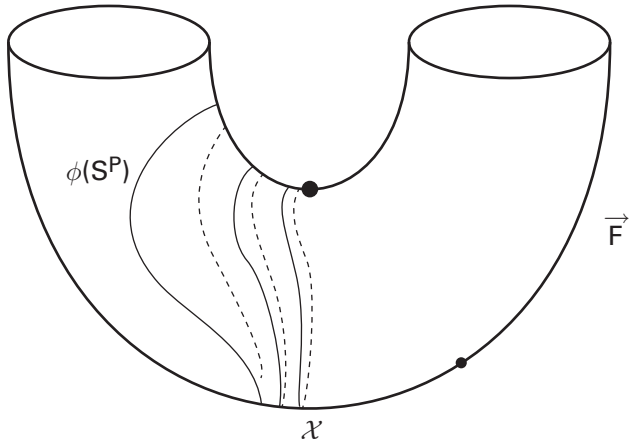

Figure 1. *Finding a critical point with a minimax sequence.*

a more complicated differential geometric object such as a map, metric, or connection. One interprets $\tau(u)$, defined as in (1), as the derivative at u of the functional $\mathcal{F}$ on a suitable infinite dimensional space $\mathcal{X}$ and the solutions of the Euler–Lagrange equation are *critical points* of $\mathcal{F}$.

A fundamental question is whether one can exploit the variational structure to establish the existence of solutions to Euler–Lagrange equations. This question came into focus at the beginning of the 20th century. Hilbert's 22nd problem from 1904 was:

Has not every regular variational problem a solution, provided certain assumptions regarding the given boundary conditions are satisfied?

If the functional $\mathcal{F}$ is bounded below one might hope to find a solution of the Euler–Lagrange equation which realises the minimum of $\mathcal{F}$ on $\mathcal{X}$. More generally, one might hope that if $\mathcal{X}$ has a complicated topology then this will force the existence of more critical points. For example, if Δ is a homotopy class of maps from the p-sphere S^p to $\mathcal{X}$ one can hope to find a critical point via a *minimax sequence*, minimising over maps $\phi \in \Delta$ the maximum of $\mathcal{F}(\phi(v))$ over points $v \in S^p$.

In Hilbert's time the only systematic results were in the case of dimension $n = 1$ and for linear problems, such as the Dirichlet problem for the Laplace equation. The development of a nonlinear theory in higher dimensions has been the scene for huge advances over the past century and provides the setting for much of Karen Uhlenbeck's work.

Harmonic Maps in Dimension 2

We begin in dimension 1 where *geodesics* in a Riemannian manifold are classical examples of solutions to a variational problem. Here we take N to be a compact, connected, Riemannian manifold and fix two points p, q in N. We take $\mathcal{X}$ to be the space of smooth paths $\gamma : [0,1] \to N$ with $\gamma(0) = p, \gamma(1) = q$, and the energy functional

$$\mathcal{F}(\gamma) = \int_0^1 |\nabla\gamma|^2,$$

where the norm of the "velocity vector" $\nabla\gamma$ is computed using the Riemannian metric on N. The Euler–Lagrange equation is the geodesic equation, in local co-ordinates,

$$\gamma_i'' - \sum_{j,k} \Gamma^i_{jk}\gamma_j'\gamma_k' = 0,$$

where the "Christoffel symbols" Γ^i_{jk} are given by well-known formulae in terms of the metric tensor and its derivatives. In this case the variational picture works as well as one could possibly wish. There is a geodesic from p to q minimising the energy. More generally one can use minimax arguments and (at least if p and q are taken in general position) the *Morse theory* asserts that the homology of the path space $\mathcal{X}$ can be computed from a chain complex with generators corresponding to the geodesics from p to q. This can be used in both directions: facts from algebraic topology about the homology of the path space give existence results for geodesics, and, conversely, knowledge of the geodesics can feed into algebraic topology, as in Bott's proof of his periodicity theorem.

The existence of a minimising geodesic between two points can be proved in an elementary way and the original approach of Morse avoided the infinite dimensional path space $\mathcal{X}$, working instead with finite dimensional approximations, but the infinite-dimensional picture gives the best starting point for the discussion to follow. The basic point is a compactness property: *any sequence $\gamma_1, \gamma_2, \ldots$ in $\mathcal{X}$ with bounded energy has a subsequence which converges in C^0 to some continuous path from p to q.* In fact for a path $\gamma \in \mathcal{X}$ and $0 \le t_1 < t_2 \le 1$ we have

$$d(\gamma(t_1), \gamma(t_2)) \le \int_{t_1}^{t_2} |\nabla\gamma| \le \mathcal{F}(\gamma)^{1/2}\, |t_1 - t_2|^{1/2},$$

where the last step uses the Cauchy–Schwartz inequality. Thus a bound on the energy gives a $\frac{1}{2}$-Hölder bound on γ and the compactness property follows from the Ascoli-Arzela theorem.

In the same vein as the compactness principle, one can extend the energy functional $\mathcal{F}$ to a completion $\overline{\mathcal{X}}$ of $\mathcal{X}$ which is an infinite dimensional Hilbert manifold, and elements of $\overline{\mathcal{X}}$ are still continuous (in fact $\frac{1}{2}$-Hölder continuous) paths in N. In this abstract setting, Palais and Smale introduced a general "Condition C" for functionals on Hilbert manifolds, which yields a straightforward variational theory. (This was extended to Banach manifolds in early work of Uhlenbeck [24].) The drawback is that, beyond the geodesic equations, most problems of interest in differential geometry do not satisfy this Palais–Smale condition, as illustrated by the case of harmonic maps.

The harmonic map equations were first studied systematically by Eells and Sampson [5]. We now take M, N to be a pair of Riemannian manifolds (say compact) and $\mathcal{X} = \mathrm{Maps}(M, N)$ the space of smooth maps. The energy of a map $u : M \to N$ is given by the same formula

$$\mathcal{F}(u) = \int_M |\nabla u|^2,$$

where at each point $x \in M$ the quantity $|\nabla u|$ is the standard norm defined by the metrics on TM_x and $TN_{u(x)}$. In local co-ordinates the Euler Lagrange equations have the form

$$\Delta_M u_i - \sum_{jk} \Gamma^i_{jk} \nabla u_j \nabla u_k = 0, \tag{2}$$

where Δ_M is the Laplacian on M. This is a quasi-linear elliptic system, with a nonlinear term which is quadratic in first derivatives. The equation is the natural common generalisation of the geodesic equation in N and the linear Laplace equation on M.

The key point now is that when $\dim M > 1$ the energy functional does *not* have the same compactness property. This is bound up with *Sobolev inequalities* and, most fundamentally, with the *scaling behaviour* of the functional. To explain, in part, the latter consider varying the metric g_M on M by a conformal factor λ. So λ is a strictly positive function on M and we have a new metric $\tilde{g}_M = \lambda^2 g_M$. Then one finds that the energy $\tilde{F}$ defined by this new metric is

$$\tilde{\mathcal{F}}(u) = \int_M \lambda^{2-n} |\nabla u|^2,$$

where $n = \dim M$. In particular if $n = 2$ we have $\tilde{\mathcal{F}} = \mathcal{F}$. Now take $M = S^2$ with its standard round metric and $\phi : S^2 \to S^2$ a Möbius map. This is a conformal map and it follows from the above that for any $u : S^2 \to N$ we have $\mathcal{F}(u \circ \phi) = \mathcal{F}(u)$. Since the space of Möbius maps is not compact we can construct a sequence of maps $u \circ \phi_i$ with the same energy but with no convergent subsequence.

We now recall the Sobolev inequalities. Let f be a smooth real valued function on $\mathbf{R}^n$, supported in the unit ball. We take polar co-ordinates (r, θ) in $\mathbf{R}^n$, with $\theta \in S^{n-1}$. For any fixed θ we have

$$f(0) = \int_{r=0}^1 \frac{\partial f}{\partial r} dr.$$

So, integrating over the sphere,

$$f(0) = \frac{1}{\omega_n} \int_{S^{n-1}} \int_{r=0}^1 \frac{\partial f}{\partial r} dr d\theta,$$

where ω_n is the volume of S^{n-1}. Since the Euclidean volume form is $d^n x = r^{n-1} dr d\theta$ we can write this as

$$f(0) = \frac{1}{\omega_n} \int_{B^n} |x|^{1-n} \frac{\partial f}{\partial r} \, d^n x.$$

The function $x \mapsto |x|^{1-n}$ is in L^q over the ball for any $q < n/n-1$. Let p be the conjugate exponent, with $p^{-1} + q^{-1} = 1$, so $p > n$. Then Hölder's inequality gives

$$|f(0)| \leq C_p \|\nabla f\|_{L^p}$$

where C_p is ω_n^{-1} times the $L^q(B^n)$ norm of $x \mapsto |x|^{1-n}$. The upshot is that for $p > n$ there is a continuous embedding of the *Sobolev space* L^p_1—obtained by completing in the norm $\|\nabla f\|_{L^p}$—into the continuous functions on the ball. In a similar fashion, if $p < n$ there is a continuous embedding $L^p_1 \to L^r$ for the exponent range $r \leq np/(n-p)$, which is bound up with the isoperimetric inequality in $\mathbf{R}^n$. The arithmetic relating the exponents and the dimension n reflects the scaling behaviour of the norms. If we define $f_\mu(x) = f(\mu x)$, for $\mu \geq 1$, then

$$\begin{aligned} \|f_\mu\|_{C^0} &= \|f\|_{C^0}, \\ \|f_\mu\|_{L^r} &= \mu^{-n/r} \|f\|_{L^r}, \\ \|f_\mu\|_{L^p_1} &= \mu^{1-n/p} \|f\|_{L^p_1}. \end{aligned}$$

It follows immediately that there can be no continuous embedding $L^p_1 \to C^0$ for $p < n$ or $L^p_1 \to L^r$ for $r > np/(n-p)$.

The salient part of this discussion for the harmonic map theory is that the embedding $L^p_1 \to C^0$ *fails* at the critical exponent $p = n$. (To see this, consider the function $\log \log r^{-1}$.) Taking $n = 2$ this means that the energy of a map from a 2-manifold does not control the continuity of the map and the whole picture in the 1-dimensional case breaks down. This was the fundamental difficulty addressed in the landmark paper [12] of Sacks and Uhlenbeck which showed that, with a deeper analysis, variational arguments can still be used to give general existence results.

Rather than working directly with minimising sequences, Sacks and Uhlenbeck introduced perturbed functionals on $\mathcal{X} = \mathrm{Maps}(M, N)$ (with M a compact 2-manifold):

$$\mathcal{F}_\alpha(u) = \int_M (1 + |\nabla u|^2)^\alpha.$$

For $\alpha > 1$ we are in the good Sobolev range, just as in the geodesic problem. Fix a connected component $\mathcal{X}_0$ of $\mathcal{X}$ (i.e. a homotopy class of maps from M to N). For $\alpha > 1$ there is a smooth map u_α realising the minimum of $\mathcal{F}_\alpha$ on $\mathcal{X}_0$. This map u_α satisfies the corresponding Euler–Lagrange equation, which is an elliptic PDE given by a variant of (2). The strategy is to study the convergence of u_α as α tends to 1. The main result can be outlined as follows. To simplify notation, we understand that α runs over a suitable sequence decreasing to 1.

- There is a finite set $S \subset M$ such that the u_α converge in C^∞ over $M \setminus S$.

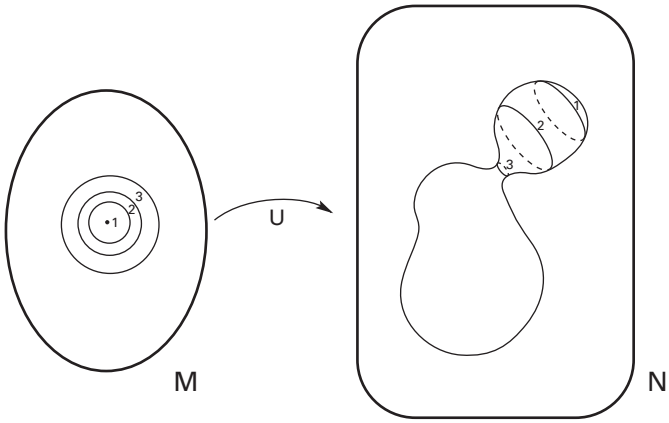

Figure 2. *Schematic representation of "bubbling."*

- The limit u of the maps u_α extends to a smooth harmonic map from M to N (which could be a constant map).
- If x is a point in S such that the u_α do *not* converge to u over a neighbourhood of x then there is a non-trivial harmonic map $v : S^2 \to N$ such that a suitable sequence of rescalings of the u_α near x converge to v.

In brief, the only way that the sequence u_α may fail to converge is by forming "bubbles," in which small discs in M are blown up into harmonic spheres in N. We illustrate the meaning of this bubbling through the example of *rational maps* of the 2-sphere. (See also the expository article [11].) For distinct points $z_1, \dots z_d$ in $\mathbf{C}$ and non-zero coefficients a_i consider the map

$$u(z) = \sum_{i=1}^{d} \frac{a_i}{z - z_i},$$

which extends to a degree d holomorphic map $u : S^2 \to S^2$ with $u(\infty) = 0$. These are in fact harmonic maps, with the same energy $8\pi d$. Take $z_1 = 0, a_1 = \epsilon$. If we make ϵ tend to 0, with the other a_i fixed, then away from 0 the maps converge to the degree $(d-1)$ map $\sum_2^d a_i(z-z_i)^{-1}$. On the other hand if we rescale about 0 by setting

$$\tilde{u}(z) = u(\epsilon z) = \frac{1}{z} + \sum_{i=2}^{d} \frac{a_i}{\epsilon z - z_i}$$

the rescaled maps converge (on compact subsets of $\mathbf{C}$) to the degree 1 map

$$v(z) = \frac{1}{z} - c$$

with $c = \sum_2^d a_i/z_i$.

A key step in the Sacks and Uhlenbeck analysis is a "small energy" statement (related to earlier results of Morrey). This says that there is some $\epsilon > 0$ such that if the energy of a map u_α on a small disc $D \subset N$ is less than ϵ then there are uniform estimates of all derivatives of u_α over the half-sized disc. The convergence result then follows from a covering argument. Roughly speaking, if the energy of the map on M is at most E then there can be at most a fixed number E/ϵ of small discs on which the map is not controlled. The crucial point is that ϵ does not depend on the size of the disc, due to the scale invariance of the energy. To sketch the proof of the small energy result, consider a simpler model equation

$$\Delta f = |\nabla f|^2, \tag{3}$$

for a function f on the unit disc in $\mathbf{C}$. Linear elliptic theory, applied to the Laplace operator, gives estimates of the schematic form

$$\|\nabla f\|_{L_1^q} \leq C\|\Delta f\|_{L^q} + \text{LOT},$$

where LOT stands for "lower order terms" in which (for this sketch) we include the fact that one will have to restrict to an interior region. Take for example $q = 4/3$. Then substituting into the equation (3) we have

$$\|\nabla f\|_{L_1^{4/3}} \leq C\||\nabla f|^2\|_{L^{4/3}} + \text{LOT} \leq C\|\nabla f\|^2_{L^{8/3}} + \text{LOT}.$$

Now in dimension 2 we have a Sobolev embedding $L_1^{4/3} \to L^4$ which yields

$$\|\nabla f\|_{L^4} \leq C\|\nabla f\|^2_{L^{8/3}} + \text{LOT}.$$

On the other hand, Hölders inequality gives the interpolation

$$\|\nabla f\|_{L^{8/3}} \leq \|\nabla f\|^{1/2}_{L^2}\|\nabla f\|^{1/2}_{L^4}.$$

So, putting everything together, one has

$$\|\nabla f\|_{L^4} \leq C\|\nabla f\|_{L^4}\|\nabla f\|_{L^2} + \text{LOT}.$$

If $\|\nabla f\|_{L^2} \leq 1/2C$ we can re-arrange this to get

$$\|\nabla f\|_{L^4} \leq \text{LOT}.$$

In other words, in the small energy regime (with $\sqrt{\epsilon} = 1/2C$) we can bootstrap using the equation to gain an estimate on a slightly stronger norm (L^4 rather than L^2) and one continues in similar fashion to get interior estimates on all higher derivatives.

This breakthrough work of Sacks and Uhlenbeck ties in with many other developments from the same era, some of which we discuss in the next section and some of which we mention briefly here.

- In *minimal submanifold theory*: when M is a 2-sphere the image of a harmonic map is a minimal surface in N (or more precisely a branched immersed submanifold). In this way, Sacks and Uhlenbeck obtained an important existence result for minimal surfaces.
- In *symplectic topology* the pseudoholomorphic curves, introduced by Gromov in 1986, are examples of harmonic maps and a variant of the Sacks–

Uhlenbeck theory is the foundation for all the ensuing developments (see, for example, [7]).

- In *PDE theory* other "critical exponent" variational problems, in which similar bubbling phenomena arise, were studied intensively (see for example the work of Brezis and Nirenberg [4]).
- In *Riemannian geometry* the Yamabe problem of finding a metric of constant scalar curvature in a given conformal class (on a manifold of dimension 3 or more) is a critical exponent variational problem for the Einstein-Hilbert functional (the integral of the scalar curvature), restricted to metrics of volume 1. Schoen proved the existence of a minimiser, completing the solution of the Yamabe problem, using a deep analysis to rule out the relevant bubbling [14].

A beautiful application of the Sacks–Uhlenbeck theory was obtained in 1988 by Micallef and Moore [8]. The argument is in the spirit of classical applications of geodesics in Riemannian geometry. Micallef and Moore considered a curvature condition on a compact Riemannian manifold N (of dimension at least 4) of having "positive curvature on isotropic 2-planes." They proved that if N satisfies this condition and is simply connected then it is a homotopy sphere (and thus, by the solution of the Poincaré conjecture, is homeomorphic to a sphere). The basic point is that a non-trivial homotopy class in $\pi_k(N)$ gives a non-trivial element of $\pi_{k-2}(\mathcal{X})$, where $\mathcal{X} = \mathrm{Maps}(S^2, N)$, which gives a starting point for a minimax argument. If N is not a homotopy sphere then by standard algebraic topology there is some k with $2 \leq k \leq \frac{1}{2}\mathrm{dim}N$ such that $\pi_k(N) \neq 0$, which implies that $\pi_{k-2}(\mathcal{X})$ is non-trivial. By developing mini-max arguments with the Sacks–Uhlenbeck theory, using the perturbed energy functional, Micallef and Moore were able to show that this leads to a non-trivial harmonic map $u : S^2 \to N$ of index at most $k-2$. (Here the index is the dimension of the space on which the second variation is strictly negative.) On the other hand the Levi–Civita connection of N defines a holomorphic structure on the pull-back $u^*(TN \otimes \mathbf{C})$ of the complexified tangent bundle. By combining results about holomorphic bundles over S^2 and a Weitzenbock formula, in which the curvature tensor of N enters, they show that the index must be at least $\frac{1}{2}\mathrm{dim}N - \frac{3}{2}$ and thus derive a contradiction.

If the sectional curvature of N is "$\frac{1}{4}$-pinched" (i.e. lies between $\frac{1}{4}$ and 1 everywhere) then N has positive curvature on isotropic 2-planes. Thus the Micallef and Moore result implies the classical sphere theorem of Berger and Klingenberg, whose proof was quite different. In turn, much more recently, Brendle and Schoen [3] proved that a (simply connected) manifold satisfying this isotropic curvature condition is in fact *diffeomorphic* to a sphere. Their proof was again quite different, using Ricci flow.

Gauge Theory in Dimension 4

From the late 1970s, mathematics was enriched by questions inspired by physics, involving *gauge fields* and the *Yang-Mills equations*. These developments were many-faceted and here we will focus on aspects related to variational theory. In this set-up one considers a fixed Riemannian manifold M and a G-bundle $P \to M$ where G is a compact Lie group. The distinctive feature, compared to most previous work in differential geometry, is that P is an auxiliary bundle not directly tied to the geometry of M. The basic objects of study are connections on P. In a local trivialisation τ of P a connection A is given by a $\mathrm{Lie}(G)$-valued 1-form A^τ. For simplicity we take G to be a matrix group, so A^τ is a matrix of 1-forms. The fundamental invariant of a connection is its curvature $F(A)$ which in the local trivialisation is given by the formula

$$F^\tau = dA^\tau + A^\tau \wedge A^\tau.$$

The Yang-Mills functional is

$$\mathcal{F}(A) = \int_M |F(A)|^2,$$

and the Euler–Lagrange equation is $d_A^* F = 0$ where d_A^* is an extension of the usual operator d^* from 2-forms to 1-forms, defined using A. This Yang-Mills equation is a nonlinear generalisation of Maxwell's equations of electromagnetism (which one obtains taking $G = U(1)$ and passing to Lorentzian signature).

In the early 1980s, Uhlenbeck proved fundamental analytical results which underpin most subsequent work in this area. The main case of interest is when the manifold M has dimension 4 and the problem is then of critical exponent type. In this dimension the Yang-Mills functional is conformally invariant and there are many analogies with the harmonic maps of surfaces discussed above. A new aspect involves gauge invariance, which does not have an analogy in the harmonic maps setting. That is, the infinite dimensional group $\mathcal{G}$ of automorphisms of the bundle P acts on the space $\mathcal{A}$ of connections, preserving the Yang-Mills functional, so the natural setting for the variational theory is the quotient space $\mathcal{A}/\mathcal{G}$. Locally we are free to change a trivialisation τ_0 by the action of a G-valued function g, which will change the local representation of the connection to

$$A^{g\tau_0} = gd(g^{-1}) + gA^{\tau_0}g^{-1}.$$

While this action of the gauge group $\mathcal{G}$ may seem unusual, within the context of PDEs, it represents a fundamental phenomenon in differential geometry. In studying Riemannian metrics, or any other kind of structure, on

a manifold one has to take account of the action of the infinite-dimensional group of diffeomorphisms: for example the round metric on the sphere is only unique up to this action. Similarly, the explicit local representation of a metric depends on a choice of local co-ordinates. In fact diffeomorphism groups are much more complicated than the gauge group $\mathcal{G}$. In another direction one can have in mind the case of electromagnetism, where the connection 1-form A^τ is equivalent to the classical electric and magnetic potentials on space-time. The $\mathcal{G}$-action corresponds to the fact that these potentials are not unique.

Two papers of Uhlenbeck [25], [26] addressed both of these aspects (critical exponent and gauge choice). The paper [25] bears on the choice of an "optimal" local trivialisation τ of the bundle over a ball $B \subset M$ given a connection A. The criterion that Uhlenbeck considers is the Coulomb gauge fixing condition: $d^*A^\tau = 0$, supplemented with the boundary condition that the pairing of A^τ with the normal vector vanishes. Taking $\tau = g\tau_0$, for some arbitrary trivialisation τ_0, this becomes an equation for the G-valued function g which is a variant of the harmonic map equation, with Neumann boundary conditions. In fact the equation is the Euler–Lagrange equation associated to the functional $\|A^\tau\|_{L^2}$, on local trivialisations τ. The Yang-Mills equations in such a Coulomb gauge form an elliptic system. (Following the remarks in the previous paragraph; an analogous discussion for Riemannian metrics involves harmonic local co-ordinates, in which the Einstein equations, for example, form an elliptic system.)

The result proved by Uhlenbeck in [25] is of "small energy" type. Specialising to dimension 4 for simplicity, she shows that there is an $\epsilon > 0$ and a constant C such that if $\|F\|_{L^2(B)} < \epsilon$ there is a Coulomb gauge τ over B in which

$$\|\nabla A^\tau\|_{L^2} + \|A^\tau\|_{L^4} \leq C\|F\|_{L^2}.$$

The strategy of proof uses the continuity method, applied to the family of connections given by restricting to smaller balls with the same centre, and the key point is to obtain *a priori* estimates in this family. The PDE arguments deriving these estimates have some similarity with those sketched in Section "Harmonic maps in dimension 2" above. An important subtlety arises from the critical nature of the Sobolev exponents involved. If $\tau = g\tau_0$ then an L^2 bound on ∇A^τ gives an L^2 bound on the second derivative of g but in dimension 4 this is the borderline exponent where we do not get control over the continuity of g. That makes the nonlinear operations such as $g \mapsto g^{-1}$ problematic. Uhlenbeck overcomes this problem by working with L^p for $p > 2$ and using a limiting argument.

In the companion paper [26], Uhlenbeck proves a renowned "removal of singularities" result. The statement is that a solution A of the Yang-Mills equations over the punctured ball $B^4 \setminus \{0\}$ with finite energy (i.e. with curvature $F(A)$ in L^2) extends smoothly over 0 in a suitable local trivialisation. One important application of this is that finite-energy Yang-Mills connections over $\mathbf{R}^4$ extend to the conformal compactification S^4. We will only attempt to give the flavour of the proof. Given our finite-energy solution A over the punctured ball let

$$f(r) = \int_{|x|<r} |F(A)|^2,$$

for $r < 1$. Then the derivative is

$$\frac{df}{dr} = \int_{|x|=r} |F(A)|^2.$$

The strategy is to express $f(r)$ also as a boundary integral, plus lower order terms. To give a hint of this, consider the case of an abelian group $G = U(1)$, so the connection form A^τ is an ordinary 1-form, the curvature is simply $F = dA^\tau$, and the Yang-Mills equation is $d^*F = 0$. Fix small $\epsilon < r$ and work on the annular region W where $\epsilon < |x| < r$. We can integrate by parts to write

$$\int_W |F|^2 = \int_W \langle dA^\tau, F\rangle = \int_W \langle A^\tau, d^*F\rangle + \int_{\partial W} A^\tau \wedge *F.$$

Since $d^*F = 0$ the first term on the right hand side vanishes. If one can show that the contribution from the inner boundary $|x| = \epsilon$ tends to 0 with ϵ then one concludes that

$$f(r) = \int_{|x|=r} A^\tau \wedge *F.$$

In the nonabelian case the same discussion applies up to the addition of lower-order terms, involving $A^\tau \wedge A^\tau$. The strategy is then to obtain a differential inequality of the shape

$$f(r) \leq \tfrac{1}{4} r \frac{df}{dr} + \text{LOT}, \tag{4}$$

by comparing the boundary terms over the 3-sphere. This differential inequality integrates to give $f(r) \leq Cr^4$ and from there it is relatively straightforward to obtain an L^∞ bound on the curvature and to see that the connection can be extended over 0. The factor $\frac{1}{4}$ in (4) is obtained from an inequality over the 3-sphere. That is, any closed 2-form ω on S^3 can be expressed as $\omega = da$ where

$$\|a\|^2_{L^2(S^3)} \leq \tfrac{1}{4}\|\omega\|^2_{L^2(S^3)}.$$

The main work in implementing this strategy is to construct suitable gauges over annuli in which the lower order, nonlinear terms $A^\tau \wedge A^\tau$ are controlled.

These results of Uhlenbeck lead to a Yang-Mills analogue of the Sacks–Uhlenbeck picture discussed in the previous section. This was not developed explicitly in Uhlenbeck's 1983 papers [25], [26] but results along those lines were obtained by her doctoral student S. Sedlacek [16]. Let c

be the infimum of the Yang-Mills functional on connections on $P \to X$, where X is a compact 4-manifold. Let A_i be a minimising sequence. Then there is a (possibly different) G-bundle $\tilde{P} \to X$, a Yang-Mills connection A_∞ on $\tilde{P}$, and a finite set $S \subset X$ such that, after perhaps passing to a subsequence i', the $A_{i'}$ converge to A_∞ over $X \setminus S$. (More precisely, this convergence is in $L^2_{1,\mathrm{loc}}$ and implicitly involves a sequence of bundle isomorphisms of P and $\tilde{P}$ over $X \setminus S$.) If x is a point in S such that the $A_{i'}$ do *not* converge to A_∞ over a neighbourhood of x then one obtains a non-trivial solution to the Yang-Mills equations over S^4 by a rescaling procedure similar to that in the harmonic map case. Similar statements apply to sequences of solutions to the Yang-Mills equations over X and in particular to sequences of Yang-Mills "instantons." These special solutions solve the first order equation $F = \pm * F$ and are closely analogous to the pseudoholomorphic curves in the harmonic map setting. Uhlenbeck's analytical results underpinned the applications of instanton moduli spaces to 4-manifold topology which were developed vigorously throughout the 1980s and 1990s—just as for pseudoholomorphic curves and symplectic topology. But we will concentrate here on the variational aspects.

For simplicity fix the group $G = SU(2)$; the $SU(2)$-bundles P over X are classified by an integer $k = c_2(P)$ and for each k we have a moduli space $\mathcal{M}_k$ (possibly empty) of instantons (where the sign in $F = \pm * F$ depends on the sign of k). Recall that the natural domain for the Yang-Mills functional is the infinite-dimensional quotient space $\mathcal{X}_k = \mathcal{A}_k/\mathcal{G}_k$ of connections modulo equivalence. The moduli space $\mathcal{M}_k$ is a subset of $\mathcal{X}_k$ and (if non-empty) realises the absolute minimum of the Yang-Mills functional on $\mathcal{X}_k$. In this general setting one could, optimistically, hope for a variational theory which would relate:

(1) The topology of the ambient space $\mathcal{X}_k$,
(2) The topology of $\mathcal{M}_k$,
(3) The non-minimal critical points: i.e. the solutions of the Yang-Mills equation which are not instantons.

A serious technical complication here is that the group $\mathcal{G}_k$ does not usually act freely on $\mathcal{A}_k$, so the quotient space is not a manifold. But we will not go into that further here and just say that there are suitable homology groups $H_i(\mathcal{X}_k)$, which can be studied by standard algebraic topology techniques and which have a rich and interesting structure.

Much of the work in this area in the late 1980s was driven by two specific questions.

- The *Atiyah-Jones conjecture* [1]. They considered the manifold $M = S^4$ where (roughly speaking) the space $\mathcal{X}_k$ has the homotopy type of the degree k mapping space $\mathrm{Maps}_k(S^3, S^3)$, which is in fact independent of k. The conjecture was that the inclusion $\mathcal{M}_k \to \mathcal{X}_k$ induces an isomorphism on homology groups H_i for i in a range $i \leq i(k)$, where $i(k)$ tends to infinity with k. One motivation for this idea came from results of Segal in the analogous case of rational maps [17].
- Again focusing on $M = S^4$: are there *any* non-minimal solutions of the Yang-Mills equations?

A series of papers of Taubes [20], [22] developed a variational approach to the Atiyah-Jones conjecture (and generalisations to other 4-manifolds). In [20] Taubes established a lower bound on the index of any non-minimal solution over the 4-sphere. If the problem satisfied the Palais–Smale condition this index bound would imply the Atiyah-Jones conjecture (with $i(k)$ roughly $2k$) but the whole point is that this condition is not satisfied, due to the bubbling phenomenon for mini-max sequences. Nevertheless, Taubes was able to obtain many partial results through a detailed analysis of this bubbling. The Atiyah-Jones conjecture was confirmed in 1993 by Boyer, Hurtubise, Mann, and Milgram [2] but their proof worked with geometric constructions of the instanton moduli spaces, rather than variational arguments.

The second question was answered, using variational methods, by Sibner, Sibner, and Uhlenbeck in 1989 [18], showing that indeed such solutions do exist. In their proof they considered a standard S^1 action on S^4 with fixed point set a 2-sphere, an S^1-equivariant bundle P over S^4 and S^1-invariant connections on P. This invariance forces the "bubbling points" arising in variational arguments to lie on the 2-sphere $S^2 \subset S^4$ and there is a dimensional reduction of the problem to "monopoles" in 3-dimensions which has independent interest.

A connection over $\mathbf{R}^4$ which is invariant under the action of translations in one direction can be encoded as a pair (A, ϕ) of a connection A over $\mathbf{R}^3$ and an additional *Higgs field* ϕ which is a section of the adjoint vector bundle $\mathrm{ad}P$ whose fibres are copies of $\mathrm{Lie}(G)$. The Yang-Mills functional induces a Yang-Mills-Higgs functional

$$\mathcal{F}(A, \phi) = \int_{\mathbf{R}^3} |F(A)|^2 + |\nabla_A \phi|^2$$

on these pairs over $\mathbf{R}^3$. One also fixes an asymptotic condition that $|\phi|$ tends to 1 at ∞ in $\mathbf{R}^3$. In 3 dimensions we are below the critical dimension for the functional, but the noncompactness of $\mathbf{R}^3$ prevents a straightforward verification of the Palais–Smale condition. Nonetheless, in a series of papers [19], [21] Taubes developed a far-reaching variational theory in this setting. By a detailed analysis, Taubes showed that, roughly speaking, a minimax sequence can always be chosen to have energy density concentrated

in a fixed large ball in $\mathbf{R}^3$ and thus obtained the necessary convergence results. In particular, using this analysis, Taubes established the existence of non-minimal critical points for the functional $\mathcal{F}(A, \phi)$.

The critical points of the Yang-Mills-Higgs functional on $\mathbf{R}^3$ yield Yang-Mills solutions over $\mathbf{R}^4$, but these do not have finite energy. However the same ideas can be applied to the S^1-action. The quotient of $S^4 \setminus S^2$ by the S^1-action can naturally be identified with the hyperbolic 3-space H^3, and S^1-invariant connections correspond to pairs (A, ϕ) over H^3. There is a crucial parameter L in the theory which from one point of view is the weight of the S^1 action on the fibres of P over S^2. From another point of view the curvature of the hyperbolic space, after suitable normalisation, is $-L^{-2}$. The fixed set S^2 can be identified with the sphere at infinity of hyperbolic space and bubbling of connections over a point in $S^2 \subset S^4$ corresponds, in the Yang-Mills-Higgs picture, to some contribution to the energy density of (A, ϕ) moving off to the corresponding point at infinity.

The key idea of Sibner, Sibner, and Uhlenbeck was to make the parameter L very large. This means that the curvature of the hyperbolic space is very small and, on sets of fixed diameter, the hyperbolic space is well-approximated by $\mathbf{R}^3$. Then they show that Taubes' arguments on $\mathbf{R}^3$ go over to this setting and are able to produce the desired non-minimal solution of the Yang-Mills equations over S^4. Later, imposing more symmetry, other solutions were found using comparatively elementary arguments [13], but the approach of Taubes, Sibner, Sibner, and Uhlenbeck is a paradigm of the way that variational arguments can be used "beyond Palais–Smale," via a delicate analysis of the behaviour of minimax sequences.

We conclude this section with a short digression from the main theme of this article. This brings in other relations between harmonic mappings of surfaces and 4-dimensional gauge theory, and touches on another very important line of work by Karen Uhlenbeck, represented by papers such as [27], [28]. In this setting the target space N is a symmetric space and the emphasis is on explicit solutions and connections with integrable systems. There is a huge literature on this subject, stretching back to work of Calabi and Chern in the 1960s, and distantly connected with the Weierstrass representation of minimal surfaces in $\mathbf{R}^3$. From around 1980 there were many contributions from theoretical physicists and any kind of proper treatment would require a separate article, so we just include a few remarks here.

As we outlined above, the dimension reduction of Yang-Mills theory on $\mathbf{R}^4$ obtained by imposing translation-invariance in one variable leads to equations for a pair (A, ϕ) on $\mathbf{R}^3$. Now reduce further by imposing translation-invariance in two directions. More precisely, write $\mathbf{R}^4 = \mathbf{R}^2_1 \times \mathbf{R}^2_2$, fix a simply-connected domain $\Omega \subset \mathbf{R}^2_1$, and consider connections on a bundle over $\Omega \times \mathbf{R}^2_2$ which are invariant under translations in $\mathbf{R}^2_2$. These correspond to pairs (A, Φ) where A is a connection on a bundle P over Ω and Φ can be viewed as a 1-form on Ω with values in the bundle $\mathrm{ad}P$. Now $A + i\Phi$ is a connection over Ω for a bundle with structure group the complexification G^c: for example if $G = U(r)$ the complexified group is $G^c = GL(r, \mathbf{C})$. The Yang-Mills instanton equations on $\mathbf{R}^4$ imply that $A+i\Phi$ is a *flat* connection. By the fundamental property of curvature, since Ω is simply-connected, this flat connection can be trivialised. The original data (A, Φ) is encoded in the reduction of the trivial G^c-bundle to the subgroup G, which amounts to a map u from Ω to the non-compact symmetric space G^c/G. For example, when $G = U(r)$ the extra data needed to recover (A, Φ) is a Hermitian metric on the fibres of the complex vector bundle, and $GL(r, \mathbf{C})/U(r)$ is the space of Hermitian metrics on $\mathbf{C}^r$. The the remaining part of the instanton equations in four dimensions is precisely the harmonic map equation for u. This is one starting point for Hitchin's theory of "stable pairs" over compact Riemann surfaces [6].

One is more interested in harmonic maps to compact symmetric spaces and, as Uhlenbeck explained in [28], this can be achieved by a modification of the set-up above. She takes $\mathbf{R}^4$ with an indefinite quadratic form of signature $(2, 2)$ and a splitting $\mathbf{R}^4 = \mathbf{R}^2_1 \times \mathbf{R}^2_2$ into positive and negative subspaces. Then the invariant instantons correspond to harmonic maps from Ω to the compact Lie group G. Other symmetric spaces can be realised as totally geodesic submanifolds in the Lie group, for example complex Grassmann manifolds in $U(r)$, and the theory can be specialised to suit. This builds a bridge between the "integrable" nature of the 2-dimensional harmonic map equations and the Penrose-Ward twistor description of Yang-Mills instantons over $\mathbf{R}^4$, although as we have indicated above much of the work on the former predates twistor theory. In her highly influential paper [28], Uhlenbeck found an action of the loop group on the space of harmonic maps from Ω to G, introduced an integer invariant "uniton number," and obtained a complete description of all harmonic maps from the Riemann sphere to G.

Higher Dimensions

In a variational theory with a critical dimension ν certain characteristic features appear when studying questions in dimensions greater than ν. In the harmonic mapping theory, for maps $u : M \to N$, the dimension in question is $n = \dim M$ and, as we saw above, the critical dimension is $\nu = 2$. A breakthrough in the higher dimensional theory was obtained by Schoen and Uhlenbeck in [12]. Suppose for simplicity that N is isometrically embedded in

some Euclidean space $\mathbf{R}^k$ and define $L_1^2(M,N)$ to be the set of L_1^2 functions on M with values in the vector space $\mathbf{R}^k$ which map to N almost everywhere on M. The energy functional $\mathcal{F}$ is defined on $L_1^2(M,N)$ and Schoen and Uhlenbeck considered an energy minimising map $u \in L_1^2(M,N)$. The main points of the theory are:

- u is smooth outside a singular set $\Sigma \subset M$ which has Hausdorff dimension at most $n-3$;
- at each point x in the singular set Σ there is a *tangent map* to u.

The second item means that there is a sequence of real numbers $\sigma_i \to 0$ such that the rescaled maps

$$u_i(\xi) = u(\exp_x(\sigma_i \xi))$$

converge to a map $v : \mathbf{R}^n \to N$ which is radially invariant, and hence corresponds to a map from the the sphere S^{n-1} to N. (Here $\exp_x$ is the Riemannian exponential map and we have chosen a frame to identify TM_x with $\mathbf{R}^n$.)

To relate this to the case $n = 2$ discussed above, the general picture is that a $\mathcal{F}$-minimising sequence in Maps (M,N) can be taken to converge outside a *bubbling set* of dimension at most $n-2$ and the limit extends smoothly over the $(n-2)$-dimensional part of the bubbling set. The new feature in higher dimensions is that the limit can have a singular set of codimension 3 or more.

Two fundamental facts which underpin these results are *energy monotonicity* and *ϵ-regularity*. To explain the first, consider a smooth harmonic map $U : B^n \to N$, where B^n is the unit ball in $\mathbf{R}^n$. For $r < 1$ set

$$E(r) = \frac{1}{r^{n-2}} \int_{|x|<r} |\nabla U|^2.$$

Then one has an identity, for $r_1 < r_2$:

$$E(r_2) - E(r_1) = 2 \int_{r_1<|x|<r_2} |x|^{2-n} |\nabla_r U|^2, \quad (5)$$

where ∇_r is the radial component of the derivative. In particular, E is an increasing function of r. The point of this is that $E(r)$ is a scale-invariant quantity. If we define $U_r(x) = U(rx)$ then $E(r)$ is the energy of the map U_r on the unit ball. The monotonicity property means that U "looks better" on a small scale, in the sense of this rescaled energy. The identity (5) follows from a very general argument, applying the stationary condition to the infinitesimal variation of U given by radial dilation. (One way of expressing this is through the theory of the stress-energy tensor.) Note that equality $E(r_2) = E(r_1)$ holds if and only if U is radially-invariant in the corresponding annulus. This is what ultimately leads to the existence of radially-invariant tangent maps.

The monotonicity identity is a feature of maps from $\mathbf{R}^n$, but a similar result holds for small balls in a general Riemannian n-manifold M. For $x \in M$ and small $r > 0$ we define

$$E_x(r) = \frac{1}{r^{n-2}} \int_{B_x(r)} |\nabla U|^2,$$

where $B_x(r)$ is the r-ball about x. Then if U is a smooth harmonic map and x is fixed the function $E_x(r)$ is increasing in r, up to harmless lower-order terms.

The ϵ-regularity theorem of Schoen and Uhlenbeck states that there is an $\epsilon > 0$ such that if u is an energy minimiser then u is smooth in a neighbourhood of x if and only if $E_x(r) < \epsilon$ for some r. An easier, related result is that if u is known to be smooth then once $E_x(r) < \epsilon$ one has *a priori* estimates (depending on r) on all derivatives in the interior ball $B_x(r/2)$. The extension to general minimising maps is one of the main technical difficulties overcome by Schoen and Uhlenbeck.

We turn now to corresponding developments in gauge theory, where the critical dimension ν is 4. A prominent achievement of Uhlenbeck in this direction is her work with Yau on the existence of Hermitian-Yang-Mills connections [29]. The setting here involves a rank r holomorphic vector bundle E over a compact complex manifold M with a Kähler metric. Any choice of Hermitian metric h on the fibres of E defines a principle $U(r)$ bundle of orthonormal frames in E and a basic lemma in complex differential geometry asserts that there is a preferred connection on this bundle, compatible with the holomorphic structure. The curvature $F = F(h)$ of this connection is a bundle-valued 2-form of type $(1,1)$ with respect to the complex structure, and we write ΛF for the inner product with the $(1,1)$ form defined by the Kähler metric. Then ΛF is a section of the bundle of endomorphisms of E. The *Hermitian-Yang-Mills equation* is a constant multiple of the identity:

$$\Lambda F = \kappa 1_E$$

(where the constant κ is determined by topology). As the name suggests, these are special solutions of the Yang-Mills equations. The result proved by Uhlenbeck and Yau is that a "stable" holomorphic vector bundle admits such a Hermitian-Yang-Mills connection. Here stability is a numerical condition on holomorphic sub-bundles, or more generally sub-sheaves, of E which was introduced by algebraic geometers studying moduli theory of holomorphic bundles. The result of Uhlenbeck and Yau confirmed conjectures made a few years before by Kobayashi and Hitchin. These extend older results of Narasimhan and Seshadri, for bundles over Riemann surfaces, and fit into a large development over the past 40 years, connecting various stability conditions in algebraic geometry with differential geometry. We will not say more about this background here but focus on the proof of Uhlenbeck and Yau.

The problem is to solve the equation $\Lambda F(h) = \kappa 1_E$ for a Hermitian metric h on E. This boils down to a second order, nonlinear, partial differential equation for h.

While this problem does not fit directly into the variational framework we have emphasised in this article, the same compactness considerations apply. Uhlenbeck and Yau use a continuity method, extending to a 1-parameter family of equations for $t \in [0,1]$ which we write schematically as $\Lambda F(h_t) = K_t$, where K_t is prescribed and $K_1 = \kappa 1_E$. They set this up so that there is a solution h_0 for $t = 0$ and the set $T \subset [0,1]$ for which a solution h_t exists is open, by an application of the implicit function theorem. The essential problem is to prove that if E is a stable holomorphic bundle then T is closed, hence equal to the whole of $[0,1]$ and in particular there is a Hermitian-Yang-Mills connection h_1.

The paper of Uhlenbeck and Yau gave two independent treatments of the core problem, one emphasising complex analysis and the other gauge theory. We will concentrate here on the latter. For a sequence $t(i) \in T$ we have connections A_i defined by the hermitian metrics $h_{t(i)}$ and the question is whether one can take a limit of the A_i. The deformation of the equations by the term K_t is rather harmless here so the situation is essentially the same as if the A_i were Yang-Mills connections. In addition, an integral identity using Chern-Weil theory shows that the Yang-Mills energy $\|F(A_i)\|_{L^2}^2$ is bounded. Then Uhlenbeck and Yau introduced a small energy result, for connections over a ball $B_x(r) \subset M$. Since the critical dimension ν is 4, the relevant normalised energy in this Yang-Mills setting is

$$E_x(r) = \frac{1}{r^{n-4}} \int_{B_x(r)} |F|^2,$$

where n is the real dimension of M. If $E_x(r)$ is below a suitable threshold there are interior bounds on all derivatives of the connection, in a suitable gauge. Then the global energy bound implies that after perhaps taking a subsequence, the A_i converge outside a closed set $S \subset M$ of Hausdorff codimension at least 4. Uhlenbeck and Yau show that if the metrics $h_{t(i)}$ do *not* converge then a suitable rescaled limit produces a holomorphic subbundle of E over $M \setminus S$. A key technical step is to show that this subundle corresponds locally to a meromorphic map to a Grassmann manifold, which implies that the subbundle extends as a coherent sheaf over all of M. The differential geometric representation of the first Chern class of this subsheaf, via curvature, shows that it violates the stability hypothesis.

The higher-dimensional discussion in Yang-Mills theory follows the pattern of that for harmonic maps above. The corresponding monotonicity formula was proved by Price [10] and a treatment of the small energy result was given by Nakajima [9]. Some years later, the theory was developed much further by Tian [23], including the existence of "tangent cones" at singular points.

This whole circle of ideas and techniques involving the dimension of singular sets, monotonicity, "small energy" results, tangent cones, etc. has had a wide-ranging impact in many branches of differential geometry over the past few decades and forms the focus of much current research activity. Apart from the cases of harmonic maps and Yang-Mills fields discussed above, prominent examples are minimal submanifold theory, where many of the ideas appeared first, and the convergence theory of Riemannian metrics with Ricci curvature bounds.

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Simon Donaldson

Credits

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Celebratio Mathematica: An Interview with Karen Uhlenbeck and Three New Volumes on Women Mathematicians

Sheila Newbery

Mathematical Sciences Publishers (`msp.org`) is pleased to announce the publication of three new volumes devoted to the careers of Karen Uhlenbeck, Joan Birman, and Dusa McDuff as part of its electronic archive of mathematicians of note, *Celebratio Mathematica*. The work was supported with funding from the Mathematical Sciences Research Institute (`msri.org`) and spearheads an intensive project now underway to develop the archive's holdings on the careers and accomplishments of women mathematicians. The support made it possible to undertake a number of special projects for these volumes, one of which we highlight here: a new interview with Karen Uhlenbeck by Allyn Jackson (the latter needing no introduction to readers of the *Notices*).

Interviews like this are especially valuable to students of mathematics for whom knowledge about the careers of other women scientists can be powerfully reinforcing. In publishing them, we aim to inspire student readers especially by showing them that there are diverse paths to a career in the sciences. What leads women to mathematics is perhaps not so different in essence from what makes men choose math, but the social and institutional realities of women's careers have been different enough from those of men to warrant thoughtful attention. This fact is certainly one of the subjects of the interview.

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The story of how Uhlenbeck became a mathematician is traced with a nuance that builds on Claudia Henrion's earlier account in *Women in Mathematics: The Addition of Difference*.[1] One detail that emerges clearly is the significance of Uhlenbeck's mother's social circle. Carolyn Keskulla was a painter, and Uhlenbeck remembers vividly as a girl that her mother's painter friends and artistic interests brought her into contact with "a lot of people who did not live normal, middle-class lives." The liveliness and eccentricity of this crowd made a lasting impression. Moreover, like her own mother (Uhlenbeck's grandmother), Carolyn was a strong, intelligent, active woman, so the value for intellectual endeavor and the uses of the imagination were early and firmly established in the family circle.

Jackson also explores the relevance of talent, and frames it instructively in the context of opportunities afforded by education: high school, first, and then a high-quality public university education. While Uhlenbeck's family background emphasized intellectual attainment, her high school gave her no specific encouragement vis-à-vis mathematics. The first seeds of mathematical inspiration were sown at university. The following excerpt from Jackson's article (see pull quote, facing page) highlights that revelatory moment, which occurred during Uhlenbeck's time as an undergraduate.

[1] *Bloomington: Indiana University Press, 1997, pp. 25–46.*

When I first read these words (while preparing the manuscript for publication), I was struck by them: for many women of Uhlenbeck's generation, the question of what one was or wasn't allowed to do was never too far from lived experience. Her word choice seemed telling. Yet the emotional impact of the episode is gloriously positive: it is one of deep intellectual excitement.

Although Uhlenbeck faced certain obstacles, such as the impossibility even as a gifted student of applying to or attending either of the nearby all-male colleges (Princeton and Rutgers), she nevertheless had access to a superb university education at the University of Michigan, and that was how she gained her first significant exposure to higher mathematics. In the post-Sputnik era of the 60s, moreover, federal agencies were issuing a clarion call to talented students, encouraging them to pursue a career in math and science after college. That posture had clear benefits for Uhlenbeck. As she puts it, "They were encouraging everybody, and women counted."

> I had a very advanced course, similar to an undergraduate real analysis course. We had done limits. I went to a help session, and the teaching assistant showed how to take a derivative....It was a moment where suddenly I realized there were all sorts of things in mathematics that you "were allowed to do."
> –Karen Uhlenbeck

Jackson touches on the importance of mathematical collaboration—in Uhlenbeck's case, with such colleagues as Jonathan Sacks, Lesley and Robert Sibner, S.-T. Yau, Richard Schoen, and Chuu-Lian Terng. The separate, detailed accounts here of Uhlenbeck's friendships with Yau and Lesley Sibner, in particular, vividly underscore the unpredictable swerves of social and intellectual opportunity that make up a career.

Jackson's keen sense of timing is one of her gifts as an interviewer, yet equally important is her ability to layer questions that illuminate the penumbra of intuition and wonder that motivate mathematical inquiry in the first place. So a question about mathematical "tastes" ("What kinds of mathematical problems appeal to you?") leads to a fascinating exchange about what Uhlenbeck refers to as one of the mysteries of mathematics: "[W]hy KdV comes up all over the place, in all sorts of geometric and physical problems."

In addition to Jackson's in-depth interview, we want to call readers' attention to several other unique contributions to these three new volumes:

- Cliff Taubes on "Karen Uhlenbeck's contributions to gauge theoretic analysis";
- Leonid Polterovich and Felix Schlenk's article on Dusa McDuff's contributions to "Symplectic embedding problems";
- Bill Menasco's essay "My work with Joan Birman";
- Dan Margalit and Rebecca R. Winarski's overview of Joan Birman and Hugh Hilden's collaboration in "The Birman–Hilden theory."

Mathematical Sciences Publishers will continue to build its archive: soon to be published are volumes on the careers of Mary Ellen Rudin and Cathleen Morawetz. We welcome suggestions from the mathematical community for future volumes on women mathematicians, and we are grateful to MSRI for its support in making our work available in perpetuity to a broad readership.

Sheila Newbery

Credits

Author photo is courtesy of Sheila Newbery.

Gertrude M. Cox and Statistical Design

Figure 1. Gertrude Cox in office.

Sharon L. Lohr

Introduction

On December 2, 1959, Gertrude Cox (Figure 1), Director of the Institute of Statistics at the consolidated University of North Carolina, responded to a query from a young woman named Pat Barber about career opportunities in statistics for women. Cox replied that the "field of statistics is certainly wide open to women" and described some of her own experiences as a statistician:

> In this area of experimental statistics, we cooperate with the research workers in other science areas with the planning and then with the evaluation and interpretation of their research results. I could give a list of a variety of interesting areas in which I have cooperated such as, the best methods of raising flowers in a greenhouse, development and selection of new varieties of corn, the nutritional problems among the Indian children in Guatemala, how to sample

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> gold in South Africa, variations in ways to make instant frosting for cakes, how to evaluate the effectiveness of fly sprays, and many others [17].

Cox's letter reflected her view of the statistician as a partner in science, a view that, in part because of her influence, is now standard in the discipline. Her pioneering contributions and example widened opportunities in statistics around the world. To list just a few of her accomplishments, Cox:

- Founded one of the world's first statistics departments (1941) at North Carolina State College.
- Became the first woman elected to membership in the International Statistical Institute (1949) and one of the first statisticians elected to the National Academy of Sciences (1975).
- Received the O. Max Gardner Award (1959) from the University of North Carolina for "contribution to the welfare of the human race."
- Served as president of the American Statistical Association (1956) and the International Biometric Society (1968), and was founding editor of the journal *Biometrics* (1947–1955).
- Co-authored one of the most influential statistics books ever written, *Experimental Designs*, first published in 1950 and still in print.
- Championed the use of electronic computers for statistical work.

Her collaborator William G. Cochran wrote, "I doubt if anyone contributed more than Gertrude Cox to building up the profession of statistics as we know it today" [11].

Early Career

Few would have predicted in 1924 that Gertrude Mary Cox would become one of the most influential statisticians of the twentieth century. She was then a 24-year-old housemother for 16 boys at a Montana orphanage, having previously taught in a one-room schoolhouse in Iowa and studied at the Iowa National Bible Training School [21].

Cox enrolled in Iowa State College to obtain the training and credentials needed for her planned career as an orphanage superintendent. She explained in a 1975 interview how she became a statistician: she took courses in math because she liked it and it was "the easiest subject," giving her time to also take the classes in psychology and crafts she would need in her chosen career. She became interested in statistics after her calculus professor, George Snedecor, invited her to work as a computer in the Mathematical Statistical Service Center. She reminisced, "As soon as I could learn to use that math knowledge with people and their orientation, it became life" [19].

Through the 1950s, a "computer" referred to a person—usually a woman—who performed calculations on a hand-operated machine such as the one seen on the table by the radiator in Figure 1. Women were hired for this work because they were thought to be more patient with the tedious calculations than men—and, incidentally, could also be paid much less [19, 20, 30].

Many of the statistical computations involved finding correlation or regression coefficients—calculations for a regression model with a large number of independent variables could take weeks [25]. The main result in Cox's first publication [13] was a table of correlations between scores on the Iowa State College Aptitude Test, high school subjects, and college courses.

Cox received her bachelor's degree in mathematics in 1929. She stayed on at Iowa State to earn the first master's degree awarded in statistics from the Department of Mathematics in 1931, with Snedecor as advisor.

In 1933, while she was midway through a doctoral program in psychology at University of California, Berkeley, Snedecor invited her to return to Iowa State, writing, "I … am rapidly being drawn into statistical responsibilities for a large part of the College. Would you like to be a part of this? I think the opportunity is great. Are you interested? Immediately you would have charge of the girls, 140 calculating machines, and all the *stray* jobs that I can rustle for you" [18].

As one of three initial faculty members in the Iowa State Statistical Laboratory, Cox supervised the computers performing data analyses. She visited laboratories and fields to see how the data were collected, which led to collaborations with researchers to develop experimental designs and analyses that would best answer the scientific questions. Her classes in experimental design attracted students from across the campus, and she soon became known as an expert in the field. Initially hired as an assistant to Snedecor, Cox was appointed Research Assistant Professor in 1939.

In 1941 Cox became the first female full professor and the first female department head at North Carolina State College, charged with developing a department that would provide statistical expertise to researchers. Her colleague Richard Anderson related how the appointment was made:

> In 1940 Snedecor was asked to recommend candidates to head the new Department of Experimental Statistics in the School of Agriculture at North Carolina State College. "Why didn't you put my name on list?" Gertrude asked when he showed her his all-male list of candidates, and her name was added to the accompanying letter in the following postscript: "If you would consider a woman for this position, I would recommend Gertrude Cox of my staff." This terse note was to have far-reaching consequences for statistics, for not only was Gertrude considered, she was selected [10].

Historian Margaret Rossiter described how unusual it was for a woman to be considered for a position as department head in the 1940s: "As for department chairmanships,

the lowest level of academic administration, women scientists still so rarely held these positions at coeducational institutions in the forties, fifties, and mid-sixties that one can almost count these exceptions on two hands." Rossiter singled out Cox as the most successful of this handful, noting that she, unlike many other university women of the time, ended up getting credit for her accomplishments in building her department and scientific discipline: "She not only managed to ride the wave of Big Science in the 1950s and 1960s but to be enough ahead of it to shape the form it took and the impact it had on her university, field, and region" [29].

Designing Experiments

Cox began her new position in North Carolina with the same energy she had shown in her work at Iowa State. She immediately started establishing training programs, hiring faculty members, collaborating with scientists, promoting statistics in the university and nationally, and teaching classes on experimental design. She expanded her mimeographed notes from the design classes into the book *Experimental Designs* [12], published with collaborator William G. Cochran in 1950.

Experimental Designs emphasized three principles:

1. Statisticians need to be involved in the research from the planning stages: the first steps, setting out the objectives of the experiment and planning the analysis, are crucial. Often, one of the statistician's most valuable contributions arises "by getting the investigator to explain clearly why he is doing the experiment, to justify the experimental treatments whose effects he proposes to compare, and to defend his claim that the completed experiment will enable its objectives to be realized." When a statistician is consulted only after the data are collected and discovers that the poorly planned experiment cannot answer the research questions, "[i]n these unhappy circumstances, about all that can be done is to indicate, if possible, how to avoid this outcome in future experiments" [12, pp. 9, 10].
2. Randomize everything that can be randomized. "Randomization is somewhat analogous to insurance, in that it is a precaution against disturbances that may or may not occur and that may or may not be serious if they do occur. It is generally advisable to take the trouble to randomize even when it is not expected that there will be any serious bias from failure to randomize. The experimenter is thus protected against unusual events that upset his expectations" [12, p. 8].
3. Use blocking whenever possible to reduce the effects of variability. Blocks are homogeneous groups of experimental units: for example, identical twins, neighboring agricultural plots, batches of raw material, cancer patients with similar demographics and disease stage, schools in the same city, or experimental runs done on the same day. When treatments are randomly assigned to experimental units within blocks—one twin is randomly chosen to receive treatment A, and the other treatment B—the block-to-block variability is removed from the treatment comparison. If blocks represent a range of experimental conditions, results from the blocked experiment have wider applicability as well as increased precision for estimating treatment effects.

The 1950 book and its second edition in 1957 set out detailed plans for Latin square, factorial, fractional factorial, split plot, lattice, balanced incomplete block, and other designs. Each design description started with examples, followed by a discussion of when the design was suitable and detailed instructions for how to perform randomization. Then came one or more detailed case studies, showing why that design had been chosen for each experiment and how it had been randomized, and taking the reader step-by-step through the calculations needed to construct the analysis of variance table and estimate the standard errors for differences of treatment means. The authors also discussed how to estimate the efficiency of the design relative to a completely randomized design and how to do the calculations for the unbalanced structure that resulted when one or more experimental runs had missing data.

The chapters on the complex designs contained tables of designs for different block sizes and numbers of treatments. Today, statistical software quickly calculates optimal designs for almost any experimental structure, but in 1950 printed design tables were needed, particularly when there was more than one blocking variable or when the number of treatments (t) exceeded the number of experimental units in a block (k).

For the latter situation, a balanced incomplete block design was recommended, where each pair of treatments occurs together in the same number of blocks. Of course such a design can always be constructed by using all combinations of the t treatments taken k at a time, but *Experimental Designs* laid out the designs that met the constraint with the smallest numbers of experimental units. For example, the smallest balanced incomplete block design with seven treatments and blocks of size four required only seven blocks and twenty-eight experimental units—one-fifth the size of the fully combinatorial design.

Cox's experience as a consulting statistician can be seen on every page of the book. Her background as a computer is also apparent: each set of instructions for calculating an analysis of variance table came with practical tips and quality checks for ensuring the calculations are accurate. Indeed, the first experiment described in *Experimental Designs* compared the speed of two calculating machines, A and B, using a cross-over design, where the same person computed the sum of squares of 10 sets (blocks) of 27 numbers on each machine; machine B turned out to be significantly faster, taking only 2 minutes 13.6 seconds, on average, to calculate the sum of squares for 27 numbers.

When Cox began her career, randomization was seldom used to protect against systematic errors or to promote valid inferences from experiments; some thought that randomization conflicted with attempts to control variation [25]. She viewed randomization as the distinguishing feature of modern statistical experimental design, and the feature that allowed proper inferences to be drawn from the results.

Cox emphasized the importance of randomization for each case study in the book. In the calculating machine experiment, for example, randomizing the machine order was essential. If the sums of squares for each block of numbers were computed first on machine A and then on machine B, and machine B turned out to be faster, one could not attribute the difference to the machines; it could have occurred because the operator became familiar with the numbers after entering them on the first machine and was able to enter them more quickly on the second. By randomly assigning machine A to be first for five of the blocks and machine B to be first for the other five, Cox could separate out the order effect and conclude that the speed difference was indeed due to the machines [12].

Cox advised the statistician to "use the simplest design that meets the needs of the experiment" [14]. In many of the experiments she consulted on, the simplest design meeting cost constraints needed blocking or other types of restricted randomization, and she and her staff tailored and developed designs for each experiment. From 1942 to 1948, all but 59 of the 6,317 experiments performed at the North Carolina Agricultural Experiment Station involved some form of blocking; 62 percent were randomized complete block designs [14]. She strove to develop ways of conducting experiments "so that the greatest amount of information can be obtained with the least expenditure of time and money" [6].

Experimental Designs is still widely used by persons designing experiments. The many experimental researchers who have recently relied on the book for guidance include Wood and Porter [33], who adopted a Latin square design to study the effects of presenting factual information to persons with strong political views, and Reeves et al. [28], who used a balanced incomplete block design to compare community and hospital eye care for persons with macular degeneration.

Designing the Statistical Profession

Of equal importance to Cox's contributions in designing experiments were her contributions in shaping the discipline of statistics.

Statistical Training

One of her earliest activities in North Carolina was establishing a summer training program in statistics. In the first six-week program, during June and July of 1941, Cox taught beginning and advanced courses on design of experiments, Snedecor taught two courses on applied statistics, and Harold Hotelling taught mathematical statistics; Ronald Fisher had been scheduled to teach, but wartime authorities in London withheld authorization for his travel. During the program, all faculty and staff members were available for individual consultations with students about statistical problems [5, 3, 21]. These courses and three affiliated one-week conferences drew 243 registrants from around the country, including many who were, or were to become, leaders of the statistics profession.

Instruction was not limited to future statisticians. The department taught multiple courses to help state government workers and other persons in the community. Cox, in addition to her administrative and other teaching duties, offered an introductory course on experimental statistics intended for tabulating clerks and computers.

In 1943, Cox intensified the department's efforts to provide trained statisticians to meet wartime and postwar needs. The summer session in 1943 offered "[f]our intensive courses in applied statistics, designed to appeal to young women who are college graduates or advanced undergraduates," including training in sampling methods. Cox said, "This training is offered because of the extreme importance of having efficient workers to help with rush work. There are numerous sampling investigations now in progress, such as those for locating sources and requirements of farm labor as well as those for studying food production and distribution problems. Training in machine problems and statistical and sampling methods is of immediate value in prosecution of the war with limited manpower" [2].

By 1946, statistics in North Carolina had grown under Cox's leadership to include the Department of Experimental Statistics at North Carolina State College and the Department of Mathematical Statistics at the University of North Carolina. Both departments were incorporated in the Institute of Statistics, which Cox directed [27]. After retiring from the university in 1960, Cox led the statistical research division at the newly formed Research Triangle Institute. She continued promoting statistics after her second "retirement" in 1965, traveling around the world to provide statistical advice and help establish statistical programs. Between travels, she served on advisory boards for the US Census Bureau, the Department of Agriculture, the National Science Foundation, and many other organizations.

When Cox established the Department of Experimental Statistics at North Carolina State in 1941, there were only a handful of statistics departments in the world: the first, Karl Pearson's Department of Applied Statistics at University College London, had been established in 1911. In general, mathematical statistics classes were taught in mathematics departments; applied statistics classes were taught in a department of agriculture, psychology, biology, or another discipline. In each, statistics was viewed as a subfield of the discipline where it was taught. Harold Hotelling, who later joined Cox's Institute of Statistics, wrote in 1940 that a great deal of the current knowledge in statistics was still in the

form of oral tradition and "the seeker after truth regarding statistical theory must make his way through or around an enormous amount of trash and downright error. The great accumulation of published writings on statistical theory and methods by authors who have not sufficiently studied the subject is even more dangerous than the classroom teaching by the same people" [23].

Cox insisted that students receive a thorough grounding in mathematical theory and applications of statistics, and that they gain experience in collaborating with scientists. The universities and organizations that consulted her about establishing statistics programs inherited this philosophy, and most statistics departments today are organized around the principles she advocated for training students.

The American Statistical Association's recent guidelines for graduate and undergraduate programs in statistics [8, 9], urging that "graduates should have a solid foundation in statistical theory and methods" as well as experience with collaborating on real problems and designing studies, repeat many of the principles for statistical education that Cox outlined in 1953:

> It is the statistician's duty to keep informed of the rapidly expanding knowledge of statistics and to make such information available to the users of statistics. This combination of a thorough knowledge of statistical theory and method along with adequate competence in the field of application requires that the consultant statistician be a person of substantial ability.... A close integration between theory and applications constitutes the best foundation for important advances in the science of statistics [15].

Statistics as a Collaborative Discipline

Cox held that statistics is by its nature collaborative. Although statisticians engage in a wide range of theoretical and applied investigations, those investigations need to be directed toward "solving problems concerned with decision making" [16]. "The cooperation involved when the statistician consults and works with researchers in other fields is an advantage to all. Also, the consulting or applied statistician in his daily use of statistics encounters new problems which call for help from the theoretical statistician. The theoretical statistician requires the stimulus of practical needs which lead him into useful developments of new techniques" [18].

Equally important was the presentation of results, and a good experimental design leads to clear findings: "close cooperation between the research worker and the statistician before the experiment is started—planning the experiment so that the statistics collected will be easily interpreted by the average reasonably intelligent person" [1].

Cox practiced what she preached. Less than a week after her arrival at North Carolina State College in 1940, "she was out trooping over a soybean field near Raleigh, helping an Experiment Station agronomist work out the best set-up for an experiment. She has also visited several of the test farms for the same purpose" [1]. Throughout her career, she tirelessly promoted statistics around the world, providing expertise and helping develop programs in statistics. Her travels included consultations in Egypt, Thailand, South Africa, Guatemala, Japan, Hong Kong, Lebanon, Malaysia, Brazil, and Honduras.

Cox encouraged collaboration and sound statistical practice in her many presentations at US and international statistical conferences. She also viewed the community as a partner in statistical activity and regularly spoke about statistics and her travels to civic organizations and women's clubs in Raleigh. The *Raleigh News and Observer* reported on many of Cox's local talks. In 1954, for example, they wrote: "A talk by Gertrude Cox, director of Statistics at the University of North Carolina, was a highlight of last night's meeting of the Lewis school PTA." Her talk was followed by a presentation of a minuet from Mrs. Hicks's fourth grade class [4].

Cox provided statistical expertise locally, as well as internationally, throughout her years in Raleigh. In 1975, for example, she was asked to evaluate a controversial statistical investigation on the effectiveness of kindergarten in North Carolina. The investigators had selected 18 schools for the assessment but had omitted one school—whose results would have changed the conclusions—from the analysis. Cox's primary recommendation was that analysts should not be selective in choosing data to be analyzed unless there is justification, and she argued that the small sample size and possible selection biases made it difficult to draw clear conclusions from the study. She concluded that the investigators "could use a great deal more help from qualified statisticians" [7]—a gentle way of saying that the controversy could have been avoided if the investigators had consulted a statistician before conducting the study.

Computation and Statistics

Cox, well aware of the importance of computation to the field of statistics, established a computing laboratory soon after moving to North Carolina. The laboratory performed computations for statistical analyses as well as for other units on campus. During World War II, the department offered classes to train women as computers for the war effort.

Perhaps because of her work as a human computer, Cox was one of the first persons in statistics to embrace the ability of "electronic computing machines" to contribute to the discipline. She immediately saw their use for regression problems and computing standard errors for complex sampling designs, and she forecast that they would soon allow statisticians to "open up even wider frontiers" in statistics [16].

Not surprisingly, Cox's department was one of the first in the country to acquire one of the new IBM-650 electronic computing machines, in 1956 [27]. Computations for large regression models could now be done in less than

20 minutes rather than taking weeks. Some of the earliest computer programs for regression and analysis of variance were written at North Carolina State College [22].

Cox's interest in computational issues continued well after her retirement from the university. In the early 1970s, she provided expertise to the Department of Health, Education, and Welfare on statistical, computational, and privacy issues relating to the proposed use of the Social Security number as a universal personal identifier and, more generally, to the large amounts of personal data that were being collected in computer-based record-keeping systems. The 1973 report of the Advisory Committee on Automated Personal Data Systems set forth principles—the Code of Fair Information Practices—that became the foundation of subsequent US privacy legislation [32, 31]. The report's recommendations reflected Cox's strong views that an individual has a right to know how his or her data are being used.

Statistical Frontiers

Cox summarized her vision for statistics in her 1956 address as President of the American Statistical Association, titled "Statistical Frontiers." She invited the audience to tour the three major continents of the statistical universe: "(1) descriptive statistics, (2) design of experiments and investigations, and (3) analysis and theory" [16]. As she visited each continent, she briefly described some of the "well developed countries" where statisticians have developed many techniques for design and analysis, and she then gave examples of frontiers needing more exploration.

The descriptive statistics continent, although having the longest history of exploration, nevertheless had multiple frontiers. Cox noted that although statistical tabulations were common, too few persons described the variability of a population or the uncertainty of an estimate. She also emphasized the statistician's contributions to the presentation of results country, where "you will be asked to swear allegiance to logical organization, preciseness, and ease of comprehension" [16, p. 3].

The longest sojourn of the tour, not surprisingly, was in the design of experiments and investigations (sampling) continent. Cox foresaw the survey sampling research problems that would arise in future decades, such as the need for statistical methods to assess and control nonsampling errors, and she anticipated the development of computer-intensive methods for estimating variances [24].

While visiting the analysis and theory continent, Cox mentioned some of the frontiers of the late 1950s such as variance component models and nonparametric methods. She also discerned the fundamental problems of inference facing future statisticians in these general frontiers. The methods of statistical inference that work for data from a designed experiment or carefully collected probability sample do not necessarily apply to data that happen to be conveniently at hand. She wrote, "How far are we justified in using statistical methods based on probability theory for the analysis of nonexperimental data? Much of the data used in the descriptive methods continent are observational or nonexperimental records" [16].

Cox's comments are relevant to many of today's frontiers in statistics. One frontier in 2019 concerns making inferences from large observational data sets such as credit card transactions, electronic medical records, sensor data, or internet activity. Statistics from "big data" are often presented without any measures of uncertainty.

Participants in the 2017 National Academies of Sciences workshop on "Refining the Concept of Scientific Inference When Working with Big Data" echoed Cox's views on the need for statistical collaboration, carefully designed experiments, and appropriate statistical inference. In their report they wrote:

- "[T]oo often statisticians become involved in scientific research projects only after experiments have been designed and data collected. Inadequate involvement of statisticians in such 'upstream' activities can negatively impact 'downstream' inference, owing to suboptimal collection of information necessary for reliable inference" [26, p. 5].
- "[B]igger data does not necessarily lead to better inferences," in part "because a lot of big data is collected opportunistically instead of through randomized experiments or probability samples designed specifically for the inference task at hand" [26, p. 14].
- "Without careful consideration of the suitability of both available data and the statistical models applied, analysis of big data may result in misleading correlations and false discoveries, which can potentially undermine confidence in scientific research if the results are not reproducible" [26, p. 1].

Most of Cox's views on statistics do not seem revolutionary to a statistician in 2019. That is because Cox helped define the profession of statistics from her entrance in the 1920s until her death in 1978. Her vision of the statistician as a partner in science—who collaborates on designing and analyzing studies, and who can develop new statistical theory as needed—characterizes the discipline today. She promoted sound statistical practice in the department and institutes she founded, in the community, and around the world.

As she said in 1940, "There is fascination about experimental work. In searching the unknown for new truths, there is mystery, and there is adventure, and there is the thrill of discovery" [1].

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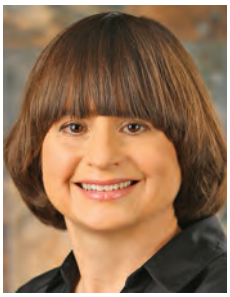

Sharon L. Lohr

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The Life and Pioneering Contributions of an African American Centenarian: Mathematician Katherine G. Johnson

Katherine G. Johnson, in her late nineties.

Johnny L. Houston

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Communicated by Notices *Associate Editor Robin Wilson.*

I first met mathematician Katherine G. Johnson when I presented her with the National Association of Mathematicians (NAM) Distinguished Service Award at NAM's Regional Conference in Norfolk, Virgina in 1996. The award celebrated her more than 50 years as a productive mathematician, most of these years having been spent with the National Aeronautics and Space Administration (NASA).

Given her necessary security clearance, NAM did not probe Johnson about the nature of her work. For an African American mathematician to have worked at NASA from the 1950s well into the 1980s was itself historic.

The full and extraordinary story of her life and pioneering contributions was revealed to the world in the *New York Times* bestseller *Hidden Figures*, published in 2016 and written by Margot Lee Shetterly. However, Johnson's contributions became best known after the Oscar®-nominated movie *Hidden Figures* was released in December 2016. I was highly impressed with Johnson from what I learned from both the book and the movie. She was a pioneer extraordinaire and a brilliant mathematician. Her work impacted the success of NASA's early space flight missions. I was extremely delighted that in her lifetime she has received the awards, honors, and recognitions that many pioneers never live long enough to witness and enjoy. I found her life story itself to be a fascinating one to know.

Katherine Coleman was born on August 26, 1918 in White Sulphur Springs (Greenbrier County), West Virginia as the fourth and youngest child of Joshua and Joylette Coleman. She had a brother Horace (b. 1912), a sister Margaret (b. 1913), and a brother Charles (b. 1915). Her mother was a schoolteacher and her father was a lumberman, farmer, and handyman who worked at the Greenbrier Hotel.

At an early age (her third birthday or younger) she began to speak very articulately and was very curious about knowing details of everyday things that she observed. Her father, with only a sixth-grade education, had an incredible ability for doing math problems. For Katherine, he was the smartest person she knew. She started to count everything she saw and attempted to emulate her father in solving math problems. For Katherine, counting things and constantly learning new information about things became her favorite daily activity. Her mother being a nurturing teacher and her father being a math whiz kept her motivated to learn. Katherine officially began attending elementary school at the age of five.

However, because of what she had learned prior to that age, she was placed into the second grade during her first year of school. When she was eight years old she should have entered the fifth grade but, being such an advanced student, she was placed in the sixth grade of a newly opened school for Blacks. With her advanced placement, she was now a grade ahead of her brother Charles who was three years older than she was. At age ten, Katherine was ready to enter high school. She was viewed by many as a child prodigy. Her father instilled in her that she was as good as anyone and could achieve whatever she desired, but she was never to think that she was better than others.

White Sulphur Springs had no high school for Black children. Because the Coleman family valued education highly and was determined that their children should have a quality education through high school and college, her parents rented a house in Institute, Kanawha County, West Virginia where their children attended high school and college. Thus, every autumn for eight years, Katherine's mother moved with her children to the rented home. In the summer they would return some 125 miles back to White Sulphur Springs where her father lived in their *home house* and worked, primarily as a farmer and at a hotel, earning about $100 per month. All four Coleman children completed high school and college under this arrangement of living in two different places during the year.

Katherine entered West Virginia State College High School before her teens and graduated at the age of 14. In high school, she excelled in mathematics, science, and English. In high school she also developed some affinity for astronomy. This was where she met another person who greatly influenced her love for math: Angie Turner King, who taught her geometry in high school. King later taught her math in college and continued to encourage her.

Katherine entered West Virginia State College (WVSC), a Historically Black College (HBCU), in her early teens. As a student at WVSC, she took every math course offered by the college. Several professors mentored her math studies, including chemist and mathematician Angie Turner King, who had also taught her in high school. Katherine said that King was "...a wonderful teacher—bright, caring, and very rigorous." James Carmichael Evans, who had BS and MS degrees from the Massachusetts Institute of Technology, also nurtured Katherine in her study of math. He was a very talented and encouraging teacher who insisted that she must major in mathematics, even though he knew of her strong interests and mentoring in French and English by others. And there was W. W. Schieffelin Claytor, the third African American to receive a PhD degree in math who took Katherine "under his wing." He was a brilliant teacher and researcher. Claytor not only taught her many of her math classes, but he also added new math courses to the curriculum just for Katherine. She recalled that Claytor told her, "You would make a good research mathematician" (after her sophomore year), and he continued, "I am going to prepare you for that career." According to a videotaped interview with Katherine, one of the courses Claytor created for her was analytic geometry, which was invaluable to her in her work at NASA. She was very fortunate to have had Claytor as a teacher. He only taught at West Virginia State College from 1934 to 1937. Katherine graduated from WVSC summa cum laude at the age of 18 in 1937 with degrees in mathematics and French; she had joined Alpha Kappa Alpha sorority while a student.

After graduation from college, she took a teaching job at a Black public school in Marion, Virginia. She was offered the job in Marion because she could teach math, teach French, and play the piano. In 1939, Katherine married James Francis Goble, who was called "Jimmy" by his friends. He worked as a high school chemistry teacher in Marion. This marriage produced three daughters: Constance, Joylette, and Katherine. All three became mathematicians and teachers.

In 1940 (before having children), Katherine enrolled in a graduate math program. She entered the graduate program at West Virginia University in Morgantown, West Virginia, the flagship university for the state of West Virginia that had been reserved for White students only. She was the first African American woman to attend the university's graduate school. This was facilitated with the courage of and assistance from WVSC's president, Dr. John W. Davis. He selected her as one of three African American students (she was the only female) to integrate the graduate school after the United States Supreme Court ruling Missouri *ex rel.* Gaines v. Canada (1938). The court ruled that states that provided public higher education for White students also had to provide it for Black students, to be satisfied either by establishing Black colleges and universities or by admitting Black students to previously White-only universities. Katherine spent a term at the University but left the program after she became pregnant. She chose to give priority, at that time, to raising a family. Jimmy and her parents supported her decision.

Katherine returned to teaching when her three daughters grew older. She taught in Morgantown and Bluefield, West Virginia. However, it was not until 1952 when a relative told her about open positions in mathematics at the all-Black West Area Computing Section at the National Advisory Committee for Aeronautics (NACA), Langley Laboratory, Hampton, Virginia that she desired a different use of her mathematical talent. The program was headed by Dorothy Vaughan, whom she had met some years earlier in West Virginia. Katherine and her husband, Jimmy, decided to move the family to Newport News, Virginia to pursue this opportunity. The NACA had stopped hiring in 1952 when they arrived and she worked as a substitute teacher for a year. Katherine was hired by Langley the next year and began work there in the summer of 1953. Just two weeks into Katherine's tenure in the office, Dorothy Vaughan assigned her to a project in the Maneuver Loads Branch of the Flight Research Division. Katherine's temporary position with the previously all White research team soon became permanent. She spent the next four years analyzing data from flight tests and worked on the investigation of a plane crash caused by wake turbulence. As she was completing this work, her husband Jimmy died from a serious medical challenge in December 1956.

Katherine sang in the choir at Carver Memorial Presbyterian Church in Newport News, Virginia for 50 years. The minister there introduced James A. Johnson to her. He had been commissioned in 1951 as a Second Lieutenant in the United States Army and was a veteran of the Korean War. In 1959, the two married. Katherine had no additional children with her second husband.

Both in the West Area Computing Section and in the Flight Research Division, Katherine worked as a "human computer," doing the complex math calculations for airplanes and space flights. NACA disbanded the "Colored

Katherine G. Johnson working at NASA in the 1960s as a "human computer," physicist, and aerospace technologist in an all-White research division of engineers.

Human Computers Group" in 1958 when it was superseded by NASA, which adopted digital computers. In the Research Flight Division, where she was the only Black, all the Whites were hired as engineers and Katherine was considered a "human computer," a mathematician, a physicist, and an aerospace technologist. During the NACA era, Katherine had to leave the Research Flight Division and go back to the West Area Colored Section to use the restroom, eat, or retrieve something out of her locker. Her questions about her daily inconveniences had a great impact on persuading NASA to eradicate its segregated facilities in the early 1960s. At NASA, she fulfilled Claytor's prophecy and vision. She became a world-class research mathematician on the stage of the largest grand challenge problem of the time: successfully conquering the frontier of space flights to other celestial bodies in space. Katherine G. Johnson made many pioneering contributions on this grand challenge stage. For the sake of brevity of this document, only 15 will be listed.

Katherine G. Johnson's primary contributions at NASA were in computational science and research. In 2017 NASA opened and named in Johnson's honor the above state-of-the-art 40,000 square-foot Computational Research Facility at NASA Langley in Hampton, Virginia.

Fifteen of Katherine G. Johnson's Major Pioneering Contributions to Space Flight History:

A. Katherine Johnson was the first African American and the first woman to work in NASA's Research Flight Division.

B. She was the first African American and the first woman to attend NASA's Research Test Flight Briefings where the fundamental problems of a space flight mission were presented, discussed, and analyzed; she specifically requested to be able attend, and they honored her requests.

C. She was the first African American and first woman to have her name placed on a Scientific Report at NASA; however, she actually did major work on many earlier reports for which she received no written credit or recognition in the report itself. The first report with Katherine's name on it was major for NASA [8]. It contained the theory necessary for launching, tracking, and returning space vehicles and was used for the famous space flight by Alan Shepard in May 1961 and the flight of John Glenn in February 1962.

D. Currently, there are more than twenty five scientific reports in the NASA archive in space flight history that Katherine authored or co-authored, the largest number by any African American or woman.

E. From 1958 until her retirement in 1986, Johnson worked as an aerospace technologist in the Spacecraft Controls Branch where all final decisions were made for space travel; she served as NASA's premier research mathematician at the time.

F. She calculated the trajectory for the May 5, 1961 flight of Alan Shepard, the first American to travel in space.

G. She also calculated the launch window for Shepard's 1961 Mercury mission.

H. She plotted backup navigation charts for astronauts in case of electronic failures.

I. When NASA used electronic computers for the first time to calculate John Glenn's orbit around the Earth, NASA's officials called on Johnson to verify the computer's numbers. Glenn specifically asked for Johnson's verifications, and he refused to fly unless she verified the calculations. These were very difficult calculations; they had to account for the gravitational pulls of celestial bodies.

J. As NASA began relying heavily on digital computers, they used Johnson's calculations to help them check the accuracy of the computers; her validations caused NASA to establish confidence in the new digital computer technology.

K. In 1961, NASA used Johnson's calculations of trajectories to help to ensure that Alan Shepard's Freedom 7 Mercury capsule would be found quickly after landing.

L. Johnson also helped to calculate the trajectory for the 1969 Apollo 11 flight to the Moon.

M. In 1970, Johnson worked on the Apollo 13 moon mission; her work on backup procedures and charts helped set a safe path for the crew's return to Earth.

N. In case of malfunctioning, Johnson had helped to create a one-star observation system that would allow astronauts to determine their location with accuracy.

O. Later in her career, Johnson worked on the Space Shuttle Program, the Earth Resources Satellite, and on plans for a mission to Mars.

In recognition of her life and contributions as a role model, a scholar, an educator, and her pioneering career as a research mathematician with NASA in space travel, Johnson has received many awards, honors, and recognitions. For the sake of brevity, only 20 will be listed.

Twenty of Katherine G. Johnson's Awards, Honors, and Recognitions:

A. 2019 (January 18) the National Association of Mathematicians, NAM's Centenarian Award

B. 2018 (August 25) West Virginia University, Morgantown, unveiled a life-size bronze statue of Katherine Johnson on campus and established a STEM scholarship in her name

C. 2018 (May 12) College of William and Mary awarded her an Honorary Doctorate Degree

D. 2017 (September 22) The Katherine G. Johnson Computational Research Facility at NASA Langley in Hampton, Virginia opened and was named in her honor (40,000 sq. feet)

E. 2017 received Daughters of the American Revolution Medal of Honor

F. 2016 Oscar®-nominated movie *Hidden Figures* profiled her life as a "colored human computer" and a research mathematician at NASA

G. 2016 received Presidential Honorary Doctorate of Humane Letters from West Virginia University, Morgantown

H. 2016 *New York Times* bestseller *Hidden Figures*, by Margot Lee Shetterly, profiled her life as a scholar, an educator, a "colored human computer," and a research mathematician at NASA

I. 2016 received the Space Flight Industry Silver Snoopy Award from Leland Melvin

J. 2016 received the Astronomical Society of the Pacific's Arthur B. C. Walker II Award

K. 2016 listed as one of the 100 most influential women worldwide by the BBC

L. 2015 received National Center for Women and Information Technology's Pioneer in Tech Award

M. 2015 received the Presidential Medal of Freedom from then president Barack Obama

N. 2014 received the De Pinza Honor from National Women History's Museum

Having recently celebrated her centennial birthday, Katherine Coleman Goble Johnson has lived to receive many awards, honors, and recognitions for her pioneering work. One such award was the Presidential Medal of Freedom from President Barack Obama in 2015.

O. 2012 selected as a Science History Maker (now archived in the Library of Congress)

P. 2010 received an Honorary Doctor of Science from Old Dominion University, Norfolk, Virginia

Q. 2006 received an Honorary Doctor of Science from Capitol University, Laurel, Maryland

R. 1999 selected as West Virginia State College Outstanding Alumnus of the Year

S. 1998 received an Honorary Doctor of Law from SUNY, Farmingdale, New York

T. 1996 received the National Association of Mathematicians Distinguished Service Award

U. 1971, 1980, 1984, 1985, and 1986 received NASA Langley Research Center Special Achievement Award

In Her Own Words: Quotes from Katherine G. Johnson

A. I like to learn. That's an art and a science.

B. Let me do it. You tell me when you want it and where you want it to land, and I'll do it backwards and tell you when to take off.

C. Girls are capable of doing everything men are capable of doing. Sometimes they have more imagination than men.

D. We will always have STEM with us. Some things will drop out of the public eye and will go away, but there will always be science, engineering, and technology. And there will always be mathematics.

E. I don't have a feeling of inferiority. I never had one. I'm as good as anybody, but not better.

F. Like what you do, and then you will do your best.

On August 26, 2018, Katherine Coleman Goble Johnson completed her 100th trip around the Sun, becoming a highly distinguished centenarian African American mathematician. Katherine G. Johnson lives in Hampton, Virginia. She continues to encourage her grandchildren and students to pursue careers in science, technology, engineering, and mathematics (STEM).

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Johnny L. Houston

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The Mathematics of Grace Murray Hopper

Asher Auel

Grace Murray Hopper (1906–1992) is well known as a pioneering computer scientist and decorated Naval officer.

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Her achievements read as a list of firsts: she was an expert at programming Harvard's Mark I, the first large-scale electromechanical computing machine; she was part of the team who developed the UNIVAC I, the first commercial computer produced in the United States, for which she wrote the first compiler; she created the first English-based data processing language FLOW-MATIC, a principal precursor for COBOL, one of the most important programming languages for business applications; and when she

Figure 1. Grace Hopper at the blackboard with students, 1957.

Figure 2. Grace Hopper teaching a COBOL class, 1961.

Figure 3. Grace Hopper with programmers at the console of UNIVAC I, 1957.

retired from the Navy as a Rear Admiral, at 79 years old, she was the oldest active-duty officer in the entire armed forces. She has been widely lauded for these accomplishments. Named in her honor are: a Naval guided missile destroyer warship; a super computer at the National Energy Research Scientific Computing Center; several buildings and a bridge on Naval bases; a park in Arlington, Virginia; a major yearly convention for women in computer science and technology; several prizes, including an early career award from the Association for Computing Machinery; and a recently renamed residential college at Yale University, among others. Her inspiring story has been the subject of many books and several upcoming film projects.

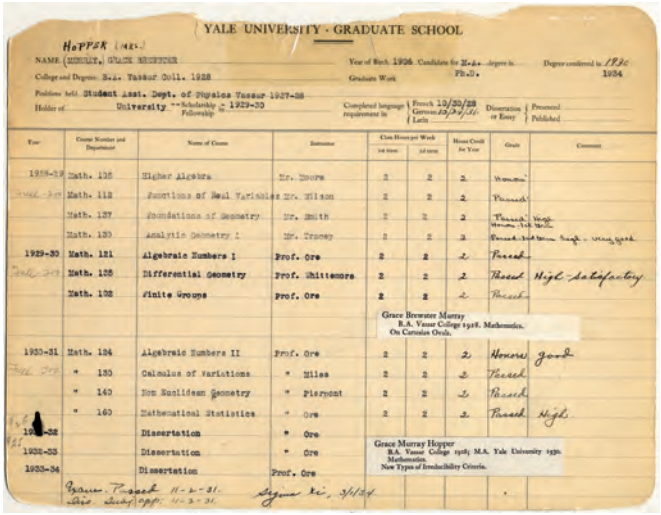
However, what is often overlooked in accounts of Hopper's life and work is her mathematical legacy. The results of her 1934 Yale PhD thesis advised by Øystein Ore (which are detailed in the section "Thesis Work") are never mentioned. Incorrect characterizations of her graduate work abound; her PhD is routinely cited as being in "mathematics and physics" or "mathematical physics" or "under computer pioneer Howard Engstrom." Her training in pure mathematics and her identity as a mathematician are often minimized or treated as a kind of incongruous early chapter in the story of the "Queen of Code."

But Grace Hopper was most certainly a mathematician. Asked in an interview [30, p. 7] later in her career what she would consider herself, she immediately replied: "Mathematician." Then adding wryly: "A rather degraded one now, because I deal with actual digits instead of letters and formulas." Her broad and rigorous mathematical education constituted what she called her "basic thinking." She was, once and forever, a mathematician: "I've been called an engineer, a programmer, systems analyst and everything under the sun but I still think my basic training is mathematics." For the first time, using archival material from Yale University's collections, this article will attempt to illuminate Hopper's foundational mathematical training as well as the specific contributions of her thesis research.

Academic Training

As both an undergraduate and a graduate student, Grace Hopper pursued a mathematical education. In 1928, she earned her BA from Vassar College, with her coursework primarily in mathematics, and secondarily split between economics and physics. She then enrolled as a graduate student in the Department of Mathematics at Yale University, receiving her MA in 1930 with a thesis titled *On Cartesian Ovals* and her PhD in 1934 with a dissertation titled *New Types of Irreducibility Criteria*. Hopper took courses in a wide variety of fields, as her graduate transcript reveals (see Figure 5). Her PhD advisor was Norwegian algebraist Øystein Ore, who had recently been recruited to Yale and "breathed new life into an aging department" [32, p. 10].

Figure 4. Grace Murray's senior portrait from the Vassar yearbook, 1928.



YALE UNIVERSITY · GRADUATE SCHOOL

Year	Course Number and Department	Name of Course	Instructor	Class Hours per Week 1st term	Class Hours per Week 2d term	Hours Credit for Year	Grade	Comment
1928-29	Math. 125	Higher Algebra	Dr. Moore	2	2	2	Honors	
	Math. 112	Functions of Real Variables	Dr. Wilson	2	2	2	Passed	
	Math. 127	Foundations of Geometry	Dr. Smith	2	2	2	Passed	High
	Math. 130	Analytic Geometry I	Dr. Tracey	2	2	2	Passed	
1929-30	Math. 121	Algebraic Numbers I	Prof. Ore	2	2	2	Passed	
	Math. 125	Differential Geometry	Prof. Whittemore	2	2	2	Passed	High-Satisfactory
	Math. 102	Finite Groups	Prof. Ore	2	2	2	Passed	
1930-31	Math. 124	Algebraic Numbers II	Prof. Ore	2	2	2	Honors	Good
	" 130	Calculus of Variations	" Miles	2	2	2	Passed	
	" 140	Non Euclidean Geometry	" Pierpont	2	2	2	Passed	
	" 160	Mathematical Statistics	" Ore	2	2	2	Passed	High
1931-32		Dissertation	" Ore					
1932-33		Dissertation	" Ore					
1933-34		Dissertation	Prof. Ore					

Grace Brewster Murray
B.A. Vassar College 1928. Mathematics.
On Cartesian Ovals.

Grace Murray Hopper
B.A. Vassar College 1928; M.A. Yale University 1930.
Mathematics.
New Types of Irreducibility Criteria.

Figure 5. Grace Murray Hopper's original Yale Graduate School transcript lists: her courses and the grades she received (on an Honors/High Pass/Pass/Fail scale), the dates of her language exams (which years later became fodder for her stories about the interchangeability of written languages [30, pp. 22–23]), as well as her yearly tuition and the date of her election to Sigma Xi, the scientific research honor society. Transcription includes full names of her professors.

Year	Course Name	Instructor
1928–29	Higher Algebra	Lucius Terrell Moore
	Foundations of Real Variables	Wallace Alvin Wilson
	Foundations of Geometry	Percey Franklyn Smith
	Analytic Geometry I	Joshua Irving Tracey
1929–30	Algebraic Numbers I	Øystein Ore
	Differential Geometry	James K. Whittemore
	Finite Groups	Øystein Ore
1930–31	Algebraic Numbers II	Øystein Ore
	Calculus of Variations	Egbert J. Miles
	Non Euclidean Geometry	James P. Pierpont
	Mathematical Statistics	Øystein Ore
1931–32	Dissertation	Øystein Ore
1932–33	Dissertation	Øystein Ore
1933–34	Dissertation	Øystein Ore

Notable on Hopper's transcript are Ore's courses on Algebraic Numbers, which had never been offered until his arrival the previous year [32, p. 10]. Hopper was awarded numerous prestigious dissertation fellowships during her years at Yale and was one of the first dozen women (going back to 1895) to earn doctoral degrees in mathematics from the university, see [16]. In 1931, while still a graduate student, Hopper started a faculty position at Vassar, eventually being promoted to assistant professor in 1939 and associate professor in 1944. During the 1941–42 academic year, Hopper was granted a half-time leave from Vassar to take courses with Richard Courant at New York University's Center for Research and Graduate Education (later to become the Courant Institute of Mathematical Sciences).

Numerous distinguished mathematicians can be counted as Hopper's mentors. At Vassar, she studied with Henry Seely White (1861–1943), a prominent American geometer who received his PhD under Felix Klein in 1891 and served as President of the AMS (1906–1908), and Gertrude Smith (1874–1965), whom Hopper declared "taught the best calculus anybody ever taught" [30, p. 21]. At Yale, she was influenced by James P. Pierpont (1866–1938) and was a close contemporary of Howard Engstrom (1902–1962), who received his PhD under Ore in 1929, five years before Hopper, and who eventually returned to Yale to take up a faculty position. In 1941, Engstrom joined the Navy and did foundational work in cryptography; later, he became a deputy director of the National Security Agency, see [12].

Engstrom encouraged several mathematicians, including Hopper and the famous group theorist Marshall Hall, Jr., to join Naval intelligence during World War II. Hall, who received his PhD under Ore in 1936, recalls that while Ore was his "nominal" advisor, he received "far more help and direction" from Engstrom, see [18]. In interviews over the years, Hopper repeatedly describes Engstrom as one of her "instructors." Though no course with Engstrom is listed on Hopper's transcript, his mentorship seems to have been as important for Hopper as it was for Hall. What is clear from the historical record is that Hopper did not "receive her PhD under Engstrom" as several authors have claimed (see [16]), perhaps in an effort to link the early histories of two pioneers in the field of computers. When Hopper enlisted in the Navy, she expected to be assigned to the Communications Supplementary Activity (Navy Communications Annex) in Washington, DC, where Engstrom led a top-secret team building cryptographic computing machines. Though she was eventually assigned to work on the Mark I at Harvard, Hopper and Engstrom stayed life-long friends.

Several of Ore's graduate students from the early 1930s tackled similar research problems on generalized irreducibility criteria for polynomials. His two male students, Harold Dorwart (PhD 1931) and Casper Shanok (PhD 1933), produced dissertations very close in subject to Hopper's work. Dorwart published a number of articles from his thesis work: in the *Annals of Mathematics* [7], in the *Duke Mathematical Journal* [8], and a survey in the *American Mathematical Monthly* [9, p. 373] that mentions and cites Hopper. Almost immediately following his graduation, Shanok's thesis [45] was published in the *Duke Mathematical Journal* (though he did not appear to continue in academia). Distressingly, Ore's two female graduate students, Hopper and Miriam Becker (PhD 1934), never published their thesis work at all. Both would, however, go on to long careers (Becker would eventually join the faculty of the City University of New York), even if their earliest work still remains unknown.

As a junior faculty member at Vassar, Hopper was given "all the courses nobody else wanted to teach." But she was such an innovative teacher that classes like technical drawing, trigonometry, calculus, probability, and finite difference method for numerical solutions of differential equations were suddenly popular [30, pp. 16–21]. On top of her demanding teaching schedule of five or six courses, she also audited two courses per year, including basic astronomy, statistical astronomy, geology, philosophy, bacteriology, biology, zoology, plant horticulture, chemistry, physics, economics, and architecture. She also took a course on cryptography sponsored by the Navy [30, p. 27].

Later in life, Hopper would reflect on the "inestimable value" of her broad education as she shaped the new field of computers [30, p. 17]. For example, it was in a chemistry course when she learned the essential concepts of round-off and truncation errors [27, p. 46]. Her years teaching technical drawing courses enabled her to invent a new method for diagramming the relay timing and associated circuitry (see Figure 6) for the Mark I (formally known as the Automatic Sequence Controlled Calculator) control manual [22], see [30, pp. 32–33]. Hopper summed it up quite neatly in a 1986 interview on *The Late Show with David Letterman* [21]. During a discussion of her Mark I days, Letterman asked, "Now, how did you know so much about computers then?" "I didn't," Hopper immediately replied, with some bemusement. "It was the first one."

But arguably, it was studying and teaching mathematics —thinking about symbolic language and how to communicate meaning with symbols—that was most pivotal in Hopper's early work on computers. Her invention of various types of early compilers enabled the translation of mathematical statements or English words into computer code.

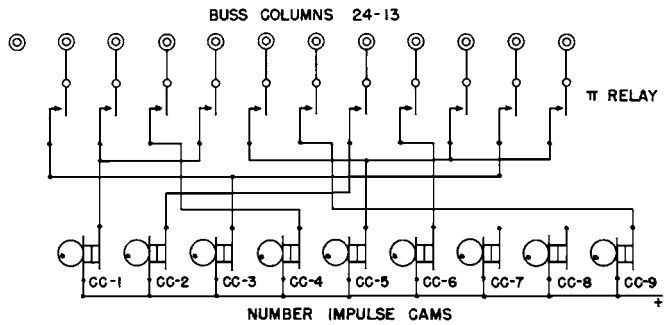


Figure 6. Wiring diagram of a Mark I table relay encoding π [22, p. 91].

> Manipulating symbols was fine for mathematicians but it was no good for data processors who were not symbol manipulators. Very few people are really symbol manipulators. If they are they become professional mathematicians, not data processors. It's much easier for most people to write an English statement than it is to use symbols. So I decided data processors ought to be able to write their programs in English, and the computers would translate them into machine code. [13, p. 3]

One of Hopper's most academically rewarding experiences was taking courses from Richard Courant at New York University in 1941–1942, during her half-time leave funded by a Vassar Faculty Fellowship. Hopper found Courant to be "one of the most delightful people to study with I've ever known in my life." It was, she recalled, "a perfectly gorgeous year. Of course, he scolded me at intervals, just as all of the others did because I kept doing unorthodox things and wanting to tackle unorthodox problems" [30, p. 28]. While there, she studied calculus of variations, differential geometry, and perhaps most fortuitously, she took a government-sponsored defense training course on methods of solutions to partial differential equations involving finite differences taught by Courant, see [30, p. 24]. Hopper later learned that her involvement in this course was in her Navy file and was one of the determining factors in her initial assignment: to program Harvard's Mark I, implementing calculations for the war effort including some for John von Neumann's work on the Manhattan Project.

The attack on Pearl Harbor, which took place during her year studying with Courant, forever changed the direction of Hopper's life. Her great grandfather had been in the Navy, and by the summer of 1942, many of Hopper's family members were joining the armed services: her husband (from whom she was already separated) and brother volunteered for the draft; her female cousins joined through the Women's Army Corps (WAC) and the Navy's Women Accepted for Volunteer Emergency Service (WAVES) program; her mother served on the Ration Board; and her retired father went back to work and served on the local Draft

Figure 7. Grace Hopper standing behind a car parked near Cruft Lab, Harvard University, ca. 1945–1947.

Board, see [47, p. 20]. Hopper was eager to enlist in the Navy, but was rejected when she failed to meet the minimum weight requirement for her height and was considered too old for enlistment. In the meantime, she taught an accelerated summer calculus course at Barnard College for women training for war-related posts. But her profession was also an impediment.

> Mathematicians were [in] an essential industry and you could not leave your job to go in the services without permission [from both the Navy and one's employer]. You couldn't even transfer jobs without permission... And I was beginning to feel pretty isolated sitting up there, the comfortable college professor—all I was doing was more teaching, and I wanted very badly to get in and so I finally gave Vassar an ultimatum that if they wouldn't release me I would stay out of work for six months because I was going into the Navy, period. [30, p. 25]

Eventually, she obtained a waiver for the weight requirement and a leave of absence from Vassar, and trained at the Naval Reserve Midshipmen's School at Smith College in Northampton, Massachusetts in the spring of 1944. After graduating first in her class, she was commissioned lieutenant junior grade.

On July 2, 1944, Hopper reported for duty at the Bureau of Ships Computation Project at Harvard under the command of Howard Aiken, and began work on the Mark I. Aside from programming the Mark I, and its successor, the Mark II, she was assigned the job of compiling notes about the operation of the Mark I into a book [22]. Hopper

Figure 8. Captain Grace Hopper, ca. 1975.

edited the volume and wrote several of its sections, including an introduction containing the first ever scholarly account of the history and development of calculating machines [22, Chapter I]. "Nobody had done this before," Hopper later said. "[The] history of computers had never been put together." It was, to use her words, "really a job" [30, p. 32].

Thesis Work

Grace Hopper's PhD thesis work with Øystein Ore concerned irreducibility criteria for univariate polynomials over the field of rational numbers. Though her work was never published, it was presented to an American Mathematical Society meeting on March 30, 1934 in New York with an abstract appearing in the *Bulletin of the AMS* [20]. The only apparent extant text of her thesis [19] remains in Yale's archives, and a detailed account of her mathematical work has never before appeared in the literature.

In this section, we provide an explanation of Grace Hopper's thesis work, the central theme of which concerns necessary conditions for the irreducibility of univariate polynomials with rational coefficients based on their Newton

polygons, see the subsection "Irreducibility via Newton polygons." The connection between the decomposability of polynomials and the slopes of their Newton polygons was initiated at the turn of the twentieth century by Dumas [10], with further refinements by Kürschak [26], Ore [35], and Rella [41]. In her work, Hopper obtains new irreducibility criteria by considering an Archimedean analogue of the Newton polygon, see the subsection "Archimedean Newton polygon." While this Archimedean Newton polygon dates back at least to an 1893 paper of Hadamard [17, §4, p.174], and was later developed further in a 1940 paper of Ostrowski [36, pp. 106, 132] and by Valiron [46, Ch. IX, pp. 193–202], its use for establishing irreducibility criteria seems to be a novel feature of Hopper's work.

Irreducibility of polynomials. A nonconstant polynomial

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

whose coefficients $a_0, \ldots, a_n$ are rational numbers is called *irreducible* if there is no way to write $f(x)$ as a product $f(x) = g(x)h(x)$ where $g(x)$ and $h(x)$ are themselves nonconstant polynomials with rational coefficients. The study of irreducible polynomials is one of the foundations of modern field theory and often involves quite a bit of number theory as well.

For example, that $x^2 - 2$ is irreducible is equivalent to the classical fact that $\sqrt{2}$ is irrational. More generally, a quadratic polynomial

$$ax^2 + bx + c$$

is irreducible if and only if its *discriminant* $b^2 - 4ac$, which appears in the quadratic formula, is not a square.

When p is an odd prime number, that the cyclotomic polynomial

$$\Phi_p(x) = x^{p-1} + x^{p-2} + \cdots + x + 1$$

is irreducible was first proved by Gauss in *Disquisitiones Arithmeticae* and is related to the arithmetic of the pth roots of unity $e^{2\pi i k/p}$ and the (non-)constructibility of the regular p-gon with compass and straightedge.

In 1929, Schur [44] proved that for $n \geq 1$, the truncated exponential series

$$1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$$

is irreducible with an argument that used a generalization of Bertrand's Postulate, whose original statement—that for any positive integer k there exists a prime number p such that $k < p \leq 2k$—was conjectured by Bertrand and proved by Chebyshev. This result, and its generalizations, implies the irreducibility of various families of orthogonal polynomials, such as those of Laguerre and Hermite type, see [9, §4].

Several standard methods for testing irreducibility are taught in a basic course on field and Galois theory. The most elementary are reduction modulo a prime number and the "rational root test." A more powerful, and yet easy to use, tool is *Eisenstein's criterion*: assuming that $f(x)$ has integer coefficients, if for some prime number p, the coefficients satisfy $p|a_i$ for all $i \neq n$, as well as $p \nmid a_n$ and $p^2 \nmid a_0$, then $f(x)$ is irreducible. In fact, a statement equivalent to Eisenstein's Criterion was first proved by Schönemann [43] in a 1846 paper that Eisenstein even cites in his own paper [11] in 1850, hence the criterion was often called the Schönemann–Eisenstein theorem in literature from the early twentieth century, see [5] for a discussion.

Irreducibility via Newton polygons. In the late nineteenth century and early twentieth century, various generalizations of the Eisenstein criterion, depending on the divisibility properties of the coefficients of $f(x)$, appeared in work of Königsberger, Netto, Bauer, Perron, Ore, and Kahan. Finally, these were all mostly subsumed by an observation of Dumas [10], that such criteria could be rephrased in terms of the irreducibility of the Newton polygon associated to $f(x)$. This history is very well summarized in the historical introduction to Hopper's thesis [19, Chapter I] and in Dorwart's survery article [9].

Given a prime number p, we consider the p-adic valuation ν_p on $\mathbb{Q}$. The *Newton polygon* $N_p(f)$ of the polynomial $f(x) = \sum_i a_i x^i \in \mathbb{Q}[x]$ with respect to p is the lower convex hull of the points $(i, \nu_p(a_i))$ in $\mathbb{R}^2$. We assume that $a_0 \neq 0$. If $a_i = 0$ for some $i \geq 0$ then by definition $\nu_p(a_i) = +\infty$, hence for the purposes of taking the lower convex hull, we can ignore such zero coefficients. Intuitively, we can imagine a large rubber band surrounding these points in $\mathbb{R}^2$, which each have small nails sticking up from them; as we stretch the rubber band up toward $+\infty$, we obtain the Newton polygon as the lower sequence of line segments formed by the stretched rubber band.

The central insight of Dumas [10, p. 217] is that the Newton polygon $N_p(g \cdot h)$ of the product of polynomials $g(x)$ and $h(x)$ is formed by composing the line segments of the Newton polygons $N_p(g)$ and $N_p(h)$ in order of increasing slope, an operation that we could denote $N_p(g) \circ N_p(h)$ and call the *Dumas sum*. This was generalized in [3] and [6], and by many later authors, including to the more general context of (multivariate) polynomials over valued fields.

If the projections to the x- and y-axes of the line segments of the Newton polygon of $f(x)$ are denoted $l_1, \ldots, l_r$ and $k_1, \ldots, k_r$, respectively, we denote by $e_i = \gcd(l_i, k_i)$ and write $l_i = e_i \lambda_i$. Then Dumas [10, p. 237] deduces a general irreducibility criterion: $f(x)$ can only have factors

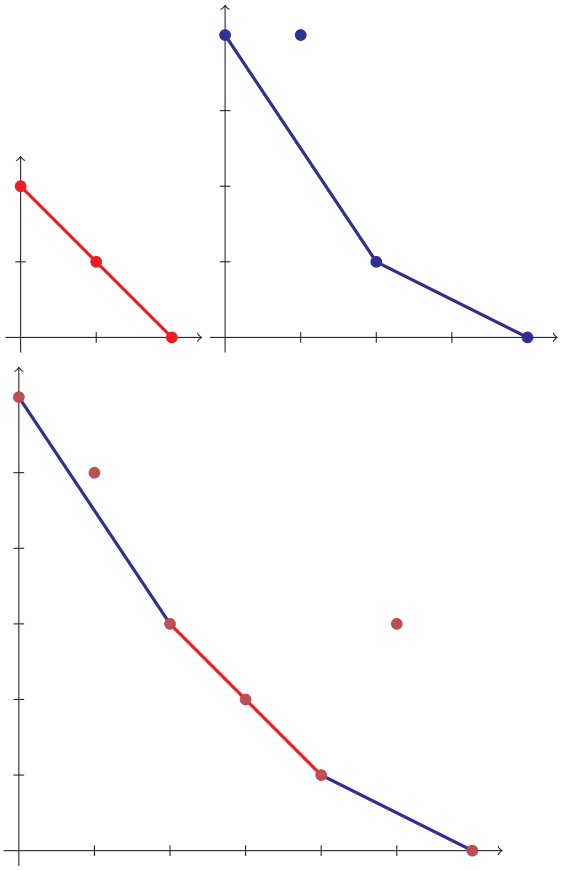


Figure 9. The Newton polygon (with $p = 2$) of $g(x) = x^2 + 6x + 4$ in red, of $h(x) = x^4 + 2x^3 + 10x^2 + 48x + 16$ in blue, and $g(x)h(x) = x^6 + 8x^5 + 26x^4 + 116x^3 + 344x^2 + 288x + 64$, with sides colored appropriately showing the Dumas sum.

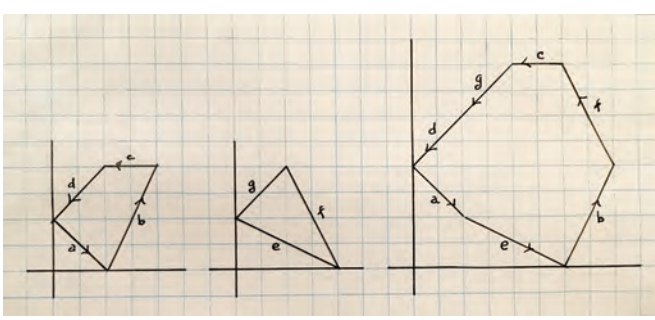


Figure 10. Graphical depiction of the Minkowski sum of two classical Newton polygons of bivariate polynomials. Hand drawn by Grace Hopper [19, p. 24].

of degree m that can be expressed in the form

$$m = \sum_{i=1}^{r} \mu_i \lambda_i$$

where $\mu_i \in \{0, 1, \dots, e_i\}$ for each $1 \le i \le r$.

For example, if $N_p(f)$ consists of a single line segment that does not pass through any lattice point in the plane, then $f(x)$ is irreducible. This immediately gives the Eisenstein criterion. Generalizations and refinements of this idea were developed by Fürtwangler, Kürschak, and Ore, see [19, Chapter I, §4].

The classical Newton polygon associated to a bivariate polynomial $f(x, y)$ over a field, defined as the convex hull of the weight vectors (i, j) in $\mathbb{R}^2$ of all monomials $x^i y^j$ appearing with nonzero coefficients in $f(x, y)$, first appears in a 1676 letter from Newton to Oldenberg [23] and was well known to Newton and his followers throughout the 18th and 19th century, cf. [4, Chapter XXX, §24, Historical Note]. Though it must have been well known, the observation that the classical Newton polygon of a product of polynomials is the Minkowski sum (see Figure 10) of their classical Newton polygons does not seem to be clearly enunciated in the literature until the theses of Shanok [45, §2, p. 103, footnote 3] and Hopper [19, Chapter II, §1].

Archimedean Newton polygon. A completely different type of irreducibility criterion depending on the relative magnitudes of the absolute values of the coefficients was introduced by Perron [40]. (We now assume that $f(x)$ is a monic polynomial with coefficients in $\mathbb{Z}$.) These criteria depend on the following simple observation: if $n - 1$ of the (complex) roots of $f(x)$ have absolute value < 1, then $f(x)$ is irreducible. Indeed, if $f(x)$ has a nonconstant factor (which by Gauss's Lemma can be taken to be monic with integer coefficients), then all of its roots will have absolute value < 1, but their product is the (integral) constant term, a contradiction. The resulting irreducibility criterion is, letting $A = |a_0| + \cdots + |a_{n-1}| + 1$: if the coefficients satisfy $|a_{n-1}| > \frac{1}{2}A$, then $f(x)$ is irreducible. There is a similar criterion if all but a pair of complex conjugate roots have absolute value < 1.

To take into account the relative magnitudes of the coefficients, Hopper [19, Chapter III] considers an *Archimedean Newton polygon* associated to a polynomial $f(x)$ with complex coefficients. Define $N_\infty(f)$ to be the lower convex hull of the set of points $(i, -\log|a_i|)$ in $\mathbb{R}^2$. As before, if $a_i = 0$ for some $0 < i < n$, then $-\log|a_i| = +\infty$, so can be ignored for the purposes of taking the lower convex hull. (In fact, Hopper defines the mirror image of this polygon.) This is a natural generalization of the Newton polygon with respect to a prime p considered above. Indeed, the negative absolute logarithm can be considered as an

Archimedean analogue of a valuation; writing $\nu_\infty(x) = -\log|x|$, then $|x| = e^{-\nu_\infty(x)}$ is in analogy with the non-Archimedean p-adic absolute value $|x|_p = p^{-\nu_p(x)}$.

Later in the twentieth century, the Archimedean Newton polygon was, in various guises, used in a variety of contexts, including: by Khovansky [25] (cf. [42]) in an algebraic reformulation of his study of exponential equations and eventually for combinatorial invariants attached to divisors on algebraic varieties; by Mueller and Schmidt [33], [34] for bounding the number of solutions to Thue equations; and by Passare and his collaborators (see e.g., [38], [37, §2.1], [1]) and Mikhalkin (see e.g., [39]) in the theory of amoebas and in tropical geometry. The genesis of the Archimedean Newton polygon going back to Hadamard, as well as most of these later uses, stems from the fact that its geometry is related to the absolute values of the roots of the polynomial.

Taking a different approach, Hopper [19, Chapter III] studies the Archimedean Newton polygon of a product of polynomials, in analogy with Dumas's result in the non-Archimedean case: how do $N_\infty(g)$ and $N_\infty(h)$ compare with $N_\infty(g \cdot h)$? Hopper remarks that if the analogue of Dumas's product result held for N_∞, then irreducibility criteria such as Perron's, which depend on the relative magnitude of the coefficients, would follow immediately. However, $-\log|x|$ is not a valuation as there is an error term in relating $-\log|x+y|$ with $\min(-\log|x|, -\log|y|)$, hence such an exact product formula is not expected. However, Hopper goes on to prove bounds on how far apart $N_\infty(g \cdot h)$ can be from $N_\infty(g) \circ N_\infty(h)$. To state these bounds, if $f(x) \in \mathbb{C}[x]$ is a polynomial of degree $n \geq 1$, we consider $N_\infty(f)$ as a piecewise-linear function of t on the real interval $[0, n]$.

Theorem 1 (Hopper [19, Chapter III, §3–5, pp. 33–38]). *Let $g(x), h(x) \in \mathbb{C}[x]$ be monic polynomials and $n = \deg(g) + \deg(h)$. Then*

$$\begin{aligned} -\log(1+\tfrac{n}{2}) &\leq (N_\infty(g \cdot h)(t) - (N_\infty(g) \circ N_\infty(h))(t) \\ &\leq \log(3 \cdot 2^{t(n-t)}) \end{aligned}$$

for all $t \in [0, n]$.

More precisely, Hopper establishes an upper bound, as in Theorem 1, that depends on the *sharpness* of the bends in $N_\infty(g) \circ N_\infty(h)$, defined as the (exponential of the) ratio of slopes of consecutive sides. Near very sharp bends, the two polygons are very close; the careful analysis [19, Chapter III, §5] of bends with small sharpness gives the upper bound. She remarks that the "result can however probably be considerably improved upon" due to certain estimates employed in the proof [19, p. 38].

The Newton–Hopper polygon. In [19, Chapter II, §2], Hopper introduces a new construction of a convex polygon associated to a monic polynomial with integer coefficients that takes into account both the divisibility (with respect to a fixed prime p) and the magnitudes of the coefficients. We call this the *Newton–Hopper* polygon $NH_p(f)$ associated to $f(x) = \sum_i a_i x^i \in \mathbb{Z}[x]$. It is defined by writing

$$f(x) = \sum_i \sum_j r_{ij} p^j x^i$$

where $r_{ij} \neq 0$ and satisfy $-p < r_{ij} < p$, and then taking the convex hull of the points (i, j) in $\mathbb{R}^2$. This construction yields a convex polygon whose "lower half" is $N_p(f)$ and whose "upper half" is the upper convex hull of the points $(i, \lfloor \log_p |a_i| \rfloor)$, so that the upper half is approximately $-N_\infty(f)$. The analogous bounds in Theorem 1 hold for the upper half of the Newton–Hopper polygon of a product.

Hopper's strategy [19, Chapter IV] is then to start with a polynomial $f(x) \in \mathbb{Z}[x]$, plot $NH_p(f)$ (in black ink), and then plot (in red and blue ink) the limits of the upper and lower bounds in Theorem 1 away from $NH_p(f)$. Finally, if one can verify that each possible polygon within the region bounded between the (red and blue) limits cannot be decomposed as a Dumas sum of Newton–Hopper polygons (where we formally apply Dumas composition to the upper half and lower half separately) of lower degrees, then $f(x)$ must be irreducible. This observation provides new irreducibility criteria that simultaneously generalize those depending on the divisibility and the magnitudes of the coefficients.

Hopper then proceeds with a careful analysis of various general situations in which this occurs, and then produces families of sparse polynomials that satisfy these criteria. Some of her families in [19, Chapter IV, §5] cannot be proven to be irreducible solely using either divisibility properties or relative magnitude properties of the coefficients on their own. For example, the polynomial

$$f(x) = x^7 \pm (p^{11} + p)x^5 \pm p^4,$$

for any prime $p > 3 \cdot 2^{49/4} > 14,612$ (e.g., $p = 14,621$ is the first such prime), is irreducible. Similarly, the polynomial

$$f(x) = x^9 \pm (p^6 + p)x^3 \pm (p^9 + p^3)x^2 \pm p^3, \quad (1)$$

for any $p > 3 \cdot 2^{81/4} > 3,740,922$ (e.g., $p = 3,740,923$ is the first such prime) is irreducible, see Figure 11. For all primes p below these bounds, a computer algebra system can verify the irreducibility of the above polynomials. Also, the following infinite family of polynomials

$$f(x) = x^n \pm kx^2 \pm lp^{2\nu+1},$$

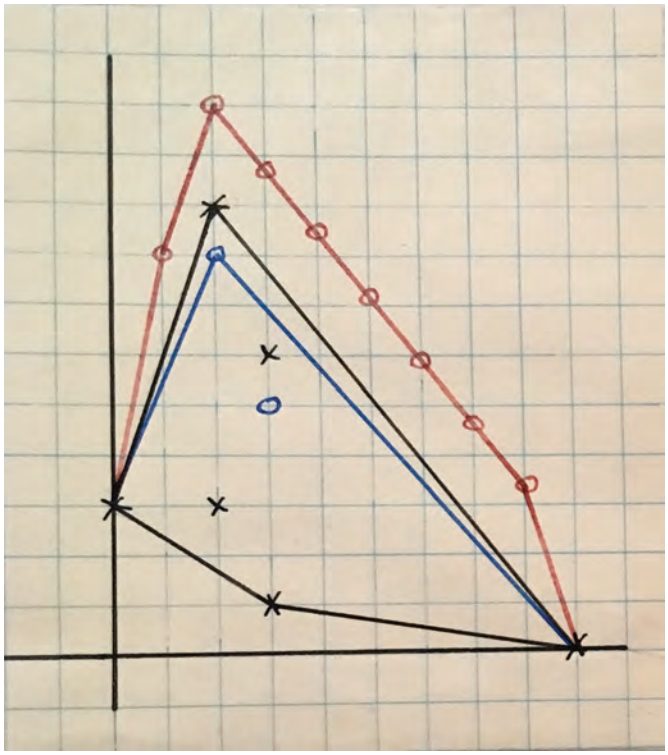


Figure 11. The Newton–Hopper polygon of the polynomial (1), in black, with the upper and lower bounds in Theorem 1 in red and blue, respectively. Hand drawn by Grace Hopper [19, p. 55].

where $n \geq 3$, $\nu \geq 3$, $0 < k < p^{2(\nu-2)}$, $p \nmid k$, $0 < l < p$, and $p > 3 \cdot 2^{n^2/4}$, are all irreducible. Similarly, the following infinite family of polynomials

$$f(x) = x^n \pm kpx \pm mp^\nu,$$

where $n \geq 2$, $\nu \geq 4$, $0 < k < p^{2(\nu-2)}$, $0 < m < p^{\nu-3}$, $p \nmid km$, and $p > 3 \cdot 2^{n^2/4}$, are all irreducible.

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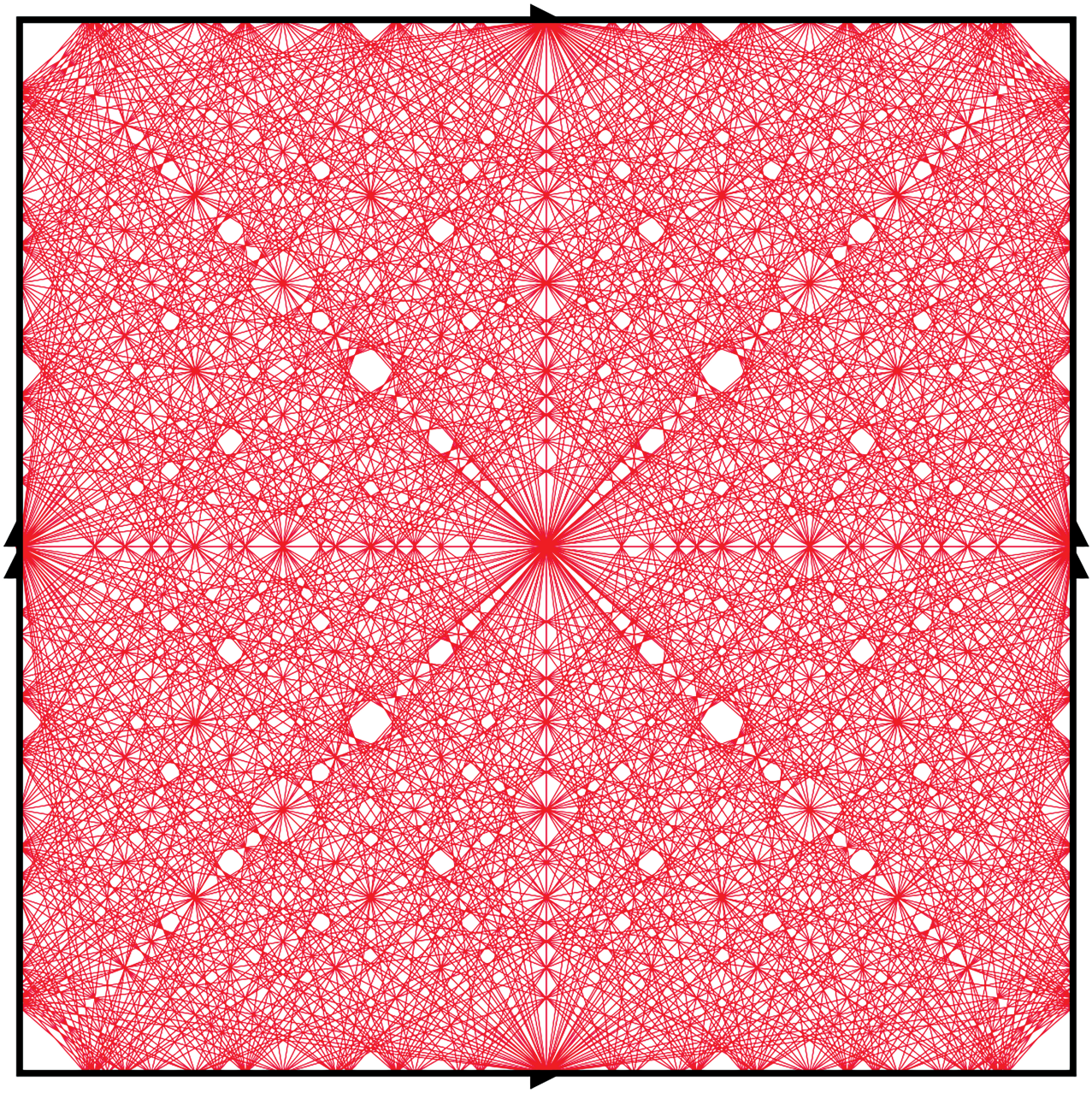
The Mathematics of Joan Birman



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Introduction

Joan Birman published her first paper, "On braid groups," in January 1969. That work introduced one of the most important tools in the study of braids and surfaces, now called the Birman exact sequence. Fifty years and more than one hundred papers later, Birman is an active researcher and has long been established as a leading figure in the field of low-dimensional topology.

The goal of this article is to give a broad overview of Birman's mathematics. In the process, we will see several related themes emerge. Time and again, Birman has shown a knack for asking the right questions, for pursuing and embracing unlikely collaborations across mathematical disciplines, and for uncovering and revitalizing hidden or forgotten fields. Because of this, her work has often been ahead of its time, with important implications and applications found years or decades after the original discoveries. For instance, her book on braids is credited with bringing that theory from the fringes to the fore. Similarly, when Birman began working on mapping class groups and Torelli groups, she was working in isolation. Now these are core topics in topology, and her contributions are of fundamental importance. In fact, Birman's work has underpinned two Fields medals.

Birman's research revolves around the theories of knots, braids, mapping class groups of surfaces, and 3-manifolds. Figure 1 shows a diagram of these topics and gives a road map for this article. We will introduce the various objects and the connections between them in the sections indicated. It is a bit of a miracle that these subjects are so closely intertwined. In what follows we will see how Birman's work has influenced and interacted with this beautiful circle of ideas.

§1 Knots

A knot is the image of a smooth embedding of the circle S^1 into $\mathbb{R}^3$. We can think of a knot as a piece of string with its ends glued together. We can draw a diagram of a knot by projecting it to a plane and indicating the over/undercrossings of the strands by putting a break in the strand that is crossing below; see Figure 2. Two knots are equivalent if they are isotopic, that is, if one knot can be continuously deformed into the other without creating any self-intersections along the way.

The fundamental problem in knot theory is to decide if two knots are equivalent. A (not really) simpler version is to decide if a given knot is equivalent to the trivial knot. The knots in Figure 2 fall into two equivalence classes (left and right trefoils). Which are equivalent?

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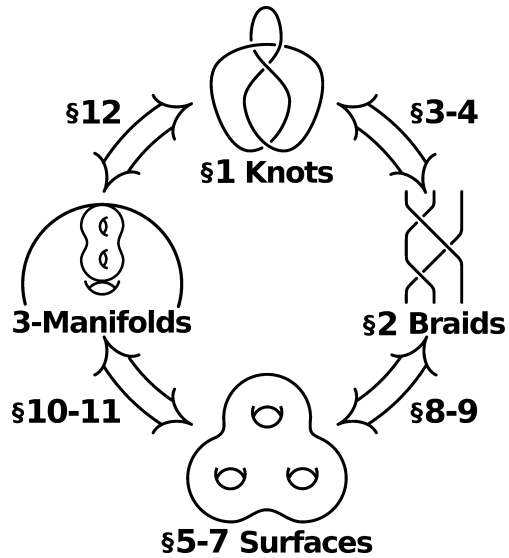

Figure 1. A road map for this article (and Birman's career).

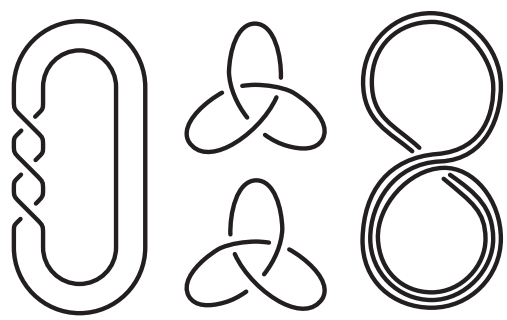

Figure 2. Some examples of knots.

As this exercise illustrates, knot theory is difficult because there are many diagrams for the same knot that are very different from one another. There is no easy way to move between two different diagrams, and there is no systematic way to choose a canonical diagram for a knot.

Among the many successes of knot theory is the discovery of knot invariants. An invariant for a knot is an object (number, polynomial, etc.) we can associate to a knot with the property that equivalent knots have the same invariant. If we find two knots with different invariants, then they are inequivalent knots.

One of the most famous and important knot invariants is the Alexander polynomial, a Laurent polynomial that can be computed from any knot diagram. The Alexander polynomial is not a complete invariant: it attains the same value on the left- and right-handed trefoil knots, and also Kinoshita and Terasaka found a nontrivial knot with the same Alexander polynomial as the trivial knot. The simplest diagram for the latter has 11 crossings. It is still an open problem to find an easily computable, complete invariant for knots (more on this later).

Knot theory has applications to statistical mechanics, molecular biology, and chemistry; see Murasugi's book [71] for a survey. Later in this article we will see several connections of knot theory with other parts of topology, group theory, dynamics, and number theory.

§2 Braids

A braid on n strands is a collection of n disjoint paths in $\mathbb{R}^2 \times [0,1]$, connecting n points in $\mathbb{R}^2 \times \{0\}$ to the corresponding points in $\mathbb{R}^2 \times \{1\}$, and intersecting each plane $\mathbb{R}^2 \times \{t\}$ in exactly n points. The n paths are called the strands of the braid.

We consider two braids to be equivalent if they are isotopic, that is, if we can continuously deform one to the other while holding the endpoints fixed and without allowing strands to pass through each other. Figure 3 shows two equivalent braids. The set of braids on n strands forms a group B_n, with the group operation given by stacking braids.

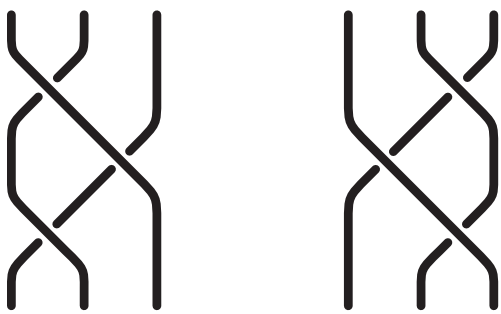

Figure 3. Two equivalent braids.

There is a more succinct (and sophisticated) way to define the braid group. Let C_n denote the configuration space of n distinct points in the plane. We have

$$B_n \cong \pi_1(C_n).$$

The isomorphism is obtained as follows. Let eta be a braid on n strands. For each t in $[0,1]$ we may consider the corresponding plane parallel to the original two planes. If we intersect this plane with the braid eta, we obtain a point in C_n. As t changes from 0 to 1, we obtain a loop in C_n, that is, an element of $\pi_1(C_n)$. This map is the desired isomorphism.

We can now see why the braid group is ubiquitous in mathematics and science: it records the motions of points in the plane. The points can be roots of polynomials, critical values of branched covers, particles in a two-dimensional medium, or autonomous vehicles moving through city streets. See the survey by Birman and her student Brendle for an excellent introduction to the theory [16].

§3 Braids and Knots

There is a simple way to obtain a knot from a braid, namely by connecting the top of the braid to the bottom by n parallel strands. Actually, in general we obtain a link, which is a disjoint union of knots. The resulting knot or link is called a closed braid; see the left-hand side of Figure 4 for an example. In 1923 Alexander proved the remarkable theorem that every knot is equivalent to a closed braid [3].

On the face of it, braids are more tractable than knots because of the group structure, and Alexander's theorem gives us hope of applying our knowledge of braid groups to the theory of knots. The immediate problem is that there are many braids giving rise to the same knot. For instance, if two braids are conjugate, then their braid closures are equivalent.

There are also nonconjugate braids with equivalent closures, and there are braids with different numbers of strands that have equivalent closures. One specific way to construct braids with different numbers of strands and equivalent closures is through stabilization, illustrated in Figure 4. In 1936 Markov announced (without proof) the following surprising theorem: if two braid closures are equivalent, then, up to conjugacy, the braids differ by a finite sequence of stabilizations, destabilizations, and exchange moves (although it was soon realized that the exchange moves were not needed).

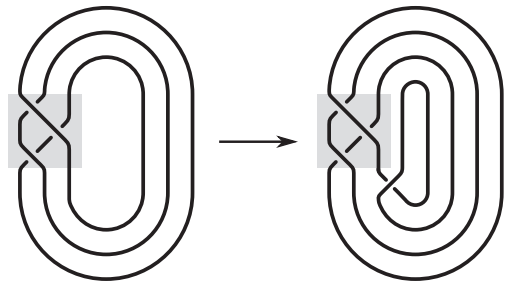

Figure 4. A closed braid and its stabilization.

Four decades later, Birman published a monograph, *Braids, links, and mapping class groups* [12], based on a graduate course she gave at Princeton University during the academic year 1971–72. Her book was the first comprehensive treatment of braid theory, and its appearance represented the birth of the modern theory. It contains in particular the first complete proof of Markov's theorem.

Our discussion of braids and knots so far points us in three natural directions:

1. the *conjugacy problem* for the braid group, namely, the problem of algorithmically determining whether or not two elements of B_n are conjugate;

2. the *algebraic link problem*, namely, the more general problem of algorithmically determining if two braids have equivalent closures; and
3. the big question of whether we can use braid theory to discover new knot invariants.

Birman's monograph focused precisely on these problems. Here we briefly touch on the first two problems, and some contributions to these made by Birman later in her career. In the next section we discuss how the book contributed to the third problem.

With respect to the conjugacy problem, Birman's work has led in two directions. In the 2000s she wrote three papers with Gebhardt and Gonzáles-Meneses [21–23] in which they expand on the Garside approach to the conjugacy problem, explored three decades earlier in Birman's book. A different approach is provided by her paper with Ko and Lee [7]. There, they introduce a new algebraic approach to the braid group, a tool now called the Birman–Ko–Lee monoid for the braid group. This is the second-most cited paper in Birman's catalog.

In the 1990s Birman and Menasco wrote a series of six papers with the title "Studying links via closed braids" [31–36]. The fourth in the series was published in *Inventiones Mathematicae*. A basic question is studied in these papers: If two braids have the same number of strands and have equivalent closures, can we find a sequence of elementary moves that pass from one braid to the other without changing the number of strands? Can we do this algorithmically?

In the end Birman and Menasco did find a "Markov theorem without stabilization," a calculus for dealing with the algebraic link problem [37]. Along the way, they developed connections and applications to the field of contact topology. In particular, they give examples where the isotopy class of a knot and the Bennequin invariant do not fully determine the transverse isotopy class [38]; see also Birman's work with her student Wrinkle [45] as well as the work of Etnyre and Honda [47].

§4 Birman's Book and the Jones Polynomial

While at Princeton, Birman's research focus was on the third problem described in the last section, namely, using braid theory to discover new knot invariants. One tool that becomes available when we have a group in hand is the subject of representation theory. This is relevant to the theory of knot invariants because conjugacy classes of matrices have many natural invariants, such as the determinant.

At the time of Birman's book, only one interesting representation of the braid group was known, namely, the Burau representation. This representation gives a knot invariant as follows: given a knot, choose a braid whose closure is that knot, apply the Burau representation, subtract this matrix from the identity, take the determinant, and then scale by $(1-t)/(1-t^n)$. This conjugacy class invariant for braids interacts nicely with stabilization, and so we indeed obtain a knot invariant.

The knot invariant arising from the Burau representation turns out to be nothing other than the Alexander polynomial. (To paraphrase one of Birman's sayings, when you discover a new knot invariant, your task is to figure out which existing invariant you have just rediscovered.) The Alexander polynomial is of fundamental importance in knot theory, but as mentioned earlier it is not a complete invariant. And without any new representations on the horizon, it seemed hopeless for Birman to use her ideas to extract knot invariants from braids.

But then in 1984, after Birman became a professor at Columbia University, Vaughan Jones asked to meet with Birman to discuss a new representation of the braid group he had discovered through his work on von Neumann algebras. His representation was a direct sum of matrix representations, one of the summands being the Burau representation. From the representation, Jones extracted a conjugacy class invariant for braids. This was not a determinant (as for the Alexander polynomial) but a weighted sum of the traces of the summands [56].

Birman explained the Markov theorem to Jones, who then realized that his conjugacy invariant for braids gave a new invariant of knots, similar to how the Burau representation gives the Alexander polynomial.

Jones' new polynomial was quickly seen to be an improvement over the Alexander polynomial, as it could distinguish the left- and right-handed trefoil knots. Even better, it evaluated nontrivially on the 11-crossing Kinoshita–Terasaka knot [58]. And so the Jones polynomial was born, and a revolution in knot theory was begun.

Jones received the Fields Medal in 1990 for this work. Fittingly, Birman gave the laudation at the International Congress of Mathematicians. See Birman's article from the proceedings [17] and also her personal recollections in this journal [1]. In his *Annals* paper [57], Jones writes, "The author would like to single out Joan Birman among the many recipients of his thanks. Her contribution to this new topic has been of inestimable importance."

Jones showed that his polynomial is not a complete knot invariant: the Conway knot and the 11-crossing Kinoshita–Terasaka knot have the same Jones polynomial. In a paper published in *Inventiones Mathematicae*, Birman further found many inequivalent closed 3-braids with the same Jones polynomial [15]. It is an open question whether or not there is a nontrivial knot with trivial Jones polynomial.

Birman and Wenzl used the theory of the Jones polynomial (specifically, the two-variable polynomial of Kauffman) to construct a new representation of the braid group

[42]. Both Jones and Birman's student Zinno [78] proved that one summand of this representation is the same as the Lawrence representation, famously proved to be faithful by Bigelow [5] and Krammer [61].

Shortly after Jones' discovery, Vassiliev discovered new invariants of knots, named for him (and sometimes called finite-type invariants). Birman and Lin gave a simplified, axiomatic, combinatorial approach to these invariants [29]. This is Birman's most cited paper and was also published in *Inventiones Mathematicae*. Birman wrote a beautiful survey paper explaining this work and the connection to the Jones polynomial [18]; this article won the Chauvenet Prize in 1996.

§5 Mapping Class Groups

We now move on from the world of knots and braids, which are one-dimensional objects, to the realm of surfaces, which are inherently two-dimensional. The theory of mapping class groups of surfaces was initiated by Dehn in the 1920s. Dehn was the doctoral advisor of Magnus who, in turn, was the advisor to Birman. As we will see, mapping class groups will play a prominent role in Birman's career.

To start at the beginning, a surface is a two-dimensional manifold. For each $g \geq 0$ there is a surface S_g of genus g, obtained as the connect sum of g tori (so S_0 is the sphere, and S_1 is the torus). The classification of surfaces says that these are all of the surfaces that are closed (compact and without boundary) and orientable.

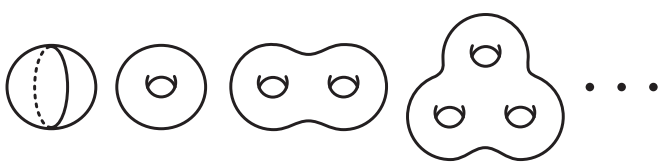

Figure 5. The first few closed, orientable surfaces.

While surfaces are completely classified, there are many open questions, and the theory of surfaces is an active area of research today. Of particular interest is the mapping class group $\mathrm{MCG}(S)$ of a surface S, the group of homotopy classes of homeomorphisms of S. This is a discrete group that encodes the symmetries of S. One source of nontrivial elements of $\mathrm{MCG}(S)$ is the set of rotations of S. For instance the surface S_3 in Figure 5 admits an obvious rotation of order 3.

An important type of infinite order element is a Dehn twist. In Figure 6 we depict a twist of the annulus. A Dehn twist on a surface is a homeomorphism that performs such a twist on some annulus and is the identity on the complement. If c is a simple closed curve in S, then the Dehn twist about an annular neighborhood of c is a well-defined element T_c of $\mathrm{MCG}(S)$.

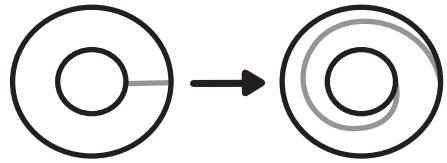

Figure 6. A twist of an annulus.

Dehn proved the foundational theorem that $\mathrm{MCG}(S_g)$ is finitely generated by Dehn twists. Dehn's point of view was motivated by the following analogy:

$$\text{linear maps} : \text{vectors} :: \text{mapping classes} : \text{curves}$$

More specifically, Dehn was interested in simple closed curves, those with no self-intersections. He referred to the set of these as the arithmetic field of the surface.

After the early work of Dehn and his student Nielsen, the subject of mapping class groups was largely forgotten. Birman reignited interest in the subject through her thesis work (see Section 8, "The Birman Exact Sequence"), her book, and her various survey articles [13, 14, 20]. The subject really exploded with the work of Thurston, which was announced shortly after Birman's book was published; see the next section.

Today, the theory of mapping class groups is a central topic, connected to many fields of mathematics and physics. For instance it can be interpreted as:

1. the outer automorphism group of the fundamental group of the surface;
2. the fundamental group of the moduli space of algebraic curves;
3. the isometry group of Teichmüller space; and
4. the classifying group for surface bundles.

See the primer by Farb and the author [48] for a modern introduction to mapping class groups.

§6 Curves on Surfaces

Birman and Series wrote a number of papers aimed at understanding the nature of the set of simple closed curves in a surface. They gave, for instance, an algorithm for determining if an element of the fundamental group of a surface has a simple representative [39]. They also described a sense in which the action of $\mathrm{MCG}(S)$ on the space of simple closed curves in S is linear, as per Dehn's analogy above [41].

The most influential result of Birman and Series [40] addresses the question, What does the set of simple closed curves look like if we draw them all at once? Precisely, they fix a surface of negative Euler characteristic and a hyperbolic metric on the surface, and they consider the (unique) geodesic representative of each homotopy class of simple closed curves. Their main theorem is that the union of all such geodesics is nowhere dense and has Hausdorff dimension 1.

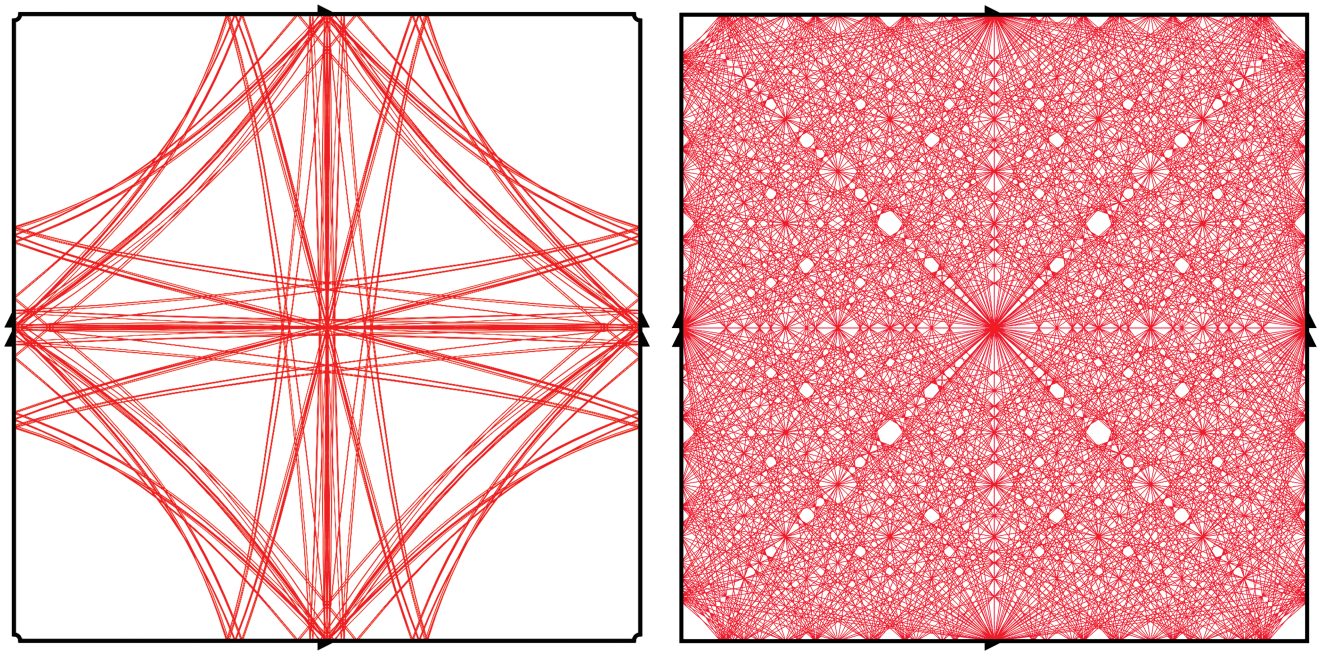

Figure 7. Left(7a): the 88 shortest geodesics on a hyperbolic punctured torus; right(7b): the 88 shortest geodesics on a Euclidean torus.

This result is illustrated by Figure 7. The left side shows a square with the four corners deleted. If we identify opposite sides, we obtain a punctured torus (a torus minus one point). The hyperbolic metric on the latter is mapped to the square by a conformal mapping. Long hyperbolic geodesics are well approximated by arcs of short ones. So even though the picture only shows the 88 shortest simple geodesics, it gives a decent approximation of the union of all simple geodesics.

Here are two striking points of contrast: (1) the union of all closed geodesics (including the ones with self-intersections) is dense; and (2) if we consider a Euclidean torus (the torus obtained by identifying opposite sides of a Euclidean square) and choose one geodesic in each homotopy class of simple closed curves, the resulting union of geodesics is dense (see the right-hand side of Figure 7).

At the end of their paper, Birman and Series suggest another interesting problem: counting the number of simple geodesics as a function of the length. They write:

> In fact the degree of the polynomial $P_0(n)$ bounding the number of simple geodesics of length n is at most $6g + 2b - 6$, where g is the genus and b the number of boundary components of M... In general the precise nature of the bound seems to be a very interesting number theoretic question.

Many years later, Mirzakhani did find the precise nature of the bound (the upper bound of Birman and Series is also a lower bound), one of the many stunning achievements in her Fields Medal work [68].

The Birman–Series result also plays a central role in the proof of the celebrated McShane identity, which states that for any hyperbolic metric on the punctured torus, we have

$$\sum_{\gamma} \frac{1}{1 + e^{\ell(\gamma)}} = 1/2,$$

where the sum is over all simple closed geodesics and $\ell(\gamma)$ denotes the hyperbolic length [66]. This theorem was also generalized by Mirzakhani [67], who used her generalization to compute the volume of moduli space in the Weil–Petersson metric.

§7 Basic Algebraic Properties of the Mapping Class Group

In this section we discuss Birman's work on the following basic algebraic questions about $\mathrm{MCG}(S_g)$:

1. What is the abelianization?
2. What is the rank of a maximal torsion-free abelian subgroup?

These are among the first questions we can ask about any infinite group.

Mumford was one of the few mathematicians who studied the mapping class group in the period between Dehn and Birman. He was interested in the applications to algebraic geometry. What he proved [70] is that any abelian quotient of $\mathrm{MCG}(S_g)$ is a quotient of $\mathbb{Z}/10$ when $g \geq 3$. Birman [11] improved the $\mathbb{Z}/10$ to $\mathbb{Z}/2$. Building on this, her student Powell further improved the $\mathbb{Z}/2$ to the trivial group [73], thus establishing the fundamental theorem

that $\mathrm{MCG}(S_g)$ is perfect for $g \geq 3$. This completely answers the first question.

The second question was answered in a joint paper by Birman, Lubotzky, and Birman's student McCarthy [30]. The three of them were working to understand Thurston's groundbreaking work on the mapping class group. As a part of his Fields Medal work, Thurston [75] gave a classification of elements of the mapping class group, now called the Nielsen–Thurston classification. This theorem states that every element of the mapping class group has a representative homeomorphism that preserves a (possibly empty) collection of disjoint curves and, on the complementary pieces, is either of finite order or pseudo-Anosov. A pseudo-Anosov map is one that locally looks like the action of the matrix $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)$ on $\mathbb{R}^2$. So there are two invariant foliations, one stretched by λ and one by λ^{-1}.

We should think of Thurston's theorem as a sort of Jordan form for mapping classes. There is one problem: he did not prove that the decomposition along curves was canonical. Birman, Lubotzky, and McCarthy addressed exactly that, by defining the canonical reduction system for a mapping class.

As a result of this work, Birman, Lubotzky, and McCarthy showed that the answer to the second question is $3g - 3$ for $\mathrm{MCG}(S_g)$. They further proved that every solvable subgroup of the mapping class group is virtually abelian.

Like the Jordan canonical form for matrices, canonical reduction systems feature prominently in modern theory of mapping class groups, especially in work on their algebraic structure. For instance, Ivanov and McCarthy used canonical reduction systems to prove that mapping class groups satisfy a Tits alternative, thus strengthening the analogy between mapping class groups and arithmetic groups [51, 65].

§8 The Birman Exact Sequence

There are many connections between the theories of braid groups and mapping class groups. The two most important are the Birman exact sequence and the Birman–Hilden theory, discussed in this section and the next. One running theme is that of group presentations for mapping class groups.

Dehn proved that the mapping class group of the torus is isomorphic to $\mathrm{SL}_2(\mathbb{Z})$, which has a well-known finite presentation. In her thesis work, Birman's goal was to find group presentations for other mapping class groups. She succeeded right away in finding an inductive procedure for computing presentations of mapping class groups of surfaces with marked points.

Let S be a surface of negative Euler characteristic, and let $p \in S$. We consider $\mathrm{MCG}(S, p)$, the group of homotopy classes of homeomorphisms of S fixing the point p (it is crucial that the homotopies fix p as well). There is a forgetful map $\mathrm{MCG}(S, p) \to \mathrm{MCG}(S)$. Birman wanted to understand the kernel.

For $[\phi] \in \mathrm{MCG}(S, p)$ to be in the kernel, this means that ϕ is homotopic to the identity as long as we allow p to move during the homotopy. If we follow the path of p throughout this homotopy, we obtain a loop in S, that is, an element of the fundamental group $\pi_1(S, p)$. Birman's theorem is that this identification is well-defined and that it gives an isomorphism of $\pi_1(S, p)$ with the kernel.

The resulting map $\pi_1(S, p) \to \mathrm{MCG}(S, p)$ is usually called the push map because we can think of the image of $\alpha \in \pi_1(S, p)$ as the element of $\mathrm{MCG}(S, p)$ obtained by pushing p along α (Birman originally called this the spin map).

Birman's result is usually stated as saying that the following sequence is exact:

$$1 \to \pi_1(S, p) \to \mathrm{MCG}(S, p) \to \mathrm{MCG}(S) \to 1.$$

Using this, she could promote a presentation of $\mathrm{MCG}(S)$ to a presentation for $\mathrm{MCG}(S, p)$. The Birman exact sequence is ubiquitous in the theory of mapping class groups, as it is used in many inductive arguments.

What is the connection to braid groups? The first step in this direction is to generalize from one point p to a finite set of points $P = \{p_1, \ldots, p_n\}$. The group $\mathrm{MCG}(S, P)$ is the group of homotopy classes of homeomorphisms of S fixing P as a set. Let $C_n(S)$ denote the space of configurations of n distinct points in S. Birman's more general exact sequence is

$$1 \to \pi_1(C_n(S), P) \to \mathrm{MCG}(S, P) \to \mathrm{MCG}(S) \to 1.$$

When $n = 1$, the space $C_n(S)$ is homeomorphic to S, and so we obtain the first exact sequence above. Recall that B_n is defined as $\pi_1(C_n(\mathbb{R}^2), P)$. The group $\pi_1(C_n(S), P)$ is known as a surface braid group. We can visualize the elements as braided strands in $S \times [0, 1]$. As a special case, when S is the disk, we conclude that B_n is isomorphic to the mapping class group of a disk with n marked points.

Birman used the more general exact sequence in her thesis to obtain presentations for the mapping class groups of the torus with any number of marked points [10]. The surface of genus 2 would have to wait for her work with Hilden.

§9 The Birman–Hilden Theory

After graduating from New York University's Courant Institute in 1968, Birman took a job at Stevens Institute of Technology, where she began a very successful collaboration with Hilden, a graduate student there at the time.

Birman and Hilden originally set out to find a presentation for $\mathrm{MCG}(S_2)$, the next natural mountain to climb. The key idea in their work is to relate $\mathrm{MCG}(S_2)$ to a braid group in the following way. The hyperelliptic involution

$\iota : S_2 \to S_2$ is the rotation by π about the axis indicated in Figure 8.

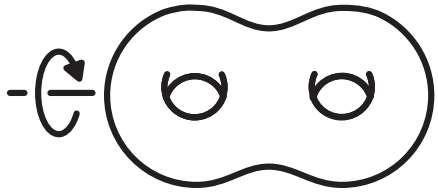

Figure 8. The hyperelliptic involution of S_2.

The quotient $S_2/\langle\iota\rangle$ is a sphere $S_{0,6}$ with six distinguished points (the images of the six fixed points of ι). Birman and Hilden proved that there is an isomorphism

$$\mathrm{MCG}(S_2)/\langle[\iota]\rangle \xrightarrow{\cong} \mathrm{MCG}(S_{0,6}).$$

Since $\mathrm{MCG}(S_{0,6})$ is closely related to a braid group (with the sphere replacing the disk), this allowed them to convert a known presentation for $\mathrm{MCG}(S_{0,6})$ into a presentation for $\mathrm{MCG}(S_2)$. This work is the subject of Birman's article, "My favorite paper" [9].

The above isomorphism is defined as follows. As observed earlier by Birman, every element of $\mathrm{MCG}(S_2)$ has a representative that commutes with ι. Such a representative descends to a homeomorphism of $S_{0,6}$ and hence gives an element of $\mathrm{MCG}(S_{0,6})$. The hard part of their theorem is showing that this map is well-defined, that is, that it interacts well with homotopies.

Birman and Hilden vastly generalized this theorem in a series of papers on hyperelliptic and symmetric mapping class groups [24–26], culminating in their most general result [28], which was published in *Annals of Mathematics*. This work was later generalized by MacLachlan and Harvey [63] and by Winarski [76], who gave Teichmüller-theoretic and combinatorial-topological points of view.

The Birman–Hilden theory gives a dictionary between the theories of braid groups and mapping class groups, with important applications on both sides. For instance it is used in the proof that $\mathrm{MCG}(S_2)$ is linear [6, 60] and also in the resolution of a question of Magnus about the action of the braid group on the fundamental group of the punctured disk [28]. We refer the reader to our survey with Winarski for a detailed discussion [64].

§10 Heegaard Splittings, Torelli Groups, and Homology Spheres

We now turn to the interface between the theories of surfaces and 3-manifolds. A 3-manifold is the three-dimensional analogue of a surface, that is, a space that locally looks like $\mathbb{R}^3$. A first example is the 3-sphere S^3. We can use stereographic projection to identify S^3 as $\mathbb{R}^3$ with one added point at infinity, in much the same way that we identify S^2 as $\mathbb{R}^2$ with a point at infinity.

In this section we will focus on one particular construction of 3-manifolds from surfaces, namely Heegaard splittings. If S_g is the surface of a donut with g donut holes, then the handlebody H_g is the donut itself. By gluing two copies of H_g along their boundaries, we obtain a 3-manifold without boundary. For each g there is a particular gluing $\psi : S_g \to S_g$ that results in the sphere S^3. (The usual embedding of H_g in $\mathbb{R}^3 \subseteq S^3$ is a realization of this gluing: the outside of H_g is another copy of H_g!) In general, the decomposition of a 3-manifold into two handlebodies glued along their boundary is called a Heegaard splitting.

If we take any homeomorphism ϕ of S_g and post-compose the gluing map ψ by ϕ, we obtain a new 3-manifold. The resulting 3-manifold only depends on the mapping class $[\phi] \in \mathrm{MCG}(S_g)$. What is more, every closed, orientable 3-manifold arises in this way. The upshot is that the theory of Heegaard splittings gives us a set map

$$\mathrm{MCG}(S_g) \to \text{3-manifolds}.$$

The mapping class group $\mathrm{MCG}(S_g)$ acts on the first homology group $H_1(S_g)$. The kernel of this action is called the Torelli group $\mathcal{I}(S_g)$. By the Mayer–Vietoris theorem, we have the restriction

$$\mathcal{I}(S_g) \to \text{homology 3-spheres}.$$

Here, a homology 3-sphere is a 3-manifold that has the same homology groups as S^3. This is an important subclass of 3-manifolds. Indeed, the fact that there exist nontrivial homology 3-spheres is the reason that the Poincaré conjecture cannot be stated in terms of homology alone (and this is what forced Poincaré to invent π_1).

Birman published a number of works on Heegaard splittings, specifically with the aim of classifying 3-manifolds through the lens of the mapping class group. For instance, with Hilden [27] she gave an algorithm to determine if a manifold with a given Heegaard splitting is homeomorphic to S^3.

§11 Birman's Work on Torelli Groups

Birman made two monumental contributions to the theory of Torelli groups. In particular, her work was aimed at the following questions:

1. What is a natural generating set for the Torelli group?
2. What are the abelian quotients of the Torelli group?
3. Is the Torelli group finitely generated?

As with mapping class groups, these are among the first properties we would like to know about a group.

There is also a connection with algebraic geometry: the Torelli group encodes the fundamental group of the Torelli space, the space of framed curves of genus g. The period

mapping takes this space to the Siegel upper half-space, sending a framed curve to its period matrix. (Torelli is the name of an Italian algebraic geometer.) As such, the above questions can be reinterpreted as basic questions about the topology of Torelli space.

Birman spent the academic year 1969–70 in Paris. By her own account, she was mathematically isolated there and discouraged [1]. But she had an idea for how to attack the first question by brute-force calculation. The starting point is that the mapping class group $\mathrm{MCG}(S_g)$ and the Torelli group $\mathcal{I}(S_g)$ fit into a short exact sequence

$$1 \to \mathcal{I}(S_g) \to \mathrm{MCG}(S_g) \to \mathrm{Sp}_{2g}(\mathbb{Z}) \to 1.$$

The group $\mathrm{Sp}_{2g}(\mathbb{Z})$ is isomorphic to the automorphism group of $H_1(S_g;\mathbb{Z}) \cong \mathbb{Z}^{2g}$; we have the symplectic group here instead of the whole general linear group because automorphisms preserve the algebraic intersection form, which is symplectic. From this point of view, we can think of $\mathrm{Sp}_{2g}(\mathbb{Z})$ as capturing the linear, easy-to-understand aspects of $\mathrm{MCG}(S_g)$ and of $\mathcal{I}(S_g)$ as encapsulating the more difficult, mysterious aspects.

Birman knew that the defining relations for $\mathrm{Sp}_{2g}(\mathbb{Z})$ correspond to generators for $\mathcal{I}(S_g)$ (this is a general principle that applies to any short exact sequence of groups). So the task then was to find a reasonable group presentation for $\mathrm{Sp}_{2g}(\mathbb{Z})$. She succeeded and obtained a presentation with three families of generators and 10 families of relations.

Birman's student Powell then gave simple descriptions of the resulting generators for $\mathcal{I}(S_g)$: they are Dehn twists about separating curves and bounding pair maps [73]. A bounding pair map is $T_a T_b^{-1}$, where a and b are disjoint, homologous, nonseparating curves; see Figure 9. Putman, who gave a geometric proof of the Birman–Powell result in his thesis [74], describes Birman's work as "absolutely heroic."

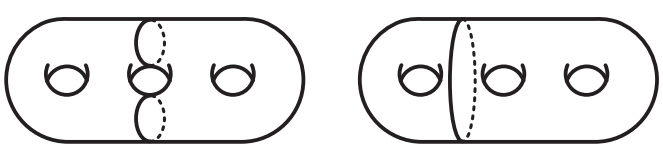

Figure 9. Left: a bounding pair; right: a separating curve.

Birman and Craggs took aim at the second and third questions, and they made a most spectacular contribution. They showed that, unlike $\mathrm{MCG}(S_g)$, the group $\mathcal{I}(S_g)$ does have nontrivial abelian quotients. They found a family of homomorphisms $\rho_\psi : \mathcal{I}(S_g) \to \mathbb{Z}/2$. Surprisingly, the definition involves the theories of 3- and 4-manifolds. One hope they had was that there would be infinitely many distinct such homomorphisms, thus proving that $\mathcal{I}(S_g)$ was not finitely generated.

In order to specify one of the Birman–Craggs homomorphisms, we need to fix some Heegaard splitting ψ of S^3. Now let $f \in \mathcal{I}(S_g)$. As in Section 10, "Heegaard Splittings, Torelli Groups, and Homology Spheres," f determines a homology 3-sphere M_f. Every homology 3-sphere is the boundary of some 4-manifold. The Rokhlin invariant of M_f is the signature of this 4-manifold, divided by 8, mod 2 (by Rokhlin's theorem, this is well-defined). This element of $\mathbb{Z}/2$ is $\rho_\psi(f)$. Miraculously, this defines a homomorphism $\mathcal{I}(S_g) \to \mathbb{Z}/2$. The proof features what is probably the first instance of a 4-manifold trisection, a tool popularized four decades later by David Gay and Robion Kirby [79].

Several years after these works, Johnson arrived on the scene. In a stunning series of deep, beautiful papers, he expanded on the work of Birman and her collaborators. He proved [55] that $\mathcal{I}(S_g)$ is finitely generated for $g \geq 3$. Also he classified the Birman–Craggs homomorphisms—showing directly that there were only finitely many—and gave a complete description of the abelianization of $\mathcal{I}(S_g)$ [52]. (Amazingly, there is still no definition of these homomorphisms that does not involve the construction of a 4-manifold.) As a byproduct, Johnson showed that $\mathcal{I}(S_g)$ cannot be generated by Dehn twists about separating curves, disproving a conjecture of Birman.

See Johnson's delightful survey for more about his work [53]. In the survey, Johnson notes that the interest in Torelli groups from topologists "was initiated principally through the work of Joan Birman" [54].

§12 Lorenz Knots

We end by discussing the work of Birman and Williams on Lorenz knots in the early 1980s. This is a fitting finale, as it combines all four of the main objects of study in this article. It is also a prime example of work that was ahead of its time, with 94 of its 106 citations on MathSciNet® coming after the year 2000.

E. N. Lorenz was a pioneer of chaos theory. He was particularly interested in the weather, and whether it was deterministic. Lorenz is perhaps most famous for coining the phrase "butterfly effect."

In order to help understand weather patterns, Lorenz devised a simplified version of the Navier–Stokes equations, a system of three ordinary differential equations in three variables [62]. This system has a strange attractor, called the Lorenz attractor, shown in the top of Figure 10. Forward trajectories of points converge to the attractor and, once there, stay forever.

A Lorenz knot is a knot obtained as a periodic orbit in the Lorenz attractor. Williams showed that Lorenz knots are exactly the ones that can be drawn on the "template" shown at the bottom of Figure 10.

A Lorenz braid is a braid consisting of strands that either go monotonically left to right or from right to left, where

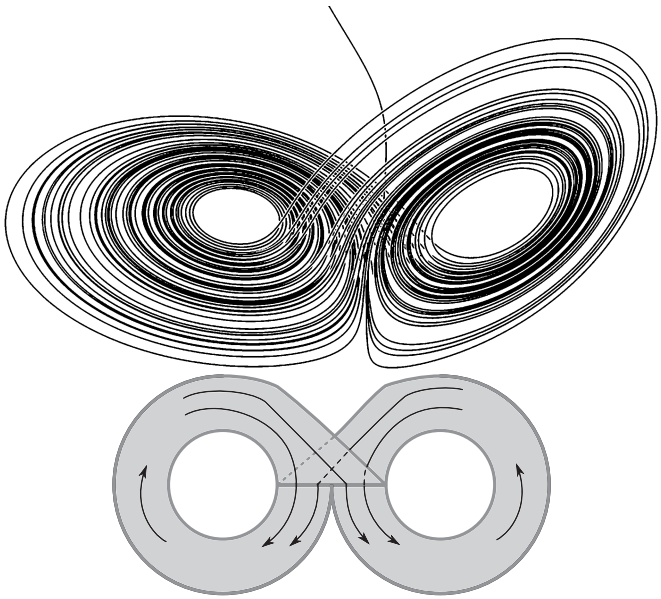

Figure 10. Top(10a): the Lorenz attractor; bottom(10b): the Lorenz template.

the strands going from left to right pass over the strands going from right to left, and where neither the left-to-right nor the right-to-left strands cross amongst themselves; see Figure 11. Lorenz knots can also be described as the closures of Lorenz braids.

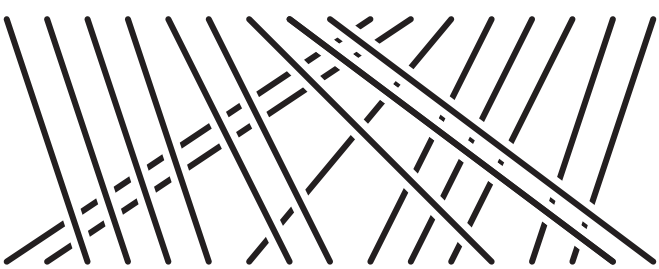

Figure 11. A Lorenz braid.

Williams approached Birman at a conference and asked her if she could identify some of the knots he was studying. She could, and their discussion quickly turned into a fruitful collaboration. In their first paper [44], Birman and Williams proved many theorems about Lorenz knots, including:

1. There are infinitely many (inequivalent) Lorenz knots.
2. Lorenz knots are prime.
3. Every algebraic knot is a Lorenz knot.
4. Every Lorenz knot is fibered.

In the third theorem, an algebraic knot is any component of the link of an isolated singularity of a complex curve. The fourth theorem requires some explanation. We can construct a 3-manifold from a surface S by the mapping torus construction: for $[\phi] \in \mathrm{MCG}(S)$, we take the product $S \times [0,1]$ and glue $S \times \{0\}$ to $S \times \{1\}$ by ϕ. The resulting 3-manifold has a natural map to S^1 with fiber S, and we say that the 3-manifold is fibered. A knot in $\mathbb{R}^3$ is said to be fibered if its complement in S^3 is a fibered 3-manifold.

Two decades after Birman and Williams, Ghys entered the picture. He was studying the manifold $M = \mathrm{PSL}_2(\mathbb{R})/\mathrm{PSL}_2(\mathbb{Z})$. The manifold M is homeomorphic to the complement in S^3 of the trefoil knot, and it can also be described as the unit tangent bundle of the modular surface (the quotient of the hyperbolic plane by $\mathrm{PSL}_2(\mathbb{Z})$). From the latter description, M has a geodesic flow. Ghys was studying the closed orbits in this flow, and he proved that the knots arising from these closed orbits are in natural bijection with the Lorenz knots (the connection was further investigated by Pinsky [72]). He further showed that the Rademacher function exactly records the linking number of each knot with the missing trefoil. We recommend Ghys's beautiful survey, written on the occasion of his plenary lecture at the International Congress of Mathematicians [50].

We next turn to the question, How common are Lorenz knots? Dehornoy, Ghys, and Jablon showed that of the 1,701,936 knots with at most 16 crossings in their diagrams, only 20 are Lorenz knots. And so from this point of view they appear to be rather rare. Birman and her postdoc Kofman took a different point of view. In order to explain it, we take a detour into hyperbolic geometry and the classification of 3-manifolds.

Thurston revolutionized the theory of 3-manifolds by showing that many knots are hyperbolic; that is, their complements in S^3 could be given complete Riemannian metrics of constant sectional curvature -1. By the Mostow rigidity theorem, hyperbolic structures on 3-manifolds are unique. In particular, a hyperbolic knot has a well-defined volume.

Thurston's work on knots eventually led him to formulate his geometrization conjecture, which shaped the field for several decades. The conjecture states that every 3-manifold can be decomposed into geometric pieces, namely, Seifert-fibered spaces (completely classified in the 1930s by Seifert) and hyperbolic manifolds. The Poincaré conjecture is a special case of Thurston's conjecture because there are no counterexamples to the latter among the Seifert-fiber
-ed spaces or the closed hyperbolic manifolds (which have infinite fundamental group).

The geometrization conjecture was famously proved by Perelman in 2003; see [46, 59, 69]. More recently, Agol and Wise proved that every closed hyperbolic 3-manifold has a finite cover that is fibered, verifying another conjecture of Thurston [2, 4, 77]. This gives a satisfying description of

the hyperbolic pieces of a 3-manifold: up to taking finite covers, they all come from surface homeomorphisms.

We return now to our story about Lorenz knots. Rather than organizing knots by the number of crossings in their diagrams, Birman and Kofman organized the hyperbolic knots by their volumes. They showed that of the 201 hyperbolic knots of smallest volume, more than half of them are Lorenz knots [8]. So among all knots, Lorenz knots are extremely rare, but among the small-volume hyperbolic knots, Lorenz knots are quite prevalent.

Birman and Williams wrote a companion paper [43] where they studied a different flow on S^3 and discovered an appropriate template in that case as well. In his gem of a thesis, Ghrist [49] showed that this flow is universal, in that it contains *all* knots as closed orbits, disproving a conjecture of Birman and Williams.

There are many other intriguing aspects to the story and tantalizing questions to answer. As Birman writes at the end of her survey [19], "There is a big world out there, and a great deal of structure, waiting to be discovered!"

Epilogue

A distinguishing feature of Birman's career is that her research has been motivated by her own vision, interests, and curiosity. There are very few instances where Birman was trying to answer someone else's question or solve someone else's problem. While this may seem like a risky way to approach a career in mathematics, it is hard to argue with the results. Besides the beautiful mathematics she has produced by herself and with her collaborators, she has had (as we have seen) a direct impact on two Fields Medals (Jones' and Mirzakhani's) and a plenary address at the International Congress of Mathematicians (Ghys'), among the many works she has helped to inspire.

As we touched on at the outset and throughout this article, Birman's work was in many cases ahead of its time, her foundational work finding applications (and appreciation) many years after the original discovery. Braid groups, mapping class groups, Torelli groups, and Lorenz knots were fringe topics when she started. With the breakthroughs of Jones, Mirzakhani, Thurston, Johnson, and Ghys we have seen the impact and validation of Birman's work.

As a recent collaborator of Birman's and as a researcher in the same field, the author has had the pleasure of seeing Birman's mathematics from up close and being inspired by her work. We eagerly look forward to the next chapters of Birman's career, including new discoveries by Birman herself and new perspectives on her prior work, yet to be uncovered.

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Dan Margalit

Credits

The opening page figure is by David Dumas (see `dumas.io/birmanseries/`).
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The Early Career Section is a new community project, featured here in the *Notices*. This column will provide information and suggestions for graduate students, job seekers, junior academics of all types, and those who mentor them. Angela Gibney serves as the editor of this section.

This month's theme is Writing. Next month's theme will be the Non-Academic Job Market.

Outward-Facing Mathematics: A Pitch

Jordan Ellenberg

We mathematicians complain a lot about the way our profession is portrayed to the general public. Come on, you've done it! And it's not like there's nothing to complain about. Topologists are always turning coffee cups into either doughnuts or bagels, number theorists spend their days gazing at prime numbers with the ultimate aim of divining a "pattern" that will let them compromise internet commerce, etc.. Math is presented as an activity that's inherently obscure, and unpleasant for all but a preselected caste of people, usually white, male, and bearded.

Why don't journalists talk about math as it really is? Because they don't know how it really is. We do. And if we want the public discourse about math to be richer, broader, and deeper, we need to tell our own stories. We have to be mathematicians where people can see us. We have to do some outward-facing mathematics.

Those are slogans. How do you actually do it?

There are a lot of ways people can math in public. You can organize outreach programs in the city where you live, you can podcast like Evelyn Lamb and Kevin Knudson, you can pitch ideas to *Quanta*, you can be a YouTube celebrity like the folks from *Numberphile*, you can run a math booth at Lollapalooza (nobody has done this last one yet but *it's actually a good idea*.)

But there's really only one form of outward-facing math I personally know well: writing about math for general-audience publications, which I've been doing for more than twenty years now. And I meet a lot of graduate students and early-career mathematicians who are interested in doing it too. So let me tell you some of the lessons I've learned.

There's no official way to start, so you can just start. There aren't degrees in math writing, and there are very few in science writing. I started by writing book reviews in a local free paper, then moved on in 2001 to writing for the online magazine *Slate* in its early days. Free local papers are

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gone and the notion of being "published" is a lot fuzzier than it used to be. So how do you write something and get people to see it? Social media drives attention, but no one has yet figured out a great way to tweet or Snap about math. That's why blogging is still alive for mathematicians, even as blogs have withered somewhat on the whole. I think best practice for getting started is to blog on a platform like Medium or WordPress, then use social media to bring readers to your writing. When you want to pitch a piece to a more formal publication, they'll want to see what your writing looks like: with the blog, you'll have something to show them. Don't let that stress you out, though! A wise veteran blogger once told me: the secret to blogging is to have pretty low standards. If you don't write until you feel extremely inspired, and don't hit publish until you're certain every word is perfect, you won't write many posts. Write enough posts and you're sure to have three really great ones for calling cards.

Editors are hungry for math content. Editors like an angle. If there's a math angle to a story in the news, pitch it! As someone with a degree in math, you have something to offer that most writers don't. Editors *love* having something other publications don't. Especially salable are pitches of the form "Everybody is saying X but because of math it's actually not-X."

Readers are more interested in pure math than you think. Not every math story has to be a mathematically inflected hot take on the headline of the week. I wrote a piece for *Slate* about Yitang Zhang's proof of bounded gaps between primes. I thought they were just being nice to a long-time writer by giving me some space to write about number theory. That piece turned out to be the most-read and most-shared thing I ever wrote for them. Those of us who teach spend a lot of hours talking about math in front of students who have been forced to be there. That makes it easy to forget that people out in the world generally admire math and are excited to learn about it, if we give them a way in! (I'd argue this is largely true of our students as well, but that's another article.)

But they're not as interested in pure math as we are. So when you write a 1000-word magazine piece about a development in pure mathematics, you're not going to give the same kind of full picture you would in a colloquium talk. You shouldn't even try. The result is the same as when you give an hour-long seminar talk in a 20-minute slot; you do everything too fast and the talk ends up conveying nothing at all. Your approach should be to transmit *one thing* about the research advance. In the case of Zhang's proof, all I explained was why number theorists *expected* bounded gaps to be true (because it's what you'd find if primes were randomly strewn). The one thing could be an idea in the proof, it could be something in the history of the problem, it could be something in the biography of the researcher, as long as it's something you can really cover in a small space. Of course there are a lot of important developments in math that really don't have *any* crisp, self-contained ideas to express. Those are probably not the right developments to write about in a short broad-circulation article.

Writing is teaching is (partly) marketing. Still feeling bad about having to oversimplify or (a phrase I mention only to object to it) "dumb down" the subject in order to talk to the public? I think of it like this. Writing is an extension of our teaching mission. Three hours a week of lecture isn't enough time to fully explain the material of a course, not even close. Our role is to convince the students that it's worth *thei*r time, spent on their own, to learn the math more fully. And the best way to do this is to let our own hot feelings for the math spill out into the way we talk and comport ourselves. Writing for the public is the same, with even more stringent time restrictions. To use crass commercial language, we are marketing. Building the brand! Your 1000 words are like a movie trailer. The trailer isn't the whole movie shown super-fast; it's a means of convincing people the movie is worth tracking down and watching in full. Most readers won't, and that's OK; they'll still have some loose sense of what the movie's about, and that's a lot better than nothing.

Writing can be occasional. For quite a while, including pre-tenure, I wrote 2-4 magazine articles a year. This is something you can definitely do without compromising your research. It just doesn't take that much time, and the more you do it, the faster you get. Writing about math doesn't typically require you to fly across the country and do interviews; you're mostly writing about things you already know and you can write in the time between other tasks. (Writing a book is a different story and I wouldn't recommend it to a junior mathematician still launching a research program.)

The chair/department/hiring committee will be OK with it. At first, I worried whether my department would mind that I was spending some of my time writing for the public, or whether the mathematical community at large would take me less seriously. Neither has happened. My experience is that public outreach is something most mathematicians simultaneously think should be done and don't want to do themselves. If you're one of those who's moved to do it, colleagues are grateful, not dismissive! And there's at least one direct professional reward: writing for the public is ace material for the "Broader Impact" section of an NSF grant proposal.

Good math writing is good writing. This is the most important part. Writing about math never succeeds unless it succeeds as writing. The same rules that apply to grumpy editorials, celebrity profiles, and sports analysis apply to us too. The words should be the right words and they should be in their right places. I once thought I might want to be a novelist, and I spent a year in creative writing grad school before getting my PhD in math. That's certainly more training than is necessary! But the lessons I learned there have served me every since. Math writing is interdisciplinary by

nature; you have to care about the math *and* the words. Whatever virtues sing to you in the writers you love (not just the math writers, all the writers) are the ones you should try to imitate. If you write a sentence you love, there will be readers who love it too. And some of them, drawn in, will learn more mathematics than they knew they wanted to.

Jordan Ellenberg

Credits

Author photo is by Mats Rudels.

To Write or Not to Write… a Book, and When?

Joseph H. Silverman

There are few academic pleasures that compare to holding in your hands a book that you've written, on a subject that you love, and seeing people teaching and learning and becoming inspired from your writing. So the answer to the initial question in the title of this article is "Yes, you should consider writing a book."

The answer to the second question, *when* to write a book, is trickier. Few methods of imparting advice are less effectual than those beginning with "Do as I say, and not as I do," but that's how I must start. I wrote my first book, a graduate-level textbook on elliptic curves that was published by Springer–Verlag, while I was a post-doc at MIT. The book was based on a course that I taught, and I foolishly believed that it would take about a year to turn my detailed course notes into a book. It took three years, which became my book-writing rule-of-thumb: books take roughly three times longer to write than the initial estimate. My book was well received, and even now, 30+ years later, it continues to be used, a source of considerable gratification. However, when hiring and promotion committees evaluate research accomplishments, textbooks, even those at an advanced level, tend to carry less weight than research articles. So I find it difficult to encourage early career mathematicians to write books. On the other hand, my own experience shows that one can go down that path and succeed, but it is important to continue publishing original research while writing your book.

What should your book be about? It should be on a subject that you are passionate about and on which you feel that you have something to say. And if that's the case, don't spend time worrying about general criticism from people who tell you about the book that you "should" be writing. As Dot admonishes Seurat in Sondheim's *Sunday in the park with George*:

> *Stop worrying if your vision is new.*
> *Let others make that decision…*
> *they usually do!*
> *You keep moving on.*

Okay, you've chosen your topic, the time is now, and you're ready to start writing your magnum opus. I generally start with a rough outline of the material that I want to cover, first by chapter, then by section; but don't become too firmly wedded to your initial plans. The preliminary table of contents for my first book included topics that eventually filled two volumes, with 20% of the topics untouched!

A great way to generate the core of a book is to use lecture notes from a course that you developed. But keep in mind that lecture notes are not books, and it can be surprisingly more difficult to clearly explain mathematics on a printed page than in a classroom. Your book will require more explanation and more detail than your notes. However, keep in mind that it's also possible to go too far in the other direction and to provide so much detail that your reader loses sight of the forest for the trees. It's a delicate balance, and a good way to get it right is to put each section aside for a few days, and then try to read it as if you were learning the subject for the first time. Which leads me to one of the keys to good writing…

Reread every paragraph and every sentence and every word, and think about whether they are necessary and/or can be rewritten to better convey your meaning. This may mean writing a paragraph or a page or an entire section,

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then throwing it out and rewriting it now that you've figured out how to do it right. Yes, this can be time consuming, but it will immeasurably improve your exposition. And to help justify the added time, you can balance the extra hours that you spend writing a page against the hundreds of hours that your (hopefully) thousands of readers will spend reading and studying it.

Once you have a draft of your book, there is no substitute for using it to teach a course. You'll find yourself making lots of changes and correcting lots of typos. And be sure to ask your students for feedback. Another way to get invaluable advice is to find a colleague who's willing to test drive your book.

It's finally time to present your work to the world. The traditional approach, and the one most authors still follow, is to sign a contract with a publisher, who then handles the myriad details of converting your source file into a physical and/or electronic book. This includes copyediting, printing, marketing, and distribution. In return, the publisher gives you a percentage of the sales price as a royalty.[1] If you go this route, you should be prepared to provide (1) an introduction to your book that explains what it contains and how it fits into the existing literature; (2) the table of contents; and (3) a couple of sample chapters. This can be done via email, starting with a brief note asking the publisher if they might be interested. But even better, if you can attend the annual Joint Mathematics Meeting of the AMS/MAA, visit the booths of potential publishers such as the AMS, Springer–Verlag, Cambridge University Press, etc., and make appointments to speak with their acquisitions editors. And note that publishing a book is different from submitting a research article; it is quite acceptable to show your book to several publishers simultaneously. Finally, once you are offered a contract, be sure to read it carefully. Some items may be negotiable, while others are not.

Back in the dark ages of the pre-21st century, there were few alternatives to using a publisher to print and distribute your book, and even now, this path offers many advantages. But there are other options, including in particular self-publishing your book online. You can do this by simply posting it on your own website, or for wider distribution, you can upload it to a website such as AMS Open Math Notes (`https://www.ams.org/open-math-notes`). But be sure that the site lets you retain the copyright to your material.

There is far more to say, and people far more experienced than me have written entire books about how to write books. So the final lesson that I will impart, and which I will now follow myself, is to resist the urge to keep adding material and to wrap up your project when it has reached a suitable length.

[1] *But don't plan on quitting your day job and living off the royalties from your advanced mathematics textbooks!*

Writing a book can be a daunting task, especially given the many other calls on your time. So in closing, I can do no better than reproduce a quote from someone who found time to write more than 30 books amidst an event-filled political career:

> *Writing a long and substantial book is like having a friend and companion at your side, to whom you can always turn for comfort and amusement, and whose society becomes more attractive as a new and widening field of interest is lighted in your mind.*
>
> —Winston Churchill

Joseph H. Silverman

Credits

Author photo is courtesy of the author.

Preparing Your Results for Publication

Julia Hartmann

You've done some work, finished a project, maybe a thesis. Now it is time to start thinking about turning your results into a paper. The main reason you want to do this is so that other people can see (and maybe even start using) your results. So make your paper as readable as possible! How well the paper is written will not only influence whether people will read the paper, but also how closely they read it, and what impression they will have of the paper (and you!) afterwards. The following are some guidelines I've given to students in the past. You may not want to follow them in exactly this order, but I think eventually you will want to think about all of these aspects.

Before we start, let me say that there are several general resources on **writing mathematics**, e.g. Paul Halmos' paper "How to Write Mathematics," or David Goss' "Some Hints on Mathematical Style." I urge you to read some of them and consider the advice given there. I will not repeat any of it here.

Your first step on the way to a paper is probably to write up the **mathematical skeleton** of your paper, i.e., the core statements and their proofs. If you have written a thesis, consider yourself lucky—you have already done this part! If not, then even if you think you know how to prove a result and have good notes, it can still take a lot of additional headache (and time!) to put all the details into writing. Sometimes not all of this will make it into the final version of the paper, but it will still be valuable to have. What you consider obvious right now after working on something for a long time may not look so obvious to you when someone emails you with a question about it ten years down the road. The order of the different pieces is likely to change as you are writing the paper, but it will be easy to move things around as needed and fill in additional text. This is also a good time to plan notation and terminology you are going to use.

Now, decide what the **story** is you'd like to tell. There is usually more than one way to present a result and the work that leads to it. If you are describing a framework developed to prove a particular theorem, keep in mind that it may be more broadly applicable, and if so, make sure to point this out. Moreover, a consequence may be more striking to a wider audience even if mathematically weaker than your main result. Talk about your work to other people. Consider giving a talk at your home institution. This will force you to come up with a way to "pitch" your story. You might also receive helpful feedback on your results and comments on connections to other existing work. During the process of writing, you may find that your conception of the story has changed, and this may change the idea of how to best present it.

Next, the **structure of your paper** needs to be well thought out. It should reflect the story you've decided to tell and help the reader understand your work. Keep the logical flow simple, and try to create a clear path to your main result. For example, avoid having to reference results that are stated later, and make it easy to skip technical details on a first reading. Breaking your paper up into sections can help the reader navigate. (In that case, every section should have its own—short—introduction.) But don't dissect your paper so much you destroy its flow. A ten or fifteen page paper is unlikely to benefit from subsubsections.

Next, think about your **audience**. In books, one often finds a little paragraph about who this is aimed at and what the reader is assumed to be familiar with. While you

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probably don't want to put it into your paper, you should ask yourself that exact same question and have a clear answer in mind while you're writing. This will help you be consistent, which in turn will make the paper easier to read. Not having thought about this is one of the more common mistakes in writing a research article. There are many papers that assume familiarity with some advanced concept but at the same time give definitions of more basic terms. It may help to think about how you would explain things to a particular person in your intended audience. Thinking about your audience is especially important when you work at the intersection of various fields. If you write only for people who are familiar with all those areas, your audience will be small. So try to write for a bigger audience. This does not necessarily mean you need to start giving lots of definitions, sometimes just adding a few references and additional explanatory text can go a long way. If your work falls into more than one field and the results are independent enough, you may also want to consider turning it into more than one paper.

From my experience, the **introduction** is best written once the main body of your paper is finished. Your introduction provides the context for your paper and explains the big picture—it is a very important part of your paper, and there is a separate article in this issue devoted to it. At the end of your introduction, consider including a little paragraph explaining the structure of the manuscript, as a quick guide to the reader. And whether here or at the end of the paper: Do not forget the acknowledgments!

Once you've written your first version, wait a week. Then read it again. You are likely to see things that could be improved, e.g., things that are not quite right, unclear, repetitive, left unsaid, should be moved around, proofs that could be simplified, etc. Fix it! You may want to do this process more than once. After that, you should **obtain feedback**. Find someone to read your draft—a fellow graduate student or postdoc, or a mentor, or just anyone who you know might be interested and knowledgeable enough to plough through your manuscript. Ask them to look for anything from general comments to typos. When I was a graduate student, my advisor (and in fact, all the postdocs working in his group) gave me drafts to comment on, and I learned a lot from that, a win-win. It's also a good idea to post papers to the arXiv preprint server at least a couple of weeks before submitting them to a journal. You might get some initial comments from the community, and those can improve your paper. For example, someone may point out a related reference that you had not looked at. (This also reduces the chances of upsetting a not-referred to researcher who may end up being your referee.)

There are many factors going into **which journal** you want to send your paper to. Among them are the importance of your work, whether it is suitable for a very general journal or better fits into a more specialized one, but also practical points like the length of the paper. Get advice from a more senior member in your community if you are unsure. Especially for early career researchers, the time aspect (from submission to report, from positive report to decision by the editors, from acceptance to publication) is important. The AMS publishes this information for a number of journals. Many journals do pre-reports, so at least if your paper is rejected for somewhat general reasons, you will find out soon and can resubmit elsewhere.

Some results are so important that people will have to use them and cite the papers they were published in, regardless of how well they are written. For most of us, these are the exceptions. So do not underestimate the importance of spending time and effort to write your papers well. The mathematicians who do read your work will thank you for it!

Julia Hartmann

Credits

Author photo is by Steven Kasich.

The Art of Writing Introductions

John Etnyre

Why should I, or anyone, read your paper or care about your results? Answering these questions is what a great introduction to a mathematics research paper should do. I am sure most people have, on numerous occasions, picked up a paper that looked promising based on its title, only to abandon it after skimming through the introduction that was either short and unenlightening, or long, confusing, and unhelpful. Some papers are so important that people will read them no matter what, but the majority will be read and appreciated, to whatever extent that they are, based on how good the introduction is. What can you do to maximize the chance people will appreciate your work?

The best way to figure this out is to read lots of introductions to papers and carefully think what works well and what does not. But let's discuss a few important ideas to get you started writing masterful introductions! First and foremost, **the introduction should clearly state, in a theorem environment, what your main results are**, so with a quick glance, experts can see what is really in the paper. If a theorem is not mentioned in the introduction, then assume that few if any will read it. The statements should be as precise as possible, though on some occasions some paraphrasing or simplifications might be appropriate if the precise statements involve new technical definitions or ideas that would be hard to work into the introduction. The goal should be that anyone can get the main idea of the result by reading your theorem and possibly the adjacent paragraphs. Just by way of contrast, I point out that there are many papers that state no theorems in their introduction, but refer to dozens sprinkled throughout countless pages of the paper, so the reader is left flipping back and forth to try to even get a sense of what is in the paper (assuming they even take the time for that, which is probably unlikely). This is not ideal, but it also brings us to an important caveat of this advice, and probably all advice: rules are made to be broken! While reading the above paragraph you might think, "Wait, I know amazing papers that don't state theorems in their introduction." This is true, such papers exist, and great writers can get away with this in some very limited situations. However, in the majority of cases it would be best not to try this.

Moving on, a common problem writers have, especially early in their career, is to overestimate what everyone will know about their work and how it fits into the research world. Assuming that everyone will understand the context of your work, and why it is really interesting, is not a good idea. Most work is focused on some part of a bigger program or problem, and even experts in a field might not, in any given moment, recall the subtleties and details to every interesting problem in their field. So tell them, and all the other readers who will have no chance of appreciating the context without some help from you. **Explain the big picture.** How does your work fit into a larger research program. Why is this a significant advance. What are new ideas or techniques that you introduce. What is surprising about the results. What is the key takeaway. All of these things should be made clear in your introduction. Don't assume the reader will sort this all out themselves. They won't. Remember, people will often interpret the results in your paper exactly the way you tell them to interpret them (or at least that will be their best possible interpretation). It is especially important to keep this in mind if you are working at the juncture of two or more fields, where it is quite likely that a reader might, at best, be familiar with one of the fields.

Another helpful point is **make sure the introduction is broken into digestible pieces**. What I mean by this is that if your introduction is more than a page or two it would be helpful if important ideas and different results are highlighted by sectioning them into meaningful pieces. This can be done through the use of subsections, theorem

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and definition environments, figures, and other such markers that indicate when new ideas or results are coming into play. Make it easy for readers to find the parts of your introduction in which they are most interested, or when they've decided they understand or don't care about one topic or idea in your paper they can move onto the next without difficulty.

While this may seem to contradict with other ideas above, **it is best to try to keep your introduction as brief as is reasonably possible**. People are busy and will only devote so much time to a paper, so do not tax their attention span. For example, when discussing context and background it is easy to get carried away. If something is a tangent to the main ideas and is not that important to the paper, then it is probably best to leave it out. If there are common, but not universally known definitions, don't define them in the introduction. Just use them and refer to a background section for details about terms that are not completely standard in the field. A rule of thumb is that if you think a person in the field has a good chance of knowing the definition, then assume they do for the purposes of writing the introduction.

Let me reiterate, the best way to improve your crafting of introductions is to read many papers and critically think about what you liked about them and what you didn't like about them. Then think about how you can use these insights in your own writing. Another important tool is to ask for constructive feedback on your papers from colleagues, especially more senior ones. But most importantly, think carefully about what you are doing while writing your introduction and what you are trying to accomplish. Good introductions can elevate the community's appreciation of your work and quite possibly you, so make sure to write great ones.

John Etnyre

Credits

Author photo is by Renay San Miguel of Georgia Tech College of Sciences.

Writing, and Reading, Referee Reports

Arend Bayer

Writing Referee Reports

You receive a referee request for the first time, and you are wondering what to do. Or you have written a few, or quite a few referee reports, and you are wondering whether your reports are actually useful. Here are some thoughts and opinions, based on my own experience as author, referee, and in the last few years as editor.

But first of all, please, please reply. I know of no editor who would get annoyed if you politely refuse a request, or if you slip the promised deadline for a report a little bit. But nothing is more detrimental to the process than not getting replies.

A referee report has various purposes. It is supposed to help the editors—some of them might be experts in the subjects, but others might not—to decide whether to accept the paper, it is supposed to verify correctness of the results and proofs, and it should point out necessary corrections or worthwhile improvements to the exposition. Last, and definitely not least, it should give feedback to the author. It should start with 1–2 paragraphs explaining the merits of the paper, and follow with detailed suggestions for corrections or improvements.

1. It is, these days, almost never good enough to write, "I recommend this article for publication in *Hilarious Mathematics*." Many more papers are written now than five years ago, or 10 years ago; as a result, even journals well below the top rank receive many more submissions than they can accept, and are forced to be selective in accepting papers. As a referee, you may have a view on the level of *Hilarious Mathematics*, but maybe the journal's backlog has recently grown so fast that the editors finally decided to be much more selective from now on. So instead, you should try to help the editors genuinely understand how strong the paper is. How to do that? It helps to summarize the results in your own words (especially if the introduction does not do a very good job of that), and put them into context. What exactly does the result add to prior work? (A theorem about graphs with up to 21 vertices becomes less impressive when it was already known for 20 vertices.) What natural questions does the result fail to answer? Is this a result others were waiting for, or that others tried to prove? How novel are the methods? Will the methods, or the results, be used by others? Does this start a new development, or does this conclude a longer program by finally giving a complete answer? Which other journals would the paper deserve to be published in? In all that, remember that a paper can be strong for many different reasons: some introduce a new idea; others prove an important technical lemma that had withstood earlier attempts; yet another will point out a simple but new connection between two different fields within mathematics. Be open-minded.
2. In my view, you should write these comments to be read by both the editors and the authors, and with the expectation that the authors will find out your identity. I don't mean this literally—authors will not normally find out the identity of referees, and guesses are wrong more often than not. And I don't mean that you should be reluctant to criticize. Instead, I mean that you should be fair in describing both the strengths and the weaknesses of the paper. Most authors appreciate a report that values what the paper adds to the literature, even if the overall

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judgment leads to a rejection. Such detailed constructive feedback can even be more motivating than strong praise from a referee that seems to have barely read the paper.

3. So what is the purpose of the box labeled "Comments for the editors"? I would only use it for comments that could identify you, or that suggest additional referees.
4. Try to avoid biases. There'd be a lot to say, but let me keep it to "Always write as if you knew the authors well." In my experience, there is no stronger bias than the one against authors we don't know personally.
5. A report doesn't have to be perfect. If you believe a proof of a lemma is wrong, you can just say so; if it turns out you misunderstood something, you probably won't be the only reader who did. If you can't understand an argument, ask the authors for clarification.
6. Restrict yourself to suggestions that can lead to clear, objective improvements. A referee report is not the place to advertise your view on the Oxford comma, whether to choose a basis, or the right way to prove a standard lemma.
7. To what extent is it your responsibility to check the correctness of the results and proofs? There are differing views on this—the ultimate responsibility for correctness is always with the authors. At the same time, referees should make an effort to convince themselves of the correctness, and to check for possible errors.
8. The paper is badly written? It certainly helps if you can make constructive suggestions for improvements, and editors and authors will appreciate your effort—especially if the author is fairly junior. But don't hesitate to recommend rejection if the exposition would need major improvements. If it's too hard to read for you, it's probably too hard to read for many others, and the paper is less likely to be influential. Enforcing a standard of exposition is part of your role as referee.

Reading Referee Reports

Let's turn to the other side. You've finally received a reply to your submission. What now? First, nothing productive can come out of trying to guess the identity of the referee, and you'd be wrong more often than not (see 2.).

It is important to take every referee remark seriously. Quite a few times my own papers were improved by referee suggestions that I initially found unconvincing. A suggested correction to a lemma doesn't make sense to you? Well, the referee has probably spent more time on this lemma than almost any other reader, so if they couldn't make sense of it… Still, referees are allowed to be wrong (see 5.), and it is fine to politely explain that. However, it is usually best to also clarify the explanation in the paper. And always avoid arguing with the referee.

Maybe the editors reject the paper. Hopefully, they include a referee report (see 2.) that helps you understand the decision. Keep in mind that it's an imperfect process—many journals have to be more selective than you might know (see 1.), and aiming for perfection would require too many resources, and make the process even slower. But sometimes, the report is infuriating; perhaps the referee completely misunderstood the main results. Well, that is unfair! At the same time—maybe they would not be the only reader to misunderstand them? You should probably rewrite (at least) the introduction before resubmitting the paper elsewhere.

But, you object, the referee really got so much wrong that you want to send an angry reply to the editors? Don't. Don't! Don't send that email!! A day or two later you still think that the referee has gotten many facts objectively wrong, and that they evidently did not give your paper a fair reading? In that hopefully exceptional case, it would certainly be fine to write a polite, friendly email (that you should ask a mentor or colleague to look at before sending) to the editors explaining your case.

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How do AMS Graduate Student Chapters support the mathematical community and beyond?

Photos courtesy of Sam Houston State University.

Sam Houston State University | Hurricane Harvey Goods Drive

As fall Semester began, Hurricane Harvey hit South Texas. To help with recovery, our student-led chapter hosted a goods drive. We provided a list of most-needed items, posted flyers, sent out emails letting faculty know, and put out collection boxes. For the goods drive, we partnered with the Good Shepherd Mission. A professor's wife, working with the Mission, delivered and distributed our goods. We had a great collection of donated items, including food, toiletries, and other household goods. We were very proud with the outcome of the drive.

Photo courtesy of Wesleyan University.

Wesleyan University | Teaching Panels

As in past fall semesters, we organized a teaching panel for graduate students, who will teach for the first time this coming semester. The event featured our more-experienced graduate students sharing their know-how teaching at Wesleyan, and their advice for classroom success.

Photo courtesy of University of Kansas.

University of Kansas | Community Workshops

Each year for Math Awareness Month, the KU math department hosts a math competition and workshops for elementary school students. These workshops bring in local elementary school classes to participate in fun interactive activities designed to encourage mathematical thinking.

Photo courtesy of University of New Orleans.

University of New Orleans | "PI Talks!" Seminars

Seminars held every semester provide a venue for current graduate students to present their work on an interesting topic to other students. Faculty and staff from the mathematics department and other related fields are also invited to present their research to students who are looking for an advisor and seeking to learn about research in the department.

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In Memory of Marina Ratner 1938–2017

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MEMORIAL TRIBUTE

Marina Ratner

S. G. Dani

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Marina Evseevna Ratner, renowned for her work in dynamics, passed away on July 7, 2017, at her home at El Cerrito, California, USA, at the age of 78. Her profound contributions, establishing the Raghunathan conjecture and its variants, from the 1990s when she was in her early fifties, have become a milestone in homogeneous dynamics and have had an impact on the study of a broad range of areas of mathematics, including dynamics, diophantine approximation, ergodic theory, geometry, and Lie group theory.

Marina was born in Moscow on October 30, 1938 to scientist parents, her father a plant physiologist and mother a chemist. As a Jewish family they had a difficult time in Russia at that time. In particular, her own mother lost her job for having corresponded with none other than her mother who was in Israel, which was considered an enemy state. Marina was educated in Moscow and fell in love with mathematics when she was in the fifth grade; "mathematics came naturally to me and I felt unmatched satisfaction solving difficult problems" she was to aver later.[1] After completing school she gained admission to the Moscow State University, which, with the dawning of the Khrushchev era, had begun to accept Jewish students on an equal footing.

After graduating from the University in 1961, Ratner worked for four years as an assistant in the Applied Statistics Group of A. N. Kolmogorov, the celebrated Russian mathematician who laid the foundations of measure-theoretic probability theory and had a great influence on her during her undergraduate years. Kolmogorov had an intensive training program for talented high school students with which Marina was actively involved. It is also during these years that she gave birth to a daughter from a short-lived marriage.

[1] *M. Cook, Marina Ratner, in:* Mathematicians: An Outer View of the Inner World, *Princeton University Press, 2009, pp. 90–91.*

In 1965 Marina took up research under the supervision of Ya. G. Sinai, a former student of Kolmogorov, renowned, in particular, for his role jointly with his advisor in the development of the very influential "entropy" invariant in ergodic theory around 1960.[2] In the context of how the theory was developing then, in Russia, the geodesic flows associated with surfaces of negative curvature had emerged as crucial examples for study from an ergodic-theoretic point of view, and Ratner also wrote her thesis on this topic.[3] Apart from the examples themselves, a general class of systems known as Anosov flows, named after D. V. Anosov who introduced and proved some deep results about them, were of interest, and Ratner worked on the asymptotic statistical properties of these flows as well. For the work she received the equivalent of the PhD degree in 1969 from Moscow State University.

After receiving the degree Ratner was employed as an assistant at the High Technical Engineering School in Moscow. In 1970 the government of USSR was led, in the face of international pressure, to increase substantially the emigration quotas, sparking an exodus of Russian Jews to Israel, of which earlier there had just been a trickle. Notwithstanding the relaxation in the policy, the government and the bureaucracy in general were highly resistant to emigration and treated those desirous of migrating with utmost severity in various ways. Thus, when Ratner applied for a visa that year (1970) to emigrate to Israel, she was dismissed from the job at the Engineering School.

[2]*A dynamical system in the present context means a one-parameter group of transformations of a space (sometimes called the phase space), with the parameter representing time (which could be continuous or discrete); the theory focuses on the long-term behavior of the trajectories of points under application of the transformations, namely, as the time parameter tends to infinity. In ergodic theory the phase space is further considered equipped with a measure of unit mass, namely, a probability is associated for points to belong to various subsets; the system is said to be measure-preserving if the probability remains unchanged when any of the transformations under consideration is applied to a point. In these instances one often focuses on trajectories of "generic points" in terms of the measure or equivalently in statistical terms with respect to the initial point. The Kolmogorov–Sinai entropy is a nonnegative number associated with each measure-preserving system, and when the entropy of two systems is different, their long-term behavior is different. The invariant thus enabled distinguishing dynamical systems on a much finer scale than was possible before.*

[3]*The "geodesic flow" consists of starting with a given point on the surface (or manifold in general) and a direction from that point and moving on, along the distance-minimizing paths corresponding to the geometry of the surface, for the desired amount of time, and noting the point of arrival and the direction of movement at that point; thus, the phase space in this case is formed of pairs consisting of a point of the surface and a direction at that point, and the above procedure describes how the transformation is defined. On the usual sphere the trajectories (paths) of such a flow would follow the great circle, and return to the original point after a fixed amount of time. However, when the surface has negative curvature, the trajectories move away from each other substantially, exponentially in time, and when the surface is compact, "almost all" of them (statistically) tend to fill up the whole space with the passage of more and more time.*

Marina Ratner giving a talk at the International Colloquium at TIFR, 1996.

Ratner landed in Israel, with her daughter, in 1971 and served as a lecturer until 1974 at the Hebrew University of Jerusalem and then at the Pre-academic School of the Hebrew University for another year. During this period Ratner continued her work on the geodesic flows, and also their generalizations in higher dimensions, and established in particular a property manifesting complete randomness of behavior of the trajectories of the Anosov flows, known as the Bernoulli property, that was much sought after in various systems.

In the West the study of flows analogous to Anosov flows, called "Axiom A" flows, was introduced by Stephen Smale, then at the University of California (UC), Berkeley. Apart from a certain generality of setting, this study involves separating the role of the measure and understanding the dynamics in terms of the construction of special kinds of partitions of the space, known as Markov partitions. Profound work was done in this direction by one of Smales's students, Robert E. Bowen, known commonly by his adopted name "Rufus" Bowen; the work led Bowen to the construction of invariant measures inherently associated to the systems in more general settings, now known as Bowen measures. Bowen completed his doctorate in 1970 and joined the Berkeley faculty in the same year. Not surprisingly, Bowen was interested in the work of Marina Ratner, and their correspondence during her Jerusalem years culminated in Ratner getting an invitation from UC Berkeley, which she joined in 1975 as acting assistant professor.

Another class of flows, called horocycle flows, are seen to have become a major love for Marina after moving to Berkeley. The geodesic flow associated with a surface of constant negative curvature has two natural companion flows, called the contracting horocycle flow and the expanding horocycle flow; they are actually twins, interchangeable through time reversal of directions at each point, so one may simply talk of *the* horocycle flow. Passing through each point of the phase space (consisting of a point of the surface together

with a direction at the point), there is a uniquely defined curve such that if we pick two points on any one of these curves and consider their trajectories under the geodesic flow we find them getting closer and closer with the passage of time, with the distance between the corresponding points of the trajectories tending to zero. Moreover, there is a natural parametrization on these curves with respect to which they can be thought of as the trajectories of a measure-preserving flow, and that is the (contracting) horocycle flow associated with the surface; the expanding horocycle flow arises similarly from consideration of trajectories of the geodesic flow in the reverse direction. These flows have historically proved to be very useful in studying the properties of the geodesic flows. While in the nature of things the horocycle flow would seem just a sidekick of the geometrically majestic geodesic flow, in the theory of dynamical systems the former has acquired a stature of its own, on account of some of its unique properties.

Marina Ratner with M. S. Raghunathan and S. G. Dani at the conference held in her honor at the Hebrew University of Jerusalem in October 2013.

In two papers published in 1978 and 1979 Ratner showed that the horocycle flows are "loosely Bernoulli" while their Cartesian squares are not "loosely Bernoulli"; the loose Bernoullicity property was introduced by J. Feldman, a colleague at Berkeley, and concerns the flow being similar to the standard winding line flows on the torus along lines with irrational slopes, if one allows the time parameter associated with the trajectories to be modified suitably. The fact, as established by Ratner, that the Cartesian square is not loosely Bernoulli for the horocycle flows is rather curious and was the first such instance to be found.

The early 1980s saw a major breakthrough in the understanding of the horocycle flows associated with compact surfaces, of constant negative curvature, at the hands of Ratner. A major question involved was the following: given two such surfaces whether the horocycle flows associated with them being isomorphic to each other as measure-preserving flows would imply that the surfaces themselves are geometrically indistinguishable. The answer to the corresponding question in the case of the geodesic flows is a definitive no since in fact the flows being Bernoullian means (by a well-known result of D. S. Ornstein) that any two of them are isomorphic to each other (up to a rescaling of the time parameter), irrespective of the specific geometries of the underlying surfaces. Ratner proved that on the other hand the horocycle flows corresponding to two distinct compact surfaces of constant negative curvature would never be isomorphic. This kind of phenomenon is referred to as rigidity. She also exhibited various variations of the rigidity property of the horocycle flows, through a series of papers, describing their factors, joinings, etc. (joining is a technical construction that enables comparing two systems with regard to the nature of their dynamics). Two of the three papers in this respect appeared, in 1982 and 1983, in the *Annals of Mathematics*. Apart from the immediate outcomes, which were striking in themselves, the work has germs of the ideas involved in the later celebrated work on the Raghunathan conjecture.

Let me now come to the Raghunathan conjecture, resolution of which was the major feat of Marina's work. Genesis of the conjecture is intricately connected with my student years at the Tata Institute of Fundamental Research, and it would be worthwhile to recall some details in that regard. I did my doctoral work in the early 1970s under the supervision of M. S. Raghunathan on flows on homogeneous spaces.[4] The thesis dealt primarily with the Kolmogorov property, which is a statistical property concerning a strong form of mixing, with no direct bearing on the behavior of individual orbits. However, in a paper written shortly after completing the thesis paper (before the award of the degree, in fact) I proved that all the orbits of actions of a class of flows, more specifically horospherical flows, are dense in the space. Around that time Jyotsna Dani (my wife) who was working under the supervision of S. Raghavan, at TIFR, had proved that for any vector whose coordinates are nonzero and not rational multiples of each other, the

[4] *For an idea of homogeneous spaces and dynamics on them let us consider a Euclidean space and agree to identify two given vectors of the space if their difference has integer coordinates, namely, we view the vectors modulo the lattice of vectors with integral coordinates; geometrically in effect we are considering a torus, and translations by vectors on any particular line define a translation flow on the torus. Similarly, when the elements of various matrix groups, more generally Lie groups, are considered modulo elements of large enough discrete subgroups (called lattices) we get what are called homogeneous spaces with a natural finite measure on them, and matrix multiplication by elements from a one-parameter subgroup of the ambient group, considered modulo the lattice, defines a flow on the homogeneous space. In the particular case when the group involved is the "modular group," the group of 2×2 matrices with real entries and determinant 1, the flows arising in this way (other than those which are periodic) in fact correspond to the geodesic and horocycle flows associated with various surfaces with constant negative curvature, via certain natural identification of the phase space of the flow with a homogeneous space of the modular group.*

Marina Ratner at the inaugural function of the International Colloquium on Lie Groups and Ergodic Theory at the Tata Institute of Fundamental Research, Mumbai, 1996. The others on the dais, from left to right are, S. G. Dani, Virendra Singh, R. Chidambaram, Hillel Furstenberg, Anatole Katok, and M. S. Raghunathan.

orbit under the action of the group of integral unimodular (determinant 1) matrices on the corresponding Euclidean space, is dense in the Euclidean space. At some point in time around 1975, which had these events in the background, when I was talking to Raghunathan about possible problems to pursue, he casually suggested a statement on the behavior of what are called unipotent flows[5] and quite nonchalantly added "call it my conjecture and prove it." He pointed out that proving it would in particular settle the conjecture of Oppenheim on density of values of indefinite forms at integral points,[6] which was one of the hallowed problems at that time in the Tata Institute precincts.

That statement of Raghunathan—the Raghunathan conjecture—first recorded in print in my *Inventiones Mathematicae* (1981) paper, is that the closure of any orbit of a unipotent one-parameter subgroup acting on a homogeneous space of finite volume is the orbit of a (possibly larger) subgroup of the ambient Lie group; in particular this means that each of these closures of orbits is a geometrically nice object—this is a remarkable thing to happen for a dynamical system, the crucial point being that the statement is being made for *every* orbit and not only the generic ones.[7] In that paper I proposed another conjecture, as a step toward proving the Raghunathan conjecture, relating to measures that are invariant under these flows, namely, that the ergodic[8] ones from among them in fact arise as measures invariant under the action the larger subgroups as above and are supported on a single orbit of the subgroup; a weak result was proved in the paper in that direction, partially vindicating the conjecture. Ratner proved the latter conjecture, which she referred to as "Raghunathan's measure conjecture," and in a separate paper deduced the original topological version.[9]

The Oppenheim conjecture itself, which had inspired the Raghunathan conjecture, was settled by G. A. Margulis in 1986 by proving a much weaker statement than the Raghunathan conjecture but in a similar spirit. Not surprisingly, proving the full conjecture led to a much broader perspective in the study of values of quadratic forms at points with integer coordinates, and many other applications, some quite immediately and many more over the years. There have been numerous results since then making use of Ratner's theorems in crucial ways, in a variety of contexts, and there is no doubt that it will serve as a mainstay for a good deal of mathematics in the coming decades.

The proofs are long and intricate and involve various ancilliary results. However, there is a beautiful key idea that concerns observing and adopting a property of the unipotent flows, which it may be worthwhile to recall. It may be informally stated as the following: if you find two trajectories of the flow having stayed quite close for reasonably long, then you can expect them to stay fairly close for substantially longer. This property of the unipotent flows, now called the Ratner property, has since acquired significance as a dynamical phenomenon.

As to be expected, Ratner gained considerable professional recognition. While her initial appointment at Berkeley had been a source of some controversy in the Department, her subsequent rise in the ranks seems to have

[5] *A flow on a homogeneous space of a matrix group as in the previous footnote is said to be unipotent when the one-parameter group involved consists of* unipotent *matrices, namely, matrices that have no eigenvalue, even in complex numbers, other than 1; for a general Lie group there is a variation of this involved. In the case of the modular group there are precisely the horocycle flows associated with surfaces of constant negative curvature and finite area.*

[6] *The conjecture originating from a paper of Alexander Oppenheim from 1929 predicted that for any nondegenerate indefinite quadratic form in at least three variables, which is not a multiple of a form with rational coefficients, the set of its values at integer tuples is dense in real numbers. It had been worked on by several notable number theorists, and by the 1980s many partial results were known, confirming the conjecture under various restrictions, but a general solution had eluded the efforts.*

[7] *For a one-parameter flow the orbit of a point is the set of all the points that can be reached by application of one of the transformations from the flow (including those corresponding to the negative value of the time parameter); similar terminology applies also to a more general group of transformations, in place of the one-parameter flows. The closure of the orbit means all the points that can be approximated by points on the orbit. In a typical dynamical system, even when the closures of almost all orbits are the whole space, for others, the exceptional ones, the closures can be very crazy. For instance, for the geodesic flows as in the above discussion there are orbit closures whose intersection with some curves transversal to the flow consists of a mess of uncountably many individual points disconnected from each other.*

[8] *Those that cannot be expressed nontrivially as a sum of two invariant measures.*

[9] *During the interim there were various partial results proved in that direction, but we shall not concern ourselves with it here.*

been smooth-sailing. She was elected in 1992 to the American Academy of Arts and Sciences, and in 1993 she was awarded the Ostrowski Prize.[10] In 1994 she won the John J. Carty Prize of the National Academy of Science. She was invited as a plenary speaker at the International Congress of Mathematicians, held in Zurich, in 1994, to become only the third woman mathematician, along with Ingrid Daubechies, to receive such an honor; Emmy Noether (in 1932) and Karen Uhlenbeck (in 1990) are the two women to have received the distinction earlier.

Marina Ratner at an excursion to Lake Louise, Alberta, Canada, with some of the delegates to the Conference on Ergodic Theory, held at the Banff International Research Station, Banff, Alberta, Canada, in July 2005. In the photo, from left to right, are Dave Witte Morris, Nimish Shah, Marina Ratner, S. G. Dani, and M. S. Raghunathan.

A conference on "Homogeneous Dynamics, Unipotent Flows, and Applications" was held at the Hebrew University of Jerusalem, October 13–17, 2013, in honor of Marina Ratner and her work, hosted by the Israel Institute for Advanced Studies and supported by the European Research Council. Earlier that year the Hebrew University of Jerusalem conferred upon her an honorary doctorate, at its Convocation held on June 16, 2013.

My personal contacts with Marina were, unfortunately, only sporadic, though they extended over a stretch of more than three decades.[11] I found her a very warm-hearted person, going out of her way to extend hospitality, which I had numerous occasions of enjoying together with my family. We had an International Colloquium on Lie Groups and Ergodic Theory at TIFR in 1996 and the pleasure of having Ratner as one of the speakers. She also contributed a paper to the proceedings of the colloquium on p-adic and S-arithmetic generalizations of the Raghunathan conjecture.

My last meeting with her was in 2015, when there was a special semester organized at the Mathematical Sciences Research Institute, Berkeley, on homogeneous dynamics. On one evening she had organized a dessert party at her home. It had been a wonderful evening thanks, apart from the sumptuous desserts of wide variety, to the warm reception by Marina that she conducted so cheerfully and energetically. My wife and I got to see some photographs from her visit to Mumbai, which she had dug out for the occasion. We also got to meet her daughter Anna and her children. On receiving the news of her sad demise, I emailed Anna a condolence message expressing shock and sadness, in which I also mentioned how energetic Marina had seemed at the party. In her response Anna added, "This is all very sudden and unexpected and difficult to comprehend. She was always so full of energy." Indeed, her sad demise was very abrupt, and we deeply miss her lively presence amongst us.

Credits

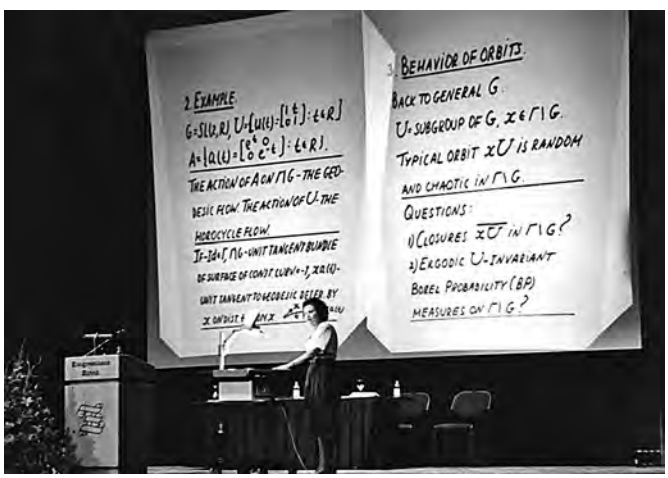
Opening photo of Marina Ratner is courtesy of Anna Ratner.
Photos of Marina Ratner at the 1996 International Colloquium at TIFR are courtesy of TIFR Archives.
Photo of Marina Ratner at the Hebrew University of Jerusalem, 2013 is courtesy of Israel Institute for Advanced Studies, The Hebrew University of Jerusalem.
Photo of Marina Ratner at Lake Louise is courtesy of Nimish A. Shah.

[10] *The prize is awarded, since 1989, by the Ostrowski Foundation every alternate year for outstanding achievements in pure mathematics or foundations of numerical analysis.*

[11] *The first of these was in the spring of 1982 when I had an opportunity to visit the University of California, Berkeley for the semester; though in anticipation of the visit I was hoping for a serious mathematical interaction with her, it turned out, much to my disappointment, that she was on sabbatical leave during the period, which she was spending at Stanford University, and we happened to meet only occasionally during her brief visits to Berkeley.*

Ratner's Work on Unipotent Flows and Its Impact

Elon Lindenstrauss, Peter Sarnak, and Amie Wilkinson



Ratner presenting her rigidity theorems in a plenary address to the 1994 ICM, Zurich.

In this note we delve a bit more into Ratner's rigidity theorems for unipotent flows and highlight some of their striking applications, expanding on the outline presented by Dani above. As the name suggests, these theorems assert that the closures, as well as related features, of the orbits of such flows are very restricted (rigid). As such they provide a fundamental and powerful tool for problems connected with these flows. The brilliant techniques that Ratner introduced and developed in establishing this rigidity have been the blueprint for similar rigidity theorems that have been proved more recently in other contexts.

We begin by describing the setup for the group of $d \times d$ matrices with real entries and determinant equal to 1 — that is, $\mathrm{SL}(d, \mathbb{R})$. An element $g \in \mathrm{SL}(d, \mathbb{R})$ is *unipotent* if $g - 1$ is a nilpotent matrix (we use 1 to denote the identity element in G), and we will say a group $U < G$ is unipotent if every element of U is unipotent. Connected unipotent subgroups of $\mathrm{SL}(d, \mathbb{R})$, in particular one-parameter unipotent subgroups, are basic objects in Ratner's work. A unipotent group is said to be a *one-parameter unipotent group* if there is a surjective homomorphism defined by polynomials from the additive group of real numbers onto the group; for instance

$$u(t) = \begin{pmatrix} 1 & t \\ & 1 \end{pmatrix} \quad \text{and} \quad u(t) = \begin{pmatrix} 1 & t & t^2/2 \\ & 1 & t \\ & & 1 \end{pmatrix}.$$

In both cases it is easy to verify directly that these polynomials do indeed define a homomorphism: i.e., for any $s, t \in \mathbb{R}$ it holds that $u(t + s) = u(t) \cdot u(s)$. While there is essentially no loss of generality in discussing only the case of $\mathrm{SL}(d, \mathbb{R})$, a more natural context is that of linear algebraic groups — subvarieties of $\mathrm{SL}(d, \mathbb{R})$ defined

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by polynomial equations that are closed under multiplications and taking inverses (this notion actually makes sense for more general fields than the real numbers; if we want to emphasize that we are working with the field of real numbers we will call such groups linear algebraic groups *over* $\mathbb{R}$). Connected unipotent subgroups of $\mathrm{SL}(d,\mathbb{R})$ are always linear algebraic groups. Another nice class of examples are the orthogonal groups. Given a quadratic form $Q(\mathbf{x})$ over $\mathbb{R}$ (positive definite or not) in d variables, one can consider the group $\mathrm{SO}(Q)$ of all matrices in $\mathrm{SL}(d,\mathbb{R})$ that preserve this form, i.e. $d\times d$-matrices M so that $Q(M\mathbf{x}) = Q(\mathbf{x})$ for all $x \in \mathbb{R}^d$. This group will be compact if and only if Q is a positive definite or a negative definite form.

Ratner's theorems on rigidity of unipotent group actions deal with the action of a unipotent group U on a quotient space of G by a discrete subgroup. An important example of such a quotient space is when $G = \mathrm{SL}(d,\mathbb{R})$ and $\Gamma = \mathrm{SL}(d,\mathbb{Z})$, in which case G/Γ can be identified with the space lattices in $\mathbb{R}^d$ that have unit covolume. A lattice in $\mathbb{R}^d$ can be specified by giving d linearly independent vectors that generate it — i.e. vectors $v_1, \ldots, v_d \in \mathbb{R}^d$ (that we prefer to think of as column vectors) so that $\Lambda = \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_d$, and the condition that the lattice has unit covolume amounts to requiring that $\det(v_1,\ldots,v_d) = 1$, or in other words that the matrix $g = (v_1,\ldots,v_d)$ obtained by joining together these d vectors be in $\mathrm{SL}(d,\mathbb{R})$. The generators of the lattice Λ are not uniquely determined: $v_1', \ldots, v_d'$ generate the same lattice as $v_1, \ldots, v_d$ if and only if $(v_1',\ldots,v_d') = (v_1,\ldots,v_d)\gamma$ for $\gamma \in \mathrm{SL}(d,\mathbb{Z})$, in other words, lattices of unit covolume in $\mathbb{R}^d$ are in one-to-one correspondence with elements of $\mathrm{SL}(d,\mathbb{R})/\mathrm{SL}(d,\mathbb{Z})$. Any matrix $h \in \mathrm{SL}(d,\mathbb{R})$ acts on this space by left multiplication; in terms of lattices this amounts to the map from the space of unit covolume lattices to itself taking a lattice $\Lambda < \mathbb{R}^d$ to the lattice $\{h.v : v \in \Lambda\}$.

This quotient space has the important property of having finite volume, or more precisely an $\mathrm{SL}(d,\mathbb{R})$-invariant *probability* measure. A subgroup Γ of a topological group G which is discrete and such that G/Γ has finite volume is called a lattice (admittedly, this can be a bit confusing at first since our basic example of such G/Γ is the space of lattices in $\mathbb{R}^d$..., though this terminology is consistent). Hermann Minkowski seems to have been the first to realize the importance of such quotients, and in particular the space of lattices in $\mathbb{R}^d$, to number theory at the turn of the 19th century. In the introduction to his book *Geometrie der Zahlen*, Minkowski writes[1]

> *This book contains a new kind of applications of analysis of the infinite to the theory of numbers or, better, creates a new bond between these two areas... Geometry of Numbers is how I have called this book, since I arrived at the methods, which deliver in it proofs of arithmetic theorems, through spatial considerations.*

Ratner's work is a remarkable contribution in the general theme of applying "analysis of the infinite" and "spatial considerations" to number theory.

So what did Ratner prove in these remarkable papers? Perhaps the easiest to explain is her Orbit Closure Classification Theorem, confirming an important conjecture of M. S. Raghunathan:

Theorem 1 (Ratner's Orbit Closure Theorem [M3]). *Let G be a real linear algebraic group as above, Γ a lattice in G and $U < G$ a connected unipotent group. Then for any point $x \in G/\Gamma$ the closure of its U-orbit is a very nice object: a single orbit of some closed connected group L that is sandwiched between U and G (and may coincide with either). Moreover, this single orbit of L has finite volume.*

Recall that the U-orbit of a point x is simply the set $\{u.x : u \in U\}$. Note that in particular this shows that any U-orbit closure has a natural U-invariant probability measure attached to it. We also remark that one can loosen the requirement that U be unipotent to U being generated by one-parameter unipotent groups — the passage from Theorem 1 to this more general statement is not very difficult. Unlike previous work towards Raghunathan's Conjecture, in particular Margulis' proof in the mid 1980s of the (then) fifty year old Oppenheim Conjecture using a special case of Raghunathan's Conjecture, Ratner's route to classifying orbit closures was not direct but by via a measure classification result:

Theorem 2 (Ratner's Measure Classification Theorem [M2, M1]). *Let G, Γ and U be as in Theorem 1. Then the only (Borel) probability measures on G/Γ that are invariant and ergodic under U are the natural measures on the orbit closures described in Theorem 1.*

This requires a bit of explanation: We equip $X = G/\Gamma$ with the Borel σ-algebra $\mathcal{B}$, and consider probability measures on the measurable space $(X,\mathcal{B})$. Such a measure μ is U-invariant if the push forward of it under left multiplication by every $u \in U$ remains the same; μ is U-ergodic if every U-invariant Borel subset of X is either null or conull. Every U-invariant probability measure can be presented as an average of ergodic ones, hence classifying the U-ergodic measures gives a description of all U-invariant probability measures on X. Dani conjectured this measure classification result in the same paper where Raghunathan's Conjecture first appeared.

It is possible to reduce both Theorem 1 and Theorem 2 to the case where U is a one-parameter unipotent group. The following theorem implies both of the theorems quoted

[1] *Translated from the original German to English.*

above in the one-parameter case, but is used by Ratner as a bridge allowing her to pass from the measure classification theorem (which, as we said in the outset, is the heart of her work on unipotent flows) to the obit closure theorem:

Theorem 3 (Ratner's Distribution Rigidity Theorem [M3]). *Let G, Γ, U be as above, and let $x \in G/\Gamma$. Then there is a U-ergodic probability measure m_x of the form given above (i.e. the uniform measure on a finite volume orbit of a connected group sandwiched between U and G) so that x is in the support of m_x and for any bounded continuous function f on G/Γ we have that the ergodic averages*

$$\frac{1}{T}\int_0^T f(u(t).x)\,dt \to \int f\,dm_x \qquad \text{as } T \to \infty. \quad (0.1)$$

The reader with some basic knowledge of ergodic theory might be fooled to think that (0.1) is an application of the Birkhoff Pointwise Ergodic Theorem. Not so! The Birkhoff Pointwise Ergodic Theorem only gives information about *almost every* point (with respect to a given ergodic measure). The whole point of Ratner's Distribution Rigidity Theorem is that it is true for each and every $x \in G/\Gamma$. Almost everywhere results are almost always much easier to prove,[2] but in a mathematical manifestation of Murphy's Law, such results might say something about virtually all points but if you are given a specific point and want to study its behaviour under a given action they tell you absolutely nothing. To give a simple analogy, it is trivial to prove that for a.e. $x \in [0,1]$ the asymptotic density of occurrence of each of the digits 0,1,2,...,9 in the decimal expansion of x is 1/10, but asking whether this holds for particular numbers of interest such as $2^{1/3}$ or π seems at present to be a hopelessly difficult question!

As it turns out, for some of the most juicy applications of these rigidity results a more general setup is required. To begin with, one may consider linear algebraic groups over other fields; and since the topological structure is very much in play here, the natural class of fields to look at are local fields, i.e. topological fields whose topology is locally compact, such as $\mathbb{R}$ or the p-adic numbers $\mathbb{Q}_p$. Both Ratner [M4] and independently Margulis and Tomanov [GAGM] extended the above results to this setting, and more generally to quotients G/Γ where $G = \prod_{i=1}^k G_i$ with each G_i a linear algebraic group over a local field of characteristic zero.[3] We shall refer to such quotient spaces G/Γ as *S-arithmetic* quotients, a terminology that probably needs some explanation which we omit to avoid too much of a digression. It would have been interesting to have such rigidity results also for local fields of positive characteristic such as $\mathbb{F}_q((t))$ — the field of formal Laurent series with coefficients in the finite field $\mathbb{F}_q$ with q elements — but there seem to be serious technical obstacles to doing so and only partial results in this direction are known.

The rigidity theorems of Ratner have had numerous applications in many areas of mathematics. A highly nontrivial special case of her general measure classification result, namely the classification of measures on a reducible product $(\mathrm{SL}(2,\mathbb{R})/\Gamma_1) \times (\mathrm{SL}(2,\mathbb{R})/\Gamma_1)$ invariant under a one-parameter unipotent group (the interesting case is classifying measures that project to the uniform measure on each $(\mathrm{SL}(2,\mathbb{R})/\Gamma_i)$ factor, or in the ergodic theoretic terminology, *joinings*) was proved by Ratner already in the early 1980s. The original motivation of Ratner in studying these flows was to understand better (and give natural examples for) a property of measure preserving systems called Loosely Bernoulli — we can view this somewhat anachronistically as an application of unipotent flows to the abstract theory of dynamical systems. Since then her work has had several other applications to abstract ergodic theory and descriptive set theory. There are very striking applications of her work to mathematical physics, for instance in the work of Marklof and Strömbergsson on the Lorentz gas, and to geometry. In this note we have chosen to highlight a couple of the many applications of her theorems (as well as the extension to products of linear groups over local fields as above) to number theory.

In making his famous conjecture, Raghunathan was motivated by the connection to the Oppenheim Conjecture, a connection that allowed Margulis to resolve this longstanding open problem by establishing a special case of the conjecture posed by Raghunathan [G]. Oppenheim conjectured in the 1930s that for any indefinite quadratic form Q in $d \geq 3$ variables that is not proportional to a quadratic form with integer coefficients, the set of values attained by Q at integer vectors, that is to say $Q(\mathbb{Z}^d)$, contains zero as a *non-isolated* point. Using Ratner's Measure Classification Theorem, and relying upon prior work by Dani and Margulis, Eskin, Margulis, and Mozes [AGS] were able not only to show that there are integer vectors $\mathbf{n} \in \mathbb{Z}^d$ for which $Q(\mathbf{n})$ is close to a given value (say 0), but to *count* the number of such vectors. More precisely, for indefinite quadratic forms as above, not of signature (1,2) or (2,2), Eskin, Margulis, and Mozes show that for any $a < b$, the number of integer vectors $\mathbf{n} \in \mathbb{Z}^d$ inside a ball of radius R for which $a < Q(\mathbf{n}) < b$ is asymptotically given by the volume of the corresponding shape cut by the two hypersurfaces $Q(\mathbf{x}) = a$ and $Q(\mathbf{x}) = b$ in this ball. Perhaps an illustration of the delicacy of the question is that this natural statement is false(!) for quadratic forms of signature

[2] *This is a slight pun—"almost everywhere" is used in the above sentence in its precise mathematical sense, whereas "almost always" is used in the ordinary, non-mathematical sense of the phrase...*

[3] *Note that our definitions of unipotent groups and one-parameter unipotent groups make sense over any field, and can be easily extended to the product case, e.g. a subgroup $U < \prod_{i=1}^k G_i$ (with each G_i defined over a different local field) is a one-parameter unipotent group if there is an i so that U is a one-parameter unipotent subgroup of G_i.*

(1,2) or (2,2), though in a follow-up paper Eskin, Mozes, and Margulis were able to prove this estimate for quadratic forms of signature (2,2) under a suitable Diophantine condition, a result which is of interest in the context of the study of the statistics of energy levels of quantization of integrable dynamical systems.

The reason unipotent dynamics is relevant to the Oppenheim Conjecture is that the symmetry group of an indefinite (real) quadratic form with ≥ 3 variables contains (indeed, is generated by) one-parameter unipotent groups. Surprisingly, there is a relatively recent application of the S-arithmetic analogue of Ratner's results to *positive definite, integral* forms.

Legendre's Three Squares Theorem says that a positive integer n can be presented as a sum of three squares if and only if it is not of the form $4^a(8b+7)$,with a, b integers. This is an example of a local-to-global principle: the quadratic form $Q(x,y,z) = x^2 + y^2 + z^2$ represents an integer n if and only if the congruences $Q(x,y,z) \equiv n \pmod{p^a}$ are solvable for any prime p and any $a \in \mathbb{N}$ (for a given p, consistency of this infinite set of congruences is equivalent to $Q(x,y,z) = n$ being solvable by p-adic integers). In this particular case, only the prime 2 can be an obstacle though there is another restriction on n implicit in the way that we set up the problem — that n is positive — which can be said to come from the "place at infinity," in other words from the necessity that $Q(x,y,z) = n$ be solvable over $\mathbb{R}$.

Legendre's Three Squares Theorem can be viewed as a special case of the following problem: Given a fixed positive definite integral quadratic form Q in many (say k) variables, which quadratic forms Q' in $\ell < k$ variables can be represented by Q? That is to say, when can we find a $k \times \ell$ integer matrix M so that as quadratic forms $Q' = Q \circ M$? For $\ell = 1$ and $Q = x^2 + y^2 + z^2$ this reduces to the question addressed by Legendre: the form $Q' = nx^2$ can be represented by Q iff n can be written as a sum of three squares. Local solvability — the existence of such matrix M with entries in $\mathbb{Z}_p$ for every p — is an obvious necessary condition that can be verified with a finite calculation.

Hsia, Kitaoka, and Kneser in 1978 established the validity of such a local-to-global principle for representing any form Q' in ℓ variables with sufficiently large square free discriminant by a given form Q in k variables once $k \geq 2\ell + 3$ by using more traditional number theoretic methods. This remained the best result on this very classical problem (essentially dating back to the work of Gauss) until Ellenberg and Venkatesh [JA] were able to use the S-arithmetic extensions to Ratner's Orbit Closure Theorem to very significantly reduce the restriction on k and ℓ to be $k \geq \ell + 5$. While we cannot get into the details of the argument, we note that even if a quadratic form Q is positive definite, hence its symmetry group over $\mathbb{R}$ is compact, over the p-adic numbers in general for $k \geq 3$ variables it would be a non-compact group with plenty of unipotents. In truth, the relevant symmetry group for this case is not the symmetry group of Q but the subgroup of this symmetry group fixing a given quadratic form in ℓ variables, but this is precisely why in this problem one needs to employ p-adics.

An even more surprising application of Ratner's work to number theory was given by Vatsal and Cornut–Vatsal (e.g. [V]). We do not give details here, but in these works families of elliptic curve L-functions, and in particular their central values (or derivatives thereof when their functional equation is odd rather than even), are considered. Using Ratner's Orbit Closure Theorem as a basic ingredient Vatsal (and in the more general cases Cornut and Vatsal) showed that all but finitely many of these values are not zero. When combined with well-known results towards the Birch and Swinnerton–Dyer Conjecture, this proves a conjecture of Mazur: essentially all the points on an elliptic curve whose coordinates lie in ring class fields with restricted ramification are generated by explicit special points first constructed by Heegner.

The impact of Ratner's work cannot be measured only by direct application of her seminal results. Techniques introduced by Ratner to study ergodic theoretic joinings in her early works on unipotent flows in the 1980s were a main inspiration in the work of the first named author on diagonalizable flows and its applications to Arithmetic Quantum Unique Ergodicity and equidistribution. Benoist and Quint were similarly inspired by Ratner's work in their breakthrough work understanding stationary measures and orbit closures for actions of thin groups on homogeneous spaces, and Eskin and Mirzakhani transformed the study of moduli spaces of abelian and quadratic differentials on Riemann surfaces by proving an analogue of Ratner's work in this setting.

The prevalence of deep and suprising applications of Ratner's Rigidity Theorems on unipotent flows is remarkable, and shows the richness of the subject of homogeneous dynamics and how interconnected it is with many other subjects. It is also a tribute to a wonderful mathematician who has left a legacy to future mathematicians for many years to come.

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Credits

Photo of Marina Ratner is courtesy of Anna Ratner.

To the Memory of Marina Ratner

Yakov Sinai

Sinai and Ratner in 1978.

I met Marina almost the same time as her elder sister, Yulia. Their father was a famous biologist. Marina started as a student of Moscow State University, and initially she was a student of A. N. Kolmogorov and later became a student of R. L. Dobrushin. When she entered graduate school she became interested in ergodic theory. This is how I became her advisor.

Marina married A. Samoilov when they both were undergraduate students of the second year. Their marriage didn't last long and soon they separated, though they maintained contact. Marina was left with her daughter, Anya. Marina was very close to the family of her daughter. During many years, Marina spent a lot of time with her grandchildren. Marina's friends knew that it was strictly forbidden to call her on Saturdays because she was always busy working the whole day with her grandson.

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Marina wrote her thesis about Markov partitions in multi-dimensional systems. At that time it was a very hot topic. One of the referees of Marina's thesis was V. A. Rokhlin, who wrote a very good report. This was important because the Scientific Council where Marina's thesis was considered was known at that time for its antisemitism. It was rather surprising that in the case of Marina the system worked well, including the voting of the PhD committee and the approval of the High Attestation Committee (VAK).

Quite soon she started to work in one of the Moscow Institutes where Marina was appointed to her first academic position. The fact that she got a position so quickly was rather unusual at the time.

In another case it would be a big step in someone's career. But not for Marina, because quite soon she decided to go in a different direction and applied to emigrate to Israel together with her daughter.

In Israel she joined the Institute of Mathematics at the Hebrew University and started teaching there. Marina did everything very well. Soon she became famous among her students. Many of them kept as souvenirs the notes left after Marina's classes.

A bit later Marina heard about some vacancies opening in Berkeley and moved there with her family. Berkeley became her home until the end of her life. She was elected as a full member of the National Academy of Sciences of the USA, and was invited as a plenary speaker to ICM-94 in Zürich.

Marina had many close friends in Berkeley and other places. One can mention Smale, Ornstein, Arnold, Fuchs, Pyatetskii-Shapiro, Kazhdan, Chorin, Zalenko, Kresin, and many others. Marina was always ready to help her friends and other people. I remember the case when our family arrived to Princeton after my son had seriously broken his leg. Marina contacted many people and eventually they helped us to find V. Golyakhowsky, who was a remarkable orthopedist. His treatment was excellent, and my son completely recovered and can freely walk now without any trace of the previous accident.

Marina was involved in many types of social and political activity, and was very strong and principled in promoting the causes that she believed in. For instance, she had strong opinions about mathematical education and education in general.

This text is a small part of what can be written about Marina. She was a great mathematician, a remarkable personality, and a close friend.

Credits

Photo is courtesy of Anna Ratner.

MEMORIAL TRIBUTE

Meetings: For Almost All of Our Lives

Boris Gurevich

Ratner with her daughter Anna, 1971.

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Ratner with her family, 1947 (left to right): mother Ksia, sister Yulia, Marina, and father Yehoshua.

I cannot claim that I have been Marina's close friend from our very first meeting. But I believe at some moment this became so. Our meetings lasted for many years, with frequency dependent on circumstances of our lives and on political events as well. I am going to remember several of these meetings in the hope that my story will shed additional light on this remarkable character.

I got to know Marina when we were about seven years old and went to the same musical school for children. By coincidence we had the same piano teacher, whose name was Anna Ratner. In one or two years I moved to another musical school, closer to my home, and we lost one another for several years.

Our next meeting occurred at the Mechanics and Mathematics Department of Moscow State University, where we entered simultaneously and quite independently. I happened to meet Marina at one of the first lectures and recognized her almost immediately, strange though it may seem. That is why when in several days our very sociable

Visiting Leningrad with fellow students—(left to right) Ilya Mindilin, Vener Galin, Lena Odnorobova (or Vera Steniushkina), Slava Perlov)—from Moscow, circa 1961.

fellow student tried to introduce me to her, this sounded funny for both of us.

During the first two years we were in different groups and met only at common lectures. But in our third year, everybody had to choose a specialization. And again, we made the same choice, which was probability, and got into the same group.

At that time the probability and statistics subdivision of the department was headed by A. N. Kolmogorov, and almost all who worked there were his former students. Kolmogorov was very active in various directions; in a few years he included Marina in a small, young team involved in his study of statistical laws in language. But her first supervisor was R. L. Dobrushin, who, as I know, liked very much her master's thesis in information theory.

Upon graduating from the university, Marina was for some time working at Kolmogorov's boarding school, a high school for gifted children from all over the country founded by Kolmogorov and later named after him. She also took part in the preparation of the principal works of Claude Shannon for publication in Russian.

About that time she married a student from our course, and I met her not too often. But in 1965 she came back to the university as a graduate student under the supervision of Y. G. Sinai, and our meetings became regular again because we both attended seminars on ergodic theory.

Once we examined an undergraduate. I began with a question, then Marina entered and I went out for some time, while she continued. When I came back, I was not pleased that she finished too fast. But later I decided that she was right: first, this student was Lenya Bunimovich, and second, Marina was very thorough in everything. Once, already in Berkeley, she observed that in one of her papers, it was written "weak convergence" instead of "weak* convergence," and she asked me to insert the asterisk by pen each time I was at a library where a journal with the paper was accessible. She told me, "I am a perfectionist," and this was the truth.

She published several papers and defended her PhD thesis on geodesic flows in 1969. Thereafter she was teaching at one of the technical universities in Moscow, but not for very long, because she applied for emigration to Israel.

As a girl of twelve, 1950.

Honestly speaking, I initially considered her intention reckless: I knew that she was going alone, with a small daughter, without language and having no relatives there.

But I had underestimated Marina: she overcame difficulties, which were indeed considerable, and in 1971 she was already working at the Hebrew University. At the time, the Soviet Union had no diplomatic relations with Israel and postal services were unreliable. Of fragmentary information from Marina I remember that in the fall of 1973, the university professors were asked to write their lectures down in order that the students, turned into soldiers for a while, could read them at the front.

When Marina moved to Berkeley, I used a possibility to hear something of her from J. Feldman, whom I met in 1977 in Warsaw at a conference on ergodic theory.

Only when Gorbachev came to power did mutual visits become possible. Marina came to Moscow more than one time in the 1990s and early 2000s. Once she left for a few days for Minsk, where a mathematical conference was conducted. Being aware of food shortage in Moscow at the time, she bought in Minsk, on her own initiative, some cheese for a small child of our friend. I appreciated her solicitous concern for her friends once again when I visited her at Berkeley in the late 1990s.

As far as I know, she came to Moscow for the last time in June of 2003 to the conference devoted to Kolmogorov's centennial, where she was an invited speaker and met many old friends.

I saw Marina for the last time in May 2014 in Oslo, where we were invited by Sinai, our common teacher, as his guests as he was awarded the Abel Prize. We walked through the city in full lilac bloom and followed Sinai, visiting the town of Stawanger for one day, where we took, together with a few friends, an excursion along a fjord; Marina took a number of snapshots there.

She was always worried about the health of others. Throughout several years she insisted that I should regularly inform her of my state of health (results of tests, etc.). Answering my questions about her own health, she would always insist that she was splendidly sound. I have kept her last message of July 1, 2017, in which she wrote about her problems, but hoped that the treatment would eventually help. I also thought so.

Credits

All photos are courtesy of Anna Ratner.

In Memory

D. Ornstein

Ratner with (left to right) Ornstein, Yakov Sinai, and Jack Feldman, 1978.

Marina Ratner was a good friend and colleague. Although we never wrote a joint paper, we did work together, and I was able to gain great appreciation for the depth of her mathematical ability. The commitment to her family was very impressive; she homeschooled her grandchildren. She was not a fan of affirmative action and made it very clear that she wanted her achievements to be rated solely on her mathematics not her gender.

I would like to call attention to Marina's work on the horocycle flow, which I hope will not be overlooked in the light of her later and more spectacular results. The study of the horocycle flow and the geodesic flow as flows on an abstract measure space began in 1938 with the work of Hedlund and Hopf. While the geodesic flow is the most random measure preserving flow on an abstract measure space, the horocycle flow has the opposite behavior. Marina elucidated its rigidity properties.

I will give just one example. If two horocycle flows on the quotient space M of $\mathrm{SL}(2,\mathbb{R})$ by a discrete subgroup are the same as measure preserving flows, then the underlying surfaces are conformally isometric. This means that even though we replaced M by an abstract measure space, the flow retains all of the geometry that we threw out.

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Credits

Photo is courtesy of Anna Ratner.

MEMORIAL TRIBUTE

Marina Ratner, *quelques évocations*

Jean-Paul Thouvenot

At my first encounter with Marina, in Jerusalem, in 1974, just after the Lavi Conference, I saw a young woman quite shy, sweet and smiling, pleased to receive a gift that a common friend had prepared for her from Paris. I met her later in Berkeley when she was already settled; I paid several visits there. Memory is sometimes strangely selective—I remember distinctly that, in an excursion which we took together with her and the Katok family to "Pebble Beach" (this I am not sure of), she had a very battered car, with the exhaust threatening to fall off at every turn.

Her mathematics, which had started in Moscow with Sinai, received the influence of the California environment, and one of her first works there was the proof that the horocycle flow is loosely Bernoulli, an abstract measure-theoretic property that was quite popular at that time in Berkeley, the impetus for it having been given by Jack Feldman. A second paper, which came quite quickly, was that the Cartesian square of the horocycle flow is not loosely Bernoulli. This was, for the group of people working in this field, quite unexpected and very strong. The non-loosely Bernoulli property all of a sudden being attached to a simple algebraic object, while all previous examples, starting with the one of Jack Feldman, required elaborate combinatorial constructions. This work of Marina is extremely difficult to read, and I remember, when I came to complain (the last time was not so long ago) to her, "But it is so simple, just follow what is written, everything is completely natural, you will not find any obstacle…" In this same work appeared for the first time the "shearing" that was going to play such an important role in her subsequent works. And then came, in an extraordinary succession across a few years of mathematical excitement, a list of papers in which she developed her theory of horocycle flows culminating in the complete description of their joinings, which entails all their rigidity properties. Strikingly, Marina always worked alone and never had coauthors. As to joinings, she managed all by herself, and to my knowledge, without trying to get too many contacts with the people close to her who were active in this field at that time.

On a visit that she paid to Paris at about the same epoch, I put her up in a nice hotel close to Jussieu (as I usually did with visitors). But almost immediately, she came to me quite pleased to have moved with her daughter to a very modest place close to Gare de l'Est, proudly announcing, "Believe me, it is the best possible place as a starting point to visit Paris." Singular Marina!

Her work took a new turning point when she got, as an elaboration of her previous ideas, her fundamental result on the Ragunathan measure conjecture. Strangely, her mathematics, so deep, which is so much alive nowadays, and in so many different directions, is most frequently used as a black box or as a model.

It was a great shock to receive the message from B. Weiss that she had died, shortly before a conference dedicated to the memory of Rufus Bowen, which she had accepted to attend.

I want to mention another memory, or more precisely, an image, because of the deep impression that it has left on me, although I cannot trace back exactly when it took place. I think that it was at a conference in Warwick: she was lecturing, so strong, so determined, and in appearance so fragile, all alone, in front of a huge audience.

In the same way as her mathematics does for our community, her presence, in the minds of those who have known her, persists with all the strength, the singularity, and the seduction of the exceptional.

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Remembering Marina Ratner

Hee Oh

Ratner with (left to right) François Ledrappier, Dmitry Kleinbock, Hillel Furstenberg, and Oh, 2005.

I first met Marina at the International Conference on Lie Groups and Ergodic Theory held at TIFR in Mumbai in January of 1996. I was a fourth-year graduate student working with Gregory Margulis.

She gave a talk on the p-adic and S-arithmetic generalizations of her earlier proof of Raghunathan's conjecture. I remember how she began her talk with the assertion that while some notations and definitions may be standard, she still needed them to know for herself what she was talking about. She then went on to spend quite a big chunk of her time introducing a long list of notations and basic definitions, such as Ad-unipotents and p-adic Lie algebras. At the time, her talk was too technical for me to follow, but her uncompromising style left a strong impression on me.

My own lecture was about my ongoing thesis work on the arithmeticity of discrete subgroups in a higher rank simple Lie group generated by lattices in a pair of opposite horospherical subgroups, which was a conjecture of Margulis based on Selberg's earlier work in the case of a product of $SL(2,\mathbb{R})$s. I had solved this conjecture for discrete subgroups of $SL(n,\mathbb{R})$ for $n \geq 4$. Ratner's theorem on orbit closures was a key ingredient of my proof. I did not get to receive any comments from her either on my talk or on my work at that time.

Seventeen years later in 2013, there was a conference in her honor titled "Homogenous Dynamics, Unipotent Flows, and Applications" at the Hebrew University. I had just finished my joint work with Amir Mohammadi on the classification of joining measures for geometrically finite subgroups of $SL(2,\mathbb{R})$ or of $SL(2,\mathbb{C})$. It was an extension of her work "Horocycle flows, joining and rigidity of products," published in *Annals of Mathematics* 1983, and our approach was to adapt her proof in the infinite-volume setting. I opened my lecture saying that I was proud of my mathematical aunt; she and Margulis shared a common advisor, Sinai. I then successfully squeezed the two subjects of discrete groups and joinings into my one-hour lecture and closed with the statement that I had started my mathematical career by applying Ratner's theorem as a black box and that I was now hoping to generalize her ideas in the infinite-volume setting. After my lecture, I asked Marina directly, "Did you like my lecture?" She said, "Yes, very much," with a big emphasis on "very," and asked, "Why don't you post your lecture notes in your webpage?" I jokingly replied to her, "Marina, who is going to read it?"

She once said in an email to me, "If a woman is good in math, she does not need encouragement or a role model. I remember when I was young, no matter what anyone would say, I knew that I would go to math. I did not need any encouragement for that. The same is probably true about you. Did you need encouragement?"

I wrote back, saying, "Marina, whether you wanted to or not, you have been a great source of pride and inspiration for female mathematicians in the area. I am very grateful to you for having been such a great role model."

Thank you Marina.

Credits

Photo is courtesy of Nimish A. Shah.

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Creating Opportunities and Building Confidence:

Clare Boothe Luce's Unexpected Support of Women in Math and Science

Della Dumbaugh

ABSTRACT. How did a woman who was a playwright and a politician advance American women in mathematics and science? This paper explores the life of Clare Boothe Luce and her pioneering—and unexpected—impact on the development of mathematics and science.

Introduction

With her death in 1987 Clare Boothe Luce bequeathed nearly $70 million[1] to establish a fund "to encourage women to enter, study, graduate and teach" in the fields of science, engineering, and mathematics. This decision seems an unlikely choice for a woman who, while alive, was widely known as a playwright, magazine editor, American ambassador to Italy, war correspondent, congresswoman, and wife of Henry Luce, who co-founded TIME Inc. Despite having no known connection to or interest in what are now STEM fields [Teltsch], Clare Boothe Luce challenged women to enter into and excel in more commonly male-dominated fields. Her vision established a foundation that has become "the most significant source of private support for women in science, math and engineering in the US [Grant Spotlight]."

The Clare Boothe Luce Program has supported more than 2300 women since awarding the first grants in 1989 [The Clare Boothe Luce Program].[2] The 30th anniversary of the initial Clare Boothe Luce Fund awards provides a timely opportunity to reflect on the life of Clare, to consider her motivation in establishing this support, and to explore the impact of her funding on women and institutions.

Clare Boothe Luce: Life Experiences Shaping a Bequest

On March 10, 1903, in New York City, Clare, born Ann Clare Boothe, began her life as she would live it—surrounded by conflict and drama. Clare was the second illegitimate child of Ann Snyder (Anglicized from Anna Clara Schneider) and William Franklin Boothe [Morris, 1997, p. 15]. William Boothe was legally married to another woman at the time. Although he subsequently divorced his first wife in 1906, William and Ann Snyder never married. After his once successful piano business dwindled, he worked as a medical salesman and, finally, as a musician. In search of work, William's musical career took the family to various cities, including Memphis, Nashville, and Chicago. Money grew increasingly scarce with each move. As William's financial resources faded, so did Ann's affection for him. She had met him as a flourishing executive and now he was an

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[1]*Roughly $156 million in 2018 dollars.*

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[2]*Interestingly, nearly twenty years before her death, Clare proposed the idea of considering a woman for a (Henry) Luce Fellowship. Specifically, in 1968, when asked her opinion on a proposed Luce Fellowship Program at Time, Inc. Clare Boothe Luce wrote mostly about "the man" or "him" in this position. Near the end of the letter, however, she dared to suggest, "Sooo—is there anything in the idea of a Time Inc. Associates Program, among whom, hopefully, the Board of Selection might annually choose a man, or* woman (please!) *worthy to be dubbed a Luce Fellow…" [Clare Boothe Luce to Andrew Heiskell, February 4, 1968, p. 8, Clare Boothe Luce papers, my emphasis].*

aging musician with too few prospects and too much of a drinking habit. Ann Snyder wanted more for her children and for herself. When Ann's father suffered a serious illness in September 1912, she took the opportunity to move her children to her parents' home in New Jersey. She eventually told acquaintances she was a widow. With the death of Ann's father in 1913, the family relocated to New York City [Morris, 1997, p. 39].

This transient lifestyle proved challenging for Clare. She had a difficult time making friends, a situation that would not improve in her lifetime. Clare spent two years at the Cathedral School of St. Mary's in Garden City, Long Island, where some students viewed her as "the most conceited girl in the school [Morris, 1997, p. 57]." Clare felt she would never succeed at St. Mary's, so she appealed to her mother to let her leave. Clare's mother subsequently enrolled her at the Castle School above Tarrytown-on-Hudson in New York. This move was intended to put Clare in a better position to find a suitable husband rather than earn a college degree. At the Castle, although Clare won the school's titles of "Most Artistic," "Cleverest," and "Prettiest," she finished second for "Most Ambitious," the only award she felt she truly deserved. As she expressed it in her diary, "[m]y whole heart and soul is wrapt [sic] up in three things: Mother, Brother and my ambition for success [Clare Boothe Luce Diary, February 6, 1919, as quoted in Morris, 1997, p. 61]."

Clare's drive for success remained with her throughout her life. She decided the best route to success was through marriage, and, in particular, marriage to a wealthy man. As she put it in a letter to a friend, "Damned if I'll ever love any mere man. Money! I need it and the power it brings, and someday you shall hear my name spoken of as—famous [Clare Boothe Luce to Ruth B. Morton, November 18, 1921, as quoted in Morris, 1997, p. 99]."[3] True to her word, Clare loved one man, but married another. At the age of twenty, she married George Tuttle Brokaw, a millionaire alcoholic more than twice her age who simultaneously doubled as New York's most eligible bachelor [Morris, 2014].

Four months after the wedding, Clare learned she was pregnant. Although she tried scalding hot baths as a way to induce an abortion, the child lived and Ann Clare Brokaw was born in 1924. The baby helped the marriage temporarily but could not save a marriage damaged from the start. Clare plotted how to exit the marriage "with minimum damage and the maximum amount of money [Morris, 1997, p. 140]." When Clare and Brokaw amicably divorced in May, 1929,[4] Clare received a settlement of a $425,000 trust fund, an annual income, and expenses for Ann. Following a difficult custody battle, each parent was allotted six months a year with Ann. For all its faults, the marriage provided Clare with plenty of money and an increased social confidence. Although "she had ample means to settle for the life of a socialite" after her divorce from Brokaw, she chose, instead, to "capitalize on her own abilities in the workplace [Morris, 2014, p. 29]."

In an attempt to give her life new direction and meaning, Clare interviewed for a position at *Vogue* magazine. After waiting all summer to hear from the magazine, Clare self-assuredly walked into the *Vogue* building and convinced an office assistant that she was a new employee. Soon enough, colleagues gave the new beautiful, professional woman sitting at an empty desk work to do. *Vogue*'s editor, Edna Woolman Chase, thought the publisher of the magazine, Condé Nast, had hired Clare. Nast, in turn, thought Chase had brought her on board the magazine's staff [Morris, 2014]. Clare received her first paycheck after one month [Morris, 1997, p. 163]. Consequently, with no formal education or experience in writing, Clare secured a job with one of the most popular magazines of the time. She soon moved down the hall to *Vanity Fair* with the title of Junior Editor. Her first piece "Talking Up—and Thinking Down: How to Be a Success in Society Without Saying a Single Word of Much Importance" appeared in 1930 [Clare Boothe Luce, "Talking Up"]. In this article, Clare encouraged readers to be conventional, predictable, safe, and even boring in order to have a successful conversation. She identified the six topics guaranteed to start a conversation: golf, the stock market, prohibition, theater, gossip, and current social activities [Clare Boothe Luce, "Talking Up," p. 39].

After the 1929 stock market crash, *Vanity Fair* struggled to adjust to the new economic conditions. Advertising revenues, for example, dropped twenty percent [Morris, 1997, p. 181]. Clare helped reestablish *Vanity Fair* as a serious magazine concerned with issues beyond the scope of fashion. Her confidence grew with the success of her public-affairs articles. She earned a promotion to associate editor. She used her candor and satire to develop her skills as a political writer. This work led her to the 1932 Democratic National Convention in Chicago, where she met Bernard Baruch, an advisor to Franklin D. Roosevelt and the fourth richest man in America. Baruch introduced Clare to many of the nation's most powerful and prominent men.

With her increasing success, Clare began to take some liberties at *Vanity Fair*. She requested weeks off for personal travel. When in the office, she often arrived late or left early. She produced fewer articles [Morris, 1997, p. 227]. Consequently, Condé Nast expressed concern over her schedule. He also questioned her ability to successfully balance her roles as an editor and author along with her

[3] *As Gore Vidal pointed out more than 75 years later, Clare expressed these thoughts fifteen years before Scarlett O'Hara leapt to and out of the pages of* Gone With the Wind *[Vidal, p. 208].*

[4] *Between 1867 and 1967, the Census Bureau measured the divorce rate by the number of divorces for every 1000 people in the population. In 1929, the rate was 1.7. See* 100 Years of Marriage and Divorce Statistics, 1867–1967.

recent aspirations to become a playwright. These circumstances prompted Clare to leave *Vanity Fair* and begin work as an independent writer. She tried short stories using her trademark satire but found her best work as a playwright. After a few unsuccessful plays, she published *The Women* in 1936 [Luce, *Women*]. *The Women* featured a group of New York's wealthiest idle women whose concerns focused on their physical appearance and the town's latest gossip. Clare worked her progressive views into the play with a conversation between the protagonist and her daughter:

Child: "What fun is there to be a lady? What can a lady do?"

Mother: "These days, ladies do all the things men do. They fly aeroplanes across the ocean, they go into politics and business [Luce, *Women*, p. 23]."

The play opened on Broadway on December 26, 1936 and reached capacity by the end of its fourth week. It ran for 657 performances in the US and 18 countries and grossed over 2 million dollars. The success of *The Women* and two other plays not only established Clare as a talented comedic writer but it also allowed her to embody the life of the modern career woman and encourage others to do the same.

Through her writing, Clare met Henry Robinson Luce, the once humble newspaper reporter on the *Chicago Daily News* now turned publishing magnate with his *Time*, *Fortune*, and *Life* magazines. Harry Luce divorced his wife of 11 years and married Clare in 1935. The marriage lasted 32 years but was not without its challenges. In his *New York Times* obituary, Alvin Krebs suggested that the "rumored difficulties" were "perhaps inevitable in a marriage between two such strongminded personalities [Krebs]."

Although Harry provided Clare with sufficient opportunities to enhance her writing career, Clare now hoped to develop her skills as a politician. In the late 1930s she traveled to Europe to observe political events firsthand. Harry joined her for part of the trip. When she returned to the States, she hastily wrote a nonfiction book titled *Europe in the Spring* to express what she called an eye- and ear-witness report of what she saw [Luce, *Europe in the Spring*]. Her book helped shape public opinion in the US as Americans tried to make sense of the growing crisis in Europe. After the outbreak of war, she accepted the position as War Correspondent for *Life* magazine and traveled again through Europe. These opportunities and her connections allowed Clare to segue into politics.

In 1942, she ran as a Republican in a largely Democratic constituency of the Connecticut district where she lived. She won by a very narrow margin. Women eager to elect the first congresswoman from Connecticut may have earned Clare her victory and she felt honored to fulfill this role. Clare acknowledged that socially established prejudices surrounding women in politics still existed, but she was eager to hold a position with (purportedly) equal opportunities for power and prestige. In Congress,

Figure 1. Congresswoman Clare Boothe Luce of Connecticut.

Clare fought for the continued military strength of the US and she supported equal employment opportunities and racial equality. Of these interests, she prioritized the nation's safety and security above the "feminist issue [Morris, 2014, p. 30]." Clare, however, found it difficult to be taken seriously. While her male colleagues were often valued for their ideas or achievements, she found that female public figures were evaluated on their looks or personalities. As a Congresswoman then, Clare must have found herself at the confluence of the theoretical and the practical, fighting for women's rights while living the reality of a woman in Congress on a daily basis.

Clare felt pressure to succeed. As she put it, "because I am a woman I must make unusual efforts to succeed. If I fail no one will say, 'She doesn't have what it takes.' They will say, 'Women don't have what it takes [Martin, p. 306, Clare's emphasis]."[5] Clare grew tired of politics because she felt politicians were overly critical and never capable of admitting a mistake. She confessed, "I always regretted that I shifted to politics. You can do nothing truly creative

[5]*Clare was inducted into the Connecticut Women's Hall of Fame in 1994 (posthumously). This quote is also featured on her biography page. See* cwhf.org/inductees/politics-government-law/Clare-boothe-luce#.W5VyKq2ZPUo

in politics by yourself [Martin, p. 272]." She continued in her position, however, because she felt she owed it to women to serve as a positive model of an ambitious and successful career woman.

In January 1944, tragedy struck and temporarily put Clare's political frustrations aside. Her daughter, Ann, was killed in a car accident while traveling back to Stanford. Although they had something of a distant relationship, Clare was overcome with grief and regret for not spending more time with Ann. While a student at Stanford, Ann had "pined for her mother to write, telephone or visit. But Clare always had excuses [Morris, 2014, p. 45]." In July 1943, Ann had blamed herself for requesting Clare's attention. "Forgive all my stupid little letters in which I normally ask you to write to me!" Ann wrote to Clare, "[s]omehow I always forget how very busy you are—and all the good you are doing—until I get a batch of clippings! Then it's always a wonder to me how you even manage to survive the work you have to do."[6] When Clare did write to Ann, her affection for her daughter was everywhere apparent. On November 7, 1943, for example, Clare opened her letter with "Annie my pudding-cake, my peach pie, and all assorted delicacies [Morris, 2014, p. 57]." Initially, Clare's grief seemed to propel her into more aggressive and combative types of politics and fueled her 1944 reelection campaign, which she won.

By September 1945, however, Ann's death combined with discouraging world events led Clare to a point of despair [Luce, The Real Reason, April, 1947]. She called (in the middle of the night) a Jesuit priest in New York who had written to her over the years. He referred Clare to Monsignor (later Bishop) Fulton Sheen in Washington, DC. Father Sheen and Clare had several conversations over the course of the next several months and, on February 16, 1946, Clare converted to Catholicism.[7] Since her Connecticut district had a very large Catholic vote, she did not want her newfound faith to be misconstrued as a political maneuver to influence her constituents. To avoid this confusion, two weeks before her conversion, Clare announced that she would not run for Congress again. This decision may have resolved the potential political issue associated with her newly adopted Catholicism, but no matter "how religious Clare became, the loss of Ann remained a persistent and tragic wound [Brenner, p. 164]."

After her two terms as a Congresswoman, she resumed her writing and suffered defeat in a Senate race in 1952. In 1953, Dwight D. Eisenhower appointed Clare ambassador to Italy, the first woman to serve as an American envoy to a major country. The ambassadorship proved mutually beneficial to Clare and to the US. On the diplomatic front, Clare accomplished her three assigned tasks, including advancing the Italian-American friendship, helping to settle the Trieste crisis, and aiding the young democracy of Italy in fighting communism [Hatch, p. 237]. Gore Vidal later went so far as to credit Clare with "single-handedly saving Italy from Communism [Vidal, p. 203]." She retired after this appointment in 1956. She and Harry settled on their ranch in Arizona, although they still traveled extensively. Harry died unexpectedly of a heart attack in 1967. In his will, Harry had established a trust for Clare that paid her interest only, "the absolute minimum he could get away with without having the will challenged [Brenner, p. 166]." The trust would revert to the Henry Luce Foundation at the time of her death. Harry Luce's son, Hank, however, allowed Clare to determine how she would like to use the trust.[8]

The Vision for the Awards: The Clare Boothe Luce Fund

With this freedom, Clare directed the majority of the proceeds of her Estate to support the Clare Boothe Luce Fund "dedicated exclusively to funding scholarships and professorships for women students and professors at educational institutions, a minimum 50% of which shall be Roman Catholic. The purpose of the Clare Boothe Luce Fund shall be to encourage women to enter, study, graduate, and teach in the following fields of endeavor: Physics, Chemistry, Biology, Meteorology, Engineering (Electrical, Mechanical, Aeronautical, Civil, Nuclear and other Engineering disciplines), Computer Science, and Mathematics [CBL Last Will and Testament, p. 12]." Her choice of scientific fields was deliberate. "I select such fields of endeavor in recognition that women today have already entered the fields of medicine, law, business and the arts, and in order to encourage more women to enter the fields of science [CBL Last Will and Testament, p. 12]." The awards were (and are) designated for scholarship and teaching in the US only.[9]

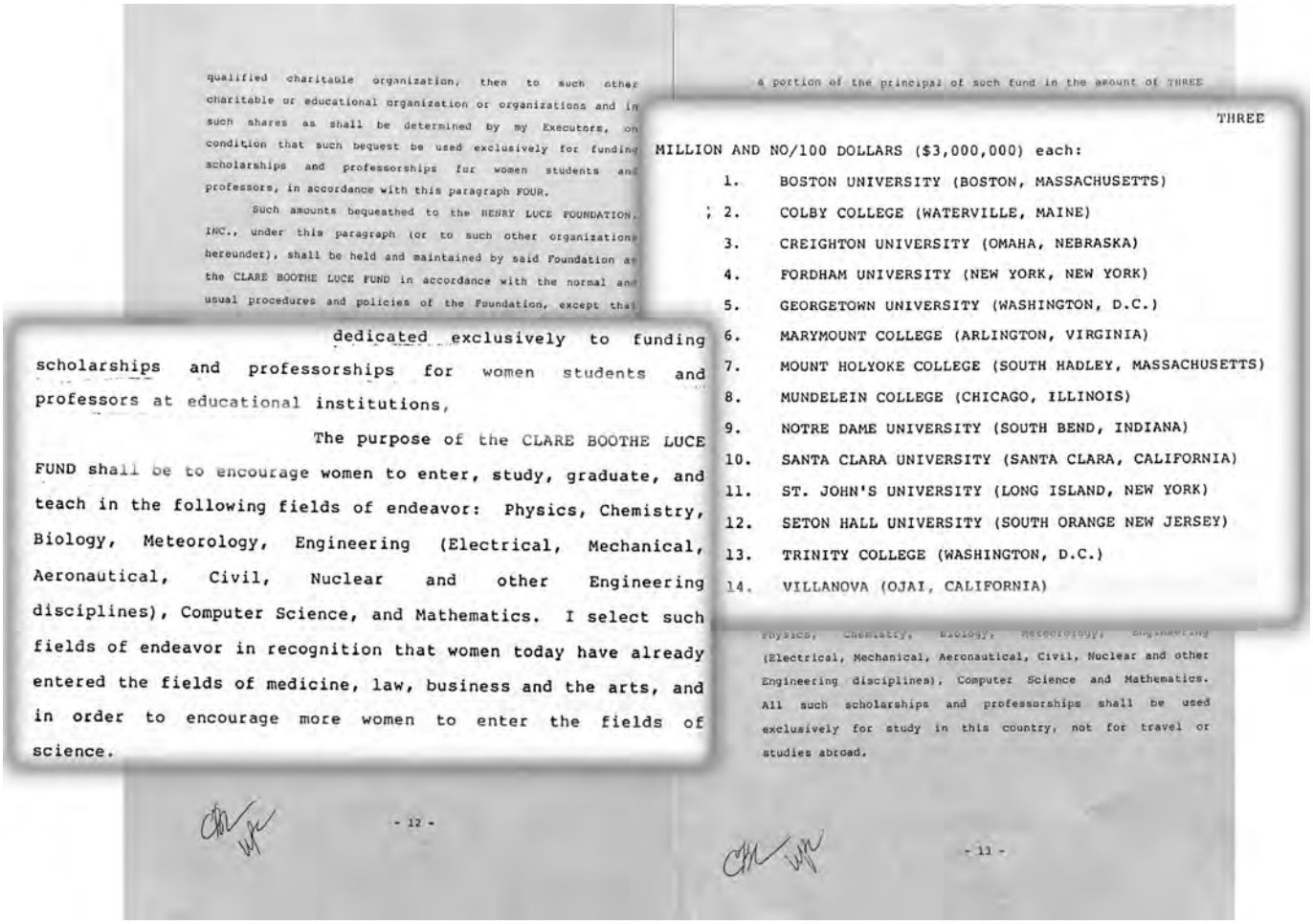
Just as Clare always hoped to accomplish more in life, she hoped other women would do the same. She would do her part to make this happen. By the time of her death in

[6] *Ann Clare Brokaw to Clare Boothe Luce, July 7, 1943, as quoted in Morris, 2014, pp. 45–46.*

[7] *Hatch chronicles the conversations between Father Sheen and Clare on pp. 176–185. Clare documented her own journey to Catholicism in "The Real Reason," an article that appeared in three installments in* McCall's *magazine in February, March and April, 1947.*

[8] *The close relationship Hank developed with Clare is everywhere evident in his tribute to her in "Clare Boothe Luce—Woman of the Century: A Son's Tribute,"* Crisis Magazine, *December 1, 1987.* https://www.crisismagazine.com/1987/clare-boothe-luce-woman-of-the-century-a-sons-tribute.

[9] *When responding to a query about an initial Visiting Assistant Professorship Program, Terrill Lautz, Program Officer of the Henry Luce Foundation, may have provided further insight into the overall aims of the still-to-be established Clare Boothe Luce Fund. "[T]he Luce Foundation wants to encourage the development of a permanent core of women faculty in fields where women have not been well represented in leadership positions at American universities [Lautz to Yu, 2 July, 1987]."*



qualified charitable organization, then to such other charitable or educational organization or organizations and in such shares as shall be determined by my Executors, on condition that such bequest be used exclusively for funding scholarships and professorships for women students and professors, in accordance with this paragraph FOUR.

Such amounts bequeathed to the HENRY LUCE FOUNDATION, INC., under this paragraph (or to such other organizations hereunder), shall be held and maintained by said Foundation as the CLARE BOOTHE LUCE FUND in accordance with the normal and usual procedures and policies of the Foundation, except that

dedicated exclusively to funding scholarships and professorships for women students and professors at educational institutions,

The purpose of the CLARE BOOTHE LUCE FUND shall be to encourage women to enter, study, graduate, and teach in the following fields of endeavor: Physics, Chemistry, Biology, Meteorology, Engineering (Electrical, Mechanical, Aeronautical, Civil, Nuclear and other Engineering disciplines), Computer Science, and Mathematics. I select such fields of endeavor in recognition that women today have already entered the fields of medicine, law, business and the arts, and in order to encourage more women to enter the fields of science.

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14. VILLANOVA (OJAI, CALIFORNIA)

(Electrical, Mechanical, Aeronautical, Civil, Nuclear and other Engineering disciplines), Computer Science and Mathematics. All such scholarships and professorships shall be used exclusively for study in this country, not for travel or studies abroad.

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Figure 2. Pages 12–13 of Clare Boothe Luce's last will and testament, noting her wish to fund women in STEM studies and careers.

1987, Clare had seen women make significant advances in some fields, but not, from her perspective, in mathematics, engineering, and certain sciences. Even though these disciplines fell outside her own areas of expertise, she recognized the need for support and funding. A comment late in her life may offer insight into Clare's choice for her legacy. In 1981, she admitted to one of her biographers, Wilfred Sheed, that she envied Sandra Day O'Connor, America's first female Supreme Court justice. Although Clare was a woman of many "firsts" herself, she told Sheed, "I don't want to be her, I [would] just like to have had that kind of chance [Sheed, p. 163]." Her creation of the Clare Boothe Luce Fund provided women with a *chance*.

She also publicly recognized the three most important breakthroughs for women that could help them achieve equal opportunity: the legal process, the female contraceptive, and opportunities for higher education [Luce, "Women Superior to Men," p. 281]. She especially valued higher education. Since Clare had only a very limited formal education, she was thrilled to witness—and advance—new opportunities for women in higher education. She was realistic, however. "Today she [a woman] is free to study for any 'masculine career' that her own ambition suggests" but… "as matters stand, her ambition is understandably dampened by the knowledge that even if she graduates at the top of her class, she will not find it easy to translate her well-earned degree into an upward-mobility job [Luce, 21st Century Woman, pp. 61–62]." Thus Clare understood that even though women had access to new educational opportunities, they still faced challenges in a male-dominated job market. Though Clare's success was not related to her level of education, she recognized that gender equality in education was a necessary precursor to job equality.

In her will, Clare designated "that the following named institutions shall be allocated a portion of the principal of such fund in the amount of…$3,000,000 each (about $6,692,000 today) [CBL Last Will and Testament, pp.12–13]." These schools included Boston University, Colby College, Creighton University, Fordham University, Georgetown University, Marymount College, Mount Holy-

oke College, Mundelein College, Notre Dame University, Santa Clara University, St. John's University (Long Island, NY), Seton Hall University, Trinity College (Washington, DC), and Villanova Preparatory School in Ojai California. [CBL Last Will and Testament, p. 13]. (Mundelein College affiliated with Loyola University Chicago in 1991 and is no longer funded from this initial allocation.) These fourteen schools had a Catholic affiliation, had awarded Clare an honorary degree, or had a sentimental attachment for her. Other schools can apply for funding through the "Clare Boothe Luce Program Invited Institution Competition." The Clare Boothe Luce Fund especially encourages Catholic institutions with strong science programs to apply [Clare Boothe Luce Program]. In 2017, by way of an example, eleven institutions received grants through the Invited Competition for funding to begin in 2018. Three decades after the initial bequest, in addition to the designated schools, more than 100 colleges and universities have benefited from the Clare Boothe Luce program [see CBL Recent Grants].

To fulfill Clare's aim "to encourage women to enter, study, graduate and teach," the Clare Boothe Luce Program administers awards in the three distinct categories of undergraduate scholarships and research awards; graduate fellowships for the first two years of a PhD program; and professorship support for the first five years of a tenure-track appointment. In the most recent year of funding for invited institutions, the Clare Boothe Luce Fund awarded grants in each of these categories [CBL Recent Grants].

Impact of a Designated Institution

Creighton University, one of the institutions designated in Clare's will, has a robust "Clare Boothe Luce Program for Women in Science [Creighton Clare Boothe Luce Program]." Through this program, Creighton funds undergraduate scholarships, graduate scholarships for women pursuing PhDs, and faculty positions. Since 1992, Creighton has rotated a Clare Boothe Luce Professorship in various fields in mathematics and science. Dr. Cynthia Farthing, who earned her PhD in mathematics from the University of Iowa, held the Clare Boothe Luce Professorship from 2007–2012. Dr. Catie Baker, an Assistant Professor in Computer Science, is currently the seventh Clare Boothe Luce Professor at Creighton. The Chair is designed to support a pre-tenure woman in a science or math field through tenure. It provides support to attend conferences, to fund undergraduate researchers, and to purchase supplies and materials.

Four of the six previous Clare Boothe Luce professors remain at Creighton and offer a strong network of support for Baker. Baker underscored the benefits of having a Clare Boothe Luce professorship at a designated institution. When she arrived at Creighton, she immediately shared a connection with the Clare Boothe Luce professors who had preceded her. This initial welcome segued into continued support and mentorship. At least two of the previous Clare Boothe Luce professors have moved into leadership positions on campus. These opportunities testify to the power of Clare's vision. Creighton's ongoing cycle of *chances* for women to earn a degree, teach others, and move into leadership positions is precisely the sort of outcome Clare aimed to achieve.

As part of her professorship, Baker oversees the selection of the Clare Boothe Luce undergraduate scholarships. Typically, Creighton offers 5–8 full tuition scholarships through the Clare Boothe Luce program. These scholarships are generally awarded to students who are actively engaged in research. Scholarship recipients take a "Women in Science" Seminar, taught by the Clare Boothe Luce Professor. This seminar focuses on issues facing women in science, including the impostor syndrome and stereotype threat.[10] The seminar fosters community and inspires conversations about graduate school, research, etc. As Dr. Baker described it, "the presence of the Clare Boothe Luce undergraduate scholarships creates an environment where women involved in undergraduate research are supported and valued [Interview with Catie Baker]." The ongoing presence of Clare Boothe Luce support at Creighton has not only advanced women at various stages in their careers but has also fostered a favorable environment on campus for women in mathematics and science to succeed as part of a broader community.[11]

Impact of Invited Institutions

Beyond the institutions designated in Clare's will, other eligible institutions of higher education can apply for awards through the "Invited Institution Competition." Sarah Spence Adams, Professor of Mathematics and Electrical & Computer Engineering at Olin College, received a Clare Boothe Luce scholarship for her final two years as a student at the University of Richmond in Richmond, Virginia in 1995–1997, for example. The award also included funding for undergraduate research. Her professor at Richmond, Dr. James Davis, called the Clare Boothe Luce opportunity to her attention and encouraged her to apply. At the time, she had no idea what "undergraduate

[10] *[Harris] includes a sample reading list for the Creighton Women in Science Seminar on pp. 109–110.*

[11] *For more on the impact of the early years of Clare Boothe Luce Funding at Creighton, see [Harris]. Although written in 1995, her insights apply to contemporary issues. As Harris puts it, "[a] topic of particular concern to students in the past 2 years has been sexual harassment…It is imperative that women not internalize harassment, whether it is called harassment or not. This is particularly true for gender-based harassment. Sexual harassment is much easier to identify, but gender-based harassment is far more common and more dangerous to the self-esteem and success of women. Examples of gender-based harassment include females being ignored in class (not called on) or, when they are called on, a female student's answer being deemed not as correct as a male student's identical response [Harris, p. 107]."*

research" meant. Davis showed her a book with an open question he had solved and helped her understand what undergraduate research might look like for her. She studied coding theory with Davis and cryptography with Dr. Gary Greenfield with her Clare Boothe Luce summer undergraduate research support.

The scholarship served as an "enormous source of confidence that I could actually be part of a mathematical research community," Adams says. "I wasn't exactly sure what that meant at the time but I understood that I had received funding to do mathematics. That was a novel idea." She presented her research at the Joint Mathematics Meetings in 1996 and won a prize for her poster. The prize was a gift certificate to select a book at a publisher. As she described it, "[to] claim the prize, I walked into the exhibit hall and was able to pick out any book I wanted. At that point, I only had books that my professors had assigned to me. Choosing my own mathematics book made me feel like a real mathematician [Interview with Sarah Spence Adams]."

Her undergraduate research experiences at Richmond made her a viable candidate for Joe Gallian's Research Experience for Undergraduates (REU) at the University of Minnesota in Duluth. She could take her Clare Boothe Luce funding with her so she did not have to rely on Gallian's NSF resources. The REU "propelled her into research" and helped her gain admission to the NSA Director's Summer Program the following year. These experiences not only improved her level of mathematics but also continued to open doors for her. She had the confidence to pursue a PhD in mathematics at Cornell, specializing in algebraic coding theory, and then to accept a faculty position at Olin College in Needham, Massachusetts, where she is now Professor of Mathematics and Electrical & Computer Engineering and a former Associate Dean of Faculty Affairs and Development. Adams says, "The Clare Boothe Luce experience taught me the value of undergraduate research so I dedicated myself to mentoring undergraduates at Olin." In her first decade at Olin, she mentored around 30 students, approximately 25 of whom continued for multiple years. All but three of these students have published professional journal articles with Adams. "I took their mentorship seriously," Adams explained. "I knew the impact it would have on them to come up with novel results, to edit, to revise, to publish, to attend a conference, to give a talk, to field questions, etc. I knew these values because I discovered them as an undergraduate myself [Interview with Sarah Spence Adams]."

Since Adams received her Clare Boothe Luce support more than twenty years ago, she provides an advantageous perspective on the long-term benefits of the program. "My Clare Boothe Luce experience was officially two years long. As the days, months and years have gone by, however, Adams notes, I have realized how much I gained from the scholarship and the opportunities that came along with it." Not surprisingly, when the Olin Development Office reached out to Adams to help craft an application for a Clare Boothe Luce grant to support undergraduate research, she was eager to help. Olin modeled their proposal around the two-year undergraduate research experience Adams had at Richmond. Olin's proposal included academic support, summer support, travel to conferences, and travel to see mentors. Adams recalls, "I had seen all of these components at Richmond and with Joe Gallian at Duluth and knew the impact they had on me." In 2011, Olin received an $180,000 award from the Clare Boothe Luce Foundation. Olin granted their first awards in 2013 [Bailey].

Epilogue

In February, 1942, Clare posed a question of possibility to her daughter Ann. "Would it amuse you," Clare asked, "to have your ma run for Congress and one day get to be a Cabinet minister, or maybe the first lady Vice President? [Morris, 1997, p. 473]." A year later, Albert P. Morano, Clare's executive assistant when she served as a Congresswoman, remarked that she "might even get to be President [Morris, 2014, p. 22]." Thus Clare and Morano at least considered the chance of Clare as the Vice President and/or President of the United States.[12] We know Clare valued a *chance*, for herself, and, as it turns out, for other women.

By the time Clare signed her will in early 1987, her experiences had more than acquainted her with the realities of life as an ambitious woman who exceeded the expectations of the existent social milieu. In perpetuity, then, she drew from these experiences to encourage women to pursue education for careers in fields, that at the time of her death, Clare viewed as primarily available to men. The last three decades testify to the continued vibrancy and veracity of her ideas.

Drawing from her two generations of experience with Clare Boothe Luce awards for mathematics, Sarah Spence Adams observed that "Clare Boothe Luce awards build confidence and create opportunities."[13] That formidable combination has advanced women not only in mathematics, but also in science and engineering, precisely what Clare Boothe Luce hoped to accomplish with her bequest and what the Clare Boothe Luce Fund aims to achieve today. Clare Boothe Luce may not have understood the intricacies of the fields she supported. She did, however, understand the necessary general framework for women to forge new

[12] *A decade after her death, Gore Vidal went so far as to say, "If born a man," she "could have easily been a president, for what that's worth these days: a cool billion, I believe." Vidal, p. 216.*

[13] *Of course, confidence is also a helpful skill for men in mathematics. University of Chicago mathematician Gilbert Ames Bliss noted the confidence his colleague, E. H. Moore, a pivotal figure in American mathematics in the late 1800s and early 1900s, acquired during his year of study in Berlin and Göttingen in 1885–1886. "There is no doubt," Bliss wrote, "that the year abroad affected greatly…Moore's career as a scholar. It established his* confidence *in his ability to take an honorable place in the … circle of mathematicians... [as quoted in Parshall and Rowe, p. 282, my emphasis]."*

pathways and find success. Clare Boothe Luce drew from her own experiences and observations to lay out the details for a foundation that would continue to promote and ensure these goals over time.

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Della Dumbaugh

Credits

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Figure 2 photos of select pages from Clare Booth Luce's last will and testament are courtesy of the Library of Congress.

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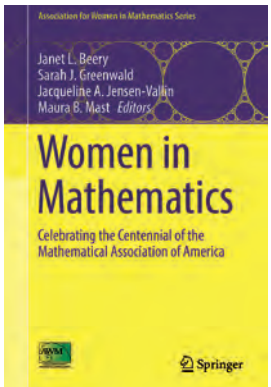
BOOK REVIEW

Women in Mathematics

A Review by Katie Spurrier Quertermous



Women in Mathematics: Celebrating the Centennial of the Mathematical Association of America
Janet L. Beery, Sarah J. Greenwald, Jacqueline A. Jensen-Vallin, and Maura B. Mast, editors
Springer International Publishing, 2017
ISBN: 978-3319666938

In the last three years,[1] at least three mass market books—Margot Lee Shetterly's *Hidden Figures*, Nathalia Holt's *Rise of the Rocket Girls*, and Liza Mundy's *Code Girls*—and a major motion picture—*Hidden Figures*, based on Shetterly's book—have captivated audiences with the previously overlooked stories of women mathematicians who worked as human computers and cryptographers for the United States government. These works have offered the public a glimpse into the ongoing efforts of mathematicians and historians to write a social history of the lived experiences and contributions of women in the mathematical sciences.[2] *Women in Mathematics* adds to this literature with a collection of twenty-one engaging articles that include biographies, historical and cultural studies, and profiles of outreach and education initiatives related to women in mathematics. Most of the articles in this collection are written for a broad mathematical audience that includes students.

This volume grew out of a contributed paper session at MAA MathFest 2015 that was sponsored by the Association for Women in Mathematics (AWM). In connection with the celebration of the one hundredth anniversary of the Mathematical Association of America, the session sought to "recognize the contributions, achievements, and progress of women mathematicians over the past 100 years" through "talks about mathematics done by women and historical or biographical presentations celebrating women in mathematics." As the editors note in their preface, the resulting collection of articles is a mix of current scholarship and exposition on a wide variety of topics related to women in mathematics as opposed to a balanced study of the participation of women in mathematics during this time. Some of the articles summarize or extend work that has appeared previously, including Judy Green and Jeanne LaDuke's detailed documentary history of all of the American women who earned PhDs in mathematics from American and European universities between 1886 and 1939 and Margaret Murray's research on American women who earned PhDs in mathematics between the years 1940 and 1959. As a result, the volume also serves as a survey of a portion of the existing literature and compellingly invites the reader to delve deeper into that work.

The first two parts of the book are dedicated to telling the stories of women mathematicians in articles that range in style from formal historical and cultural studies to personal reflections and collections of interviews. These articles include more than eighty biographical profiles of women mathematicians and statisticians as well as numerous more concise descriptions of the experiences and contributions of women in these fields. The profiles are a mix of short sketches grouped within larger discussions of the mathematical and social context of a particular time, place, or culture and more in-depth studies of the professional and personal lives of individual women. Most of the profiles

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[1] *This number of years is correct through April 5, 2019.*

[2] *Margaret Murray describes the concept of social history in more detail in Chapter 5 of* Women in Mathematics.

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focus on women from the United States, Canada, and Europe whose career paths are connected to academia or secondary education during at least a portion of their professional careers. Several chapters highlight women from underrepresented groups in mathematics, although the total number of biographical profiles of women from these groups is still small. The authors can be commended for including profiles of women whose stories are not widely known, so readers should expect to encounter at least a few unfamiliar names in these pages.

By presenting many of the profiles in groups, the articles emphasize both the connections between the individual women and the diversity of professional paths that they pursued even in times of limited career options. In their article on Girton College, Cambridge, Shawnee McMurran and James Tattersall profile a group of ten women who studied at Girton between 1880 and 1900 and achieved honors on the Mathematical Tripos exam despite the fact that women could not earn degrees from Cambridge at this time. These women applied their mathematical training to achieve success in numerous areas. As examples, Charlotte Angas Scott and Hertha Ayrton, the founders of Girton's Mathematical Club, had widely recognized research careers, Scott in algebraic geometry and Ayrton in engineering. Kate Knight Gale taught for many years, eventually becoming co-owner and joint headmistress of a school in South Africa, and Margaret Frances Evans was Mathematical Mistress of St Leonards School before ending her professional career to focus on family life. Emily Perrin and Beatrice Mabel Cave-Browne-Cave were both computers in Karl Pearson's statistical research lab at University College, London. In the 1930s, Girton College's Yarrow Research Fellowship supported the early work of Olga Taussky-Todd and Mary Lucy Cartwright, who both became prolific research mathematicians, Taussky-Todd in matrix theory and number theory and Cartwright in function theory and differential equations. Cartwright was elected Mistress of Girton College in 1948, and she led the college in this role for nineteen years while continuing her active involvement in the mathematical community.

Erica Walker explores the history of Black women in mathematics in the United States in a reflective essay that draws upon research she conducted for her 2014 book, *Beyond Banneker: Black Mathematicians and the Paths to Excellence*. As part of this essay, Walker juxtaposes the stories of Euphemia Lofton Haynes (PhD 1943) and Evelyn Boyd Granville (PhD 1949), the first two Black women to be awarded doctorates in mathematics in the United States.[3] Although they were born thirty-four years apart, both women were raised in Washington, DC, attended the same segregated high school, now called Dunbar High School, and earned undergraduate degrees from Smith College.

Euphemia Rosalie Lofton from Washington DC. Class Book, Smith College, 1914, Smith College Archives, Special Collections, Smith College.

Evelyn Boyd, Lawrence House, Smith College Yearbook, 1945, Smith College Archives, Special Collections, Smith College.

Figure 1. Euphemia Lofton Haynes (Smith College Class of 1914), left, and Evelyn Boyd Granville (Smith College Class of 1945), shown here in their college yearbook photos, attended the same Washington, DC, high school and the same college before earning PhDs in mathematics.

Dunbar was known for having committed, highly educated teachers who encouraged their students to attend college. The influence of these teachers seems to have played a strong role in both Haynes's and Granville's enrollments at Smith. Despite their many commonalities, Granville has said that she did not learn about Haynes until 1999, almost twenty years after Haynes's death. Haynes taught in the DC public school system, established the mathematics department at Miner Teachers College (now part of the University of the District of Columbia), served as president of the DC Board of Education, and was a key advocate for integration of the DC public schools. Granville's career in industry at IBM and other NASA contractors was bookended by academic positions, including faculty positions at Fisk University and the University of Texas at Tyler. Continuing the chain of influence begun by the teachers at Dunbar, two Black women taught by Granville, Etta Zuber Falconer and Vivienne Malone-Mayes, earned PhDs in mathematics and mentored additional generations of mathematicians.

The celebration of women's often under-recognized contributions in areas such as mentoring, teaching, professional service, academic administration, and advocacy is a theme that runs throughout most of the articles. It is impossible to categorize all of these contributions here, so I will just mention a few that I learned about for the first time while reading this book. An article by Emelie Agnes Kenney explores the vital roles that women played in keeping mathematics alive in Nazi-occupied Poland by studying and teaching in the clandestine education system despite the danger they faced if caught. Kenney recounts how Irena Gołąb disguised her math classes as crochet circles to avoid detection. Patti Hunter discusses Gertrude Cox's efforts to support statistics training around the world, efforts that are

[3] *Marjorie Lee Browne also finished the requirements for a PhD in mathematics in 1949, but her degree was not awarded until 1950.*

not as widely known as her achievements in building statistics programs in the United States or her service as president of the American Statistical Association. As part of this work, Cox was a program specialist at Cairo University's Institute of Statistical Studies and Research in 1964–1965. She was a strong advocate for the importance of statistical consulting and personally consulted on numerous projects in Cairo. Norma Hernandez earned a PhD in mathematics education in 1970 and was a faculty member at the University of Texas at El Paso for thirty years, serving as dean of the College of Education for six of those years. Hernandez was born and raised in El Paso, and Luis Ortiz-Franco investigates how Hernandez's life experiences in this multicultural city likely influenced her research on the relationships between culture and mathematics in the context of the K–12 mathematics education of Latinx students.

Woven throughout the historical accounts of women's contributions are discussions of some of the challenges the women faced in their pursuit of mathematical careers, especially those they encountered prior to the 1970s. Each woman's story is different, but common obstacles include barriers to advanced training, bias against women in hiring and promotion practices, and a lack of recognition of women's accomplishments. Many of these obstacles reflect the prevailing social norms of the women's times, and this context is important for helping readers understand the significance of individual and collective contributions as well as the dedication and perseverance of the women who made these contributions.

Multiple authors comment anecdotally on the shifts in mathematical culture that have occurred during their lifetimes, but a formal analysis of these changes is not the focus of this volume. Readers are, however, offered glimpses into a few of the advocacy efforts that championed changes in culture. Jacqueline Dewar describes outreach activities to encourage middle and high school girls to study mathematics that grew out of regional organizing meetings of the AWM during the 1970s. Laura Turner explores discussions at Canadian Mathematical Society meetings in the late 1980s and early 1990s that highlighted the underrepresentation of women on journal editorial boards and as plenary lecturers at meetings, and Sue Geller discusses skits presented at the Summer and Winter Joint Mathematics Meetings from 1990 to 1994 that used humor to draw attention to micro-inequities.

The third part of the book focuses on outreach and educational efforts. In a joint article, Jacqueline Dewar and Sarah Greenwald outline courses they have developed on women and mathematics that combine history, mathematical work, and equity issues and suggest opportunities for readers to experiment with these ideas through shorter-term outreach activities. Karl Schaffer describes the process of creating *Daughters of Hypatia*, a full-length dance performance that shares the stories of historical and contemporary women in mathematics, showcases mathematical thinking embedded in arts and crafts, and challenges stereotypes through dance and music. In an example of the connections between articles, *Daughters of Hypatia* includes an adaption of one of Sue Geller's skits on micro-inequities mentioned above. Sylvia Bozeman, Susan D'Agostino, and Rhonda Hughes discuss the EDGE Program (Enhancing Diversity in Graduate Education), which supports women mathematicians from diverse backgrounds through an annual summer session for beginning graduate students and ongoing mentoring networks. As part of this article, the authors describe aspects of the program that are designed to increase students' ability to successfully navigate the academic, cultural, and social transitions they will encounter in graduate school.

Readers with a wide variety of mathematical, educational, and historical interests will find the articles in this collection engaging and inspiring. Most of the articles are accessible to undergraduate and graduate students although a faculty mentor may be required at times to help students contextualize the importance of contributions described in terms of journal titles and professional committees. The material is well suited for inclusion in existing courses and math club activities, and it provides opportunities not only to teach students about the history of women in mathematics but also to introduce them to important elements of mathematical culture through stories in which women play central roles. For example, Amy Shell-Gellasch's article on Mina Rees provides an excellent starting point for discussing both research funding and the value of conference attendance. Rees was instrumental in shaping federal funding of mathematics research through her work with the US government's Applied Mathematics Panel and the Office of Naval Research, and this article recounts, in Rees's own words, the importance she placed on the personal connections and broad understanding of

Figure 2. *Daughters of Hypatia* uses dance to share the stories of women mathematicians. In this 2015 photo, Laurel Shastri, left, and Lila Salhov perform "A Circle Has No Sides," one of several pieces that employ circular imagery.

the current mathematical landscape that she established at conferences as preparation for these positions.

I highly recommend reading *Women in Mathematics*. The articles are compellingly written and contain a wealth of information about the lives and accomplishments of women mathematicians and the efforts of women and men to share this history and advocate for inclusion and diversity. They encourage readers to reflect on the experiences and contributions of women mathematicians, not only those profiled in the volume but also those in the readers' own communities. The articles challenge readers to celebrate the achievements, recognize the challenges, and use both to inform their understanding of the history and culture of mathematics. This volume invites us to continue learning about the fascinating past and present of women in mathematics and provides inspiration as we think about the future.

Katie Spurrier Quertermous

Credits

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CALL FOR APPLICATIONS & NOMINATIONS

Secretary
of the American Mathematical Society

The American Mathematical Society is seeking candidates for the position of Secretary, one of the most important and influential positions within the Society. The Secretary participates in formulating policy for the Society, participates actively in governance activities, plays a key role in managing committee structures, oversees the scientific program of meetings, and helps to maintain institutional memory.

The first term of the new AMS Secretary will begin February 1, 2021, with initial appointment expected in January 2020 in order that the Secretary-designate may observe the conduct of Society business for a full year before taking office.

All necessary expenses incurred by the Secretary in performance of duties for the Society are reimbursed, including staff support. The Society is prepared to negotiate a financial arrangement with the successful candidate and the candidate's employer in order that the new Secretary be granted sufficient release time and provided staff support to carry out the many functions of the office.

QUALIFICATIONS

The Secretary should be a research mathematician and must have substantial knowledge of Society activities. Although the AMS Secretary is appointed by the Council for a term of two years, candidates should be willing to make a long-term commitment, as it is expected that the new Secretary will be reappointed for subsequent terms pending successful performance reviews.

DUTIES

- Organize and coordinate the Council and its committees.
- Serve as a member of the Council, Executive Committee, and several other committees, including the five policy committees, the Development Committee, and the Editorial Boards Committee.
- Work closely with the President to coordinate and administer the activities of committees.
- Oversee, together with the Associate Secretaries, the scientific programs of all AMS meetings.

APPLICATIONS & NOMINATIONS

A Search Committee, with Tara Holm as Chair, has been formed to seek and review applications. Persons wishing to apply should do so through MathPrograms.Org. Nominations and questions should be directed to the Chair of the Search Committee: ssc-chair@ams.org.

For full consideration, applications, nominations, and supporting documentation should be received by **May 15, 2019**.

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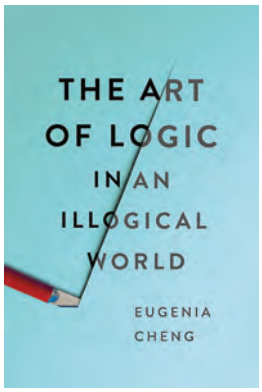
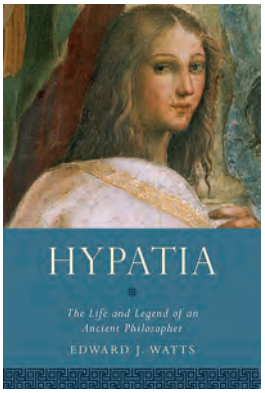


The Art of Logic in an Illogical World by Eugenia Cheng
(Basic Books, 2018, 320 pages)

Eugenia Cheng is well known for her previous popular mathematics books *How to Bake Pi* and *Beyond Infinity*. Her latest book seeks to remedy the apparent demise of rationality in modern life by introducing the general public to logical thought.

There are plenty of diagrams throughout the book and the level is appropriate for almost any reader. The book is ideal for non-mathematicians who struggle to make sense of an apparently irrational world of internet memes, fake news, and propaganda. Social justice and equity are constant themes throughout the examples. Indeed, the back cover tells us "Cheng shows us how to use logic and alogic together to navigate a world awash in bigotry, mansplaining, and manipulative memes."

The book is divided into three long sections: "The Power of Logic," "The Limits of Logic," and "Beyond Logic." Each of these is divided into several chapters. The first section argues for the necessity of logic and rational thought. It covers many of the basic aspects of logic, such as implications, negations, and contrapositives. These are always grounded with plain-English examples. For example, how does one assign blame for the United Airlines passenger-beating scandal? Which factors are relevant and which are irrelevant? The second section, which is the most math-oriented, discusses a few of the seminal puzzles and paradoxes that confronted mathematicians over the years. Hilbert's hotel, Russell's paradox, Zeno's paradox, Gödel's theorems, and the Prisoner's dilemma are each touched upon. However, these are necessarily discussed at a superficial level and only hint at the great depth behind these topics. The final section concerns axioms (what are "axioms" for everyday life?), analogies, and the relationship between emotions, intelligence, and rationality. For example, Cheng groups her personal axioms into three main groups: "kindness," "knowledge," and "existence" and discusses how these core axioms inform her decisions in everyday life.



Hypatia: the Life and Legend of an Ancient Philosopher
by Edward J. Watts
(Oxford University Press, 2017, 224 pages)

Any book about Hypatia of Alexandria is guaranteed to be of interest to the mathematical community, even if its primary focus is not on her mathematical career. Watts, a historian who studies religion and philosophy in late antiquity, devotes only a few pages near the beginning of the book to mathematics. However, he does paint a vivid picture of both daily and intellectual life in the Alexandria of Hypatia's time. In addition, Hypatia's philosophical views and teachings are discussed in great detail.

Given the paucity of original sources, it is not surprising that this is a short book. The main text ends on page 155 and the remainder consists mostly of detailed endnotes. The final chapter recounts and reflects upon the various representations of Hypatia that have appeared over the centuries in art and literature. Although the mathematical content of this book is minimal, *Notices* readers will still likely enjoy reading *Hypatia* to learn about the historical Hypatia and the world in which she lived.

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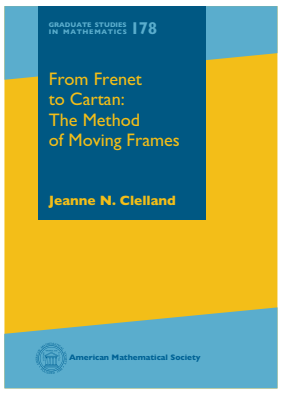

From Frenet to Cartan: The Method of Moving Frames
(GSM/178,2017)[1]
By Jeanne N. Clelland

Jeanne Clelland's book *Moving Frames* introduces graduate level differential geometry using the highly intuitive notion of frames. The approach has a particularly natural appeal to the kinetic learner: one can imagine walking along a hilly landscape watching the view change as the road navigates the contours of the hillside. Clelland takes the reader through the background, definitions and properties of this powerful tool, and includes examples and computational methods that allow the reader to explore and experience its effectiveness.

This book assumes some basic knowledge of linear algebra and the theory of differentiable manifolds. After a review of differential forms, tensors and Lie algebras, Clelland introduces moving frames for curves and surfaces in Euclidean space, and then proceeds to other geometries including Minkowski, affine and projective spaces. Using this language, she builds the theory of minimal surfaces and proves Bäcklund's theorem on pseudo-spherical line congruences. At the end of the book, she turns to the case of non-flat Riemannian manifolds and defines the Levi-Civita connection.

Exercises are interspersed throughout the text to enhance understanding and fill in details that will be useful in later sections. Coding is also a big feature of this book. Clelland includes many exercises and projects to do explicit calculations using Maple's Cartan package.

[1]The Graduate Studies in Mathematics Series *contains textbooks aimed at graduate level courses at the pre-qual and post-qual level.*

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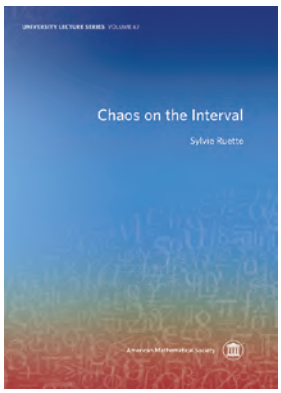


Chaos on the Interval
(ULECT/67, 2017)[2]
By Sylvie Ruette

In *Chaos on the Interval* Sylvie Ruette beautifully demonstrates the importance of the particular in understanding the universal. The book begins by laying out the language of continuous dynamical systems using continuous maps from the unit interval as its main example. It then proceeds to show that interval maps have startlingly intricate structure, which also lead to insights for the cases of more general and higher dimensional systems.

The overarching theme of the book is the interplay between properties of transitivity, periodic points, topological mixing, and sensitivity to initial conditions, and how these relate to chaos and entropy. In the first half of the book Ruette presents two foundational results: Sharkovsky's Theorem on orders of periodic points, and the Misiurewicz Theorem on existence of horseshoes for positive entropy maps. The second half of the book is devoted to two notions of chaotic interval maps: one, due to Li-Yorke, rests on the existence of an uncountable scrambled subset, another due to Devaney, is one that is transitive, sensitive to initial conditions, and has a dense set of periodic points.

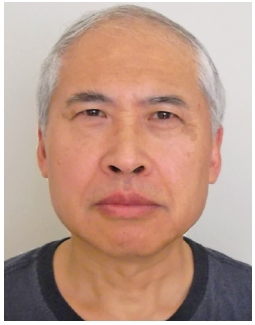
The book contains results that are scattered in the literature (as well as some well known theorems), with full proofs and details not always found in the original articles in which they appeared.

For graduate students and researchers who are seeing the theory of interval maps and chaos for the first time, this book provides a self-contained, well-motivated, and highly accessible introduction. For those seeking a review of the theory, this book provides a delightful refresher.

[2]The AMS University Lecture Series *features self-contained expositions of currently active research areas in mathematics.*

SECTIONAL SAMPLER


Grigoriy Blekherman


Ming Liao


Aaron Naber


Deanna Needell


Katherine E. Stange

In this sampler, the speakers below have kindly provided introductions to their Invited Addresses at the following meetings: AMS Spring Southeastern Sectional taking place March 15–17 at Auburn University, Auburn, Alabama, and the AMS Spring Central and Western Joint Sectional taking place March 22–24 at University of Hawai'i at Mānoa, Honolulu, Hawaii.

*CORRIGENDUM

On page 412 of Katherine E. Stange's article "An Illustration in Number Theory," a density one result concerning curvatures in an Apollonian circle packing is attributed to Bourgain and Fuchs (cited as [2]). In fact, this result is due to Jean Bourgain and Alex Kontorovich, *On the local-global conjecture for integral Apollonian gaskets*, with an appendix by Péter Varjú, in *Invent. Math.*, 196 (2014), no. 3, 589–650.

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Do Sums of Squares Dream of Free Resolutions?

Grigoriy Blekherman

A real polynomial in n variables is called nonnegative if it is greater than or equal to 0 at all points in $\mathbb{R}^n$. A sum of squares of polynomials is obviously nonnegative. It is a central question in real algebraic geometry whether a nonnegative polynomial can be written in a way that makes its nonnegativity "obvious", e.g., as a sum of squares of polynomials (or more general objects). A familiar case is quadratic forms, which can be represented by real symmetric matrices. Diagonalization of real symmetric matrices shows that any nonnegative quadratic form is a sum of squares. Starting with the work of Lasserre [Las00] and Parrilo [Par00] in the early 2000s, sums of squares methods found a wide array of applications, ranging from applied areas (e.g., robotics, computer vision, optimization) to theoretical applications (e.g., extremal combinatorics, theoretical computer science) [BPT12]. I will discuss in my talk a recent line of work that shows an intrinsic connection of this area to classical algebraic geometry. To discover this connection it was crucial to consider convex geometry of the cones of nonnegative polynomials and sums of squares.

A rich history. The story begins in 1885 at the PhD defense of Hermann Minkowski, where David Hilbert was one of the examiners. During his defense Minkowski claimed that there exist nonnegative polynomials that are not sums of squares, although he did not provide an example or a proof. The years later Hilbert published a seminal paper [Hil88] classifying all cases in terms of number of variables n and degree $2d$ for which all n-variate nonnegative forms are sums of squares. Without loss of generality we may consider the case of homogeneous polynomials (forms). Besides quadratic forms there are only two further cases: bivariate forms (univariate polynomials) of even degree and ternary forms of degree four. It should be noted that Hilbert did not provide an explicit nonnegative polynomial that is not a sum of squares of polynomials. The first explicit example appeared only much later and is due to Motzkin [Mot67]. The main difficulty of constructing an example is certifying that a given polynomial is globally nonnegative without relying on a sums of squares decomposition. Motzkin's idea was to do this via known inequalities! The Motzkin form

$$M(x,y,z) = x^2y^4 + x^4y^2 + z^6 - 3x^2y^2z^2$$

is nonnegative by applying the arithmetic mean/geometric mean inequality, and a simple argument shows that it is not a sum of squares. Since then many other examples of nonnegative polynomials that are not sums of squares have appeared, particularly in the work of Choi, Lam, and Reznick; see [Rez00] for a nice overview.

Hilbert's 17th problem asked whether any nonnegative form is a sum of squares of rational functions. In 1927 E. Artin solved this problem in the affirmative [Art27] using the Artin–Schreier theory of real closed fields. However, the complexity of the underlying squares in the rational sums of squares representations is still not well understood. A philosophical interpretation of Artin's result is that a nonnegative polynomial, deep in its soul, is a sum of squares, but if one wants to find the sum of squares decomposition then it might be necessary to look for a very long time. Starting with the seminal work of Artin and Schreier the paths of real algebraic and classical algebraic geometry sharply diverged. The goal of my talk is to show that questions of nonnegativity and sums of squares are actually intimately connected to classical algebraic geometry.

Modern perspective. A natural approach, generalizing questions of global nonnegativity, is to consider sums of squares and nonnegative forms on a real projective variety $X \subseteq \mathbb{P}^n$. For instance, let X be the second Veronese embedding of $\mathbb{P}^2$, i.e., X is the image of the map $\nu_2 : \mathbb{P}^2 \to \mathbb{P}^5$ given by $\nu_2([x : y : x]) = [x^2 : y^2 : z^2 : xy : xz : yz]$. Observe that quadratic forms on X are simply ternary quartics. Therefore, globally nonnegative ternary quartics correspond to quadratic forms nonnegative on X, and similarly, ternary quartics that are sums of squares correspond to quadratic forms on X, which are sums of squares of linear forms. We will use P_X to denote the set of nonnegative quadratic forms on X and Σ_X to denote the set of quadratic forms on X, which are sums of squares of linear forms. We observe that P_X and Σ_X are both closed convex cones inside the finite-dimensional real vector space of quadratic forms on X. Understanding their convex geometry has been crucial in proving our results.

It is natural to restrict our attention to *totally real* varieties, i.e., varieties X whose real points are Zariski dense in the complexification $X_{\mathbb{C}}$. Equivalently, the ideal I of

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X is a real radical ideal. In a series of papers we showed the following result, settling the question of equality of P_X and Σ_X.

Theorem ([BSV16, BSV17, BPSV16]). *Let $X \subseteq \mathbb{P}^n$ be a totally real nondegenerate variety. All nonnegative quadratic forms on X are sums of squares of linear forms if and only if X has Castelnuovo–Mumford regularity two. Moreover, any nonnegative quadratic form is a sum of at most* $\dim X$ *squares.*

Castelnuovo–Mumford regularity is an algebraic property of the equations defining X. It can be read off from the minimal free resolution of the ideal of X, but varieties of regularity two are also well-understood geometrically. Irreducible varieties of regularity two are varieties of minimal degree [EG84], i.e., varieties satisfying $\operatorname{degree} X = \operatorname{codimension} X + 1$, which were classically classified by Del Pezzo and Bertini; see [EH87] for a modern account. Reducible varieties of regularity two are varieties of minimal degree, which are glued together in a very special way [EGHP06].

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Grigoriy Blekherman

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Invariant Markov Processes Under Actions of Lie Groups

Ming Liao

In probability theory, the invariance of probability distributions under various transformations has played an important role. In many examples, the transformations form a Lie group and the distributions are associated to a Markov process.

Roughly speaking, a Markov process is defined by the Markov property, which says that given the present state, the future distribution of the process is independent of the past. This is a little like a classical dynamical system whose future evolution is completely determined by the present state. A Markov process may thus be regarded as a family of processes, one for each starting point, all governed by the same transition probabilities. It is said to be invariant under the action of a group G, or G-invariant, if its probability distribution is G-invariant. This means that if x_t is the process starting at x_0, then for any $g \in G$, gx_t is equal in distribution to the same Markov process starting at gx_0.

In the classical theory, the translation-invariant Markov processes x_t in a Euclidean space can be identified with Lévy processes, which are characterized by independent and stationary increments, in the sense that for any $s < t$, $x_t - x_s$ is independent of the process up to time s and its distribution depends only on $t - s$. The celebrated Lévy–Khinchin formula is the Fourier transform of a Lévy process, which is expressed by a triple of a drift vector, a covariance matrix (of diffusion), and a Lévy measure (of jumps). This provides a useful representation for a Lévy process because its probability distribution is determined by the triple, and to any such triple, there is an associated Lévy process, unique in distribution. Lévy processes include Brownian motion but may have jumps, so they provide a useful model for many applications. The purpose of this talk is to present a representation theory for more general invariant Markov processes under the action of a Lie group, in the spirit of the classical Lévy–Khinchin representation.

Invariant Markov processes under Lie group actions may be considered at three different levels of generality. First, we may consider Markov processes in a Lie group G that are invariant under the left (or right) translations. Such processes are direct extensions of classical Lévy processes in $\mathbb{R}^n$ and can be identified with processes x_t in G that have independent and stationary increments of the form $x_s^{-1}x_t$ (or $x_t x_s^{-1}$) for $s < t$. At the second level, we may consider a Markov process x_t in a manifold X that is invariant under the transitive action of a Lie group G on X. In this case, X may be identified with a homogeneous space G/K and x_t with a Markov process in G/K invariant under the natural G-action. Hunt (1956) obtained a formula for the generator of an invariant Markov process in G or G/K (when K is compact), which allows us to represent such a process in distribution by a triple of a drift vector, a covariance matrix, and a Lévy measure, just as for a Lévy process in $\mathbb{R}^n$.

At the third level of generality, we may consider a Markov process x_t in a manifold X that is invariant under the non-transitive action of a Lie group G. In this case, X is a collection of G-orbits. Under certain conditions, x_t may be decomposed into a radial part and an angular part. The radial part can be an arbitrary Markov process in a subspace that is transversal to G-orbits, whereas under the conditional distribution given the radial part, the angular part is a (time) inhomogeneous Markov process in a G-orbit that is invariant under the G-action. Note that a G-orbit is a homogeneous space G/K. For example, the Brownian motion in $\mathbb{R}^n$ is invariant under the orthogonal group $O(n)$, and the corresponding decomposition is usual spherical polar decomposition.

This naturally leads us to consider inhomogeneous Markov processes in G and G/K that are invariant under the G-action. Such processes may be characterized by independent, but not necessarily stationary, increments. Feinsilver (1978) obtained a representation of such processes in G by a time-dependent triple via a martingale property. Liao (2009) extended the representation to G/K.

The original motivation to study the representation on G/K is to obtain a skew-product decomposition for an invariant Markov process. For Brownian motion in $\mathbb{R}^n$ with the usual spherical polar decomposition (r_t, θ_t), the angular part θ_t may be written as $w_{a(t)}$, where w_t is a Riemannian Brownian motion in the unit sphere (a G-orbit), independent of the radial process r_t, and $a(t)$ is a random time change determined by r_t. This is the well-known skew-product decomposition of the Brownian motion. Using the representation on G/K, we can extend this skew-product to any continuous Markov process that is invariant under a non-transitive G-action with irreducible orbits (such as spheres).

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Ming Liao

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Analysis of Nonlinear Geometric Equations

Aaron Naber

One of the primary properties of solutions to linear elliptic equations is that they are smooth, and indeed effectively so. For instance, if we consider a H^1 weak solution to

$$\Delta u = 0 \tag{1}$$

on a ball $B_2(0^n) \subseteq \mathbb{R}^n$, then on B_1 we have that u is smooth and in fact for all $k \in \mathbb{N}$

$$||u||_{H^k(B_1)} \leq C(n,k)||u||_{H^1(B_2)}. \tag{2}$$

Nonlinear analogues of the elliptic equation $\Delta u = 0$ are abundant in mathematics. Typical examples would include nonlinear harmonic maps, minimal surfaces, Yang–Mills connections, Einstein manifolds, etc. Much as linear harmonic maps arise as critical points of the Dirichlet functional, often times solutions of nonlinear equations arise as critical points of energy-type functionals.

It is not surprising that when one moves to the nonlinear world the smoothing property (2) breaks down, and singularities can form inside a typical solution. However, for many nonlinear equations it turns out that these singular sets themselves come with a great deal of structure. In essence, solutions to many nonlinear equations can only *break* in very special ways.

The analysis of these singularities begins with stratification theory. In essence, it is important not just to distinguish *singular* versus *regular* points but also to ask *how* singular a given point is in order to define the stratification

$$S^0 \subseteq S^1 \subseteq \cdots \subseteq S^{n-1} = Sing. \tag{3}$$

The sets S^k are defined in terms of the symmetries of tangent cones of the solution. Slightly more carefully, a tangent cone at a point x roughly describes the behavior of the solution at very small scales at the point. Tangent cones are only nontrivial at singular points. The notion of symmetry tends to depend on the equation; however, roughly a solution is said to be k-symmetric if it is independent of k-variables. The set S^k is defined as the set of points for which no tangent cone is $k+1$-symmetric. In words, it is the set of points so that at very small scales the solution does not have too much symmetry. On the contrapositive side, points not in S^k must have small scales for which the solution looks very symmetric, and thus it is good to show that the set S^k is small.

This sort of analysis has existed since Federer in the 60's in the context of minimal surfaces, where he proved $\dim S^k \leq k$ through his dimension reduction methods. Since then these methods have also been applied to nonlinear harmonic maps by Schoen/Uhlenbeck and Ricci curvature by Cheeger/Colding, among many dozens of other setups.

A more refined understanding of S^k has only come into light in the last few years. More modern understanding involves both the structure and size of the stratums S^k. These questions turn out to be the key points in proving *a priori* estimates on solutions of nonlinear equations in forms analogous to (2). One is able to show that the sets S^k are k-rectifiable, which roughly means they are k-manifolds away from measure zero subsets. Effective estimates on the singular sets and solutions are derived by introducing and studying the *quantitative stratifications*. In combination with other methods these tools play a central role in the resolution of various other problems and conjectures—for instance, the energy identity conjecture for Yang–Mills and the $n-4$ finiteness conjecture for Ricci curvature.

Aaron Naber

Credits

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He is supported by NSF Grant 1809011. *Aaron Naber works in geometric analysis at Northwestern University with a focus on the regularity of nonlinear equations.*

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Data Analysis Techniques with Applications to Lyme Data

Deanna Needell

Deep learning is an exploding area of machine learning based on data representations using multiple levels of abstraction. Deep neural network algorithms have recently obtained state-of-the-art results for classification of large datasets due to advancement in computing power, availability of massive amounts of observed data, and the development of new techniques. Some drawbacks to these approaches include that they are not always convergent, are not well understood mathematically, and the output classifications can randomly fail without warning. Other strategies for data classification and feature extraction, such as topic-modeling-based strategies, have also recently progressed. Topic models combine data modeling with optimization to learn interpretable and consistent feature structures in data. We propose a novel approach that combines the interpretability and predictability of topic modeling learned representations with some of the attributes of deep neural networks, introducing a deep non-negative matrix factorization (NMF) framework capable of producing reliable, interpretable, and predictable hierarchical classification of large-scale data, far exceeding existing approaches. We showcase our techniques on recently acquired Lyme disease data, obtained from a large-scale patient survey by lymedisease.org.

Suppose you have an $M \times N$ data matrix X whose columns represents patients and whose rows represent words that the patient used when answering a specific question. The problem of non-negative matrix factorization is to factor the matrix X into $X \approx AS$, where A is a non-negative $M \times T$ matrix and S is a non-negative $T \times N$ matrix. The parameter T corresponds to the number of *topics* to represent the data, the matrix A then gives a topic representation for each question, and S is a topic representation for each patient. Concretely, we can view A_{ij} as an indicator of how important the ith question is for the jth topic and S_{jk} as how important the jth topic is to patient k. This structure implicitly reveals topics in the data, which can be interpreted by looking at which questions are important for which topics, via the matrix A. The matrix S can then be used to understand which patients are involved in which topics.

Often, one wishes to perform classification from the data X. For example, based on a patient's answer to question(s), we may want to predict labels from previous observations, e.g., determine whether they are well or unwell. We can use the raw data X as input to a classification method (e.g., support vector machines, linear regression, logistic regression, or artificial neural networks), or we can instead use the topic representations as input. With these goals in mind, we combine the NMF approach with ideas from multilayered or deep networks. Though multilayer networks have been around for decades, the recent surge of interest can be attributed to the increased availability of large amounts of data and computational hardware. Motivated by this success and the power of topic models, we introduce an innovative *deep NMF* method that combines two extensions of NMF.

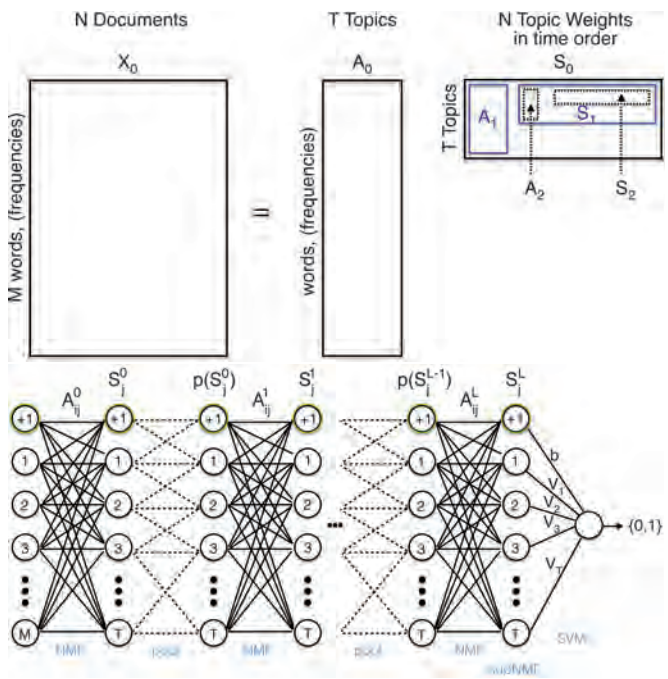

Figure 1. Top: Multilayered NMF. Bottom: Proposed deep non-negative matrix factorization.

Our proposed multilayered NMF method begins with standard NMF and then repeatedly factors the resulting topic weight matrices, $S^{(i)} \approx A^{(i+1)}S^{(i+1)}$. This creates a multilayered factorization of the data matrix X as illustrated in Figure 1, which shows our multilayered NMF model and the nested decomposition, $X \approx A^{(0)}(A^{(1)}(A^{(2)}S^{(2)}))$. One challenge of standard NMF is choosing

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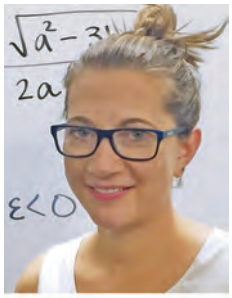
the correct number of topics, or rank restriction, for the factorization. This layered approach can start with a large overestimate and then reduces the number of topics at each level. Adding a supervised layer allows the algorithm to find the optimal number of topics and subtopics across all layers. We also incorporate pooling and several other variations. We utilize this approach on the Lyme disease data and other real data to showcase the advantages of such an approach. In doing so, we also uncover interesting patterns and observations on the Lyme data that will be important directions for future study.



Deanna Needell

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Figure 1 produced by J. Flenner and B. Hunter.
Author photo is courtesy of Blake Hunter.

An Illustration in Number Theory

Katherine E. Stange

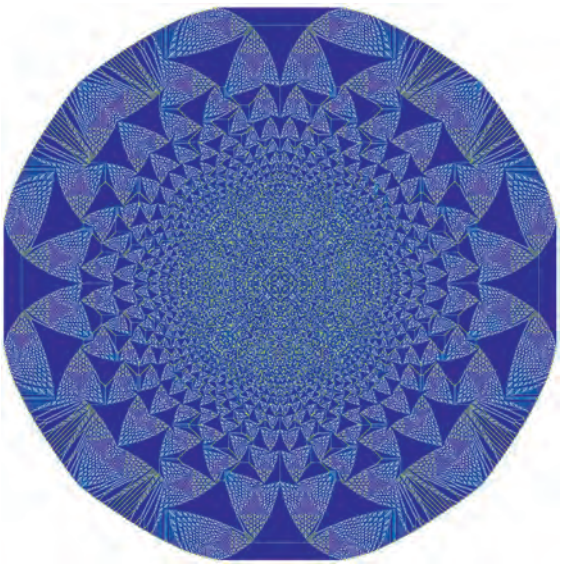

Figure 1. Stabilized sandpile for $500,000$ chips at the origin of $\mathbb{Z}^2$. Dark blue, light blue, green, and white represent 3, 2, 1, and 0 chips, respectively.

In the spirit of ICERM's upcoming semester on illustrating mathematics in fall 2019, my talk is a celebration of the symbiosis that can exist between research and illustration. I hereby invite you on a number-theoretic "explore" (as Pooh and Piglet would have it).

The story begins with a sandpile. A discrete sandpile on the square grid $\mathbb{Z}^2$ is a dynamical system in which each vertex contains some integer number of *chips*. Just as grains of sand will tumble and spread out, if these chips number four or greater, then the position will *topple*, sending one chip to each of its four neighbors. In this way, a stack of N chips initially placed at the origin will spread out across the grid until each position contains either 0, 1, 2, or 3 chips.

The result, for $N = 500,000$ chips is shown in Figure 1. Surprisingly, the patterns evident in Figure 1 do not fade as N tends to infinity. In fact, by rescaling to a constant width, one can define a scaling limit of this discrete sandpile, and its boundary is not even circular [11]. In evidence are curvilinear triangular regions of regularity in which periodic patterns of sand appear. This model, introduced by Bak, Tang, and Wiesenfeld, is an archetypical example of *self-organized criticality* [1]; see [10] for an overview.

Consider the following discrete Laplacian, which acts on functions $g : \mathbb{Z}^2 \to \mathbb{Z}$:

$$\Delta g(x) = \sum_{y \sim x} \left(g(y) - g(x)\right),$$

where $x \sim y$ indicates adjacency of vertices. If we start with a barren empty grid and think of $g : \mathbb{Z}^2 \to \mathbb{Z}$ as an *odometer*, that is, a local count of topples that occur, then Δg is the resulting distribution of sand (we must allow topplings that result in a negative number of chips).

This is the discrete analogue to the continuous Laplacian $\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$, which has the property that $\Delta f(x, y) \equiv 0$ if and only if $f : \mathbb{R}^2 \to \mathbb{R}$ is harmonic. Levine, Pegden, and Smart therefore asked, *which odometers are superharmonic* [9]? More specifically, for which symmetric 2×2 real matrices A does there exist $g : \mathbb{Z}^2 \to \mathbb{Z}$ so that

$$g(x) = x^t A x + o(|x|^2), \quad \Delta g \leq 0?$$

They defined Γ to be the set of such A, as a subset of the parameter space $\mathbb{R}^3$ given by the entries of the matrix. It is not hard to show the set should be a union of cones (this property is downward closed), but when they approximated this set experimentally, they obtained a surprising fractal boundary; see Figure 2.

The cones were arranged according to an *Apollonian circle packing*. This is an iterative fractal generated from a quadruple of mutually tangent circles by filling in the triangular region between any three tangent circles with a *daughter* circle tangent to its ancestors (Figure 3). A remarkable

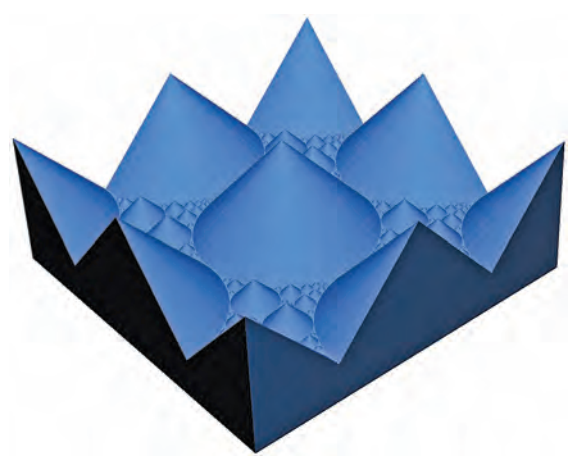

Figure 2. A part of the set Γ, which is a union of cones subtended by circles of an Apollonian circle packing.

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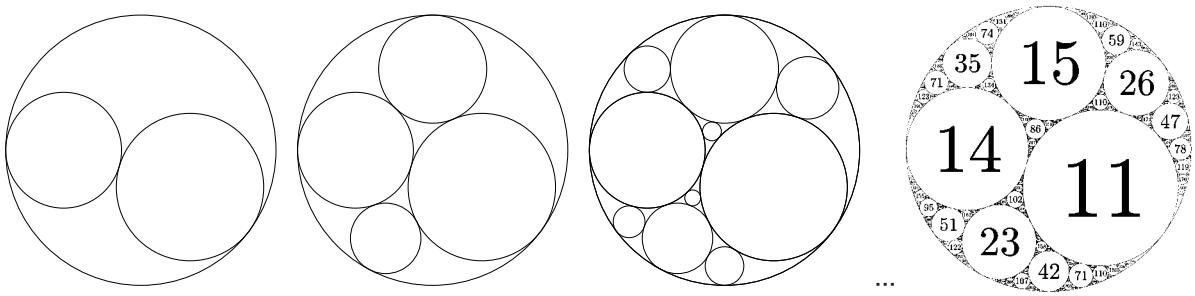

Figure 3. The iterative process generating an Apollonian circle packing, with curvatures shown in the final stage.

property of Apollonian circle packings, explained by René Descartes, is that if one begins with four mutually tangent circles with integral curvature (the curvature of a circle is 1 over its radius), then all circles in the packing will have integral curvature [5].

The collection of curvatures of an Apollonian circle packing has generated recent attention among number theorists. The collection is generated by the action of a thin matrix group. If G is an algebraic group, a subgroup H of $G(\mathbb{Z})$ is *thin* if it is Zariski dense but of infinite index; such groups are much less accessible than lattices such as congruence subgroups. The collection of curvatures is conjectured to include all but finitely many integers satisfying a congruence condition modulo 24 (a *local-to-global principle* for circle packings) [6, 7]. Bourgain and Fuchs applied the Hardy–Littlewood circle method to prove a density one result [2]. For an overview of this approach to thin groups, see [8].

Computer evidence for the shape of Γ was a surprise, and it turned the investigation toward the "Apollonian" peaks. These odometers g correspond to very special Laplacians Δg, which can be observed, experimentally, as periodic sandpiles (after stabilization) appearing in regions of Figure 1; see Figure 4. David Wilson of Microsoft Research

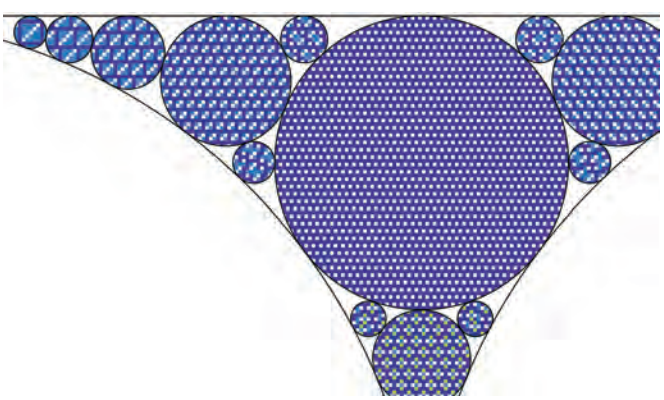

Figure 4. A selection of circles of the Apollonian circle packing filled with the corresponding stabilized sandpile patterns. Dark blue, light blue, green, and white represent 1, 0, -1, and -2 chips, respectively.

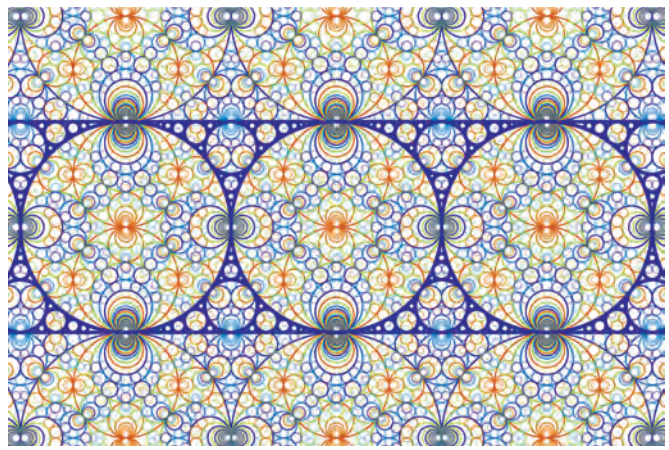

Figure 5. The Schmidt Arrangement of the Gaussian integers, with an Apollonian circle packing highlighted in dark blue. Circles fade as an increasing function of their curvatures. Light blue, dark blue, orange, and green represent curvatures 0, 1, 2, and 3 modulo 4, respectively. The picture has periodicity modulo $\mathbb{Z}[i]$ and is centered on $i/2$.

created an interactive computer program, wherein the user could mouse over a circle C in an Apollonian circle packing to obtain a basis for $\Lambda_C \subseteq \mathbb{Z}^2$, the lattice of periodicity of the corresponding sandpile. The covolume of Λ_C was observed to match the curvature of the circle.

From this computer data, one can glean a recursive rule for generating the lattices, from ancestors to daughters in the packing (just as one generates curvatures). The resulting theory is reminiscent of the *topograph*, which is a fractal tree demarcating $\mathbb{P}^1(\mathbb{Z})$, and which Conway and Fung use to elegant effect to classify quadratic forms [4]. For a fanciful paper comparing the two, see [13]. These ideas figure in the proof of Γ's shape by Levine, Pegden, and Smart [9].

The Apollonian circle packing is an orbit of circles under the Möbius action of a thin subgroup of $\mathrm{PSL}_2(\mathbb{Z}[i])$. If one computes the orbit of the entire Bianchi group $\mathrm{PSL}_2(\mathbb{Z}[i])$, the result is Figure 5. Each circle in this arrangement represents an element of the Bianchi group, and the lattice Λ_C is generated by $\mathbb{Z}$-linear combinations of the lower entries of the corresponding matrix [14]. This gives

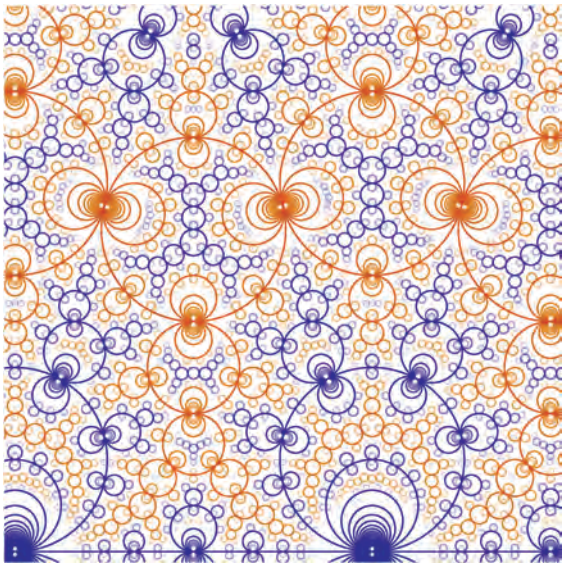

Figure 6. The Schmidt Arrangement of $\mathbb{Q}(\sqrt{-15})$ in blue, together with the remainder of the orbit of the extended Bianchi group, in yellow. The two orbits represent the two elements of the class group of $\mathbb{Q}(\sqrt{-15})$.

a bijection between sublattices of $\mathbb{Z}^2$ (or, if you prefer, certain ideals of orders in $\mathbb{Z}[i]$), and the circles of Figure 5 (up to certain translations and rotations). The geometric arrangement of the circles is reflected in algebraic relationships among the lattices.

Why not draw similar pictures for other Bianchi groups $\mathrm{PSL}_2(\mathcal{O}_K)$, where K is an imaginary quadratic field? Each has its own iterative structure and contains Apollonian-like circle packings [15]. An example is shown in Figure 6. What aspects of the arithmetic of the field control the variation in corresponding geometry? One example is that the picture is connected if and only if the ring of integers is Euclidean [14]. These images are called *Schmidt Arrangements* for Asmus Schmidt, who used them to define complex continued fractions before the arrangements could be drawn by computer [12]; see [3] for another perspective on continued fractions and their relationship to Figure 5.

Aided by computer experimentation, we have now wandered from sandpiles, through Apollonian circle packings, to arithmetic geometry. In fact, a general theory connecting sandpiles and arithmetic geometry is now emerging by way of tropical geometry, where the odometers of our story will correspond to tropical theta functions.

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Maria Gaetana Agnesi
1718–1799

Della Dumbaugh

On March 8, 2018, the government of Italy issued a set of four stamps celebrating what they described as "Italian Female Genius: Depicting Italian Women who have Distinguished Themselves in the Arts, Sciences & Culture." Among the four women celebrated was Maria Gaetana Agnesi, described in the announcement as a "mathematician."[1] On September 6, 2018, the Vatican issued a stamp of Maria Agnesi in the category of "Science and Faith."[2] Both stamps commemorate the 300th anniversary of Agnesi's birth on May 16, 1718 and call attention to the two main aspects of her life: the intellectual and the pious.

The first child of a prosperous Italian family, Agnesi's father hoped to advance in social circles in part by educating and exhibiting the education of his children to Milanese society. From an early age, multiple tutors, mostly from religious orders, fostered Agnesi's knowledge and faith within the broader purview of the Catholic Enlightenment (Mazzotti, 2001, 2008; Petrunic). Agnesi was recognized as a child prodigy by the age of five owing in part to her proficiency with languages. Although shy by nature, she joined in the seminars and discussions that took place in her family home in the evenings.

In 1738, at the age of 20, she published her *Philosophical Propositions*, which included 191 propositions that she purportedly debated with leading scholars. A decade later, she published her *Analytical Institutions for the Use of Italian Youth* (*Instituzioni analitiche ad uso della gioventu italiana*), which contained principles and methods of algebra, geometry, and differential and integral calculus. She wrote in Italian (not Latin) with an eye towards clarity, initially aiming to provide a text for her younger siblings (there were 20 of them, after all). She dedicated the book to Empress Maria Theresa, who expressed her gratitude with diamond jewels. Her family's largess allowed her to print the book on "1044 pages of thick, creamy paper that must have cost her father a pretty penny" (Findlen, 258). Her book garnered the attention of Pope Benedict XIV and earned her an election to the Bologna Academy of Sciences and an appointment to the faculty at the University of Bologna.[3]

Agnesi is best known for the curve that has now come to be known as the "Witch of Agnesi." Modern forms of this curve are given by $yx^2 = a^2\ (a-y)$ or $y = \frac{a^3}{(x^2+a^2)}$. This curve did not originate with Agnesi, however, as Pierre de Fermat and Guido Grandi had both studied it. When John Colson, Lucasian Professor of Mathematics at Cambridge,

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[1] *See* www.virginstamps.com/browse_item.html?category_id=3123&item_id=16835&page=80.

[2] *See* www.vaticanstate.va/content/vaticanstate/en/servizi/ufficio-filatelico-e-numismatico/emissioni-filateliche/ultime-emissioni/scienza-e-fede-maria-gaetana-agnesi-padre-angelo-secchi.html.

[3] *Despite this appointment, which included her name on the rolls of the university for forty-five years, Agnesi never went to Bologna so she never assumed the duties of this position (Truesdell, 1989, p. 127).*

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translated Agnesi's *Analytical Institutions* before his death in 1760 (published posthumously in 1801), he confused the Italian name of the curve "la versiera," which means "to turn," with "l'aversiera," which means "the witch" or "the she-devil" (Truesdell, 1992, 386; Mazzotti, 2008, 116–117). Both stamps include a sketch of the "witch of Agnesi," although this title is inaccurate both in attributing the figure to her and in its association of her with the notion of a witch.

After her father's death in 1752, Agnesi devoted her life to "great and pious acts of charity" that mostly offered aid to neglected, ill, and aging women (Findlen, 271; Osen, 47). She transformed her home into a refuge for the poor and, when Milan's newest hospital, the Pio Albergo Trivulzio, opened in 1771, she accepted the appointment as director of the women's ward. She hoped to offer women and orphans faith and dignity for a useful life or for the final steps of their earthly journey. Agnesi died in 1799.

In Agnesi's time, as Mazzotti notes, the "intellect was necessary for being a good Christian. If you work[ed] on strengthening your intellect, you [were] doing a good thing for your spiritual life as well" (Lamb). Mazzotti's insight suggests a more holistic view of Agnesi's life, one joined rather than divided by her two primary interests in intellectual and pious pursuits. Reflecting on Agnesi as a woman in mathematics (to use a modern term), Agnesi's family provided her with an education and a chance to study mathematics. She took risks with her *Analytical Institutions*, writing in Italian and focusing on the mathematics itself, rather than linking it with physical applications as previous texts had done. She also took risks with her father, asking to step back from evening scholarly discourses after her mother died and gradually pursuing more religious studies and activities over time. Findlen may rightly note that Agnesi's life did not follow a "modern trajectory" in that she had the opportunity to do science but then gave "it all up for love of God" (Lamb). But there is something more. Agnesi's life, even at 300 years old, calls attention to some enduring qualities associated with advancing women in mathematics. She had the opportunity to study mathematics, she had educators and family who supported her work, she had a strong enough spirit to take risks, and she received recognition for her work (Fenster & Parshall, 230). In these ways, the stamps commemorating Agnesi reflect a very modern life.

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A Mathematician Hosts *NOVA Wonders*

Sophia D. Merow

When, in 2015, Harvey Mudd College professor Talithia Williams got an email from a producer of the public television science program *NOVA*, it felt too good to be true. Might she have any interest in being involved with an episode on prediction? Would she want to host it? What about hosting a six-part series exploring the frontiers of human knowledge?

The conversation continued for months—via email, by phone, and in person over meals in Claremont, CA, and Baltimore, MD. "The entire time I'm just like, '*Yeah*,' '*Sure*,' 'I'll host a show,'" Williams recalls, her intonation conveying some unsubtle skepticism. "It didn't seem real because there were so many things that needed to line up to make it happen."

NOVA needed to raise three million dollars, for one thing, to make the six-episode series a reality, and Williams had a hard time—at first, anyway—believing that the producers wanted *her*. "Like, really," she says, "you just found me off my TED talk and now I'm going to host a show?"

Williams had also been skeptical when one of her students, Elly Schofield, approached her about participating in the third TEDxClaremontColleges event, held in February 2014. Her initial reaction to the event's theme, "Unexpected Narratives," was "I'm a statistician. I don't tell stories." But under pressure from Schofield—and with assistance from her and TED veteran and fellow Harvey Mudd professor Art Benjamin—Williams developed a talk around the idea of telling stories with data. In "Own Your Body's Data," Williams used graphs, personal anecdotes about childbirth and emergency room visits, and a "Show me the data!" refrain to highlight health insights potentially gained by tracking biodata like waking temperature or blood pressure.

Well-received at the Claremont Colleges event, Williams's talk was picked up by TED in August 2014 and featured on the TED website (`https://bit.ly/1SIctkx`). *NOVA*'s talent scouts saw it, recognized in Williams the makings of an on-screen science communicator, and reached out by email.

Though slow in solidifying, the partnership *NOVA* proposed eventually worked out. Williams did host that episode on prediction. An hour-long examination of how predictions underpin many aspects of modern life and why some succeed spectacularly while others fail, "Prediction by the Numbers" (`https://to.pbs.org/2EHCdCr`) included appearances by Billy Beane of *Moneyball* fame;

In *NOVA*'s "Prediction by the Numbers," Talithia Williams demonstrates the power of the wisdom of the crowd...with jelly beans.

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FiveThirtyEight founder Nate Silver; and mathematicians such as Keith Devlin, Jordan Ellenberg, and Rebecca Goldin.

Williams, who holds an undergraduate degree in mathematics from Spelman College, a masters in mathematics from Howard University, and a PhD in statistics from Rice University, also landed a gig less focused on her field of study. *NOVA Wonders* (https://to.pbs.org/2O-MOx9k) devotes one hour to each of six big questions about life and the cosmos. One episode tackles "What's the universe made of?" Another digs into "Can we build a brain?" In "Are we alone?" viewers meet SETI researchers scanning the skies for signs of extraterrestrial intelligence. In "What's living in you?" an entomologist shows science museum visitors the arachnids living on their faces. Mathematics plays only a supporting role in *NOVA Wonders*, with mathematical physics and data analytics making cameo appearances. But in the title sequence of each show, there's host Talithia Williams, introducing herself: "I'm a mathematician," she says, "using Big Data to understand our modern world."

NOVA employs a team of scientists and other professionals to frame questions, track down experts, and script episodes, so Williams wasn't doing any of that as host of *NOVA Wonders*. Her responsibility, she says, was twofold: to "serve as a scientific authority" and to "enthusiastically communicate the beauty of science."

Enlisting scientists as hosts—as opposed to Tom Cruise or Morgan Freeman—gives the program authenticity, Williams says. Each host—Williams was joined by co-hosts André Fenton, a neuroscientist, and Rana el Kaliouby, a computer scientist—could step in and inject authority into the show whenever her or his specialty came up. "So anything that had to do with math, statistics, or data, I became sort of the expert who was brought to the table," Williams explains.

Williams and her co-hosts received segment scripts in advance, and were encouraged to provide feedback. Sticklers for accuracy, the *NOVA* crew didn't mind when, even as filming was under way, Williams asked that statistics be verified or wanted to tweak a phrase because "a mathematician wouldn't say it like that."

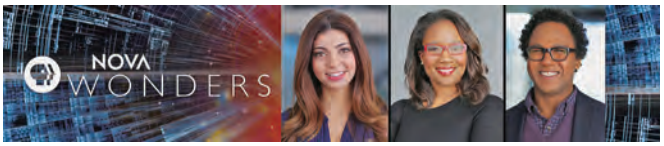
"We definitely weren't hired to be a pretty face and read the script," Williams recalls.

There's a lot more to science communication than reading a script, Williams says. You have to want people to understand what you're talking about, and to convey that. "Your actions and your gestures and your tone have to really invite people to the table and invite them to take on your enthusiasm for a particular topic," she explains.

But it's a modulated enthusiasm that's required. However jazzed you are about the potential benefits of gene therapy, for instance, you have to temper your excitement when reporting that a clinical trial killed an eighteen-year-old suffering from a hereditary liver condition. Exuberant by default, Williams found the sadder segments the most challenging to shoot, often needing multiple takes to nail the appropriate level of engagement.

"I did musical theater in high school and some acting," Williams says. "I think I leaned more on those skills than my skills as a mathematician."

Williams watched public television as a child, but didn't see herself reflected there. *NOVA* readily admits, according to Williams, that in the 1970s, 80s, and 90s, the experts they presented to the viewing public were, overwhelmingly, white men. "They feel like they did a great job of inspiring other white boys to be scientists, and so I think they're trying to make a course correction," she says. "They've been intentional to make sure that the constant presence you see throughout every episode is two women and three people of color."



Mathematician Talithia Williams (center) cohosts *NOVA Wonders* with computer scientist Rana el Kaliouby (left) and neuroscientist André Fenton (right).

Los Angeles Times television critic Robert Lloyd, for one, caught the show's drift (https://lat.ms/2Pl6Dyt): "Without making a point of it," he wrote in April 2018, shortly after the show aired, *NOVA Wonders* "sends the message that scientists come in a range of ages, genders, colors and ethnicities, with a range of hairstyles, clothing and research specialties."

Williams knew going into *NOVA Wonders* that the series wasn't going to leave viewers with an understanding (or even an awareness) of her research area. (She develops statistical models that emphasize the spatial and temporal structure of data, and applies them to problems in the environment.) Her goal, rather, was to give girls of color an image of someone who looks like them who is excited about math and science—something she never had. Until the book and movie came out, Williams had not heard of *Hidden Figures* heroines Katherine Johnson and Dorothy Vaughan, even though she spent four summers at NASA as a NASA scholar. If anybody should have known about these women, she should have. "And what a difference it would have made to know that they came before me," Williams says. She thinks *NOVA Wonders* is "trying to highlight to a really diverse country that lots of diverse people can do science and are doing science."

Involvement with *NOVA Wonders* has, indirectly, enabled Williams to spread the word about the likes of Johnson and Vaughan. The visibility that came with hosting the show has garnered Williams more speaking invitations, and she uses these to promote her book, *Power In Numbers: The Rebel Women of Mathematics*, which came out in May 2018 (and was reviewed in the February 2019 issue of *Notices*). "I like that I can go and say 'I did *NOVA Wonders*, it was amazing, and there are also these women who do math, and math is awesome too,'" Williams says. "I think the two of those together makes for a really powerful way to engage the public."

Whether *NOVA Wonders* continues to engage the public will depend (as does so much) on funding. The 2018 series attracted a cohort of viewers younger than *NOVA*'s typical demographic of fifty- or sixty-something life-long learners, and there's interest in serving this audience going forward. But since a show with the scope of *NOVA Wonders* takes years to realize, a better near-term prospect is a stand-alone show more like "Prediction by the Numbers." Nothing is confirmed yet, but a math-only hour hosted by Williams is a distinct possibility. Something about the origins of zero and infinity, perhaps, or the Fibonacci numbers in nature? Stay tuned.

Sophia D. Merow

Credits

Photo of Talithia Williams is courtesy of WGBH/Cara Feinberg.
NOVA Wonders banner is courtesy of WGBH.
Author photo is by Craig Merow.

Washington Update

Karen Saxe

The American Mathematical Society has four physical facilities: our headquarters in Providence, RI; our Printing and Distribution Department in Pawtucket, RI; Mathematical Reviews/MathSciNet® Division in Ann Arbor, MI; and our Office of Government Relations (https://www.ams.org/government/government) in Washington, DC. This is the first in what will be a quarterly column highlighting the work of the DC office.

The staff based in DC run several AMS programs and activities that occur on an annual or bi-annual schedule. We also follow and respond in a timely manner to legislation and policies under consideration in Washington. In all of our activities, we work to get math "at the table" for important discussions with decision-makers that have the potential to impact mathematics research and mathematicians.

In this first column, I will present some of our regular activities, including an example response to a current government policy discussion.

Recent Activities of the Committee on Education

The AMS has five policy committees, each taking a long-range view of their areas of expertise, to provide advice to the leadership of the society, coordinate with other professional organizations, and communicate with membership. This office supports the work of two of these committees—the Committee on Education and the Committee on Science Policy. I will discuss activities of the Committee on Science Policy in a later column.

This past October the Committee on Education hosted a daylong Mini-conference. The Mini-conference was open to the public, and invitations had been sent to department chairs at bachelor, masters, and PhD granting institutions. About seventy attended, including mathematics faculty members from all over the country; representatives from the National Science Foundation and National Academies of Sciences, Engineering, and Medicine; colleagues from other mathematics professional societies; and at least one Congressional staff member. Next year's Mini-conference will be held on October 25; save the date and watch our website for details.

Mathematicians, other academics, and policy-makers gave presentations at the October Mini-conference. Christopher Edlay, Jr. (The Opportunity Institute, and former Dean of the University of California, Berkeley School of Law) shared perspectives on law, education policy, and equity. Ellen Hildreth (Wellesley College) described the role of mathematics in the study of the brain's ability to process information, and Sonin Kwon (MassMutual Financial Group) described why mathematics education matters to the financial industry. Rachel Levy (Deputy Executive Director, MAA) spoke about how departments can best support our students as they pursue careers in all areas of business, industry, and government. Uri Treisman (Charles A. Dana Center, University of Texas at Austin) and I gave a presentation on the creation, evolution, and agenda of the Transforming Post-Secondary Education in Mathematics initiative (tpsemath.org). Jake Steel (US Department of Education) gave us an overview of the current administration's priorities for STEM education, and discussed what

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we can all do better to support mathematics educators. Manil Suri (University of Maryland, Baltimore County) shared his perspective on a STEAM-related approach to general education mathematics and informal math outreach (STEAM refers to an expansion of STEM—Science, Technology, Engineering, Arts, and Mathematics). A full program and links to presentation slides are found at the Mini-conference website (https://www.ams.org/meetings/coeminiconference).

Involvement in Golden Goose Awards

From 1975 until 1988, Senator William Proxmire offered the Golden Fleece Awards; the Senator from Wisconsin thought it a good idea to publish a monthly bulletin highlighting what he viewed as the most frivolous and wasteful uses of taxpayers' dollars. The June 1987 award, as just one example, called out the Executive Office of the President for spending $611,623 to fit one room with gold trim.

Many of the Golden Fleece Awards have gone to scientists, and these researchers' intentions have been presented unfairly, making them appear frivolous. Golden Fleece Awards have raised suspicion and hostility towards the notion of government spending on science and, indeed, toward science itself. Maybe you've heard of the shrimp on the treadmill (https://www.npr.org/2011/08/23/139852035/shrimp-on-a-treadmill-the-politics-of-silly-studies)? Or smiling bowlers (https://www.psychologicalscience.org/observer/why-bowlers-smile)?

In response, Representative Jim Cooper suggested attempting to temper growing animosity toward federal funds going to science. He aimed to give higher visibility to the great return on our federal investment in scientific research, and also show that successes in science are often unpredictable. The Golden Goose Award (https://www.goldengooseaward.org) was first awarded in 2012 and aims "to celebrate scientists whose federally funded research seemed odd or obscure but turned out to have a significant, positive impact on society."

The AMS is a contributing sponsor of the Golden Goose Award, and this brings good visibility to the AMS with Congressional members and other key players in DC who are science supporters. We attend a relatively exclusive Congressional lunch celebrating the winners, and also attend the award ceremony in the evening, hosted at the Library of Congress.

Mathematicians have been honored with this award. In 2013, Lloyd Shapley and David Gale (together with economist Al Roth) were recognized for their work on an algorithm to address the stable marriage problem (www.ams.org/publicoutreach/feature-column/fc-2015-03). This algorithm could be employed to match equal numbers of men and women with a partner (of the opposite sex) so that no two people of the opposite sex would prefer each other over their algorithmically-assigned partner. Their original research was funded by the U.S. Office of Naval Research and may sound silly. However, results of this research have saved lives because it has now been used to improve our national kidney exchange program. The algorithm has also been adapted by urban school systems for school choice programs, and has been honored with a Nobel Prize. The scientific community makes nominations, and I encourage you to nominate a colleague; nomination instructions are found at the Golden Goose website (https://www.goldengooseaward.org).

Incidentally, the Golden Fleece Award has been resurrected (since 2015) by Representative French Hill and his most recent award went out in July. Not all of these highlight work of individual scientists, and many point to waste akin to the aforementioned gold trim. There are other congressional members who are challenging the value of the federal investment in scientific research. Senator Rand Paul recently introduced the BASIC Research Act (https://www.congress.gov/bill/115th-congress/senate-bill/1973) as part of his effort to convince others that there is a lot of "silly research" being done that does not deserve federal funding (www.sciencemag.org/news/2017/10/rand-paul-takes-poke-us-peer-review-panels). Supporting the Golden Goose Award fits in with the AMS Office of Government Relations' mission to advocate for robust and dependable federal funding for scientific research and to stress the value of the NSF's peer review system for evaluating proposals.

The AMS Response to Recent Government Policies on Open Access

This past fall was an active one on the "open access" front. In 2013, then presidential science adviser John Holdren issued a memo on open access requiring that published results of federally funded research be freely available to the public within a one year "embargo period" of publication (https://obamawhitehouse.archives.gov/blog/2013/02/22/expanding-public-access-results-federally-funded-research). Agencies have taken various paths to achieving open access. The Department of Energy, Department of Defense, and NSF—which together account for nearly three-quarters of all federally sponsored non-biomedical research articles—use the CHORUS data infrastructure service (https://www.chorusaccess.org) to comply with the 2013 memo requirement. Principal Investigators on NSF grants are responsible for meeting the public access requirements. The NSF maintains a useful website addressing many questions you may have about your obligations (https://www.nsf.gov/pubs/2016/nsf16009/nsf16009.jsp#q3).

The AMS has created two open access journals—mirror journals of the *Transactions* and *Proceedings*—so that authors

who want to publish in the open access setting can do so. In our model, an author submits to the editorial board of the parent journal (*Transactions* or *Proceedings*) and, following acceptance, is offered the choice to publish in the parent journal or the open access mirror journal. This separates the editorial decision from the business decision. The AMS website "About Open Access" describes which of our journals are open access and the various types of open access models available to AMS journal authors (https://www.ams.org/publications/journals/open-access/open-access).

The current administration is reviewing the 2013 directive and considering updating it. As it is now on the table, the AMS Associate Executive Director for Publishing Robert Harington and I have spent time with White House Office of Science and Technology Policy (OSTP) staff discussing the AMS as a publisher, and how policies—such as on embargo periods for federally funded research—can affect society publishers and AMS members. Surplus funds that the AMS makes from publishing activities—including subscription revenues from journals—go directly back into our programs to fortify the U.S. mathematical sciences community. These programs include the Joint Mathematics Meetings (jointmathematicsmeetings.org/jmm) and the Mathematics Research Communities (https://www.ams.org/programs/research-communities/mrc). As a society publisher, we are concerned that cutting back or eliminating the embargo period would reduce incentives for universities and other institutions to pay for journal subscriptions.

Open access is a global concern with a range of models in play, including emerging models such as the European Plan S (https://www.scienceeurope.org/coalition-s). Our "one-pager" expands on open access and mathematics research (https://www.ams.org/government/dc-collaborations). Open Science provides a good primer on open access (https://openscience.com/green-gold-gratis-and-libre-open-access-brief-overview-for-beginners). The Scholarly Kitchen blog (of which AMS Director of Publishing Robert Harington is a "chef") is a source of opinion pieces on open access (https://scholarlykitchen.sspnet.org/tag/open-access).

The AMS Office of Government Relations writes a biweekly blog: https://blogs.ams.org/capitalcurrents. Posts include timely information on activities in Congress that affect mathematicians and the broader science community, historical tidbits on federal science policy, and opportunities for engaging with Congress and other policy makers.

Karen Saxe

Credits

Author photo is courtesy of Macalester College/David Turner.

An Interview with Immediate Past President Kenneth A. Ribet

Evelyn Lamb

Every other year, when a new AMS president takes office, the *Notices* publishes interviews with the outgoing and incoming presidents. What follows is an edited version of an interview with Kenneth A. Ribet, whose two-year term as president ended on January 31, 2019. Ribet is a professor of mathematics at the University of California, Berkeley. The interview was conducted in fall 2018 by freelance writer Evelyn Lamb.

An interview with incoming president Jill C. Pipher will appear in the April 2019 issue of the *Notices*.

Kenneth A. Ribet

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Notices: *I'd like to start with something that I think is on the minds of a lot of people, which is how the US political situation in the past few years has affected mathematics and the AMS.*

Ribet: Many of Trump's actions have been harmful to science and mathematics. My very first act when I was becoming President of the AMS was to draft a statement protesting Trump's travel ban and explaining its impact on the mathematical community.

Actions like that by the President of the United States have forced the AMS, through its officers, to make political statements, whereas in the past, this was almost never necessary. Some AMS members are disturbed by this turn of events. Of course, there are many people who are delighted that the AMS, along with other scientific societies in the US, has taken a stand against what the President is doing.

Fortunately, the US Congress has been a lot more sympathetic to scientific research than the White House. Proposals by the White House to drastically cut budgets of the National Science Foundation and other agencies that fund mathematics were ignored by Congress.

Notices: *What role do you feel the AMS should take in protecting mathematicians who are affected by things like the travel ban, which has targeted researchers from some Muslim-majority countries, or other immigration policies that affect people from, for example, Latin America?*

Ribet: I would say that the AMS needs to take global positions against the sort of problems that have befallen

many mathematicians as a result of this administration's policies. When it comes to intervening in individual cases, there is no easy answer. The AMS has intervened in some individual cases in the past; while this can call attention to an individual case, the Society lacks the resources to do much more. Recently, there have been active discussions, starting in the Committee on the Profession and the Committee on Human Rights, as to what the appropriate role of the AMS is in individual cases.

To me, this is really an ongoing question.

Notices: *In your interview with the* Notices *at the beginning of your term as president, you talked about some shifts in the strategic plan of the AMS. Can you talk about what those shifts have been and how the AMS is moving forward in different directions?*

Ribet: For one thing, there are new people on the job. Catherine Roberts took over from Donald McClure about six months before I became president. Karen Saxe, who heads the Washington office (now known as the Office of Government Relations) began her work a month before I became president. There's a new Director of Membership, Megan Turcotte, who is trying to revitalize the Society in ways that matter to members.

Beyond personnel changes, the AMS [began a rebranding effort to strengthen our members' and our potential members' understanding of what the AMS does and has to offer to the math community. As part of that rebrand] the AMS also designed an awesome new logo!

Additionally, we launched a major fundraising effort called the Campaign for the Next Generation. It aims to secure a $3 million endowment that can be used for people early in their careers. So this is a lot of good change that's going on in the AMS.

The Society's 2016–2020 strategic plan calls for us to increase our publishing activity. As a non-profit, the AMS takes the proceeds from our profitable publishing operations, such as MathSciNet®, the academic journals, and the book program, and puts them back into the mathematical profession.

For example, publishing profits subsidize AMS regional conferences, which are extremely popular but expensive to run—registration fees fall short of the cost of running the conference by about $25 per person on average. This is just one of many ways in which the AMS uses its publishing revenue to support mathematicians.

Recently, the AMS purchased the entire program of books—monographs and textbooks—from the Mathematical Association of America [MAA]. All of the series like the Carus Mathematical Monographs that are associated with the MAA are now [owned] by the AMS. And so in one fell swoop, the AMS has substantially increased its book publishing program.

Notices: *Related to the relationship between the AMS and MAA, recently it was announced that in the near future, the Joint Meetings are no longer going to have the same form, where the AMS and MAA jointly organize and fund them. Can you comment at all on that decision, and what, if any, practical effects people will see from that change?*

Ribet: Starting in 2022, the MAA will shift a lot of the activities that it has been conducting at the winter meeting, the Joint Mathematics Meetings [JMM], to the summer meeting, MathFest. One thing that has emerged in the comments about this decision is that AMS members who think of themselves as primarily research mathematicians love going to the JMM in order to attend both research talks and talks about teaching. The aggregate AMS–MAA community at JMM has always been really great. I think what is likely to happen is that the AMS will start organizing special sessions that are analogs of the special sessions that the MAA is now thinking of moving to the summer. The footprint of the AMS will change a little bit, that we will start being more explicit, for example, about our commitment to undergraduate education. This was always part of the AMS, but the organization of those activities at the JMM was done by the MAA, and I think the AMS will step in.

Another interesting theme from the comments about the change is that many MAA members feel that the employment services at the JMM—and those are run completely by the AMS—were a very important reason for their attendance at JMM. Perhaps this will inspire MAA members to continue attending JMM, and it is the job of the AMS to make them feel at home.

I should add that there are a number of other organizations that participate in JMM. There are a number of other organizations, including the Association for Women in Mathematics and the Society for Industrial and Applied Mathematics, that are partners in JMM. A likely scenario is that those other organizations will start being more active in JMM as the MAA shifts its focus to the summer meeting.

Notices: *It seems like publishing is still exploding, as it has been for a while. There is more and more information that researchers are trying to take in, and more and more demands on them for refereeing and so on. A lot of new open access journals are starting, debates about publishing with companies like Elsevier are continuing. What do you see as the future of publishing in mathematics?*

Ribet: There are lots of things going on. Open Access is certainly one of them. The exploding mathematical literature is another. The question of how long people are going to be walking around with physical books, as opposed to tablets, these are all questions in real flux. Mathematicians tends to be more conservative than people in some other sciences. For example, if you have a publisher like Springer,

which is publishing in medicine and all across the sciences, from that broad perspective, they see virtually no demand for printed books. But you go in the mathematical community, and people really like to browse through books on paper.

Optimally, what people would like is the hard copy that they can keep open and refer to while they're working and a soft copy they can use for searching. In the near term, the future will probably be that people will buy a bundle consisting of paper and PDF so that they can have access to both depending on what they want to do.

The Society began to prepare for open access five or six years ago by creating open access partners to the *Proceedings of the AMS* and the *Transactions of the AMS*, and we still anticipate that there will be a big demand for publishing in open access journals. But at least in the United States, so far, that hasn't exactly materialized. People are still publishing in journals that are behind paywalls, at least for a year, and at the same time, they're putting up their final version on arXiv and on their webpages. In practical terms, this gives everyone access to the content and still keeps the publishers happy that they have some special product that they're charging access for, for some limited embargo period.

Notices: *Another thing you talked about in your previous interview was the increasing visibility of math popularization and news stories about math. Do you feel like that's still continuing? How do you think controversies, such as the status of the ABC conjecture, have played to the larger science news consuming public?*

Ribet: Oh, I think they're wonderful. There is more and more interest in mathematics matched with a just-as-rapid increase in the ways mathematics can be accessed. It's just astonishing how many views a Numberphile video gets, almost as soon as it's published. The *Hidden Figures* book and movie drew a tremendous amount of attention to mathematics. There was a lot of media attention on the Fields Medals when they were awarded in August. There is a lot of hunger for high-level mathematics from people kind of interested in mathematics but who are not professional mathematicians.

I think it's good for the general public to see how mathematicians struggle and how progress is incremental, how people attempt things and consult with their colleagues and learn that their proposed proof might not correct. But on the other hand, incorrect proofs often contain interesting ideas that turn out to be useful later on for a correct proof or in different contexts.

Notices: *What do you see as some of the big challenges to the mathematical profession right now?*

Ribet: The one that comes to mind is the increased focus on hiring people in research departments whose primary function is going to be teaching. They are often called lecturers, though other titles are used at different universities. Many lecturers were trained in research, and they are part of research departments, but they might be devoting most of their time to teaching large calculus or pre-calculus courses. An important challenge to the society is how to serve this large community, this increasing community, that was not part of the mathematics profession before.

Notices: *And some of this might be related to the job market, which has not completely recovered after the economic downturn. Can you talk a little bit about what you see as the challenges of the job market and how that will shape the profession?*

Ribet: The obvious answer is what Mark Green has called the mathematical diaspora—having mathematicians spread across areas that are not pure academic mathematics. It's completely obvious that the majority of people who are now training in graduate programs in mathematics, 10 years from now, 15 years from now, will not have positions as professors of mathematics in academic institutions. A lot of them were going to be working in in the BIG area, B-I-G for business, industry and government.

Some will be working for the NSA [National Security Agency]; some will be working in biotech; some will be working for insurance companies. There is a great deal of demand for hiring people with mathematical skills, especially if they include programming, statistics, and probability or machine learning.

There's a tremendous difference between the culture and expectations in mathematics and in neighboring subjects, like statistics and computer science. At Berkeley, at least, from day one, many graduate students in statistics are aiming to work in Silicon Valley as soon as they get their PhDs. They're not thinking about academic jobs. The same is true in computer science. But in mathematics, students still seem to be motivated by the idea that they're going to work in the "Ivory tower." What is beginning to happen is that after an initial post-PhD academic position, mathematicians realize that there are so many opportunities outside the university: they can live in an exciting geographic area, increase their salaries, and have fun if they are willing to expand their mindset to be more aligned with their colleagues in statistics and computer science. I believe that graduate programs will start offering resources to students who want to think in that direction before getting their PhDs so they can get a little bit of training as part of their PhD program that will help them if they look for non-academic jobs.

Notices: Do you think the AMS is responding to this change in the proportion of people who get PhDs who continue in the academic career path?

Ribet: The AMS is certainly aware of this trend, that people are moving outside academia.

The Society is prodding departments to start offering resources to such students when they are training for non-academic careers. Of course, the AMS wants to serve these mathematicians once they go into industry. The membership department understands that mathematicians who joined the AMS as graduate students will stay as members and contribute to the Society if the Society continues to provide a good community and relevant support.

Notices: Is there something you've been most proud of about your work for the AMS for the past two years?

Ribet: A little known responsibility of the AMS President is to name mathematicians to committees. The AMS has binders full of committees, scores of committees, and every year, there's a meeting of the Committee on Committees. They spend several hours going through every committee in the binders and generate a list of credible candidates to appoint. After that session, the President, the Secretary, and other people from the secretariat office make the final selections. I'm really proud of the large number of committees that I've done a good job of populating well. In making these selections, I've thought about diversity of many kinds: ethnic diversity, gender diversity, small institutions, large institutions, teaching institutions, research institutions, geographic distribution. People I've selected for the committees have been delighted to serve and have done fantastic jobs.

A frequent response to an AMS invitation to join a committee is "It will be an honor to serve." The AMS committees have made vast contributions to the mathematical profession.

Notices: Is there anything else that you wanted to talk about, related to the future of mathematics or the AMS?

Ribet: I am just amazed by the large organization that is the AMS. The AMS includes over 200 staff members working in four different locations. I'm overwhelmed by the breadth of activities and the dedication and expertise of the staff, who work behind the scenes to produce AMS publications, run meetings and conferences, and provide essential services to the mathematics community.

Notices: Finally, in your previous interview, you mentioned that you are quite the oenophile. Is there a special bottle of wine that you've been saving to celebrate the end of your term as president?

Ribet: What an interesting question! I have a couple of bottles that are really ready for some special occasion. One of them is an apparently very expensive bottle of pinot noir that one of my wife's colleagues gave us about two years ago. It's getting to be time to drink it. Another is a strange bottle of Bordeaux from 1961 that I bought at the end of my first year in Paris, in 1976. I'm very curious to see what that wine is like. It may just be complete vinegar by this point, but 1961 was one of the absolute great vintages of the 20th century, and there have only been a couple of times in my life that I have had the opportunity to taste 1961s.

Notices: Thanks for taking the time to talk with me.

Ribet: Thank you.

Evelyn Lamb

Credits

Photo of Kenneth A. Ribet is courtesy of the AMS.
Photo of Evelyn Lamb is by Jon Chaika.

Lafforgue Awarded 2019 Breakthrough Prize in Mathematics

Vincent Lafforgue

Vincent Lafforgue of the National Center for Scientific Research (CNRS) and Institut Fourier, Université Grenoble Alpes, has been awarded the 2019 Breakthrough Prize in Mathematics "for groundbreaking contributions to several areas of mathematics, in particular to the Langlands program in the function field case."

The prize committee provided the following statement: "The French lay claim to some of the greatest mathematical minds ever—from Descartes, Fermat and Pascal to Poincaré. More recently, Weil, Serre, and Grothendieck have given new foundations to algebraic geometry, from which arithmetic geometry was coined. Vincent Lafforgue is a leader of this latter school, now at the heart of new discoveries into cryptography and information security technologies. He makes his professional home in Grenoble at the CNRS, the largest fundamental science agency in Europe, where he has a tenured position and free rein to ponder the imponderable. Deeply concerned about the ecological crisis, Lafforgue is now focused on operator algebras in quantum mechanics and devising new materials for clean energy technologies."

"Vincent Lafforgue, in the so called 'function field case,' found a beautifully simple direct argument," said Richard Taylor, the chair of the selection committee. "After seeing it you ask yourself how the rest of us could have missed it for so long. You finally see why Langlands correspondence has to exist—it no longer seems an unmotivated miraculous consequence of complicated computations."

Lafforgue tells the *Notices*: "I am interested in maths for ecology and especially quantum mechanics for new materials for clean energy (because my first subject was operator algebras). I discovered that global public investment in clean energy RD&D is incredibly low: of the order of 0.02% of global GDP. Corporate investment is even lower, of the order of 0.01% of global GDP.

"Governments should really invest more in basic research and in RD&D for clean energy (including nuclear energy, e.g., thorium molten salt reactors). Scientists can organize themselves by helping collaborations between mathematicians, physicists, chemists, biologists on subjects useful for ecology: multidisciplinary institutes could be dedicated to this purpose.

"More modestly, I am interested in creating a website where physicists, chemists, biologists working on subjects useful for ecology could explain the mathematical problems they have."

Biographical Sketch

Vincent Lafforgue was born in Antony, France, in 1974. He received his PhD from Ecole Normale Supérieure (ENS) in 1999 under the supervision of Jean-Benôit Bost. His thesis work concerned the Baum-Connes conjecture, and in his research he introduced original Banach algebra techniques to solve new cases of the conjecture. His honors include the 2000 EMS Prize and the 2015 CNRS Silver Medal for his work on global fields of positive characteristics, which contributed significantly to the Langlands program. He was an invited speaker at the 2002 International Congress of Mathematicians in Beijing. He likes to spend his time hiking in foothills of the Alps.

About the Prize

The Breakthrough Prize in Mathematics was created by Mark Zuckerberg and Yuri Milner in 2013. It recognizes major advances in the field, honors the world's best mathematicians and supports their future endeavors, and aims to communicate the excitement of mathematics to the general public. Breakthrough Prize Foundation sponsors are Sergey Brin, Priscilla Chan and Mark Zuckerberg, Ma Huateng, Yuri and Julia Milner, and Anne Wojcicki.

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The prizewinners were recognized at a ceremony that was broadcast live on *National Geographic*, YouTube, and Facebook Live. The prize is accompanied by a cash award of US$3 million. Previous winners of the Breakthrough Prizes are:

- 2015: Simon Donaldson, Maxim Kontsevich, Jacob Lurie, Terence Tao, and Richard Taylor
- 2016: Ian Agol
- 2017: Jean Bourgain
- 2018: Christopher Hacon and James McKernan

New Horizons in Mathematics

The New Horizons in Mathematics Prizes are awarded to promising early career researchers who have already produced important work in mathematics. The honorees for 2019 are:


Chenyang Xu


Karim Adiprasito


June Huh


Kaisa Matomäki


Maksym Radziwill

Chenyang Xu of the Massachusetts Institute of Technology and Beijing International Center for Mathematical Research "for major advances in the minimal model program and applications to the moduli of algebraic varieties."

Xu tells the *Notices*: "The best part of being a mathematician is that during the process of trying to solve a mathematical problem, I earn myself a peaceful mind. This becomes more and more important lately."

Karim Adiprasito of the Hebrew University of Jerusalem and June Huh of the Institute for Advanced Study "for the development, with Eric Katz, of combinatorial Hodge theory leading to the resolution of the log-concavity conjecture of Rota."

Kaisa Matomäki of the University of Turku, Finland, and Maksym Radziwill of the California Institute of Technology "for fundamental breakthroughs in the understanding of local correlations of values of multiplicative functions."

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Kaisa Matomäki of the University of Turku, Finland, and Maksym Radziwill of the California Institute of Technology "for fundamental breakthroughs in the understanding of local correlations of values of multiplicative functions."

The prizes carry a cash award of US$100,000. Collaborators share the award.

—Elaine Kehoe with Breakthrough Prize Committee and CNRS announcements

Credits

Photo of Vincent Lafforgue is by Gérard Laumon, CNRS.
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Photo of Karim Adiprasito is courtesy of Archives of the MFO.
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Photo of Maksym Radziwill was provided by Petra Lein, © MFO.

Mathematics People

Daubechies Awarded Benter and Fudan–Zhongzhi Prizes


Ingrid Daubechies

Ingrid Daubechies of Duke University has been awarded the 2018 William Benter Prize in Applied Mathematics and the 2018 Fudan-Zhongzhi Science Award. The Benter Prize, given by the City University of Hong Kong, carries a cash award of US$100,000 and recognizes "outstanding mathematical contributions that have had a direct and fundamental impact on scientific, business, finance and engineering applications." The Fudan-Zhongzhi Award of the Fudan Science and Innovation Forum carries a cash award equivalent to US$435,000. It recognizes Daubechies' leadership in wavelet theory and time-frequency analysis, which fundamentally changed image and signal processing. The Fudan–Zhongzhi Science Award recognizes scientists who make fundamental achievements in the fields of mathematics, physics, and biomedicine.

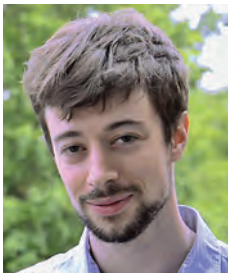
Daubechies has made exceptional contributions to a wide spectrum of scientific and mathematical subjects, including profound work with wavelets. Wavelets are mathematical functions used in processing digital signals and shrinking digital photos and movies. The US. Federal Bureau of Investigation, for example, has used her digital compression techniques for managing its huge amount of data pertaining to fingerprints. In addition, she has created and applied mathematical algorithms for spotting art forgeries and analyzing damaged paintings. Her algorithms are used to compare the different styles of artists, pinpoint when a work was painted, and to restore artwork that has cracked, faded, or been damaged during conflict, without actually touching the artwork.

Daubechies was born in Belgium and received her PhD in physics from the Free University Brussels in 1980. After two years of postdoctoral work in the United States, she returned to join the faculty of the Free University. In 1987 she joined AT&T Bell Laboratories in Murray Hill, New Jersey. She later served as professor of mathematics at Rutgers University. In 1994 she became professor of mathematics at Princeton University. She was director of the program in applied and computational mathematics from 1997 to 2001. She joined the faculty at Duke University in 2011.

Her honors include the Steele Prize for Mathematical Exposition (1994), the Satter Prize in Mathematics (1997), the National Academy of Sciences Award in Mathematics (2000), the Gold Medal of the Flemish Royal Academy of Arts and Sciences, Belgium (2005), the ICIAM Pioneer Prize (2007), the Steele Prize for Seminal Contribution to Research (2011), and the Nemmers Prize in Mathematics (2012). In 2005 she gave the AMS Gibbs Lecture and the AWM-SIAM Sonia Kovalevsky Lecture; in 2006 she gave the Emmy Noether Lecture at the Joint Mathematics Meetings in San Antonio. She was the SIAM John von Neumann Lecturer in 2011. She is a member of the American Academy of Arts and Sciences (1993), the US National Academy of Sciences (1998), and the Royal Netherlands Academy of Arts and Sciences (1999). She was elected a Fellow of the Institute for Electrical and Electronics Engineers (IEEE) in 1998. She was the first woman to serve as president of the International Mathematical Union (2011–2014). She was a member of the inaugural class of AMS Fellows (2012). She is also the first woman to win the Benter Prize.

—From Benter Prize and Fudan–Zhongzhi Prize announcements

Pym and Walton Awarded Lichnerowicz Prize


Brent Pym

Brent Pym of McGill University and Chelsea Walton of the University of Illinois at Urbana–Champaign have been awarded the 2018 Lichnerowicz Prize, awarded for notable contributions to Poisson geometry.

Brent Pym received his PhD from the University of Toronto in 2013, under the direction of Marco Gualtieri. He has held postdoctoral positions at McGill, Oxford, and Edinburgh universities before becoming assistant professor at McGill. In his thesis work, Pym classified the noncommutative deformations of complex projective 3-space, proved

the 4-dimensional case of the Bondal conjecture about Fano Poisson manifolds, and, jointly with Gualtieri and Li, developed the theory of the Stokes groupoids on Riemann surfaces. In recent work, Pym developed the notion of an elliptic singularity for a holomorphic Poisson structure and used it to obtain some of the only available classification results in dimension greater than 3. He has also developed the notion of a holonomic Poisson manifold (joint with Schedler), bringing the theory of perverse sheaves into the mainstream of Poisson geometry. In additional joint works, Pym has contributed to the enumerative geometry of noncommutative spaces and to the theory of Dirac structures and Courant algebroids as objects in shifted symplectic geometry. Pym tells the *Notices*: "Outside of mathematics, I enjoy cooking, playing jazz saxophone, and spending time with my infant daughter."


Chelsea Walton

Chelsea Walton completed her PhD in 2011 at the University of Michigan under the direction of Toby Stafford and Karen Smith. Following postdoctoral stays at the University of Washington, the Mathematical Sciences Research Institute, and the Massachusetts Institute of Technology, she became assistant professor at Temple University in 2015. In July 2018, she joined the faculty of the University of Illinois at Urbana–Champaign. Walton has written several important works in Poisson geometry, in addition to being a well-established expert in noncommutative algebra and quantum groups. Her work in Poisson geometry includes a deep investigation of the 3-D and 4-D Sklyanin algebras, especially those that are module-finite over their center. Joint with Wang and Yakimov, Walton showed that these are close analogues of Poisson algebras, namely Poisson Z-orders, which carry Poisson structures on the center. Walton, in joint work with several collaborators, has written a deep series of works on actions of Hopf algebras on commutative and noncommutative domains, showing that semisimple Hopf actions generally factor through group algebra actions, and also investigating the difficult non-semisimple case. She also gave a negative answer to the long-standing conjecture about whether the universal enveloping algebra of the Witt algebra is noetherian (joint with Sierra). Walton tells the *Notices*: "Outside of mathematics, I enjoy watching TV to turn my brain off. In fact, I have a documentary about the Backstreet Boys on pause as I type this. But I could very well start working if I get an idea. This is the way creativity works, I guess… Nick Carter and Hopf algebras."

The André Lichnerowicz Prize in Poisson geometry is awarded every two years at the International Conference on Poisson Geometry in Mathematics and Physics to researchers who completed their doctorates at most eight years before the year of the conference.

—Giovanni Forni, Chair, Prize Selection Committee

Lim and Mehrmann Awarded Hans Schneider Prize

Lek-Heng Lim of the University of Chicago and Volker Mehrmann of Technische Universität Berlin have been chosen as recipients of the 2019 Hans Schneider Prize.


Lek-Heng Lim

Lim and collaborators have made several fundamental contributions to linear and multilinear algebra, matrix theory, and their applications. These include his work on eigen and singular values of tensors, on tensor ranks, on a Perron-Frobenius theorem for nonnegative tensors, and on ill-posedness and NP-hardness of some tensor problems. A striking result of his shows that every $n \times n$ matrix is a product of at most $2n + 5$ Toeplitz matrices. Another lays the theoretical foundation for measuring the distance between subspaces of different dimensions. This work finds applications in statistics, data analysis, optimization, and computational mathematics. At the same time, it makes use of diverse ideas of techniques from several areas of mathematics. Lim received his PhD in 2007 from Stanford University under Gene Golub and Gunnar Carlsson. He was assistant professor at the University of California Berkeley from 2007 to 2010, when he joined the faculty at Chicago. In 2017 he received the Stephen Smale Prize from the Society for the Foundations of Computational Mathematics and the Wilkinson Prize in Numerical Analysis and Scientific Computing from the Society for Industrial and Applied Mathematics (SIAM).


Volker Mehrmann

Mehrmann was one of the early contributors to the development of efficient and reliable numerical algorithms for systems and control theory. He has made fundamental contributions to the solution of algebraic and differential algebraic equations arising in optimal control and to efficient numerical codes for implementing these solutions. He has also made very significant contributions to numerical methods for linear algebra problems with special structure (such as Hamiltonian or symplectic) and developed techniques that preserve this structure when

computing spectra. He has played an important role in the development of software that implements these algorithmic ideas. His theoretical results and numerical software are being used in industry. In addition, he has played a very important role in mentoring younger mathematicians, in the working of many mathematical societies, and as an editor of several major journals. He was elected president of the European Mathematical Society (EMS) in 2018. Mehrmann tells the *Notices*: "I like to do long bike tours (this year from Berlin to Copenhagen), and I love to knit. I am still knitting my own socks, and one of my upcoming projects is to knit a Klein bottle."

The Hans Schneider Prize in Linear Algebra is awarded every three years by the International Linear Algebra Society (ILAS) for research, contributions, and achievements at the highest level of linear algebra. The prize may be awarded for an outstanding scientific achievement or for lifetime contribution.

—Peter Semrl
President, ILAS

Prizes of the New Zealand Math Society

The New Zealand Mathematical Society (NZMS) has announced several awards for 2018.


Alex James


Carlo Laing

Alex James of the University of Canterbury and Carlo Laing of Massey University are the recipients of the 2018 Research Award. James was recognized for her contributions in mathematical modeling, ranging from the theoretical, such as Lévy walks and complex ecological systems, to the very applied, such as masting and snail dynamics. She is a part-time mathematician and spends the rest of her time riding cargo bikes, building forts, and rescuing friendly bears with her two small children.

Laing was recognized for his sustained contributions to the field of mathematical neuroscience and pioneering work in the study of coupled oscillator networks. He tells the *Notices*: "I have played percussion in samba bands for the last eighteen years and performed with a New Zealand group in the Asakusa Samba Carnival held in Tokyo this past August."

Fabien Montiel of the University of Otago was chosen the recipient of the Early Career Award for "outstanding


Left to right: Andre Nies, Dan Turetsky, Noam Greenberg

contributions to the development of mathematical and computational methods in wave scattering theory and his innovative approach to modelling the propagation of ocean waves in ice-covered seas."

Noam Greenberg of the Victoria University Wellington, Andre Nies of the University of Auckland, and Dan Turetsky of Victoria University Wellington were awarded the 2018 Kalman Prize for Best Paper for the paper Laurent Bienvenu, Noam Greenberg, Antonin Kucera, Andre Nies, and Dan Turetsky, "Coherent Randomness Tests and Computing the *K*-Trivial Sets," *Journal of the European Mathematical Society* **18** (2016), 773–812.

The Student Prize for the best contributed talk by a student at the annual New Zealand Mathematics Colloquium was awarded to Pascal Eun Sig Cheon of the University of Auckland for the talk "Domain Truncation in Pipeline Monitoring Problems."

—From an NZMS announcement

IEEE Awards Announced

The Institute of Electrical and Electronics Engineers (IEEE) has honored two researchers whose work involves the mathematical sciences. Éva Tardos of Cornell University has been awarded the 2019 John von Neumann Medal "for contributions to the field of algorithms, including foundational new methods in optimization, approximation algorithms, and algorithmic game theory."

Pramod P. Khargonekar of the University of California, Irvine was honored with the 2019 Award in Control Systems "for contributions to robust and optimal control theory."

—From an IEEE announcement

Rhodes Scholars Announced

The Rhodes Trust has announced the names of the American scholars chosen as Rhodes Scholars for 2019. Following are the names and brief biographies of the scholars whose work involves the mathematical sciences.

Alaleh Azhir of New York City, is a senior at Johns Hopkins University with a triple major in biomedical engineering, computer science, and applied mathematics and statistics. She has a perfect GPA (including an A+ in twenty-two courses). A Goldwater Scholar, she has many publications in genomics and various biomedical subjects in major national and international journals and has done research at Harvard University, the Massachusetts Institute of Technology, the National Institutes of Health, and laboratories in Switzerland, as well as at Johns Hopkins. She mentors middle school students, edits a philosophy journal, runs arts programs for children in underresourced neighborhoods, and provides cooked meals for a shelter for survivors of domestic abuse. She immigrated from Iran when she was fourteen. She will do the MSc in women's and reproductive health at Oxford.

Jennifer Huang of Granger, Indiana, graduated in 2017 from Indiana University with a perfect 4.0 GPA, majoring in mathematics and social and cultural analysis. As an undergraduate, she conducted cross-disciplinary research, including an ethnography of Iceland's renewable energy industry and a computational text analysis of *Poetry* magazine's archives. She coauthored a paper in a leading science journal, *Proceedings of the National Academy of Sciences*, and wrote stories that received Indiana University's top undergraduate fiction award. After graduating, she has redirected her focus toward community-based political and policy work. She is currently working both as the civic engagement program coordinator at the Institute of Politics at the University of Chicago and as a policy associate in the Office of the Mayor of South Bend. She will read for an MSc in social science of the Internet and a master's in public policy.

Lia Petrose of Laurel, Maryland, graduated from the University of Pittsburgh in 2017 with a BS in neuroscience and economics with a minor in chemistry. A Truman Scholar, she has a vision of improving how data are used to facilitate health care delivery. She wrote three first-authored papers in leading medical journals and is currently a research assistant for Dr. Heidi Williams at the Massachusetts Institute of Technology. As an undergraduate, she was elected to the executive board of the student government and served as the student member of the Board of Trustees Committee on Academic Affairs. She was born and raised until adolescence in Ethiopia. She will read for a BA in computer science and philosophy at Oxford.

James W. Brahm of Huntsville, Alabama, is a senior at the US. Air Force Academy, where he majors in computer science and minors in Chinese, as well as nuclear weapons and strategy. A Truman Scholar, he does research at the intersection of cybersecurity and computer science and has a computer-science-related patent pending with the US Patent Office. He commands the Wing Information Services Team, which is responsible for ensuring informational technology support to over 4,000 cadets. Throughout his undergraduate career, he has worked extensively in cybersecurity, including as an intern at the National Security Agency, where he engineered reusable software to support US intelligence efforts. He is an Eagle Scout and enjoys freefall parachuting. At Oxford, he will pursue an MSc in computer science.

Kristina M. Correa of Robstown, Texas, is a senior at Stanford University, where she majors in biology and minors in computer science. She is the daughter of Mexican immigrants and was raised by a single mother. She has never received a grade below an A across her demanding undergraduate and graduate courses. She has done extensive research in four different laboratories, and her senior thesis relates to cellular glycans and cancer. She has been an active volunteer working with children whose parents have cancer and as a tutor for low-income and/or minority students. She plans a career in computational immunology and is committed to Latino empowerment in the sciences. At Oxford, Kristina intends to do master's degrees in integrated immunology and computer science.

Vidal M. Arroyo of Rancho Santa Margarita, California, is a senior at Chapman University, where he is pursuing a BS in biochemistry and molecular biology with a minor in computational science and integrated educational studies. He is Chapman University's first Rhodes Scholar. He has maintained a perfect GPA. He has researched the link between cancer and obesity, as well as outcome disparities in survivors in childhood cancers. His studies have also focused on the use artificial intelligence to strengthen and personalize cancer treatments. As the founder and president of Chapman STEMtors, he has worked to expose at-risk youth to careers in science. At Oxford, he will read for a DPhil in engineering science.

Madison L. Tung of Santa Monica, California, is a senior at the US Air Force Academy, where she is majoring in mathematics and humanities and minoring in Chinese. A Truman Scholar, she researches the use of artificial intelligence and other mathematical techniques to develop tools for decision makers. Ultimately, she believes in the power of artificial intelligence to improve people's lives. She is a six-time All-American, national champion in women's wrestling and holds a black belt in hapkido (Korean martial art). Her hobbies include ice climbing and jiu-jitsu. At Oxford, she will read for master's degrees in computer science and in global governance and diplomacy.

—From a Rhodes Trust announcement

NEWS

AAAS Fellows Elected

The American Association for the Advancement of Science (AAAS) has elected its new fellows for 2018.

The new Fellows of the Section on Mathematics are:

- Eric M. Friedlander, University of Southern California
- Ilse C. F. Ipsen, North Carolina State University
- George Em Karniadakis, Brown University
- C. T. Kelley, North Carolina State University
- David E. Keyes, King Abdullah University of Science and Technology (Saudi Arabia)
- Yi Li, John Jay College of Criminal Justice, City University of New York

The new Fellows of the Section on Statistics are:

- A. John Bailer, Miami University
- Song Xi Chen, Peking University
- Dianne M. Finkelstein, Massachusetts General Hospital/ Harvard T. H. Chan School of Public Health
- Edward L. Ionides, University of Michigan
- David A. Marker, Westat
- Sharon-Lise Teresa Normand, Harvard Medical School
- Giovanni Parmigiani, Dana-Farber Cancer Institute

—From an AAAS announcement

ACM Fellows

The Association for Computing Machinery (ACM) has elected its new Fellows for 2018. Below are the names, affiliations, and citations for the new Fellows whose work involves the mathematical sciences.

- Tamal Dey, Ohio State University, "for contributions to computational geometry and computational topology."
- Mohammad T. Hajiaghayi, University of Maryland, College Park, "for contributions to the fields of algorithmic graph theory and algorithmic game theory."
- Dan Halperin, Tel Aviv University, "for contributions to robust geometric computing and applications to robotics and automation."
- Sanjeev Khanna, University of Pennsylvania, "for contributions to approximation algorithms, hardness of approximation, and sublinear algorithms."
- Tom Leighton, Akamai Technologies, "for his leadership in the establishment of content delivery networks and his contributions to algorithm design."
- Toniann Pitassi, University of Toronto, "for contributions to research and education in the fields of computational and proof complexity."
- Avi Wigderson, Institute for Advanced Study, "for contributions to theoretical computer science and mathematics."

—From an ACM announcement

Credits

Photo of Ingrid Daubechies is courtesy of Les Todd: Duke Photography.
Photo of Brent Pym is courtesy of Brent Pym.
Photo of Chelsea Walton is courtesy of University of Illinois at Urbana–Champaign.
Photo of Lek-Heng Lim is by Sou-Cheng Choi.
Photo of Volker Mehrmann is courtesy of Volker Mehrmann.
Photo of Alex James is courtesy of Kerry Weston.
Photo of Carlo Laing is courtesy of Christopher Tuffley.
Photo of Noam Greenberg, Dan Turetsky, and Andre Nies is courtesy of Brown Westrick.

2019 Class of Fellows of the AMS

Sixty-five mathematical scientists from around the world have been named Fellows of the American Mathematical Society (AMS) for 2019.

The Fellows of the American Mathematical Society program recognizes members who have made outstanding contributions to the creation, exposition, advancement, communication, and utilization of mathematics. Among the goals of the program are to create an enlarged class of mathematicians recognized by their peers as distinguished for their contributions to the profession and to honor excellence.

The 2019 class of Fellows was honored at a dessert reception held during the Joint Mathematics Meetings in Baltimore, Maryland. Names of the individuals who are in this year's class, their institutions, and citations appear below.

The nomination period for Fellows is open each year from February 1 to March 31. For additional information about the Fellows program, as well as instructions for making nominations, visit the web page www.ams.org/profession/ams-fellows.

Then AMS President Kenneth A. Ribet welcoming the new Class of Fellows of the AMS at JMM in Baltimore.

Joseph A. Ball, Virginia Polytechnic Institute and State University
For contributions to operator theory, analytic functions, and service to the profession.

John S. Baras, University of Maryland
For contributions to the mathematical foundations and applications of systems theory, stochastic systems, stochastic control, network security and trust, mentoring and academic leadership.

Saugata Basu, Purdue University
For contributions to algorithmic and quantitative real algebraic geometry, computational complexity, and o-minimal structures.

Bonnie Berger, Massachusetts Institute of Technology
For contributions to computational biology, bioinformatics, algorithms and for mentoring.

Christopher J. Bishop, Stony Brook University
For contributions to the theory of harmonic measures, quasiconformal maps and transcendental dynamics.

Martin Bridgeman, Boston College
For contributions to hyperbolic geometry and low-dimensional topology and service to the mathematical community.

Fioralba Cakoni, Rutgers The State University of New Jersey New Brunswick
For contributions to analysis of partial differential equations especially in inverse scattering theory.

Hector D. Ceniceros, University of California, Santa Barbara
For contributions to numerical analysis, fluid mechanics, and soft materials and for professional leadership, outreach and guidance of postdoctoral, graduate and undergraduate students.

Mark Andrea De Cataldo, Stony Brook University
For contributions to algebraic geometry, especially questions involving the topology of algebraic varieties and mappings.

Tommaso de Fernex, University of Utah
For contributions to algebraic geometry, in particular his work on birational rigidity of hypersurfaces.

Alicia Dickenstein, University of Buenos Aires
For contributions to computational algebra and its applications, especially in systems biology, and for global leadership in supporting underrepresented groups in mathematics.

Richard S. Falk, Rutgers The State University of New Jersey New Brunswick
For contributions to numerical analysis and for service to the mathematical community.

Paul M. N. Feehan, Rutgers The State University of New Jersey New Brunswick
For contributions to gauge theory.

Shmuel Friedland, University of Illinois at Chicago
For contributions to the theory of matrices, tensors, and their applications to other areas.

Stephan Ramon Garcia, Pomona College
For contributions to operator theory and leadership in undergraduate research and mentoring.

Skip Garibaldi, Center for Communications Research, La Jolla
For contributions to group theory and service to the mathematical community, particularly in support of promoting mathematics to a wide audience.

Rebecca F. Goldin, George Mason University
For contributions to differential geometry and service to the mathematical community, particularly in support of promoting mathematical and statistical thinking to a wide audience.

Daniel Groves, University of Illinois at Chicago
For contributions to geometric group theory and low-dimensional topology.

Larry Guth, Massachusetts Institute of Technology
For contributions to harmonic analysis, combinatorics and geometry, and for exposition of high level mathematics.

Michael Harris, Columbia University
For contributions to arithmetic geometry, particularly the theory of automorphic forms, L-functions and motives.

Harald Andrés Helfgott, Mathematics Institute, Georg-August University Göttingen and CNRS
For contributions to analytic number theory, additive combinatorics and combinatorial group theory.

Susan Hermiller, University of Nebraska-Lincoln
For contributions to combinatorial and geometric group theory and for service to the profession, particularly in support of underrepresented groups.

Jeffrey Hoffstein, Brown University
For contributions to analytic number theory and to the development of lattice-based quantum resistant cryptography.

Fern Yvette Hunt, National Institute of Standards and Technology
For outstanding applications of mathematics to science and technology, exceptional service to the US government, and for outreach and mentoring.

Sergei V. Ivanov, University of Illinois, Urbana-Champaign
For contributions in combinatorial and geometric group theory.

Stephen C. Jackson, University of North Texas
For contributions to set theory and its applications.

Dihua Jiang, University of Minnesota-Twin Cities
For contributions to automorphic forms, L-functions, representation theory, and the Langlands Program.

Matthew Kahle, Ohio State University, Columbus
For contributions to stochastic topology, mentoring and service to the mathematics community.

Efstratia (Effie) Kalfagianni, Michigan State University
For contributions to knot theory and 3-dimensional topology, and for mentoring.

Andrew Knyazev, University of Colorado, Denver
For contributions to numerical partial differential equations, computational mathematics and linear algebra, with industrial applications.

W.B. Raymond Lickorish, University of Cambridge
For contributions to knot theory and low-dimensional topology.

Dan Margalit, Georgia Institute of Technology
For contributions to low-dimensional topology and geometric group theory, exposition, and mentoring.

John Edward McCarthy, Washington University
For contributions to operator theory and functions of several complex variables.

Steven Joel Miller, Williams College
For contributions to number theory and service to the mathematical community, particularly in support of mentoring undergraduate research.

Ngaiming Mok, The University of Hong Kong
For contributions to complex differential and algebraic geometry.

Elchanan Mossel, Massachusetts Institute of Technology
For contributions to probability, combinatorics, computing, and especially the interface between them.

Camil Muscalu, Cornell University
For contributions to multi-linear harmonic analysis with applications to partial differential equations and for expository writing in modern harmonic analysis.

Andrew Neitzke, University of Texas at Austin
For contributions to research on the boundary of geometry and physics.

Lenhard Ng, Duke University
For contributions to Floer homology and low-dimensional topology and service to the mathematical community.

Claudia Polini, University of Notre Dame
For contributions to commutative algebra and for service to the profession.

Alex Poltoratski, Texas A&M University
For contributions to harmonic analysis, operator and spectral theory.

Vladimir Retakh, Rutgers The State University of New Jersey New Brunswick
For contributions to noncommutative algebra and noncommutative algebraic geometry.

J. Maurice Rojas, Texas A&M University
For contributions to algorithmic algebraic geometry, complexity theory, and scientific computation, and mentoring of student research at all levels.

Min Ru, University of Houston
For contributions to complex analysis and geometry, particularly Nevalinna theory and the theory of minimal surfaces.

Thomas Schick, Mathematics Institute, Georg-August University Göttingen
For contributions to index theory and the geometry and topology of manifolds.

Anne Schilling, University of California, Davis
For contributions to algebraic combinatorics, combinatorial representation theory, and mathematical physics and for service to the profession.

Stefan Schwede, Universität Bonn
For contributions to homotopy theory.

Julius L. Shaneson, University of Pennsylvania
For contributions to topology.

Members of the 2019 Class of Fellows of the AMS who attended the reception at JMM in Baltimore, pictured with then AMS President Kenneth A. Ribet and then AMS President-Elect Jill C. Pipher.

Ratnasingham Shivaji, University of North Carolina at Greensboro
For contributions to the theory of semipositone elliptic questions applied to reaction diffusion systems, for mentoring and for providing leadership for the inception of a doctoral program in mathematics.

Aravind Srinivasan, University of Maryland
For contributions to theoretical computer science, discrete probability, network science and applications and for service to the profession.

Irena Swanson, Reed College
For contributions to commutative algebra, exposition, service to the profession and mentoring.

Murad S. Taqqu, Boston University
For contributions to self-similar random processes and their applications to real world phenomena such as diverse internet traffic and hydrology.

Valerio Toledano Laredo, Northeastern University
For contributions to the representation theory of quantum groups.

Valentino Tosatti, Northwestern University
For contributions in geometric analysis and complex geometry.
Burt Totaro, University of California, Los Angeles
For contributions to algebraic geometry, Lie theory and cohomology and their connections and for service to the profession.

Peter E. Trapa, University of Utah
For contributions to Lie theory and for service to his university and to math circles initiatives.

Moshe Y. Vardi, Rice University
For contributions to the development and use of mathematical logic in computer science.

Eric Vigoda, Georgia Institute of Technology
For contributions to theoretical computer science, in particular through its interactions with probability, combinatorics and statistical physics and for service to the profession.

Alexander A. Voronov, University of Minnesota-Twin Cities
For contributions to mathematical physics, operad theory and homotopical algebra.

Zhenghan Wang, University of California, Santa Barbara
For contributions to quantum computing and topological quantum field theory.

Olof B. Widlund, New York University, Courant Institute
For contributions to numerical analysis of domain decompositions within computational mathematics and for incubation through his writing and mentorship of a broad international, creative community of practice applied to highly resolved systems simulations.

Tonghai Yang, University of Wisconsin, Madison
For contributions to the theory of Shimura varieties, L-functions, automorphic forms, and complex multiplication.

Zhiwei Yun, Massachusetts Institute of Technology
For contributions to geometry, number theory, and representation theory, including his construction of motives with exceptional Galois groups.

Chongchun Zeng, Georgia Institute of Technology
For contributions to applied dynamical systems and nonlinear partial differential equations.

Wei Zhang, Massachusetts Institute of Technology
For contributions to number theory, algebraic geometry and geometric representation theory.

Credits

Photos by Kate Awtrey, Atlanta Convention Photography.

CALL FOR APPLICATIONS & NOMINATIONS

Treasurer

of the American Mathematical Society

The American Mathematical Society is seeking applications and nominations of candidates for the position of Treasurer. The Treasurer is an officer of the Society and is appointed by the Council for a two-year term. The first term of the new Treasurer will begin February 1, 2021, with initial appointment expected in January 2020 in order that the Treasurer-designate may observe the conduct of Society business for a full year before taking office.

All necessary expenses incurred by the Treasurer in performance of duties for the Society are reimbursed.

QUALIFICATIONS

The Treasurer should be a research mathematician and must have substantial knowledge of Society activities. Although the Treasurer is appointed by the Council for a term of two years, candidates should be willing to make a long-term commitment, as it is expected that the new Treasurer will be reappointed for subsequent terms pending successful performance reviews.

DUTIES

- Administer or supervise the administration of fiscal policies in the interest of the mathematical community, as laid down by the Trustees.
- Monitor the receipt and expenditure of funds and the care of investments.
- Monitor budgets and trends of finance over periods of years.
- Review salary policy for AMS employees and its applications to individuals.
- Serve *ex officio* as a member of the Board of Trustees, Council, and several other committees; the Treasurer chairs the Audit and Risk, the Investment, and the Salary committees.

APPLICATIONS & NOMINATIONS

A Search Committee, with Alejandro Adem as Chair, has been formed to seek and review applications. Persons wishing to apply should do so through MathPrograms.Org. Nominations and questions should be directed to the Chair of the Search Committee: tsc-chair@ams.org.

For full consideration, applications, nominations, and supporting documentation should be received by **May 15, 2019**.

AMS AMERICAN MATHEMATICAL SOCIETY

Advancing research. Creating connections.

The American Mathematical Society is committed to diversity. All qualified applicants will receive consideration without regard to race, color, religion, gender, gender identity or expression, sexual orientation, national origin, genetics, disability, age, or veteran status.

Community Updates

AWM Research Symposium


Mariam Manuel

The Association for Women in Mathematics (AWM), with partial support from the AMS, will hold its 2019 Research Symposium at Rice University on April 6–7, 2019. The symposium will showcase research from women across the mathematical sciences working in academia, government, and industry, as well as feature women across the career spectrum: undergraduates, graduate students, postdocs, and professionals. Plenary lectures will be given by Susanne Brenner, Kristin Lauter, and Chelsea Walton. Mariam Manuel will be the keynote speaker. A poster session, research sessions, and a career panel on women in industry will be held. For more information, see `https://awm-math.org/meetings/awm-research-symposium/`.

—From an AWM announcement,

Mathematics Opportunities

Listings for upcoming mathematics opportunities to appear in Notices *may be submitted to* notices@ams.org.

Early Career Opportunity

Project NExT 2019–2020

MAA Project NExT (New Experiences in Teaching) is a year-long professional development program of the Mathematical Association of America (MAA) for new or recent PhDs in the mathematical sciences. The program is designed to connect new faculty with master teachers and leaders in the mathematics community and to address the three main aspects of an academic career: teaching, research, and service. The program welcomes and encourages applications from new and recent PhDs in postdoctoral, tenure-track, and visiting positions. Applicants who will be starting new academic positions in the fall of 2019 should apply by **April 15, 2019**. For applications and further information, see www.maa.org/programs/faculty-and-departments/project-next.

—From an MAA announcement

Early Career Opportunity

CAS-TWAS President's PhD Fellowship Program

The World Academy of Sciences (TWAS) and the Chinese Academy of Sciences (CAS) are collaborating on the CAS-TWAS President's Fellowship Program, which provides non-Chinese students and scholars an opportunity to pursue doctoral degrees at the University of Chinese Academy of Sciences (UCAS), the University of Science and Technology of China (USTC), or institutes of CAS around China. The deadline for application is March 31, 2019. For more information and instructions for applying, see https://twas.org/opportunity/cas-twas-presidents-phd-fellowship-programme.

—From a TWAS announcement

Call for Nominations for Graham Wright Award

The Graham Wright Award for Distinguished Service of the Canadian Mathematical Society (CMS) recognizes individuals who have made sustained and significant contributions to the Canadian mathematical community and, in particular, to the CMS. The deadline for nominations is **April 30, 2019**. See cms.math.ca/Prizes/dis-nom.

—From a CMS announcement

Early Career Opportunity

International Mathematical Union Breakout Graduate Fellowships

The International Mathematical Union (IMU) Breakout Graduate Fellowship program offers a limited number of complete grants, with a duration of up to four years, for excellent students from developing countries to earn a doctoral degree.

The IMU invites professional mathematicians to nominate highly motivated and mathematically talented students from developing countries who plan to complete a doctoral degree in mathematical sciences in a developing country. The deadline for the 2019 Call for Nominations is **May 30, 2019.** More information is available at https://www.mathunion.org/cdc/scholarships/graduate-scholarships.

—From information posted by IMU/CDC to IMU-Net

NEWS

Early Career Opportunity

Call for Nominations for Aisenstadt Prize

The Centre de Recherches Mathématiques (CRM) seeks nominations for the André Aisenstadt Mathematics Prize, which recognizes outstanding research by a young Canadian mathematician. Deadline for nominations is **March 1, 2019**. See `www.crm.umontreal.ca/prix/prixAndreAisenstadt/prix_attributionAA_an.shtml`.

—From a CRM announcement

Classified Advertising

Employment Opportunities

NEW YORK

Cornell University
Department of Mathematical Sciences

The Mathematics Department of Cornell University invites applications for a potential non-tenure track renewable 3-year Lecturer position beginning July 1, 2019. Responsibilities include teaching four courses per year, serving on committees and contributing to overall the educational mission of the Department. A PhD in mathematics is required. The Department actively encourages applications from women and minority candidates. Applicants must apply electronically at `www.mathjobs.org`. Deadline May 15, 2019.

06

CHINA

Tianjin University, China
Tenured/Tenure-Track/Postdoctoral Positions at the Center for Applied Mathematics

Dozens of positions at all levels are available at the recently founded Center for Applied Mathematics, Tianjin University, China. We welcome applicants with backgrounds in pure mathematics, applied mathematics, statistics, computer science, bioinformatics, and other related fields. We also welcome applicants who are interested in practical projects with industries. Despite its name attached with an accent of applied mathematics, we also aim to create a strong presence of pure mathematics. Chinese citizenship is not required.

Light or no teaching load, adequate facilities, spacious office environment and strong research support. We are prepared to make quick and competitive offers to self-motivated hard workers, and to potential stars, rising stars, as well as shining stars.

The Center for Applied Mathematics, also known as the Tianjin Center for Applied Mathematics (TCAM), located by a lake in the central campus in a building protected as historical architecture, is jointly sponsored by the Tianjin municipal government and the university. The initiative to establish this center was taken by Professor S. S. Chern. Professor Molin Ge is the Honorary Director, Professor Zhiming Ma is the Director of the Advisory Board. Professor William Y. C. Chen serves as the Director.

TCAM plans to fill in fifty or more permanent faculty positions in the next few years. In addition, there are a number of temporary and visiting positions. We look forward to receiving your application or inquiry at any time. There are no deadlines.

Please send your resume to `mathjobs@tju.edu.cn`.

For more information, please visit `cam.tju.edu.cn` or contact Ms. Erica Liu at `mathjobs@tju.edu.cn`, telephone: 86-22-2740-6039.

01

New Books Offered by the AMS

Algebra and Algebraic Geometry

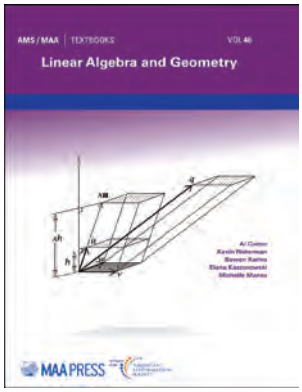
Linear Algebra
Concepts and Applications

Przemyslaw Bogacki, *Old Dominion University, Norfolk, VA*

Providing a complete coverage of core linear algebra topics, this text is designed to be used in a first linear algebra course taken by mathematics and science majors. Unusual features of the text include a pervasive emphasis on the geometric interpretation and viewpoint as well as a very complete treatment of the singular value decomposition.

AMS/MAA Textbooks, Volume 47
March 2019, 383 pages, Hardcover, ISBN: 978-1-4704-4384-9, LC 2018041789, 2010 *Mathematics Subject Classification:* 15Axx, 15A03, 15A04, 15A06, 15A09, 15A15, 15A18, 97–01, 97H60, List US$99, **AMS Individual member US$74.25**, AMS Institutional member US$79.20, **MAA members US$74.25**, Order code TEXT/47

bookstore.ams.org/text-47



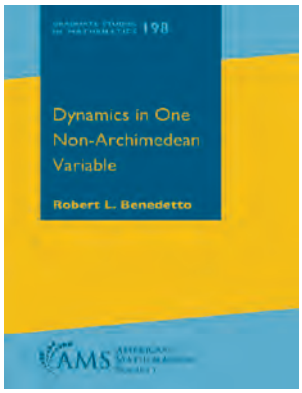
Linear Algebra and Geometry

Al Cuoco, **Kevin Waterman**, **Bowen Kerins**, and **Elena Kaczorowski**, *all of Education Development Center, Inc., Waltham, MA*, and **Michelle Manes**, *University of Hawaii, Honolulu, HI*

Requiring only high school algebra, this text uses elementary geometry to build the beautiful edifice of results and methods that make linear algebra such an important field. It is organized around carefully sequenced problems that help students build both the tools and the habits that provide a solid basis for further study in mathematics.

AMS/MAA Textbooks, Volume 46
April 2019, approximately 603 pages, Hardcover, ISBN: 978-1-4704-4350-4, LC 2018037261, 2010 *Mathematics Subject Classification:* 08–01, 15–01, 15A03, 15A04, 15A06, 15A09, 15A15, 15A18, 60J10, 97–01, List US$99, **AMS Individual member US$74.25**, AMS Institutional member US$79.20, **MAA members US$74.25**, Order code TEXT/46

bookstore.ams.org/text-46



Dynamics in One Non-Archimedean Variable

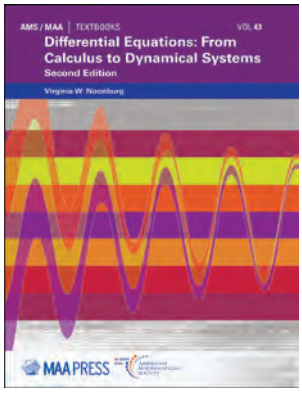
Robert L. Benedetto, *Amherst College, MA*

This textbook presents the fundamentals of non-archimedean dynamics, including a unified exposition of Rivera-Letelier's classification theorem, as well as results on wandering domains, repelling periodic points, and equilibrium measures. The presentation is accessible to graduate students with only first-year courses in algebra and analysis under their belts, although some previous exposure to non-archimedean fields is recommended. The book should also be a useful reference for more advanced students and researchers in arithmetic and non-archimedean dynamics.

Graduate Studies in Mathematics, Volume 198
March 2019, approximately 418 pages, Hardcover, ISBN: 978-1-4704-4688-8, LC 2018044194, 2010 *Mathematics Subject Classification:* 37P40; 37P30, 37P50, List US$95, **AMS members US$76**, **MAA members US$85.50**, Order code GSM/198

bookstore.ams.org/gsm-198

Differential Equations



Differential Equations: From Calculus to Dynamical Systems

Second Edition

Virginia W. Noonburg, *University of Hartford, West Hartford, CT*

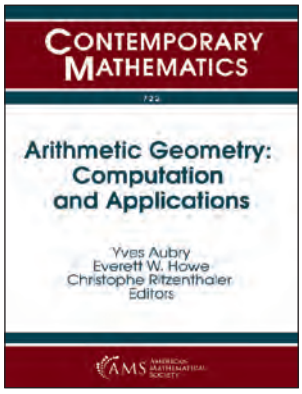
This second edition of Virginia Noonburg's best-selling textbook includes two new chapters on partial differential equations, making the book usable for a two-semester sequence in differential equations. It includes exercises, examples, and extensive student projects taken from the current mathematical and scientific literature.

AMS/MAA Textbooks, Volume 43
February 2019, 402 pages, Hardcover, ISBN: 978-1-4704-4400-6, LC 2018025154, 2010 *Mathematics Subject Classification:* 34–01, 35–01, List US$75, **AMS Individual member US$56.25**, AMS Institutional member US$60, **MAA members US$56.25**, Order code TEXT/43

bookstore.ams.org/text-43

New in Contemporary Mathematics

Algebra and Algebraic Geometry



Arithmetic Geometry: Computation and Applications

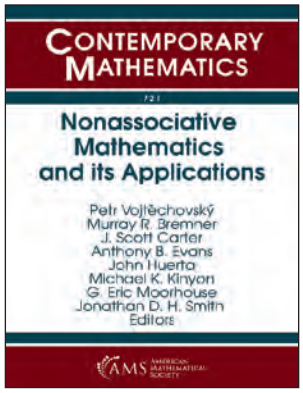
Yves Aubry, *Institut de Mathématiques de Toulon-IMATH, La Garde, France, and Institut de Mathématiques de Marseille - I2M, France*, **Everett W. Howe**, *Center for Communications Research, La Jolla, CA*, and **Christophe Ritzenthaler**, *Institut de Recherche Mathématiques de Rennes (IRMAR), France*, Editors

The papers in this volume are original research articles covering a large range of topics, including weight enumerators for codes, function field analogs of the Brauer–Siegel theorem, the computation of cohomological invariants of curves, the trace distributions of algebraic groups, and applications of the computation of zeta functions of curves. Despite the varied topics, the papers share a common thread: the beautiful interplay between abstract theory and explicit results.

This item will also be of interest to those working in number theory.

Contemporary Mathematics, Volume 722
February 2019, 175 pages, Softcover, ISBN: 978-1-4704-4212-5, LC 2018037194, 2010 *Mathematics Subject Classification:* 11G20, 11G30, 11G32, 11G40, 11T71, 14G10, 14H40, 14Q05, 20C20, 20G41, List US$117, **AMS members US$93.60**, **MAA members US$105.30**, Order code CONM/722

bookstore.ams.org/conm-722



Nonassociative Mathematics and its Applications

Petr Vojtěchovský, *University of Denver, CO*, **Murray R. Bremner**, *University of Saskatchewan, Saskatoon, SK, Canada*, **J. Scott Carter**, *University of South Alabama, Mobile, AL*, **Anthony B. Evans**, *Wright State University, Dayton, OH*, **John Huerta**, *University of Lisbon, Portugal*, **Michael K. Kinyon**, *University of Denver, CO*, **G. Eric Moorhouse**, *University of Wyoming, Laramie, WY*, and **Jonathan D. H. Smith**, *Iowa State University, Ames, IA*, Editors

This volume contains the proceedings of the Fourth Mile High Conference on Nonassociative Mathematics, held from July 29–August 5, 2017, at the University of Denver, Denver, Colorado. Included are research papers covering active areas of investigation, survey papers covering Leibniz algebras, self-distributive structures, and rack homology, and a sampling of applications ranging from Yang-Mills theory to the Yang-Baxter equation and Laver tables.

Contemporary Mathematics, Volume 721
February 2019, approximately 299 pages, Softcover, ISBN: 978-1-4704-4245-3, LC 2018037208, 2010 *Mathematics Subject Classification:* 11R52, 17A32, 17C40, 20N02, 20N05, 52A01, 55N35, 57M27, 81T60, 83E50, List US$117, **AMS members US$93.60**, **MAA members US$105.30**, Order code CONM/721

bookstore.ams.org/conm-721

New in Memoirs of the AMS

Algebra and Algebraic Geometry

Variations on a Theorem of Tate

Stefan Patrikis, *Princeton University, NJ*

Memoirs of the American Mathematical Society, Volume 258, Number 1238
February 2019, 156 pages, Softcover, ISBN: 978-1-4704-3540-0, 2010 *Mathematics Subject Classification:* 11R39, 11F80, 14C15, List US$81, **AMS Individual member US$48.60**, AMS Institutional member US$64.80, **MAA members US$72.90**, Order code MEMO/258/1238

bookstore.ams.org/memo-258-1238

Analysis

Crossed Products of Operator Algebras

Elias G. Katsoulis, *East Carolina University, Greenville, NC*, and **Christopher Ramsey**, *University of Manitoba, Winnipeg, Manitoba, Canada*

Memoirs of the American Mathematical Society, Volume 258, Number 1240
February 2019, 83 pages, Softcover, ISBN: 978-1-4704-3545-5, 2010 *Mathematics Subject Classification:* 46L07, 46L08, 46L55, 47B49, 47L40, 47L65, List US$81, **AMS Individual member US$48.60**, AMS Institutional member US$64.80, **MAA members US$72.90**, Order code MEMO/258/1240

bookstore.ams.org/memo-258-1240

Fusion of Defects

Arthur Bartels, *Westfälische Wilhelms-Universität Münster, Germany*, **Christopher Douglas**, *University of Oxford, United Kingdom*, and **André Henriques**, *Universiteit Utrecht, The Netherlands*

This item will also be of interest to those working in mathematical physics.

Memoirs of the American Mathematical Society, Volume 258, Number 1237
February 2019, 102 pages, Softcover, ISBN: 978-1-4704-3523-3, 2010 *Mathematics Subject Classification:* 81T05, 46L37, 46M05; 81T40, 46L60, 81R10, List US$81, **AMS Individual member US$48.60**, AMS Institutional member US$64.80, **MAA members US$72.90**, Order code MEMO/258/1237

bookstore.ams.org/memo-258-1237

Generalized Mercer Kernels and Reproducing Kernel Banach Spaces

Yuesheng Xu, *Syracuse University, NY*, and **Qi Ye**, *South China Normal University, Guangzhou, China*

Memoirs of the American Mathematical Society, Volume 258, Number 1243

February 2019, 122 pages, Softcover, ISBN: 978-1-4704-3550-9, 2010 *Mathematics Subject Classification:* 68T05, 94A20; 46E22, 68P30, List US$81, **AMS Individual member US$48.60**, AMS Institutional member US$64.80, **MAA members US$72.90**, Order code MEMO/258/1243

bookstore.ams.org/memo-258-1243

Differential Equations

Extended States for the Schrödinger Operator with Quasi-Periodic Potential in Dimension Two

Yulia Karpeshina and **Roman Shterenberg**, *both of University of Alabama at Birmingham*

This item will also be of interest to those working in mathematical physics.

Memoirs of the American Mathematical Society, Volume 258, Number 1239
February 2019, 139 pages, Softcover, ISBN: 978-1-4704-3543-1, 2010 *Mathematics Subject Classification:* 35J10, 35P15, 35P20, 81Q05, 81Q10; 81Q15, 37K55, 47F05, List US$81, **AMS Individual member US$48.60**, AMS Institutional member US$64.80, **MAA members US$72.90**, Order code MEMO/258/1239

bookstore.ams.org/memo-258-1239

Geometry and Topology

Geodesics, Retracts, and the Norm-Preserving Extension Property in the Symmetrized Bidisc

Jim Agler, *University of California at San Diego, CA*, **Zinaida Lykova**, and **Nicholas Young**, *both of Newcastle University, Newcastle upon Tyne, United Kingdom*

Memoirs of the American Mathematical Society, Volume 258, Number 1242
February 2019, 108 pages, Softcover, ISBN: 978-1-4704-3549-3, 2010 *Mathematics Subject Classification:* 32A07, 53C22, 54C15, 47A57, 32F45; 47A25, 30E05, List US$81, **AMS Individual member US$48.60**, AMS Institutional member US$64.80, **MAA members US$72.90**, Order code MEMO/258/1242

bookstore.ams.org/memo-258-1242

CR Embedded Submanifolds of CR Manifolds

Sean N. Curry and **A. Rod Gover**, *both of The University of Auckland, New Zealand*

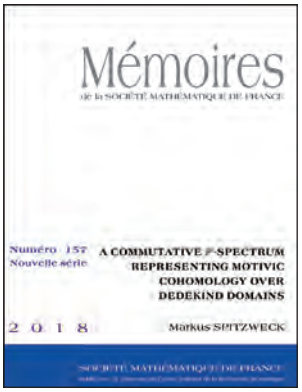
Memoirs of the American Mathematical Society, Volume 258, Number 1241
February 2019, 81 pages, Softcover, ISBN: 978-1-4704-3544-8, 2010 *Mathematics Subject Classification:* 32V05, 53B15; 32V30, 53B25, 53A30, List US$81, **AMS Individual member US$48.60**, AMS Institutional member US$64.80, **MAA members US$72.90**, Order code MEMO/258/1241

bookstore.ams.org/memo-258-1241

New AMS-Distributed Publications

Algebra and Algebraic Geometry

A Commutative $\mathbb{P}^1$-Spectrum Representing Motivic Cohomology Over Dedekind Domains



Markus Spitzweck, *Universität Osnabrück, Germany*

The author constructs a motivic Eilenberg–Mac Lane spectrum with a highly structured multiplication over general base schemes which represents Levine's motivic cohomology, defined via Bloch's cycle complexes, over smooth schemes over Dedekind domains. The author's method involves gluing p-completed and rational parts along an arithmetic square. Hereby, the finite coefficient spectra are obtained by truncated étale sheaves (relying on the now proven Bloch-Kato conjecture) and a variant of Geisser's version of syntomic cohomology, and the rational spectra are the ones which represent Beilinson motivic cohomology.

As an application, the arithmetic motivic cohomology groups can be realized as Ext-groups in a triangulated category of motives with integral coefficients. The author's spectrum is compatible with base change giving rise to a formalism of six functors for triangulated categories of motivic sheaves over general base schemes, including the localization triangle.

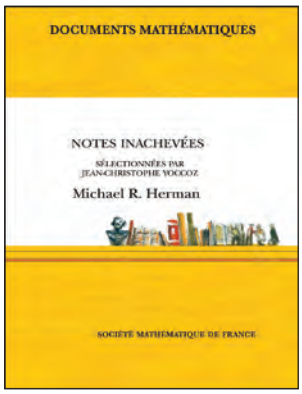
Further applications are a generalization of the Hopkins-Morel isomorphism and a structure result for the dual motivic Steenrod algebra in the case where the coefficient characteristic is invertible on the base scheme.

A publication of the Société Mathématique de France, Marseilles (SMF), distributed by the AMS in the US, Canada, and Mexico. Orders from other countries should be sent to the SMF. Members of the SMF receive a 30% discount from list.

Mémoires de la Société Mathématique de France, Number 157
October 2018, 114 pages, Softcover, ISBN: 978-2-85629-890-9, 2010 *Mathematics Subject Classification:* 14F42, 14C25, 19E20, 19E99, 14F05, 14F20, 14F30, 55P43, List US$48, **AMS members US$38.40**, Order code SMFMEM/157

bookstore.ams.org/smfmem-157

Analysis



Notes Inachevées
Sélectionnées Par Jean-ChristopheYoccoz
Michael R. Herman

Michael R. Herman was a leading specialist of dynamical system theory. After his sudden death, he left many handwritten notes, some of them of high quality and never published.

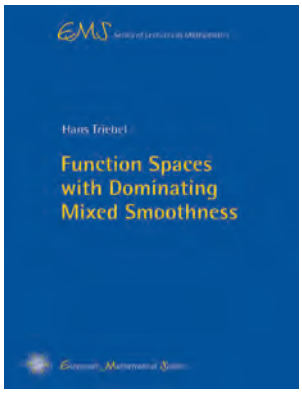
Jean-Christophe Yoccoz, who was not only his scientific executor but also one of his first students and one of his most cherished mathematical companions, decided to assemble the most important notes and make them available to the community. He gathered a team of colleagues acquainted with Herman's research domains to classify and investigate these notes and then to type the selected ones.

The guideline of this collective work was to adhere as strictly as possible to the original manuscript, adding, if necessary, some helpful corrections or comments. The result is this volume of unpublished notes in which readers will discover or rediscover some facets of the mathematical mind of Michael R. Herman.

A publication of the Société Mathématique de France, Marseilles (SMF), distributed by the AMS in the US, Canada, and Mexico. Orders from other countries should be sent to the SMF. Members of the SMF receive a 30% discount from list.

Documents Mathématiques, Number 16
October 2018, 200 pages, Softcover, ISBN: 978-2-85629-867-1, 2010 *Mathematics Subject Classification:* 37, List US$75, **AMS members US$60**, Order code SMFDM/16

bookstore.ams.org/smfdm-16



Function Spaces with Dominating Mixed Smoothness
Hans Triebel, *University of Jena, Germany*

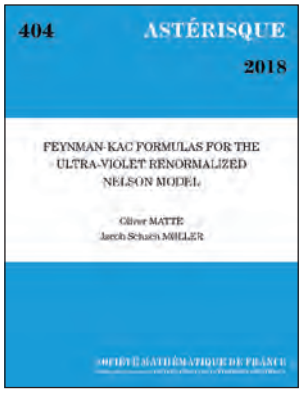
The first part of this book is devoted to function spaces in Euclidean n-space with dominating mixed smoothness. Some new properties are derived and applied in the second part where weighted spaces with dominating mixed smoothness in arbitrary bounded domains in Euclidean n-space are introduced and studied. This includes wavelet frames, numerical integration, and discrepancy measuring the deviation of sets of points from uniformity.

These notes are addressed to graduate students and mathematicians having a working knowledge of basic elements of the theory of function spaces, especially of Besov–Sobolev type. In particular, the book will be of interest to researchers dealing with approximation theory, numerical integration, and discrepancy.

A publication of the European Mathematical Society (EMS). Distributed within the Americas by the American Mathematical Society.

EMS Series of Lectures in Mathematics, Volume 30
January 2019, 210 pages, Softcover, ISBN: 978-3-03719-195-8, 2010 *Mathematics Subject Classification:* 46–02, 46E35, 42C40, 42B35, 41A55, List US$42, **AMS members US$33.60**, Order code EMSSERLEC/30

bookstore.ams.org/emsserlec-30



Feynman-Kac Formulas for the Ultra-Violet Renormalized Nelson Model

Oliver Matte and **Jacob Schach Møller**, *both of Institut for Matematik, Aarhus Universitet, Denmark*

The authors derive Feynman-Kac formulas for the ultra-violet renormalized Nelson Hamiltonian with a Kato decomposable external potential and for corresponding fiber Hamiltonians in the translation invariant case. They simultaneously treat massive and massless bosons. Furthermore, they present a non-perturbative construction of a renormalized Nelson Hamiltonian in a non-Fock representation defined as the generator of a corresponding Feynman-Kac semi-group.

The authors' novel analysis of the vacuum expectation of the Feynman-Kac integrands shows that if the external potential and the Pauli principle are dropped, then the spectrum of the N-particle renormalized Nelson Hamiltonian is bounded from below by some negative universal constant times g^4n^3 for all values of the coupling constant g. A variational argument also yields an upper bound of the same form for large g^2N.

The authors further verify that the semi-groups generated by the ultra-violet renormalized Nelson Hamiltonian and its non-Fock version are positivity improving with respect to a natural self-dual cone if the Pauli principle is ignored. In another application, they discuss continuity properties of elements in the range of the semi-group of the renormalized Nelson Hamiltonian.

This item will also be of interest to those working in probability and statistics.

A publication of the Société Mathématique de France, Marseilles (SMF), distributed by the AMS in the US, Canada, and Mexico. Orders from other countries should be sent to the SMF. Members of the SMF receive a 30% discount from list.

Astérisque, Number 404
October 2018, 110 pages, Softcover, ISBN: 978-2-85629-893-0, 2010 *Mathematics Subject Classification:* 47D08, 60H30, 81T16, 81V99, List US$52.50, **AMS members US$42**, Order code AST/404

bookstore.ams.org/ast-404

Large KAM Tori for Perturbations of the Defocusing NLS Equation

Massimiliano Berti, *Scuola Internazionale Superiore di Studi Avanzati (SISSA), Trieste, Italy*, **Thomas Kappeler**, and **Riccardo Montalto**, *both of Institut für Mathematik, Universität Zürich, Switzerland*

The authors prove that small, semi-linear Hamiltonian perturbations of the defocusing nonlinear Schrödinger (dNLS) equation on the circle have an abundance of invariant tori of any size and (finite) dimension which support quasi-periodic solutions. When compared with previous results, the novelty consists in considering perturbations which do not satisfy any symmetry condition (they may depend on x in an arbitrary way) and need not be analytic. The main difficulty is posed by pairs of almost resonant dNLS frequencies.

The proof is based on the integrability of the dNLS equation, in particular, the fact that the nonlinear part of the Birkhoff coordinates is one smoothing.

The authors implement a Newton-Nash-Moser iteration scheme to construct the invariant tori. The key point is the reduction of linearized operators, coming up in the iteration scheme, to 2×2 block diagonal ones with constant coefficients together with sharp asymptotic estimates of their eigenvalues.

This item will also be of interest to those working in differential equations.

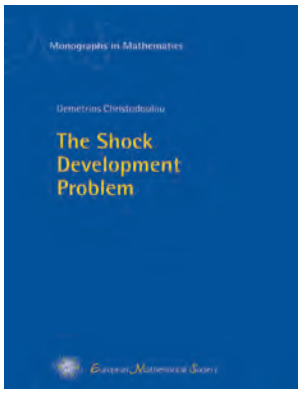
A publication of the Société Mathématique de France, Marseilles (SMF), distributed by the AMS in the US, Canada, and Mexico. Orders from other countries should be sent to the SMF. Members of the SMF receive a 30% discount from list.

Astérisque, Number 403
October 2018, 148 pages, Softcover, ISBN: 978-2-85629-892-3, 2010 *Mathematics Subject Classification:* 37K55, 35Q55, List US$67, **AMS members US$53.60**, Order code AST/403

bookstore.ams.org/ast-403

NEW BOOKS

Differential Equations



The Shock Development Problem

Demetrios Christodoulou, *Eidgen Technische Hochschule, Zurich, Switzerland*

This monograph addresses the problem of the development of shocks in the context of the Eulerian equations of the mechanics of compressible fluids. The mathematical problem is that of an initial-boundary value problem for a nonlinear hyperbolic system of partial differential equations with a free boundary and singular initial conditions.

The free boundary is the shock hypersurface and the boundary conditions are jump conditions relative to a prior solution, conditions following from the integral form of the mass, momentum and energy conservation laws. The prior solution is provided by the author's previous work which studies the maximal classical development of smooth initial data. New geometric and analytic methods are introduced to solve the problem.

Geometry enters as the acoustical structure, a Lorentzian metric structure defined on the spacetime manifold by the fluid. This acoustical structure interacts with the background spacetime structure. Reformulating the equations as two coupled first order systems, the characteristic system, which is fully nonlinear, and the wave system, which is quasilinear, a complete regularization of the problem is achieved. Geometric methods also arise from the need to treat the free boundary.

These methods involve the concepts of bivariational stress and of variation fields. The main new analytic method arises from the need to handle the singular integrals appearing in the energy identities. Shocks are an ubiquitous phenomenon and also occur in magnetohydrodynamics, nonlinear elasticity, and the electrodynamics of nonlinear media. The methods developed in this monograph are likely to be found relevant in these fields as well.

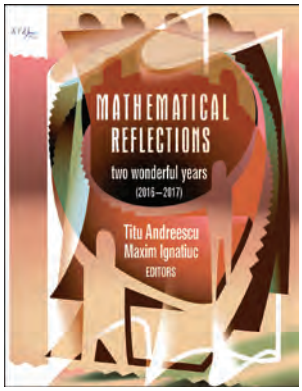
This item will also be of interest to those working in mathematical physics.

A publication of the European Mathematical Society (EMS). Distributed within the Americas by the American Mathematical Society.

EMS Monographs in Mathematics, Volume 8
January 2019, 932 pages, Hardcover, ISBN: 978-3-03719-192-7, 2010 *Mathematics Subject Classification:* 35L67, 35L04, 35L65, 76L05, 76N15, List US$148, **AMS members US$118.40**, Order code EMSMONO/8

bookstore.ams.org/emsmono-8

Math Education



Mathematical Reflections

Two Wonderful Years (2016–2017)

Titu Andreescu and **Maxim Ignatiuc**, *both of University of Texas at Dallas, TX*, Editors

Mathematical Reflections: Two Wonderful Years is a compilation and revision of the 2016 and 2017 volumes from the online journal of the same name. This book is aimed at high school students, participants in math competitions, undergraduates, and anyone who has a fire for mathematics. Passionate readers submitted many of the problems, solutions, and articles and all require creativity, experience, and comprehensive mathematical knowledge.

This book is a great resource for students training for advanced national and international mathematics competitions such as USAMO and IMO.

A publication of XYZ Press. Distributed in North America by the American Mathematical Society.

XYZ Series, Volume 32
November 2018, 515 pages, Hardcover, ISBN: 978-0-9993428-2-4, 2010 *Mathematics Subject Classification:* 00A05, 00A07, 97U40, 97D50, List US$59.95, **AMS members US$47.96**, Order code XYZ/32

bookstore.ams.org/xyz-32

NEW BOOKS

Probability and Statistics

Poisson Ensembles of Loops of One-Dimensional Diffusions

Titus Lupu, *Sorbonne Université, Paris, France*

There is a natural measure on loops (time-parametrized trajectories that, in the end, return to the origin) which one can associate to a wide class of Markov processes. The Poisson ensembles of Markov loops are Poisson point processes with intensity proportional to these measures. In wide generality, these Poisson ensembles of Markov loops are related, at intensity parameter 1/2, to the Gaussian free field, and at intensity parameter 1, to the loops done by a Markovian sample path.

Here, the author studies the specific case when the Markov process is a one-dimensional diffusion. After a detailed description of the measure, the author studies the Poisson point processes of loops, their occupation fields, and explains how to sample these Poisson ensembles of loops out of diffusion sample path perturbed at their successive minima.

Finally, the author introduces a couple of interwoven determinantal point processes on the line, which is a dual through Wilson's algorithm of Poisson ensembles of loops, and studies the properties of these determinantal point processes.

A publication of the Société Mathématique de France, Marseilles (SMF), distributed by the AMS in the US, Canada, and Mexico. Orders from other countries should be sent to the SMF. Members of the SMF receive a 30% discount from list.

Mémoires de la Société Mathématique de France, Number 158
October 2018, 162 pages, Softcover, ISBN: 978-2-85629-891-6, 2010 *Mathematics Subject Classification:* 60–02, 60G15, 60G17, 60G55, 60G60, 60J55, 60J60, 60J65, 60J80, List US$48, **AMS members US$38.40**, Order code SMFMEM/158

bookstore.ams.org/smfmem-158

Meetings & Conferences of the AMS
March Table of Contents

The Meetings and Conferences section of the *Notices* gives information on all AMS meetings and conferences approved by press time for this issue. Please refer to the page numbers cited on this page for more detailed information on each event.

Invited Speakers and Special Sessions are listed as soon as they are approved by the cognizant program committee; the codes listed are needed for electronic abstract submission. For some meetings the list may be incomplete. Information in this issue may be dated.
The most up-to-date meeting and conference information can be found online at: `www.ams.org/meetings`.

Important Information About AMS Meetings: Potential organizers, speakers, and hosts should refer to page 127 in the January 2019 issue of the *Notices* for general information regarding participation in AMS meetings and conferences.

Abstracts: Speakers should submit abstracts on the easy-to-use interactive Web form. No knowledge of LaTeX is necessary to submit an electronic form, although those who use LaTeX may submit abstracts with such coding, and all math displays and similarily coded material (such as accent marks in text) must be typeset in LaTeX. Visit `www.ams.org/cgi-bin/abstracts/abstract.pl`. Questions about abstracts may be sent to `abs-info@ams.org`. Close attention should be paid to specified deadlines in this issue. Unfortunately, late abstracts cannot be accommodated.

Associate Secretaries of the AMS

Central Section: Georgia Benkart, University of Wisconsin-Madison, Department of Mathematics, 480 Lincoln Drive, Madison, WI 53706-1388; email: `benkart@math.wisc.edu`; telephone: 608-263-4283.

Eastern Section: Steven H. Weintraub, Department of Mathematics, Lehigh University, Bethlehem, PA 18015-3174; email: `steve.weintraub@lehigh.edu`; telephone: 610-758-3717.

Southeastern Section: Brian D. Boe, Department of Mathematics, University of Georgia, 220 D W Brooks Drive, Athens, GA 30602-7403, email: `brian@math.uga.edu`; telephone: 706-542-2547.

Western Section: Michel L. Lapidus, Department of Mathematics, University of California, Surge Bldg., Riverside, CA 92521-0135; email: `lapidus@math.ucr.edu`; telephone: 951-827-5910.

Meetings in this Issue

See `https://www.ams.org/meetings` *for the most up-to-date information on the meetings and conferences that we offer.*

Meetings & Conferences of the AMS

IMPORTANT INFORMATION REGARDING MEETINGS PROGRAMS: AMS Sectional Meeting programs do not appear in the print version of the *Notices*. However, comprehensive and continually updated meeting and program information with links to the abstract for each talk can be found on the AMS website. See `www.ams.org/meetings/`.

Final programs for Sectional Meetings will be archived on the AMS website accessible from the stated URL .

Auburn, Alabama

Auburn University

March 15–17, 2019

Friday – Sunday

Meeting #1146

Southeastern Section
Associate secretary: Brian D. Boe

Announcement issue of *Notices*: January 2019
Program first available on AMS website: January 31, 2019
Issue of *Abstracts*: Volume 40, Issue 2

Deadlines

For organizers: Expired
For abstracts: Expired

The scientific information listed below may be dated. For the latest information, see `www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Grigoriy Blekherman, Georgia Institute of Technology, *Do sums of squares dream of free resolutions?*.
Carina Curto, Pennsylvania State University, *Graphs, network motifs, and threshold-linear algebra in the brain*.
Ming Liao, Auburn University, *Invariant Markov processes under actions of Lie groups*.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at `www.ams.org/cgi-bin/abstracts/abstract.pl`.

Algebraic and Discrete Methods in Mathematical Biology, **Carina Curto**, The Pennsylvania State University, **Katherine Morrison**, University of Northern Colorado, and **Nora Youngs**, Colby College.

Applications of Algebraic Geometry, **Greg Blekherman**, Georgia Institute of Technology, **Michael Burr**, Clemson University, and **Tianran Chen**, Auburn University at Montgomery.

Clustering Methods and Applications, **Benjamin McLaughlin**, Naval Surface Warfare Center Panama City Division (NSWCPCD), and **Sung Ha Kang**, Georgia Institute of Technology.

Combinatorial Matrix Theory, **Zhongshan Li**, Georgia State University, and **Xavier Martínez-Rivera**, Auburn University.

Commutative and Combinatorial Algebra, **Selvi Kara Beyarslan**, University of South Alabama, and **Alessandra Costantini**, Purdue University.

Developments in Commutative Algebra, **Eloísa Grifo**, University of Michigan, and **Patricia Klein**, University of Kentucky.

Differential Equations in Mathematical Biology, **Guihong Fan**, Columbus State University, **Zhongwei Shen**, University of Alberta, and **Xiaoxia Xie**, Idaho State University.

Discrete and Convex Geometry, **Andras Bezdek**, Auburn University, **Ferenc Fodor**, University of Szeged, and **Wlodzimierz Kuperberg**, Auburn University.

Evolution Equations and Applications, **Dmitry Glotov**, **Wenxian Shen**, and **Paul G. Schmidt**, Auburn University.

Experimental Mathematics in Number Theory, Analysis, and Combinatorics, **Amita Malik**, Rutgers University, and **Armin Straub**, University of South Alabama.

Geometric Flows and Minimal Surfaces, **Theodora Bourni**, University of Tennessee, and **Giuseppe Tinaglia**, King's College London and University of Tennessee.

Geometric Methods in Representation Theory, **Jiuzu Hong** and **Shrawan Kumar**, University of North Carolina, Chapel Hill, and **Yiqiang Li**, University at Buffalo, the State University of New York.

Geometric and Combinatorial Aspects of Representation Theory, **Mark Colarusso**, University of South Alabama, and **Jonas Hartwig**, Iowa State University.

Geometry and Topology of Low Dimensional Manifolds, and Their Invariants, **John Etnyre**, Georgia Institute of Technology, **Bulent Tosun**, University of Alabama, and **Shea Vela-Vick**, Louisiana State University.

Graph Theory in Honor of Robert E. Jamison's 70th Birthday, **Robert A Beeler**, East Tennessee State University, **Gretchen Matthews**, Virginia Tech, and **Beth Novick**, Clemson University.

Hopf Algebras and Their Applications, **Robert Underwood**, Auburn University at Montgomery, and **Alan Koch**, Agnes Scott College.

Mapping Class Groups, **Joan Birman**, Columbia University, and **Kevin Kordek** and **Dan Margalit**, Georgia Institute of Technology.

Mathematical Analysis and Control Theory of Coupled Partial Differential Equation Models, **George Avalos** and **Pelin Gu˝ven Geredeli**, University of Nebraska-Lincoln, and **László Kindrat**, University of New Hampshire.

Nonlinear Reaction-Diffusion Equations and Their Applications, **Jerome Goddard,II**, Auburn University at Montgomery, **Nsoki Mavinga**, Swarthmore College, **Quinn Morris**, Appalachian State University, and **R. Shivaji**, University of North Carolina at Greensboro.

Probability and Stochastic Processes, **Ming Liao**, **Erkan Nane**, and **Jerzy Szulga**, Auburn University.

Random Discrete Structures, **Lutz P Warnke**, Georgia Institute of Technology, and **Xavier Pérez-Giménez**, University of Nebraska-Lincoln.

Recent Advances in Coarse Geometry, **Jerzy Dydak**, University of Tennessee.

Recent Advances in Numerical Methods for PDEs and PDE-constrained Optimization, **Yanzhao Cao**, **Thi-Thao-Phuong Hoang**, and **Junshan Lin**, Auburn University.

Recent Developments in Graph Theory, **Xiaofeng Gu**, **Jeong-Hyun Kang**, **David Leach**, and **Rui Xu**, University of West Georgia.

Representations of Lie Algebras, Algebraic Groups, and Quantum Groups, **Joerg Feldvoss**, University of South Alabama, **Lauren Grimley**, Spring Hill College, and **Cornelius Pillen**, University of South Alabama.

The Modeling and Analysis of Spatially Extended Structures, **Shibin Dai**, University of Alabama, **Keith Promislow**, Michigan State University, and **Qiliang Wu**, Ohio University.

Topological Data Analysis, Statistics and Applications, **Yu-Min Chung**, University of North Carolina at Greensboro, and **Vasileios Maroulas**, University of Tennessee.

Honolulu, Hawai'i

University of Hawai'i at Mānoa

March 22–24, 2019

Friday – Sunday

Meeting #1147

Central Section
Associate secretaries: Georgia Benkart and Michel L. Lapidus

Announcement issue of *Notices*: January 2019
Program first available on AMS website: February 7, 2019
Issue of *Abstracts*: Volume 40, Issue 2

Deadlines

For organizers: Expired
For abstracts: Expired

The scientific information listed below may be dated. For the latest information, see `www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Barry Mazur, Harvard University, *On the arithmetic of curves* (Einstein Public Lecture in Mathematics).
Aaron Naber, Northwestern University, *Analysis of geometric nonlinear partial differential equations*.
Deanna Needell, University of California, Los Angeles, *Simple approaches to complicated data analysis*.
Katherine Stange, University of Colorado, Boulder, *An Illustration in Number Theory*.
Andrew Suk, University of California, San Diego, *On the Erdos-Szekeres convex polygon problem*.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at `www.ams.org/cgi-bin/abstracts/abstract.pl`.

Advances in Iwasawa Theory, **Frauke Bleher**, University of Iowa, **Ted Chinburg**, University of Pennsylvania, and **Robert Harron**, University of Hawai'i at Mānoa.

Advances in Mathematical Fluid Mechanics, **Kazuo Yamazaki**, University of Rochester, and **Adam Larios**, University of Nebraska - Lincoln.

Algebraic Groups, Galois Cohomology, and Local-Global Principles, **Raman Parimala**, Emory University, **Andrei Rapinchuk**, University of Virginia, and **Igor Rapinchuk**, Michigan State University.

Algebraic Number Theory and Diophantine Equations, **Claude Levesque**, University of Laval.

Algebraic Points, **Barry Mazur** and **Hector Pasten**, Harvard University.

Algebraic and Combinatorial Structures in Knot Theory, **Sam Nelson**, Claremont McKenna College, **Natsumi Oyamaguchi**, Shumei University, and **Kanako Oshiro**, Sophia University.

Algebraic and Geometric Combinatorics, **Andrew Berget**, Western Washington University, and **Steven Klee**, Seattle University.

Analysis of Nonlinear Geometric Equations, **Aaron Naber**, Northwestern University, and **Richard Bamler**, University of California Berkeley.

Analytic and Probabilistic Methods in Convex Geometry, **Alexander Koldobsky**, University of Missouri, **Alexander Litvak**, University of Alberta, **Dmitry Ryabogin**, Kent State University, **Vladyslav Yaskin**, University of Alberta, and **Artem Zvavitch**, Kent State University.

Applications of Ultrafilters and Nonstandard Methods, **Isaac Goldbring**, University of California, Irvine, and **Steven Leth**, University of Northern Colorado.

Arithmetic Dynamics, **Andrew Bridy**, Texas A&M University, **Michelle Manes**, University of Hawai'i at Mānoa, and **Bianca Thompson**, Harvey Mudd College.

Arithmetic Geometry and Its Connections, **Laura Capuano**, University of Oxford, and **Amos Turchet**, University of Washington.

Arithmetic and Transcendence of Special Functions and Special Values, **Matthew A. Papanikolas**, Texas A&M University, and **Federico Pellarin**, Université Jean Monnet, St. Étienne.

Coarse Geometry, Index Theory, and Operator Algebras: Around the Mathematics of John Roe, **Erik Guentner**, University of Hawai'i at Mānoa, **Nigel Higson**, Penn State University, and **Rufus Willett**, University of Hawai'i at Mānoa.

Coding Theory and Information Theory, **Manabu Hagiwara**, Chiba University, and **James B. Nation**, University of Hawai'i.

Combinatorial and Experimental Methods in Mathematical Phylogeny, **Sean Cleary**, City College of New York and the CUNY Graduate Center, and **Katherine St. John**, Hunter College and the American Museum of Natural History.

Commutative Algebra and its Environs, **Olgur Celikbas** and **Ela Celikbas**, West Virginia University, and **Ryo Takahashi**, Nagoya University.

Computability, Complexity, and Learning, **Achilles A. Beros** and **Bjørn Kjos-Hanssen**, University of Hawai'i at Mānoa.

Computational and Data-Enabled Sciences, **Roummel Marcia**, **Boaz Ilan**, and **Suzanne Sindi**, University of California, Merced.

Constructive Aspects of Complex Analysis, **Ilia Binder** and **Michael Yampolsky**, University of Toronto, and **Malik Younsi**, University of Hawai'i at Mānoa.

Differential Geometry, **Vincent B. Bonini**, Cal Poly San Luis Obispo, **Jie Qing**, University of California, Santa Cruz, and **Bogdan D. Suceava**, California State University, Fullerton.

Dynamical Systems and Algebraic Combinatorics, **Maxim Arnold**, University of Texas at Dallas, **Jessica Striker**, North Dakota State University, and **Nathan Williams**, University of Texas at Dallas.

Emerging Connections with Number Theory, **Katherine Stange**, University of Colorado, Boulder, and **Renate Scheidler**, University of Calgary.

Equivariant Homotopy Theory and Trace Methods, **Andrew Blumberg**, University of Texas, **Teena Gerhardt**, Michigan State University, **Michael Hill**, UCLA, and **Michael Mandell**, Indiana University.

Factorization and Arithmetic Properties of Integral Domains and Monoids, **Scott Chapman**, Sam Houston State University, **Jim Coykendall**, Clemson University, and **Christopher O'Neill**, University California, Davis.

Generalizations of Symmetric Spaces, **Aloysius Helminck**, University of Hawai'i at Mānoa, **Vicky Klima**, Appalachian State University, **Jennifer Schaefer**, Dickinson College, and **Carmen Wright**, Jackson State University.

Geometric Approaches to Mechanics and Control, **Monique Chyba**, University of Hawai'i at Mānoa, **Tomoki Ohsawa**, The University of Texas at Dallas, and **Vakhtang Putkaradze**, University of Alberta.

Geometry, Analysis, Dynamics and Mathematical Physics on Fractal Spaces, **Joe P. Chen**, Colgate University, **Lũ(Tim) Hùng**, Hawai'i Pacific University, **Machiel van Frankenhuijsen**, Utah Valley University, and **Robert G. Niemeyer**, University of the Incarnate Word.

Homotopy Theory, **Kyle Ormsby** and **Angélica Osorno**, Reed College.

Interactions between Geometric Measure Theory, PDE, and Harmonic Analysis, **Mark Allen**, Brigham Young University, **Spencer Becker-Kahn**, University of Washington, **Max Engelstein**, Massachusetts Institute of Technology, and **Mariana Smit Vega Garcia**, University of Washington.

Interactions between Noncommutative Algebra and Noncommutative Algebraic Geometry, **Garrett Johnson**, North Carolina Central University, **Bach Nguyen** and **Xingting Wang**, Temple University, and **Daniel Yee**, Bradley University.

Lie Theory in the Representations of Groups and Related Structures - dedicated to the memory of Kay Magaard, **Christopher Drupieski**, DePaul University, and **Julia Pevtsova**, University of Washington.

Mapping Class Groups, **Asaf Hadari**, University of Hawai'i.

Mathematical Analysis of Nonlinear Phenomena, **Mimi Dai**, University of Illinois at Chicago.

Mathematical Methods and Models in Medicine, **Monique Chyba**, University of Hawai'i, and **Jakob Kotas**, University of Hawai'i and University of Portland.

New Trends in Geometric Measure Theory, **Antonio De Rosa**, Courant Institute of Mathematical Sciences, New York University, and **Luca Spolaor**, Massachusetts Institute of Technology.

New Trends on Variational Calculus and Non-Linear Partial Differential Equations, **Craig Cowan**, University of Manitoba, **Michinori Ishiwata**, Osaka University, **Abbas Moameni**, Carleton University, and **Futoshi Takahashi**, Osaka City University.

Nonlinear Wave Equations and Applications, **Boaz Ilan**, University of California, Merced, and **Barbara Prinari**, University of Colorado, Colorado Springs.

Numerical Methods for Partial Differential Equations, **Evan Gawlik**, **Michael Holst**, and **Martin Licht**, University of California, San Diego.

Real and Complex Singularities, **Leslie Charles Wilson**, University of Hawai'i, Mānoa, **Goo Ishikawa**, Hokkaido University, and **David Trotman**, Aix-Marseille University.

Recent Advances and Applications of Modular Forms, **Amanda Folsom**, Amherst College, **Pavel Guerzhoy**, University of Hawai'i at Mānoa, **Masanobu Kaneko**, Kyushu University, and **Ken Ono**, Emory University.

Recent Advances in Lie and Related Algebras and their Representations, **Brian D. Boe**, University of Georgia, and **Jonathan Kujawa**, University of Oklahoma.

Recent Advances in Numerical Methods for PDEs, **Hengguang Li**, Wayne State University, and **Sara Pollock**, University of Florida.

Recent Advances in Numerical Methods for PDEs, **Hengguang Li**, Wayne State University, and **Sara Pollock**, University of Florida.

Recent Developments in Automorphic Forms, **Solomon Friedberg**, Boston College, and **Jayce Getz**, Duke University.

Recent Trends in Algebraic Graph Theory, **Sebastian Cioaba**, University of Delaware, and **Shaun Fallat**, University of Regina.

SYZ Mirror Symmetry and Enumerative Geometry, **Siu Cheong Lau**, Boston University, **Naichung Leung**, The Chinese University of Hong Kong, and **Hsian-Hua Tseng**, Ohio State University.

Several Complex Variables, **Peter Ebenfelt**, University of California, San Diego, **John Erik Fornaess**, University of Michigan and Norwegian University of Science and Technology, **Ming Xiao**, University of California, San Diego, and **Yuan Yuan**, Syracuse University.

Spaces of Holomorphic Functions and Their Operators, **Mirjana Jovovic** and **Wayne Smith**, University of Hawai'i.

Sparsity, Randomness, and Optimization, **Deanna Needell** and **Jamie Haddock**, University of California, Los Angeles.

Spectral Geometry: The Length and Laplace Spectra of Riemannian Manifolds, **Benjamin Linowitz**, Oberlin College, and **Jeffrey S. Meyer**, California State University at San Bernardino.

Stability and Singularity in Fluid Dynamics, **Tristan Buckmaster**, Princeton University, **Steve Shkoller**, University of California, Davis, and **Vlad Vicol**, Princeton University.

Structural Graph Theory, **Zixia Song**, University of Central Florida, **Martin Rolek**, College of William and Mary, and **Yue Zhao**, University of Central Florida.

The Mathematics of Cryptography, **Shahed Sharif**, California State University, San Marcos, and **Alice Silverberg**, University of California, Irvine.

Three-dimensional Floer Theory, Contact Geometry, and Foliations, **Joan Licata**, Australian National University, and **Robert Lipshitz**, University of Oregon.

Topics at the Interface of Analysis and Geometry, **Alex Austin** and **Sylvester Eriksson-Bique**, University of California, Los Angeles.

Valuations on Algebraic Function Fields and Their Subrings, **Ron Brown**, University of Hawai'i, **Steven Dale Cutkosky**, University of Missouri, and **Franz-Viktor Kuhlmann**, University of Szczecin.

What is Happening in Mathematical Epidemiology? Current Theory, New Methods, and Open Questions, **Olivia Prosper**, University of Kentucky.

Hartford, Connecticut

University of Connecticut Hartford (Hartford Regional Campus)

April 13–14, 2019

Saturday – Sunday

Meeting #1148

Eastern Section
Associate secretary: Steven H. Weintraub

Announcement issue of *Notices*: February 2019
Program first available on AMS website: February 21, 2019
Issue of *Abstracts*: Volume 40, Issue 2

Deadlines

For organizers: Expired
For abstracts: Expired

The scientific information listed below may be dated. For the latest information, see `www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Olivier Bernardi, Brandeis University, *Percolation on triangulations and a bijective path to Liouville quantum gravity*.

Brian Hall, Notre Dame University, *Eigenvalues of random matrices in the general linear group in the large-N limit*.

Christina Sormani, Lehman College and CUNY Graduate Center, *Compactness Theorems for Sequences of Riemannian Manifolds*.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at `http://www.ams.org/cgi-bin/abstracts/abstract.pl`.

Algebraic Number Theory, **Harris Daniels**, Amherst College, and **Alvaro Lozano-Robledo** and **Erik Wallace**, University of Connecticut.

Analysis, Geometry, and PDEs in Non-smooth Metric Spaces, **Vyron Vellis**, University of Connecticut, **Xiaodan Zhou**, Worcester Polytechnic Institute, and **Scott Zimmerman**, University of Connecticut.

Banach Space Theory and Metric Embeddings, **Mikhail Ostrovskii**, St. John's University, and **Beata Randrianantoanina**, Miami University.

Chip-firing and Divisor Theory, **Caroline Klivans**, Brown University, and **David Perkinson**, Reed College.

Cluster Algebras and Related Topics, **Emily Gunawan** and **Ralf Schiffler**, University of Connecticut.

Combinatorial Commutative Algebra and Polyhedral Geometry, **Elie Alhajjar**, US Military Academy, and **McCabe Olsen**, Ohio State University.

Computability Theory, **Damir Dzhafarov** and **Reed Solomon**, University of Connecticut, and **Linda Brown Westrick**, Pennsylvania State University.

Convergence of Riemannian Manifolds, **Lan-Hsuan Huang** and **Maree Jaramillo**, University of Connecticut, and **Christina Sormani**, City University of New York Graduate Center and Lehman College.

Discrete Dynamical Systems and Applications, **Elliott J. Betrand**, Sacred Heart University, and **David McArdle**, University of Connecticut.

Invariants of Knots, Links, and Low-dimensional Manifolds, **Patricia Cahn**, Smith College, and **Moshe Cohen** and **Adam Lowrance**, Vassar College.

Knot Theory, the Colored Jones Polynomial, and Khovanov Homology, **Adam Giambrone**, Elmira College, and **Katherine Hall**, University of Connecticut.

Mathematical Cryptology, **Lubjana Beshaj**, United States Military Academy, and **Jaime Gutierrez**, University of Cantabria, Santander, Spain.

Mathematical Finance, **Oleksii Mostovyi**, University of Connecticut, **Gu Wang**, Worcester Polytechnic Institute, and **Bin Zhou**, University of Connecticut.

Modeling and Qualitative Study of PDEs from Materials Science and Geometry., **Yung-Sze Choi**, **Changfeng Gui**, and **Xiaodong Yan**, University of Connecticut.

Recent Advances in Structured Matrices and Their Applications, **Maxim Derevyagin**, University of Connecticut, **Olga Holz**, University of California, Berkeley, and **Vadim Olshevsky**, University of Connecticut.

Recent Development of Geometric Analysis and Nonlinear PDEs, **Ovidiu Munteanu**, **Lihan Wang**, and **Ling Xiao**, University of Connecticut.

Representation Theory of Quantum Algebras and Related Topics, **Drew Jaramillo**, University of Connecticut, **Garrett Johnson**, North Carolina Central University, and **Margaret Rahmoeller**, Roanoke College.

Special Session on Regularity Theory of PDEs and Calculus of Variations on Domains with Rough Boundaries, **Murat Akman**, University of Connecticut, and **Zihui Zhao**, University of Washington.

Special Values of L-functions and Arithmetic Invariants in Families, **Ellen Eischen**, University of Oregon, **Yifeng Liu**, Yale University, **Liang Xiao**, University of Connecticut, and **Wei Zhang**, Massachusetts Institute of Technology.

Stochastic Analysis and Related Fields, **Fabrice Baudoin**, University of Connecticut, and **Cheng Ouyang**, University of Illinois at Chicago.

Stochastic Processes, Random Walks, and Heat Kernels, **Patricia Alonso Ruiz**, University of Connecticut, and **Phanuel Mariano**, Purdue University.

Sub-Riemannian and CR Geometric Analysis, **Fabrice Baudoin**, University of Connecticut, and **Luca Capogna**, Worcester Polytechnic Institute.

Quy Nhon City, Vietnam

Quy Nhon University

June 10–13, 2019

Monday – Thursday

Meeting #1149

Associate secretary: Brian D. Boe
Announcement issue of *Notices*: April 2019
Program first available on AMS website: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: Expired
For abstracts: To be announced

The scientific information listed below may be dated. For the latest information, see `www.ams.org/amsmtgs/internmtgs.html`.

Invited Addresses

Henry Cohn, Microsoft Research, *To be announced*.
Robert Guralnick, University of Southern California, *To be announced*.
Le Tuan Hoa, Hanoi Institute of Mathematics, *To be announced*.
Nguyen Dong Yen, Hanoi Institute of Mathematics, *To be announced*.
Zhiwei Yun, Massachusetts Institute of Technology, *To be announced*.
Nguyen Tien Zung, Toulouse Mathematics Institute, *To be announced*.

Madison, Wisconsin

University of Wisconsin–Madison

September 14–15, 2019

Saturday – Sunday

Meeting #1150

Central Section
Associate secretary: Georgia Benkart
Announcement issue of *Notices*: June 2019
Program first available on AMS website: July 23, 2019
Issue of *Abstracts*: Volume 40, Issue 3

Deadlines

For organizers: February 14, 2019
For abstracts: July 16, 2019

The scientific information listed below may be dated. For the latest information, see `www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Nathan Dunfield, University of Illinois, Urbana-Champaign, *Title to be announced*.
Teena Gerhardt, Michigan State University, *Title to be announced*.
Lauren Williams, University of California, Berkeley, *Title to be announced* (Erdős Memorial Lecture).

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at `www.ams.org/cgi-bin/abstracts/abstract.pl`.

Algebraic and Geometric Combinatorics (Code: SS 12A), **Benjamin Braun**, University of Kentucky, **Marie Meyer**, Lewis University, and **McCabe Olsen**, Ohio State University.

Analysis and Probability on Metric Spaces and Fractals (Code: SS 10A), **Guy C. David**, Ball State University, and **John Dever**, Bowling Green State University.

Association Schemes and Related Topics – in Celebration of J.D.H. Smith's 70th Birthday (Code: SS 8A), **Kenneth W. Johnson**, Penn State University Abington, and **Sung Y. Song**, Iowa State University.

Computability Theory in honor of Steffen Lempp's 60th birthday (Code: SS 6A), **Joseph S. Miller**, **Noah D. Schweber**, and **Mariya I. Soskova**, University of Wisconsin–Madison.

Extremal Graph Theory (Code: SS 14A), **Józef Balogh**, University of Illinois, and **Bernard Lidický**, Iowa State University.

Geometry and Topology of Singularities (Code: SS 13A), **Laurentiu Maxim**, University of Wisconsin–Madison.

Hodge Theory in Honor of Donu Arapura's 60th Birthday (Code: SS 11A), **Ajneet Dhillon**, University of Western Ontario, **Kenji Matsuki** and **Deepam Patel**, Purdue University, and **Botong Wang**, University of Wisconsin–Madison.

Homological and Characteristic $p > 0$ Methods in Commutative Algebra (Code: SS 1A), **Michael Brown**, University of Wisconsin–Madison, and **Eric Canton**, University of Michigan.

Model Theory (Code: SS 5A), **Uri Andrews** and **Omer Mermelstein**, University of Wisconsin-Madison.

Recent Developments in Harmonic Analysis (Code: SS 3A), **Theresa Anderson**, Purdue University, and **Joris Roos**, University of Wisconsin–Madison.

Recent Work in the Philosophy of Mathematics (Code: SS 4A), **Thomas Drucker**, University of Wisconsin–Whitewater, and **Dan Sloughter**, Furman University.

Several Complex Variables (Code: SS 7A), **Hanlong Fang** and **Xianghong Gong**, University of Wisconsin–Madison.

Special Functions and Orthogonal Polynomials (Code: SS 2A), **Sarah Post**, University of Hawai'i at Mānoa, and **Paul Terwilliger**, University of Wisconsin–Madison.

Uncertainty Quantification Strategies for Physics Applications (Code: SS 9A), **Qin Li**, University of Wisconsin–Madison, and **Tulin Kaman**, University of Arkansas.

Binghamton, New York

Binghamton University

October 12–13, 2019

Saturday – Sunday

Meeting #1151

Eastern Section
Associate secretary: Steven H. Weintraub

Announcement issue of *Notices*: August 2019
Program first available on AMS website: August 29, 2019
Issue of *Abstracts*: Volume 40, Issue 3

Deadlines

For organizers: March 12, 2019
For abstracts: August 20, 2019

The scientific information listed below may be dated. For the latest information, see `www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Richard Kenyon, Brown University, *Title to be announced.*

Tony Pantev, University of Pennsylvania, *Title to be announced.*

Lai-Sang Young, New York University, *Title to be announced.*

Gainesville, Florida

University of Florida

November 2–3, 2019

Saturday – Sunday

Meeting #1152

Southeastern Section
Associate secretary: Brian D. Boe
Announcement issue of *Notices*: September 2019

Program first available on AMS website: September 19, 2019
Issue of *Abstracts*: Volume 40, Issue 4

Deadlines

For organizers: April 2, 2019
For abstracts: September 10, 2019

The scientific information listed below may be dated. For the latest information, see `www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Jonathan Mattingly, Duke University, *To be announced*.
Isabella Novik, University of Washington, *To be announced*.
Eduardo Teixeira, University of Central Florida, *To be announced*.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at `www.ams.org/cgi-bin/abstracts/abstract.pl`.

Fractal Geometry and Dynamical Systems (Code: SS 2A), **Mrinal Kanti Roychowdhury**, University of Texas Rio Grande Valley.

Geometric and Topological Combinatorics (Code: SS 1A), **Bruno Benedetti**, University of Miami, **Steve Klee**, Seattle University, and **Isabella Novik**, University of Washington.

Riverside, California

University of California, Riverside

November 9–10, 2019

Saturday – Sunday

Meeting #1153

Western Section
Associate secretary: Michel L. Lapidus
Announcement issue of *Notices*: September 2019

Program first available on AMS website: September 12, 2019
Issue of *Abstracts*: Volume 40, Issue 4

Deadlines

For organizers: April 9, 2019
For abstracts: September 3, 2019

The scientific information listed below may be dated. For the latest information, see `www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Mohsen Aliabadi, University of Illinois at Chicago, Chicago, IL, *A connection between matchings in field extensions and the fundamental theorem of algebra*.
Jonathan Novak, University of California, San Diego, *Title to be announced*.
Anna Skripka, University of New Mexico, Albuquerque, *Title to be announced*.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at `www.ams.org/cgi-bin/abstracts/abstract.pl`.

Inverse Problems (Code: SS 3A), **Hanna Makaruk**, Los Alamos National Laboratory, and **Robert Owczarek**, University of New Mexico, Albuquerque and University of New Mexico, Los Alamos.

Random Matrices and Related Structures (Code: SS 2A), **Jonathan Novak**, University of California, San Diego, and **Karl Liechty**, De Paul University.

Topics in Operator Theory (Code: SS 1A), **Anna Skripka** and **Maxim Zinchenko**, University of New Mexico.

Denver, Colorado

Colorado Convention Center

January 15–18, 2020

Wednesday – Saturday

Meeting #1154

Joint Mathematics Meetings, including the 126th Annual Meeting of the AMS, 103rd Annual Meeting of the Mathematical Association of America (MAA), annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association of Symbolic Logic (ASL), with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM)

Associate secretary: Michel L. Lapidus
Announcement issue of *Notices*: October 2019
Program first available on AMS website: November 1, 2019
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: April 1, 2019
For abstracts: To be announced

Charlottesville, Virginia

University of Virginia

March 13–15, 2020

Friday – Sunday
Southeastern Section
Associate secretary: Brian D. Boe
Announcement issue of *Notices*: To be announced

Program first available on AMS website: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced
For abstracts: To be announced

The scientific information listed below may be dated. For the latest information, see `www.ams.org/amsmtgs/sectional.html`.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at `http://www.ams.org/cgi-bin/abstracts/abstract.pl`.

Curves, Jacobians, and Abelian Varieties (Code: SS 1A), **Andrew Obus**, Baruch College (CUNY),, **Tony Shaska**, Oakland University, and **Padmavathi Srinivasan**, Georgia Institute of Technology.

Medford, Massachusetts

Tufts University

March 21–22, 2020

Saturday – Sunday
Eastern Section
Associate secretary: Steven H. Weintraub
Announcement issue of *Notices*: To be announced

Program first available on AMS website: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced
For abstracts: To be announced

Fresno, California

California State University, Fresno

May 2–3, 2020

Saturday – Sunday
Western Section
Associate secretary: Michel L. Lapidus
Announcement issue of *Notices*: To be announced

Program first available on AMS website: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced
For abstracts: To be announced

El Paso, Texas

University of Texas at El Paso

September 12–13, 2020

Saturday – Sunday
Central Section
Associate secretary: Georgia Benkart
Announcement issue of *Notices*: To be announced

Program first available on AMS website: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced
For abstracts: To be announced

State College, Pennsylvania

Pennsylvania State University, University Park Campus

October 3–4, 2020

Saturday – Sunday
Eastern Section
Associate secretary: Steven H. Weintraub
Announcement issue of *Notices*: To be announced

Program first available on AMS website: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced
For abstracts: To be announced

Salt Lake City, Utah

University of Utah

October 24–25, 2020

Saturday – Sunday
Western Section
Associate secretary: Michel L. Lapidus
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced

Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced
For abstracts: To be announced

Washington, District of Columbia

Walter E. Washington Convention Center

January 6–9, 2021

Wednesday – Saturday

Joint Mathematics Meetings, including the 127th Annual Meeting of the AMS, 104th Annual Meeting of the Mathematical Association of America (MAA), annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association of Symbolic Logic (ASL), with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM).

Deadlines

For organizers: April 1, 2020
For abstracts: To be announced

Associate secretary: Brian D. Boe
Announcement issue of *Notices*: October 2020
Program first available on AMS website: November 1, 2020
Issue of *Abstracts*: To be announced

Grenoble, France

Université Grenoble Alpes

July 5–9, 2021

Monday – Friday
Associate secretary: Michel L. Lapidus
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced

Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced
For abstracts: To be announced

Buenos Aires, Argentina

The University of Buenos Aires

July 19–23, 2021

Monday – Friday
Associate secretary: Steven H. Weintraub
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced

Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced
For abstracts: To be announced

Omaha, Nebraska

Creighton University

October 9–10, 2021

Saturday – Sunday
Central Section
Associate secretary: Georgia Benkart
Announcement issue of *Notices*: To be announced

Program first available on AMS website: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced
For abstracts: To be announced

Seattle, Washington

Washington State Convention Center and the Sheraton Seattle Hotel

January 5–8, 2022

Wednesday – Saturday
Associate secretary: Georgia Benkart
Announcement issue of *Notices*: October 2021
Program first available on AMS website: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced
For abstracts: To be announced

Boston, Massachusetts

John B. Hynes Veterans Memorial Convention Center, Boston Marriott Hotel, and Boston Sheraton Hotel

January 4–7, 2023

Wednesday – Saturday
Associate secretary: Steven H. Weintraub
Announcement issue of *Notices*: October 2022
Program first available on AMS website: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced
For abstracts: To be announced

Upcoming Features and Memorial Tributes

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by Michael Robinson

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POSITION AVAILABLE

DIRECTOR OF EDUCATION

AMERICAN MATHEMATICAL SOCIETY

The American Mathematical Society (AMS) is dedicated to advancing research and connecting the diverse global mathematical community through our publications, meetings and conferences, MathSciNet®, professional services, advocacy, and awareness programs.

Applications are invited for the position of Director of Education. This is a new, full-time position in the Government Relations Division at the Washington, DC office, with a preferred start date of July 1, 2019.

This position will oversee the AMS education portfolio, with a focus on undergraduate and graduate education in the mathematical sciences (including the preparation of students to enter graduate programs, the mentoring of students for success in graduate school, the preparation for careers both inside and outside of academia, and the promotion of diversity and inclusiveness in all mathematics education).

RESPONSIBILITIES:

- Advance the Society's involvement in student preparation for, and success in, graduate programs leading to an advanced degree in the mathematical sciences, with a focus on underrepresented groups, including women.
- Provide leadership for AMS efforts that support education in the mathematical sciences.
- Contribute to advocacy work focusing on education, engaging in discussions with policymakers and organizations, such as the National Academies and the US Department of Education.
- Interact with academic departments at the undergraduate and graduate levels.
- Work closely with the AMS Committee on Education.

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Direct specific and confidential inquiries about this position to Karen Saxe, Associate Executive Director for Government Relations (kxs@ams.org), or Catherine Roberts, Executive Director (exdir@ams.org).

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Virginia W. Noonburg, *University of Hartford, West Hartford, CT*

This second edition of Noonburg's best-selling textbook includes two new chapters on partial differential equations, making the book usable for a two-semester sequence in differential equations.

AMS/MAA Textbooks, Volume 43; 2019; 402 pages; Hardcover; ISBN: 978-1-4704-4400-6; List US$75; AMS members US$56.25; MAA members US$56.25; Order code TEXT/43

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Robert L. Benedetto, *Amherst College, MA*

This textbook presents the fundamentals of non-archimedean dynamics, including a unified exposition of Rivera-Letelier's classification theorem, as well as results on wandering domains, repelling periodic points, and equilibrium measures.

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AMS / **MAA** Press

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