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Perturbation Theorems for Maximal L_p -Regularity

PEER CHRISTIAN KUNSTMANN – LUTZ WEIS

Abstract. In this paper we prove perturbation theorems for R -sectorial operators. Via the characterization of maximal L_p -regularity in terms of R -boundedness due to the second author we obtain perturbation theorems for maximal L_p -regularity in UMD-spaces. We prove that R -sectoriality of A is preserved by A -small perturbations and by perturbations that are bounded in a fractional scale and small in a certain sense. Here, our method seems to give new results even for sectorial operators.

We apply our results to uniformly elliptic systems with bounded uniformly continuous coefficients, to Schrödinger operators with bad potentials, to the perturbation of boundary conditions, and to pseudo-differential operators with non-smooth symbols.

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1. – Introduction

Maximal regularity of type L_p is an important tool when dealing with quasi-linear and non-autonomous equations of parabolic type (see, e.g., [1], [3]). If the closed linear operator $-A$ is the generator of a bounded analytic C_0 -semigroup (T_t) in a Banach space X and $p \in (1, \infty)$ then we say that A has maximal L_p -regularity (which we denote by $A \in MR_p(X)$) if for any $f \in L_p((0, \infty), X)$ the solution $u = T * f$ of the equation $u' + Au = f$, $u(0) = 0$, satisfies $u' \in L_p((0, \infty), X)$ and $Au \in L_p((0, \infty), X)$. By the closed graph theorem this is equivalent to the existence of a constant $C > 0$ such that

$$(1) \quad \|u'\|_{L_p((0,\infty),X)} + \|Au\|_{L_p((0,\infty),X)} \leq C \|f\|_{L_p((0,\infty),X)}.$$

If X is a UMD-space and the operator A has bounded imaginary powers or – even stronger – an H^∞ -calculus of a suitable angle then A has maximal L_p -regularity by the well-known Dore-Venni result.

Recently, the second named author obtained — in UMD-spaces — a characterization of the maximal L_p -regularity property in terms of R -boundedness of resolvents of A (see [22]). Using the quantities defined below, it says that an operator A in a UMD-space X has maximal L_p -regularity if and only if A is R -sectorial with angle $\theta_r(A) < \pi/2$ (recall that A generates a bounded analytic semigroup in X if and only if A is sectorial with angle $\theta_\sigma(A) < \pi/2$). This characterization unifies the different approaches for checking maximal L_p -regularity, and provides a new convenient tool to check maximal regularity for concrete operators (see, e.g., [23]). As we will show in this paper, this is true in particular for perturbation theorems.

There are perturbation theorems known for operators A having bounded imaginary powers (cf. [19, Sect. 3]) or an H^∞ -calculus (cf. [2, Sect. 2]) but they require further properties of the perturbation in addition to A -smallness.

In this paper we show perturbation theorems for R -sectorial operators. By using the characterization of [22] this yields perturbation theorems for maximal L_p -regularity in UMD-spaces.

The first main result of our paper (Theorem 1 in Section 2, announced in [22], [23]) states that, in a general Banach space, R -sectoriality is preserved under A -small perturbations. Hence, in UMD-spaces, maximal L_p -regularity is preserved under A -small perturbations. We give an application of this result to elliptic operators with non-smooth coefficients in Section 3. Another application may be found in [13].

The second main result of our paper (Theorem 8 in Section 4) treats perturbations that are bounded in the scale $(X_\alpha)_{|\alpha| \leq 1}$ of domains of the fractional powers of A (actually, $X_\alpha := D(A^\alpha)$ only for $\alpha \geq 0$ whereas the definition for $\alpha < 0$ is somewhat different, see Section 4). Motivation comes from the form method in Hilbert space: If the operator A is given by a symmetric closed form q with form domain V in a Hilbert space H and r is another symmetric form with form domain V which is a form-small perturbation of q then the perturbation operator B associated with r is bounded $V \rightarrow V'$. If we write $V = D(A^{1/2}) =: H_{1/2}$ and $V' = (H_{1/2})' =: H_{-1/2}$ (where duality is taken with respect to the scalar product in H) then, for $\alpha = 1/2$, the perturbation B is a bounded operator $H_\alpha \rightarrow H_{\alpha-1}$. We generalize this method to general Banach spaces (and general $\alpha \in [0, 1]$) by considering perturbations by operators which are bounded $X_\alpha \rightarrow X_{\alpha-1}$ and have small norm. The case $\alpha = 1$ is of course already covered by Theorem 1. Perturbations for the case $\alpha = 0$ which is in some sense dual to the case $\alpha = 1$ have been considered before (we refer to [10, Sect. III.3.a]) but for generation of semigroups not for R -boundedness or maximal L_p -regularity.

We want to underline that, in the case $0 < \alpha < 1$, our arguments also prove a perturbation result for sectorial operators, and in particular for generators of analytic semigroups, which seems not to be stated explicitly in the literature although it has implicitly been used in [24, p. 223], see Remark 17 below.

In Section 5 we give a number of applications of the results in Section 4 that demonstrate the usefulness of our extended perturbation result with $\alpha \in [0, 1]$. In

particular we consider Schrödinger operators, elliptic operators of higher order, and pseudodifferential operators on $L_p(\Omega)$. Here X_α is usually a Sobolev space or a Bessel potential space and norm estimates for (perturbation) operators $X_\alpha \rightarrow X_{\alpha-1}$ are available in the literature. We also show how our result may be used to perturb the boundary conditions of a generator, and indicate that, even in a Hilbert space, it may be helpful to be able to treat situations with $\alpha \neq 1/2$.

In this paper, all Banach spaces are complex, and we use the following notation.

Let $(\varepsilon_j)_{j=0}^\infty$ be a sequence of independent symmetric $\{-1, 1\}$ -valued random variables, defined on the probability space (Ω, \mathcal{A}, P) . If X is a Banach space and $\tau \subset L(X)$ then τ is called *R-bounded* if there is a constant C such that

$$\left\| \sum_{j=0}^n \varepsilon_j T_j x_j \right\|_{L^2(\Omega; X)} \leq C \left\| \sum_{j=0}^n \varepsilon_j x_j \right\|_{L^2(\Omega; X)}$$

for all finite families $(T_j)_{j=0}^n$ in τ and $(x_j)_{j=0}^n$ in X . The infimum of all such constants C is denoted by $R(\tau)$ and called the *R-bound of τ* . Clearly, *R*-boundedness of τ implies boundedness in operator norm. For more information on this notion and workable criteria to check it in L_p -spaces we refer to [4], [5], [22], [23].

For any $\psi \in [0, \pi)$, let γ_ψ denote the path given by $\gamma_\psi(t) := |t|e^{i\psi \operatorname{sgn} t}$ for $t \neq 0$, and let $\Sigma_\psi := \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg(\lambda)| < \psi\}$. Recall that a closed linear operator A in X is called *sectorial* if $(-\infty, 0) \subset \rho(A)$ and $\sup_{t>0} \|t(t+A)^{-1}\| < \infty$. Then $M_A(\theta) := \sup\{\|\lambda(\lambda + A)^{-1}\| : \lambda \in \Sigma_{\pi-\theta}\} < \infty$ for some $\theta \in [0, \pi)$ and the *sectoriality angle* $\theta_\sigma(A)$ is defined by

$$\theta_\sigma(A) := \inf\{\theta \in [0, \pi) : M_A(\theta) < \infty\}.$$

One may replace the constant $M_A(\theta)$ in the definition by $\tilde{M}_A(\theta)$ where

$$\tilde{M}_A(\theta) := \sup\{\|A(\lambda + A)^{-1}\| : \lambda \in \Sigma_{\pi-\theta}\}.$$

If, in these definitions, we replace uniform boundedness by *R*-boundedness we are led to the concept of *R-sectorial operators*, i.e. sectorial operators A such that $R(\{\lambda(\lambda + A)^{-1} : \lambda > 0\}) < \infty$. The *R-sectoriality angle* $\theta_r(A)$ is defined by

$$\theta_r(A) := \inf\{\theta \in [0, \pi) : R_A(\theta) := R(\{\lambda(\lambda + A)^{-1} : \lambda \in \Sigma_{\pi-\theta}\}) < \infty\}.$$

Again, $R_A(\theta)$ may be replaced by

$$\tilde{R}_A(\theta) := R(\{A(\lambda + A)^{-1} : \lambda \in \Sigma_{\pi-\theta}\}).$$

Note that $M_A(\theta) \leq R_A(\theta)$ and $\tilde{M}_A(\theta) \leq \tilde{R}_A(\theta)$ always.

2. – Perturbations that are A -small

The following is our first main result. It states that — just as sectoriality is preserved by A -small perturbations — R -sectoriality is preserved as well. The perturbation theorem for sectorial operators implies a perturbation theorem for generators of analytic semigroups and the perturbation theorem for R -sectorial operators implies a perturbation theorem for maximal L_p -regularity in UMD-spaces.

THEOREM 1. *Let A be an R -sectorial operator in the Banach space X and $\theta > \theta_r(A)$. Let B be a linear operator satisfying $D(B) \supset D(A)$ and*

$$\|Bx\| \leq a\|Ax\| \quad (x \in D(A)).$$

If $a < 1/\tilde{R}_A(\theta)$ then $A + B$ is again R -sectorial and

$$R_{A+B}(\theta) \leq \frac{R_A(\theta)}{1 - a\tilde{R}_A(\theta)}.$$

In particular, we have $\theta_r(A + B) \leq \theta$.

If X is a UMD-space, $-A$ generates a bounded analytic semigroup and A has maximal L_p -regularity, and B satisfies the assumptions above then $-(A + B)$ generates a bounded analytic semigroup and $A + B$ has maximal L_p -regularity.

PROOF. We prove the first assertion. Then the second follows by the characterization result in [22].

For $|\arg \lambda| \leq \pi - \theta$ we obtain that

$$\|B(\lambda + A)^{-1}x\| \leq a\|A(\lambda + A)^{-1}x\| \leq a\tilde{M}_A(\theta)\|x\| \quad (x \in X).$$

Hence $I + B(\lambda + A)^{-1}$ is invertible by the assumption and

$$(\lambda + A + B)^{-1} = (\lambda + A)^{-1}(I + B(\lambda + A)^{-1})^{-1} = (\lambda + A)^{-1} \sum_{k=0}^{\infty} (-B(\lambda + A)^{-1})^k.$$

This representation implies

$$M_\theta(A + B) \leq \frac{M_A(\theta)}{1 - a\tilde{M}_A(\theta)},$$

hence $A + B$ is a sectorial operator and $\sigma(A + B) \subset \Sigma_\theta$. From the definition of R -boundedness and the assumption it is clear that, for any subset Λ of $\rho(-A)$, we have

$$R\{B(\lambda + A)^{-1} : \lambda \in \Lambda\} \leq a \cdot R\{A(\lambda + A)^{-1} : \lambda \in \Lambda\}.$$

Applying this to the series representation of $(\lambda + A + B)^{-1}$ we obtain

$$R_\theta(A + B) \leq \frac{R_A(\theta)}{1 - a\tilde{R}_A(\theta)}$$

as asserted. □

COROLLARY 2. *Let A be an R -sectorial operator in X with $\theta > \theta_r(A)$. Let B be a linear operator satisfying $D(B) \supset D(A)$ and*

$$(2) \quad \|Bx\| \leq a\|Ax\| + b\|x\| \quad (x \in D(A))$$

for some $a, b \geq 0$. If $a < (\tilde{M}_A(\theta)\tilde{R}_A(\theta))^{-1}$ then $A + B + \lambda$ is R -sectorial for any

$$(3) \quad \lambda > bM_A(\theta)\tilde{R}_A(\theta)/(1 - a\tilde{M}_A(\theta)\tilde{R}_A(\theta)).$$

Consequently, if X is a UMD-space, $-A$ generates an analytic semigroup, A has maximal L_p -regularity, and B satisfies the assumptions above, then $A + B + \mu$ has maximal regularity for any $\mu > \omega(A + B)$.

PROOF. For any $\lambda > 0$ and $x \in X$ we have

$$\begin{aligned} \|B(\lambda + A)^{-1}x\| &\leq a\|A(\lambda + A)^{-1}x\| + b\|(\lambda + A)^{-1}x\| \\ &\leq (a\tilde{M}_A(\theta) + bM_A(\theta)/\lambda)\|x\| =: c(\lambda)\|x\|. \end{aligned}$$

This means that B satisfies the assumptions of Theorem 1 for $A + \lambda$ in place of A if $c(\lambda) < \tilde{R}_A(\theta)^{-1}$ (note that $\tilde{R}_{A+\lambda}(\theta) \leq \tilde{R}_A(\theta)$ for $\lambda > 0$). Since $a\tilde{M}_A(\theta)\tilde{R}_A(\theta) < 1$ by assumption, the condition $c(\lambda)\tilde{R}_A(\theta) < 1$ is equivalent to (3). \square

REMARK 3. In [8, Thm. 6.1], a perturbation theorem for maximal regularity is given via the operator sum method: Suppose that A has maximal regularity in X . Define \mathcal{A} on $L_p(I, X)$ by $(\mathcal{A}f)(t) := A(f(t))$ and denote the inverse of $d/dt + \mathcal{A}$ by \mathcal{M} . Then $A + B$ has maximal regularity if, in addition to (2), we have $b\|\mathcal{M}\| + a\|\mathcal{A}\mathcal{M}\| < 1$ which holds in particular if $b = 0$ and a is sufficiently small.

The smallness condition $a\|\mathcal{A}\mathcal{M}\| < 1$ should be compared with the assumption $a\tilde{R}_A(\theta) < 1$ in Theorem 1. In principle, i.e. modulo an absolute constant (for $L_p(I, X)$ with $p = 2$ and $X = L_q(\Omega)$ we may take 1 and we may take 4 in the general case), the R -bound is smaller than the constant $\|\mathcal{A}\mathcal{M}\|$ of maximal regularity (see [5]). It seems that in many cases the R -bound is more accessible, and in general $\|\mathcal{A}\mathcal{M}\|_{L_p(I, X)} \rightarrow \infty$ for $p \rightarrow \infty$. Furthermore, R -boundedness is useful in other situations besides maximal L_p -regularity, e.g. in situations where $\theta_r > \pi/2$.

In UMD-spaces, Theorem 1 yields a perturbation theorem for maximal L_p -regularity, but if the space X is not UMD, the results are not comparable. For perturbations $B : X_\alpha \rightarrow X$, $0 \leq \alpha < 1$, see Corollary 12 below and [8, Thm. 6.2].

Finally we give a perturbation theorem in the style of Miyadera-Voigt (see, e.g., [10, Sect. III.3.c]). Since we consider analytic semigroups, we can even weaken the usual assumptions.

COROLLARY 4. *Let A be a densely defined R -sectorial operator in a reflexive space X , $\theta_r(A) < \theta \leq \pi/2$, and let (T_t) be the bounded analytic semigroup generated by $-A$. Assume that B is a closed and densely defined linear operator in X such that $D(B^*)$ norms X and*

$$C(a) := \sup \left\{ \int_0^a |\langle B^*x^*, T_t x \rangle| dt : x^* \in D(B^*), \|x^*\| \leq 1, \|x\| \leq 1 \right\} < \eta(\mu, a)$$

for some $a, \mu > 0$ where

$$\eta(\mu, a) := (1 - e^{-\mu a}) \left(\sup_{t \geq 0} \|T_t\| \tilde{R}_A(\theta) \right)^{-1}.$$

Then $D(B) \supset D(A)$ and $A + B + \mu$ is R -sectorial with $\theta_r(A + B + \mu) \leq \theta$.

PROOF. Let $M := \sup_{t \geq 0} \|T_t\|$. For $x \in X$ and $x^* \in D(B^*)$ we have by $\|e^{-\mu a n} T_{a n} x\| \leq M e^{-\mu a n} \|x\|$ the following estimate

$$\begin{aligned} |\langle B^*x^*, (\mu + A)^{-1}x \rangle| &\leq \int_0^\infty |\langle B^*x^*, T_t x \rangle| dt \\ &\leq \sum_{n=0}^\infty \int_0^a |\langle B^*x^*, T_t e^{-\mu n a} T_{n a} x \rangle| dt \\ &\leq \|x^*\| \|x\| C(a) M \sum_{n=0}^\infty e^{-\mu a n} \\ &< \|x^*\| \|x\| (\tilde{R}_A(\theta))^{-1}. \end{aligned}$$

Since X is reflexive and $D(B^*)$ norms X the closed operator $B(\mu + A)^{-1}$ is bounded. Hence $D(B) \supset D(A)$, and the estimate shows that the assumptions of Theorem 1 hold for $A + \mu$ in place of A , note again $\tilde{R}_{A+\mu}(\theta) \leq \tilde{R}_A(\theta)$. \square

3. – An application to elliptic operators

We consider in this section uniformly (M, θ) -elliptic systems in the sense of [2], [9], and first recall this notion. Let $n, m, N \in \mathbb{N}$. Let $M \geq 1$ and $\theta \in [0, \pi)$. The differential operator

$$(4) \quad \mathcal{A} := \sum_{|\alpha| \leq 2m} a_\alpha D^\alpha$$

of order $2m$ with measurable coefficients $a_\alpha : \mathbb{R}^n \rightarrow \mathbb{C}^{N \times N}$ is called uniformly (M, θ) -elliptic if

$$\begin{aligned} \sum_{|\alpha|=2m} \|a_\alpha\|_\infty &\leq M, \\ \sigma(\mathcal{A}_\pi(x, \xi)) &\subset \Sigma_\theta \setminus \{0\}, \\ |(\mathcal{A}_\pi(x, \xi))^{-1}| &\leq M, \quad x \in \mathbb{R}^n, |\xi| = 1, \end{aligned}$$

where $\mathcal{A}_\pi(x, \xi)$ denotes the principle symbol of the operator \mathcal{A} , i.e.

$$\mathcal{A}_\pi(x, \xi) := \sum_{|\alpha|=2m} a_\alpha(x) \xi^\alpha \quad (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n.$$

We assume that the highest order coefficients are bounded and uniformly continuous, i.e. $a_\alpha \in BUC(\mathbb{R}^n, \mathbb{C}^{N \times N})$ if $|\alpha| = 2m$. For $1 < p < \infty$, the $L_p(\mathbb{R}^n, \mathbb{C}^N)$ -realization A_p of \mathcal{A} is a closed linear operator with domain $W_p^2(\mathbb{R}^n, \mathbb{C}^N)$. For simplicity we assume here that $a_\alpha = 0$ if $|\alpha| \leq 2m - 1$. Otherwise, one may apply Corollary 12 below (for suitable conditions on the coefficients see, e.g., [19], [2], [9], [13]). Under an additional assumption on the modulus of continuity of the a_α with $|\alpha| = 2m$ it was shown in [2] that, for any $\theta < \tilde{\theta} < \pi$ a suitable translate of A_p has a bounded H^∞ -calculus of angle θ . If $\theta < \pi/2$ this implies in particular that the set $\{it(it + A + \nu)^{-1} : t \in \mathbb{R} \setminus \{0\}\}$ is R -bounded and hence that $A + \nu$ has maximal regularity of angle $\leq \theta$ for some $\nu > 0$. The assumption on the modulus of continuity was removed in [9, Thm. 6.1].

We now show that R -boundedness and hence maximal L_p -regularity for elliptic systems may be proved using Theorem 1. The crucial part in the proof is the second step where the perturbation is of the same order of differentiation as the unperturbed operator. The proof of [9, Thm. 6.1] uses the particular form of the operators involved and relies heavily on Calderon-Zygmund theory and the $T(1)$ -theorem for singular integral operators due to David and Journé. Theorem 1 is a much simpler device, but, on the other hand, R -boundedness is a weaker property than having an H^∞ -calculus.

THEOREM 5. *Let $1 < p < \infty$ and $M \geq 1, 0 \leq \theta < \omega < \pi$ be given. Then there are constants $\nu > 0$ and $K \geq 1$ such that, for any (M, θ) -sectorial operator A_p satisfying the assumptions above, the operator $A_p + \nu$ is R -sectorial with*

$$R_{A_p + \nu}(\omega) \leq K.$$

PROOF. We fix p and write A for A_p . As in [2] and [9] the proof is carried out in three steps. In a first step one treats homogeneous constant coefficient operators by Fourier multiplier arguments, e.g. the Mihlin multiplier theorem. One has $R_A(\theta') \leq C$ for all (M, θ) -elliptic differential operators A with constant coefficients where $\pi \geq \theta' > \theta$ and C only depends on M and θ' .

In a second step one treats homogeneous operators which are small perturbations of constant coefficient operators. Finally one uses a localization and (lower order) perturbation procedure (cf. [19], [2], [13]) and the remark below.

We concentrate on the second part of the proof which consists in showing the following perturbation result (cf. [9, Thm. 4.2]) because it is here that our method introduces a simplification of the argument. Here A is the L_p -realization of an operator of the form (4) with a_α constant ($a_\alpha = 0$ for $|\alpha| \leq 2m - 1$), and B is an operator of the form (4) with coefficients b_α instead of a_α where b_α is assumed to be measurable and bounded ($b_\alpha = 0$ for $|\alpha| \leq 2m - 1$).

PROPOSITION 6. *Let $\pi \geq \omega > \theta$ be given. Then there are constants $\varepsilon > 0$ and $K \geq 1$ such that $A + B$ is an R -sectorial operator satisfying*

$$R_{A+B}(\omega) \leq K$$

for all (M, θ) -elliptic operators A with constant coefficients and all differential operators B with $\|b\|_\infty \leq \varepsilon$ where

$$\|b\|_\infty := \sum_{|\alpha|=2m} \|b_\alpha\|_\infty.$$

PROOF. We can apply Theorem 1 directly since we have

$$\|Bf\|_p \leq \sum_{|\alpha|=2m} \|b_\alpha\| \|D^\alpha f\|_p \leq \varepsilon \sum_{|\alpha|=2m} \|D^\alpha f\|_p \leq \varepsilon \|Af\|_p \sum_{|\alpha|=2m} \|D^\alpha A^{-1}\|_{p \rightarrow p}$$

for $f \in W_p^2(\mathbb{R}^n, \mathbb{C}^N)$ and $\sum \|D^\alpha A^{-1}\| \leq L$ where the constant L only depends on M and p (the operator $D^\alpha A^{-1}$ has symbol $\xi \mapsto \xi^\alpha \mathcal{A}_\pi(\xi)^{-1}$ which is homogeneous of degree 0 and one can apply Mihlin's theorem).

Finally one uses a localization and (lower order) perturbation procedure (cf. [2, Sect. 3], [9, Sect. 6]). □

REMARK 7. The constant ν in Theorem 5 stems from the fact that the localization procedure brings in lower order perturbations (cf. [2]).

By arguing in the third step of the proof as in [13] (a combination of the ideas from [19] and [2]) one can show that sectoriality and bounded invertibility of $A_p + \nu$ implies R -sectoriality with optimal angle in the sense that $\theta_\sigma(A_p + \nu) = \theta_r(A_p + \nu)$. In particular A_p is R -sectorial whenever it is sectorial and boundedly invertible, and the respective angles coincide.

We want to remark that maximal L_p -regularity of A_p can also be achieved using [8, Thm. 6.1]. By considering suitable rotations $e^{i\phi} A_p$ of A_p this also holds for the optimality of θ_r (which may be alternatively defined in terms of the maximal regularity constants of such rotations).

4. – Perturbations in the fractional scale

Our next perturbation result is formulated in terms of the fractional powers of the sectorial operator A with $\theta_\sigma(A) < \pi$. We assume from now on that A is densely defined. Fix $\pi > \theta > \theta_\sigma(A)$. Then for $\alpha > 0$ and $\lambda \in \Sigma_{\pi-\theta}$ the (bounded) operator $(\lambda + A)^{-\alpha}$ is given by

$$(5) \quad (\lambda + A)^{-\alpha} := \frac{1}{2\pi i} \int_{\Gamma_{\varepsilon,\psi}} (\lambda - z)^{-\alpha} (z + A)^{-1} dz$$

where the curve $\Gamma_{\varepsilon,\psi}$ is parametrized by

$$(6) \quad \gamma(t) := \begin{cases} te^{i\psi}, & t \geq \varepsilon \\ \varepsilon e^{i\psi t/\varepsilon}, & |t| \leq \varepsilon \\ -te^{-i\psi}, & t \leq -\varepsilon \end{cases}$$

and $\pi - \theta_\sigma(A) > \psi > \pi - \theta$, $0 < \varepsilon < |\lambda|$.

The operator $(\lambda + A)^\alpha$ is defined as the inverse of the operator $(\lambda + A)^{-\alpha}$. It is a closed operator whose domain is independent of λ . The operator A^α is defined as the limit $\lim_{\delta \rightarrow 0+} (\delta + A)^\alpha$. It is again a closed operator and has domain $D(A^\alpha) = D((\delta + A)^\alpha)$ where $\delta > 0$. In case $0 \in \rho(A)$ one may take $\varepsilon = 0$ in (6), and the operator A^α is the inverse of the bounded operator $A^{-\alpha}$ which is given by (5) for $\lambda = 0$ (we could have used this as a definition if $0 \in \rho(A)$).

The fractional powers give rise to a scale $(X_\alpha)_{|\alpha| \leq 1}$ of Banach spaces in the following way (where $\pi > \theta > \theta_\sigma(A)$ is fixed):

For $\alpha \geq 0$, the space X_α is $D(A^\alpha)$ equipped with the graph norm (one could use any norm $\|(\lambda + A)^\alpha \cdot\|$, $\lambda \in \Sigma_{\pi-\theta}$, instead), and the space $X_{-\alpha}$ is the completion of X for the norm $x \mapsto \|(\lambda + A)^{-\alpha}x\|$ where one may take any $\lambda \in \Sigma_{\pi-\theta}$, all giving rise to equivalent norms.

Then $X_0 = X$ and, for $\alpha > 0$ and $\lambda \in \Sigma_{\pi-\theta}$, $(\lambda + A)^{-\alpha}$ is an isomorphism $X_0 \rightarrow X_\alpha$ and extends to an isomorphism $J_{\lambda,\alpha} : X_{-\alpha} \rightarrow X_0$ whose inverse $J_{\lambda,\alpha}^{-1}$ is an extension of $(\lambda + A)^\alpha$ from $D(A^\alpha) = X_\alpha$ to $X = X_0$. In each of the spaces X_α we have an operator A_α similar to the operator A in X , given by the part of A in X_α if $\alpha \geq 0$, and by $A_\alpha := J_{\lambda,\alpha}^{-1}AJ_{\lambda,\alpha}$ if $\alpha > 0$ (the definition does not depend on $\lambda \in \Sigma_{\pi-\theta}$).

Then $J_{\lambda,\alpha} = (\lambda + A_{-\alpha})^{-\alpha}$ for $\alpha \geq 0$, $\lambda \in \Sigma_{\pi-\theta}$. We refer to [10, Ch. II, Sect. 5] and [1, Ch. V] for constructions of this type.

We first assume $0 \in \rho(A)$ for simplicity. Then the above assertions also hold for the case $\lambda = 0$. In particular $D((A_\alpha)^\alpha) = X_0 = X$ and $(\lambda + A_\alpha)^{-\alpha}(A_\alpha)^\alpha = A^\alpha(\lambda + A)^{-\alpha}$ is an isomorphism of X . Our second main result now reads as follows.

THEOREM 8. *Let A be a densely defined R -sectorial operator in X with $0 \in \rho(A)$ and $\pi > \theta > \theta_r(A)$. Let $\alpha \in [0, 1]$ and assume that $B : X_\alpha \rightarrow X_{\alpha-1}$ is a linear operator satisfying*

$$(7) \quad \|(A_{\alpha-1})^{\alpha-1}BA^{-\alpha}\| \leq \eta$$

where $\eta < (\tilde{R}_A(\theta, 1 - \alpha)\tilde{R}_A(\theta, \alpha))^{-1}$ and $\tilde{R}_A(\theta, \beta)$ is defined in the following lemma. Then $C := (A_{\alpha-1} + B)|_X$, the part of $A_{\alpha-1} + B$ in X , is R -sectorial and

$$R_C(\theta) \leq \frac{R_A(\theta, \alpha)R_A(\theta, 1 - \alpha)}{1 - \eta\tilde{R}_A(\theta, \alpha)\tilde{R}_A(\theta, 1 - \alpha)},$$

in particular $\theta_r(C) \leq \theta$.

REMARK 9. The assumption (7) can be restated as a relative boundedness condition in $X_{\alpha-1}$:

$$(8) \quad \|Bx\|_{\alpha-1} \leq \eta\|Ax\|_{\alpha-1} \quad (x \in X_\alpha = D(A_{\alpha-1})),$$

but it is not clear, if the application of Theorem 1 would give a suitable operator in X . The following proof will avoid this difficulty. In Section 5 we give applications where the X_α are mostly Sobolev spaces and (8) becomes a natural estimate. For a non-negative selfadjoint operator A in a Hilbert space and $\alpha = 1/2$ the theorem is closely related to the well known KLMN perturbation theorem (see Remark 16 below).

In the proof we shall make use of the following lemma. Note that we do not assume $0 \in \rho(A)$ here.

LEMMA 10. Suppose that A is R -sectorial and that $\pi > \theta > \theta_r(A)$. Let $\alpha \in (0, 1)$. Then the following subsets of $L(X)$

$$\{\lambda^\alpha(\lambda + A)^{-\alpha} : \lambda \in \Sigma_{\pi-\theta}\} \quad \text{and} \quad \{A^\alpha(\lambda + A)^{-\alpha} : \lambda \in \Sigma_{\pi-\theta}\}$$

are R -bounded. The R -bounds of these sets are denoted $R_A(\theta, \alpha)$ and $\tilde{R}_A(\theta, \alpha)$, respectively.

PROOF. We fix $\pi - \theta_r(A) > \psi > \pi - \theta$ and use the representation formula (5) for $(\lambda + A)^{-\alpha}$, $\lambda \in \Sigma_{\pi-\theta}$ with $\varepsilon = |\lambda|/2$. By [4, Lemma 3.2] (the absolute convex hull of an R -bounded set is R -bounded), the first set in the assertion is R -bounded if we have proved that

$$(9) \quad \int_{\Gamma_{|\lambda|/2, \psi}} \left| \frac{\lambda^\alpha}{(\lambda - z)^\alpha z} \right| |dz| \leq C$$

where C does not depend on $\lambda \in \Sigma_{\pi-\theta}$.

Using the parametrization γ we have to estimate the integrals

$$\int_\varepsilon^\infty \frac{|\lambda|^\alpha}{|\lambda - te^{\pm i\psi}|^\alpha t} dt, \quad \int_{-\varepsilon}^\varepsilon \frac{|\lambda|^\alpha}{|\lambda - \varepsilon e^{i\psi t/\varepsilon}|^\alpha \varepsilon} dt$$

for $\varepsilon = |\lambda|/2$. Then $|\lambda - \varepsilon e^{i\psi t/\varepsilon}| \geq |\lambda|/2$ and we can estimate the integral $\int_{-\varepsilon}^\varepsilon \dots$ by

$$2\varepsilon \cdot |\lambda|^\alpha \cdot \frac{2^\alpha}{|\lambda|^\alpha \varepsilon} = 2^{\alpha+1}.$$

For the integral $\int_\varepsilon^\infty \dots$ we observe that

$$(10) \quad |\lambda - te^{\pm i\psi}| \geq c(\psi, \theta)(|\lambda| + t), \quad \lambda \in \Sigma_{\pi-\theta}, t \geq |\lambda|/2.$$

Indeed, we have $|\mu - se^{\pm i\psi}| \geq c_1(\psi, \theta)$ for all $\mu \in \Sigma_{\pi-\theta}$ with $|\mu| = 1$ and $s \geq 0$. On the other hand, $|\mu - se^{\pm i\psi}| \geq s - 1$ for the same range of μ and s . Multiplying with $|\lambda|$ yields (10) with $c(\psi, \theta) := c_1(\psi, \theta)/(2 + c_1(\psi, \theta))$. Hence we can estimate the integral $\int_\varepsilon^\infty \dots$ by

$$c(\psi, \theta)^{-\alpha} \int_{|\lambda|/2}^\infty \frac{|\lambda|^\alpha}{(|\lambda| + t)^\alpha t} dt = c(\psi, \theta)^{-\alpha} \int_{1/2}^\infty \frac{ds}{(1 + s)^\alpha s} ds =: C(\psi, \theta, \alpha),$$

which means that (9) holds with $C := 2C(\psi, \theta, \alpha) + \psi 2^{\alpha+1}$.

For the second set in the assertion we write

$$A^\alpha(\lambda + A)^{-\alpha} = (A^\alpha - (\lambda + A)^\alpha)(\lambda + A)^{-\alpha} + I =: B_\alpha(\lambda)(\lambda + A)^{-\alpha} + I$$

and use the representation

$$B_\alpha(\lambda) = -\alpha \int_0^1 \lambda(s\lambda + A)^{\alpha-1} ds.$$

This yields

$$\begin{aligned} A^\alpha(\lambda + A)^{-\alpha} &= I + \frac{B_\alpha(\lambda)}{\lambda^\alpha} \lambda^\alpha (\lambda + A)^{-\alpha} \\ &= I - \alpha \int_0^1 s^{\alpha-1} (s\lambda)^{1-\alpha} (s\lambda + A)^{\alpha-1} ds \cdot [\lambda^\alpha (\lambda + A)^{-\alpha}]. \end{aligned}$$

Noting $\alpha \int_0^1 s^{\alpha-1} ds = 1$ and using [4, Lemma 3.2] again we obtain

$$\tilde{R}_A(\theta, \alpha) \leq 1 + R_A(\theta, 1 - \alpha)R_A(\theta, \alpha)$$

which proves the second assertion. □

PROOF OF THEOREM 8. If $\lambda \in \Sigma_{\pi-\theta}$ then

$$\begin{aligned} &(\lambda + A_{\alpha-1})^{\alpha-1} B(\lambda + A)^{-\alpha} \\ &= (\lambda + A_{\alpha-1})^{\alpha-1} (A_{\alpha-1})^{1-\alpha} [(A_{\alpha-1})^{\alpha-1} B A^{-\alpha}] A^\alpha (\lambda + A)^{-\alpha} \\ &= A^{1-\alpha} (\lambda + A)^{\alpha-1} [(A_{\alpha-1})^{\alpha-1} B A^{-\alpha}] A^\alpha (\lambda + A)^{-\alpha}, \end{aligned}$$

which yields

$$R(\{(\lambda + A_{\alpha-1})^{\alpha-1} B(\lambda + A)^{-\alpha} : \lambda \in \Sigma_{\pi-\theta}\}) \leq \tilde{R}_A(\theta, 1 - \alpha)\eta \tilde{R}_A(\theta, \alpha).$$

By the assumption on η the operator $I + (\lambda + A_{\alpha-1})^{\alpha-1} B(\lambda + A)^{-\alpha}$ is invertible in $L(X)$ for $\lambda \in \Sigma_{\pi-\theta}$ and the series in

$$S_\lambda := (\lambda + A)^{-\alpha} \left(\sum_{k=0}^{\infty} \left(-(\lambda + A_{\alpha-1})^{\alpha-1} B(\lambda + A)^{-\alpha} \right)^k \right) (\lambda + A)^{\alpha-1}$$

converges. By the preceding it is clear that

$$R(\{\lambda S_\lambda : \lambda \in \Sigma_{\pi-\theta}\}) \leq R_A(\theta, \alpha) \frac{1}{1 - \tilde{R}_A(\theta, 1 - \alpha)\eta\tilde{R}_A(\theta, \alpha)} R_A(\theta, 1 - \alpha).$$

We have

$$S_\lambda^{-1} = (\lambda + A)^{1-\alpha} (I + (\lambda + A_{\alpha-1})^{\alpha-1} B(\lambda + A)^{-\alpha}) (\lambda + A)^\alpha$$

with domain the set of all $x \in D(A^\alpha)$ such that $y := (\lambda + A)^\alpha x + (\lambda + A_{\alpha-1})^{\alpha-1} Bx \in D(A^{1-\alpha})$, i.e. the set of all $x \in D(A^\alpha)$ such that $z := (\lambda + A_{\alpha-1})^{1-\alpha} y \in X$. But $z = (\lambda + A_{\alpha-1})x + Bx$ and, on the other hand, $z = S(\lambda)^{-1}x$. Hence $D(S_\lambda^{-1}) = \{x \in D(A^\alpha) : A_{\alpha-1}x + Bx \in X\}$ and $S_\lambda^{-1} = \lambda + (A_{\alpha-1} + B)|_X$ which finishes the proof. \square

REMARK 11. As in Section 2, the assertion of Theorem 8 yields a perturbation theorem for maximal L_p -regularity if X is a UMD-space.

We can allow for perturbations B which are arbitrarily large in norm if they are bounded $X_\alpha \rightarrow X_{\beta-1}$ for some $0 \leq \alpha < \beta \leq 1$. For $\beta = 1$ and $\alpha < 1$, this corresponds to a well known perturbation theorem for generators of analytic semigroups and extends it to maximal regularity in UMD-spaces (for this special case see also [8, Thm. 6.2]).

COROLLARY 12. *Let A be an R -sectorial operator in X with $\pi > \theta > \theta_r(A)$. Let $0 \leq \alpha < \beta \leq 1$ and assume that $B : X_\alpha \rightarrow X_{\beta-1}$ is a bounded linear operator. Then, for $\alpha \leq \gamma \leq \beta$, the operator $(A_{\gamma-1} + B)|_X + \lambda$ is R -sectorial with angle $\leq \theta$ if $\lambda > 0$ is sufficiently large.*

PROOF. Without loss of generality we assume that $0 \in \rho(A)$. If $\alpha \leq \gamma \leq \beta$, then B is bounded $X_\gamma \rightarrow X_{\gamma-1}$. We write, for $\lambda > 0$,

$$(\lambda + A_{\gamma-1})^{\gamma-1} B(\lambda + A)^{-\gamma} = (\lambda + A)^{-(\beta-\gamma)} [(\lambda + A_{\beta-1})^{-(1-\beta)} (A_{\beta-1})^{1-\beta}]$$

$$[(A_{\beta-1})^{\beta-1} B A^{-\alpha}] [A^\alpha (\lambda + A)^{-\alpha}] (\lambda + A)^{-(\gamma-\alpha)}$$

and observe $(\lambda + A_{\beta-1})^{-(1-\beta)} (A_{\beta-1})^{1-\beta} = A^{1-\beta} (\lambda + A)^{1-\beta}$. If we denote, for $\sigma \in [0, 1]$, the norm bounds of the (by Lemma 10 bounded) sets $\{\lambda^\sigma (\lambda + A)^{-\sigma} : \lambda > 0\}$ and $\{A^\sigma (\lambda + A)^{-\sigma} : \lambda > 0\}$ by $M(\sigma)$ and $\tilde{M}(\sigma)$, respectively, this implies the estimate

$$\begin{aligned} & \|(\lambda + A_{\gamma-1})^{\gamma-1} B(\lambda + A)^{-\gamma}\| \\ & \leq \lambda^{\gamma-\beta} M(\beta - \gamma) \tilde{M}(1 - \beta) \|A^{\beta-1} B A^{-\alpha}\| \tilde{M}(\alpha) \lambda^{\alpha-\gamma} M(\gamma - \alpha) \\ & = \lambda^{\alpha-\beta} c(\alpha, \beta, \gamma, B). \end{aligned}$$

Choosing $\lambda > 0$ sufficiently large we can apply Theorem 8 for $\lambda + A$ in place of A .

We now want an analogue of Corollary 2, and we want to get rid of the assumption $0 \in \rho(A)$. It is well known and easy to see that, for $\alpha = 0$ the assumption (7) can be reformulated in terms of the dual operators A' and B' in X' , in a way that avoids the assumption $0 \in \rho(A)$ and is similar to the assumption in Theorem 1. The case $\alpha \in (0, 1)$ is more complicated and we shall make use of the following remark, assuming from now on in addition that X is reflexive. Since our main interest are perturbation theorems for maximal L_p -regularity and our method thus requires X to be a UMD-space which is always reflexive, this restriction is not essential.

Remark 13. If the space X is reflexive we may obtain the spaces $X_{-\alpha}$ for $\alpha \in (0, 1]$ also via the following approach: Denoting the dual space of $(X, \|\cdot\|)$ by $(X', |\cdot|)$ and the dual operator of A by A' (which is again sectorial) we let $(X')_\alpha := D((A')^\alpha)$ with the graph norm (again, one may take any norm $|(\lambda + A')^\alpha \cdot|$, $\lambda \in \Sigma_{\pi-\theta}$, instead). Then it is not hard to see that $X_{-\alpha} = ((X')_\alpha)'$ for the duality pairing (X', X) (we again refer to [1, Ch. V]).

A bounded linear operator $B : X_\alpha \rightarrow X_{\alpha-1}$ then corresponds to the bounded bilinear form $[D(A^\alpha)] \times [D((A')^{1-\alpha})] \rightarrow \mathbb{C}$, $(x, x') \mapsto \langle Bx, x' \rangle$ where both spaces are equipped with the graph norm.

We thus are led to the following version of Theorem 8.

THEOREM 14. *Let A be a densely defined R -sectorial operator in a reflexive space X with $\pi > \theta > \theta_r(A)$. Let A be injective with dense range and $\alpha \in [0, 1]$. Assume that $B : X_\alpha \rightarrow X_{\alpha-1}$ is a linear operator satisfying*

$$(11) \quad |\langle Bx, x' \rangle| \leq \eta \|A^\alpha x\| \|(A')^{1-\alpha} x'\| \quad (x \in D(A^\alpha), x' \in D((A')^{1-\alpha}))$$

where $\eta < (\tilde{R}_A(\theta, 1-\alpha)\tilde{R}_A(\theta, \alpha))^{-1}$. Then $C := (A_{\alpha-1} + B)|_X$ is R -sectorial and

$$R_C(\theta) \leq \frac{R_A(\theta, \alpha)R_A(\theta, 1-\alpha)}{1 - \eta\tilde{R}_A(\theta, \alpha)\tilde{R}_A(\theta, 1-\alpha)},$$

in particular $\theta_r(C) \leq \theta$.

PROOF. We have to work with bilinear forms. For any $\lambda \in \Sigma_{\pi-\theta}$, the bilinear form $(x, x') \mapsto \langle B(\lambda + A)^{-\alpha}x, (\lambda + A')^{\alpha-1}x' \rangle$ which corresponds to the operator $K(\lambda) := (\lambda + A_{\alpha-1})^{\alpha-1}B(\lambda + A)^{-\alpha} : X \rightarrow X$ is bounded on $X \times X'$ with norm $\leq \eta\tilde{M}_A(\theta, \alpha)\tilde{M}_A(\theta, 1-\alpha) < 1$ (observe $\tilde{M}_{A'}(\dots) = \tilde{M}_A(\dots)$).

Next we observe that the assumptions imply that A^{-1} is a densely defined R -sectorial operator which has, together with its dual, fractional powers satisfying $A^\alpha(A^{-1})^\alpha x = x$ for $x \in D((A^{-1})^\alpha)$ and $(A')^{1-\alpha}((A^{-1})')^{1-\alpha}x' = x'$ for $x' \in D(((A^{-1})')^{1-\alpha})$. We write $A^{-\alpha}$ and $(A')^{\alpha-1}$ for $(A^{-1})^\alpha$ and $((A^{-1})')^{1-\alpha}$, respectively. The assumption then yields that the bilinear form $(x, x') \mapsto \langle BA^{-\alpha}x, (A')^{\alpha-1}x' \rangle$ which is originally defined on the dense subset $D(A^{-\alpha}) \times D((A')^{\alpha-1})$ is bounded on $X \times X'$ with norm $\leq \eta$. We denote the corresponding operator $X \rightarrow X'' = X$ by K .

Now $K(\lambda) = A^{1-\alpha}(\lambda + A)^{\alpha-1} K A^\alpha (\lambda + A)^{-\alpha}$ since we have for $x \in X$ and $x' \in D((A')^{1-\alpha})$ that

$$\begin{aligned} & \langle A^{1-\alpha}(\lambda + A)^{\alpha-1} K A^\alpha (\lambda + A)^{-\alpha} x, x' \rangle \\ &= \langle K A^\alpha (\lambda + A)^{-\alpha} x, (A')^{1-\alpha} (\lambda + A')^{\alpha-1} x' \rangle \\ &= \langle B(\lambda + A)^{-\alpha} x, (\lambda + A')^{\alpha-1} x' \rangle. \end{aligned}$$

Hence $R(\{K(\lambda) : \lambda \in \Sigma_{\pi-\theta}\}) \leq \tilde{R}_A(\theta, \alpha) \eta \tilde{R}_A(\theta, 1-\alpha)$. We now proceed as in the proof of Theorem 8 letting

$$S_\lambda := (\lambda + A)^{-\alpha} \left(\sum_{k=0}^{\infty} (-K(\lambda))^k \right) (\lambda + A)^{\alpha-1}$$

and observing $S_\lambda^{-1} = \lambda + (A_{\alpha-1} + B)|_X$. □

This formulation allows for the following corollary of Theorem 8 which is an analogue of Corollary 2. By $M_A(\theta, \alpha)$ and $\tilde{M}_A(\theta, \alpha)$ we denote the uniform norm-bounds of the sets in Lemma 10.

COROLLARY 15. *Let A be a densely defined R -sectorial operator in a reflexive space X with $\pi/2 \geq \theta > \theta_r(A)$. Let $\alpha \in [0, 1]$. Assume that $B : X_\alpha \rightarrow X_{\alpha-1}$ is a linear operator satisfying*

$$(12) \quad \begin{aligned} | \langle Bx, x' \rangle | &\leq a(\|A^\alpha x\| + b\|x\|)(\|(A')^{1-\alpha} x'\| + b\|x'\|) \\ &(x \in D(A^\alpha), x' \in D((A')^{1-\alpha})) \end{aligned}$$

for some $a, b \geq 0$ satisfying $a < (\tilde{M}_A(\theta, \alpha) \tilde{R}_A(\theta, \alpha) \tilde{M}_A(\theta, 1-\alpha) \tilde{R}_A(\theta, 1-\alpha))^{-1}$. Then $(A_{\alpha-1} + B)|_X + \lambda$ is R -sectorial with R -sectoriality angle $\leq \theta$ for $\lambda > 0$ sufficiently large.

PROOF. If $\lambda > 0$ the assumption (12) implies for all $x \in X, x' \in X'$

$$\begin{aligned} & | \langle (\lambda + A)^{\alpha-1} B(\lambda + A)^{-\alpha} x, x' \rangle | \\ &\leq a(\|A^\alpha (\lambda + A)^{-\alpha} x\| + b\|(\lambda + A)^{-\alpha} x\|)(\|(A')^{1-\alpha} (\lambda + A')^{\alpha-1} x'\| \\ &\quad + b\|(\lambda + A')^{\alpha-1} x'\|) \\ &\leq a(\tilde{M}_A(\theta, \alpha) + \frac{b}{\lambda^\alpha} M_A(\theta, \alpha))(\tilde{M}_A(\theta, 1-\alpha) \\ &\quad + \frac{b}{\lambda^{1-\alpha}} M_A(\theta, 1-\alpha))\|x\| \|x'\| \\ &=: \eta(\lambda)\|x\| \|x'\|. \end{aligned}$$

Here we used that $M_{A'}(\dots) = M_A(\dots)$. By the assumption on a we find λ large such that

$$a \tilde{M}_A(\theta, \alpha) \tilde{M}_A(\theta, 1-\alpha) < \eta(\lambda) < (\tilde{R}_A(\theta, \alpha) \tilde{R}_A(\theta, 1-\alpha))^{-1}.$$

Now we apply Theorem 8 with $\lambda + A$ in place of A and $\eta := \eta(\lambda)$ (observe that $R_{A+\lambda}(\dots) \leq R_A(\dots)$ and $\tilde{R}_{A+\lambda}(\dots) \leq \tilde{R}_A(\dots)$). □

We now show that our perturbation method extends the KLMN perturbation theorem for forms in Hilbert spaces.

REMARK 16. If A is a non-negative selfadjoint operator in a Hilbert space H , then $A = A'$, and $M_A(\pi/2, \alpha) = \tilde{M}_A(\pi/2, \alpha) = R_A(\pi/2, \alpha) = \tilde{R}_A(\pi/2, \alpha) = 1$ for all $\alpha \in [0, 1]$. A symmetric quadratic form β in H with form domain $D(\beta) \supset D(A^{1/2}) =: H_{1/2}$ corresponds to a symmetric operator $B : H_{1/2} \rightarrow H_{-1/2}$ (symmetric in the sense that $(H_{\pm 1/2})' = H_{\mp 1/2}$ and the adjoint $B' : H_{1/2} \rightarrow H_{-1/2}$ equals B). The assumption of the KLMN theorem ([20, Thm. X.17]) then reads

$$|(Bx, x)| \leq a' \|A^{1/2}x\|^2 + b' \|x\|^2 \quad (x \in D(A^{1/2}))$$

for some $a' < 1$. By the argument that proved Corollary 15 we obtain

$$|((\lambda + A)^{-1/2} B (\lambda + A)^{-1/2} x, x)| \leq \eta$$

for some $\eta < 1$. Since the bounded operator $(\lambda + A)^{-1/2} B (\lambda + A)^{-1/2}$ is symmetric the assumption of Theorem 8 holds for $\lambda + A$ in place of A and λ large. Thus we obtain that $C := (A_{1/2} + B)|_H$ generates an analytic semigroup of angle $\leq \pi/2$. By the symmetry of $A_{1/2}$ and B it is easy to see that C is symmetric. Hence C is selfadjoint. The KLMN theorem also gives $D((C + b')^{1/2}) = D(A^{1/2})$ which is more than one can hope for in the general situation, even in Hilbert space (cf. [24, Sect. 4]).

REMARK 17. We want to emphasize that the results of this section also hold if we require A merely to be sectorial, take $\pi > \theta > \theta_\sigma(A)$, replace $R_A(\dots)$, $\tilde{R}_A(\dots)$ by $M_A(\dots)$, $\tilde{M}_A(\dots)$, respectively, in the assumptions, the conclusion being sectoriality of the perturbed operator (instead of R -sectoriality) of angle $\theta_\sigma \leq \theta$ (we get an estimate for $M_C(\theta)$ if we replace the $R_\star(\dots)$ -constants in the estimate for $R_C(\theta)$ by the corresponding $M_\star(\dots)$ -constants).

We could not find those results in the literature although an argument of this kind had implicitly been used in [24, p. 223].

5. – Applications to partial differential equations

5.1. – Schrödinger operators

There is an extensive literature on operators of the form $A := H = -\Delta + V$ where V is a potential. Usually, H is defined in $L_2(\mathbb{R}^N)$ by a form perturbation of the form $q(u, v) := \int \overline{\nabla v}^t \nabla u$ with form domain $W_2^1(\mathbb{R}^N)$, associated with $-\Delta$.

For certain potentials V one may check by the Beurling-Deny criteria that e^{-tH} is a positive semigroup of contractions in each $L_p(\mathbb{R}^N)$, $1 \leq p \leq \infty$,

(w^* -continuous for $p = \infty$). If we denote its generator by A_p , then A_p has maximal regularity in $L_p(\mathbb{R}^N)$, $1 < p < \infty$, by a result due to Lambertson ([14]). For a larger class of potentials V it is known that the operators e^{-tH} are integral operators satisfying Gaussian type bounds. Then, for some $\mu \geq 0$, $A + \mu$ has maximal regularity in $L_p(\mathbb{R}^N)$, $1 < p < \infty$, by a result due to Hieber and Prüss ([11], see also [23]).

These results cannot be applied for bad potentials V for which the semigroup e^{-tH} acts boundedly in $L_p(\mathbb{R}^N)$ only for $p_0 < p < p'_0$ where $p_0 \in (1, 2)$ (see, e.g., [16]). Theorem 8 enables us to treat those situations directly in the corresponding L_p -spaces. We give a simple concrete example.

EXAMPLE. Let $N \geq 3$. The operator $A := -\Delta$ has domain $D(A) = W_p^2(\mathbb{R}^N)$ and $D(A^{1/2}) = W_p^1(\mathbb{R}^N)$. Moreover $A' = -\Delta$ in $L_{p'}(\mathbb{R}^N)$, $D(A') = W_{p'}^2(\mathbb{R}^N)$, and $D((A')^{1/2}) = W_{p'}^1(\mathbb{R}^N)$. We consider the potential $V(x) := C|x|^{-2}$ (cf. [16]).

By Hardy’s inequality we know

$$(13) \quad \left\| \frac{f}{|x|} \right\|_q \leq C_q \|\nabla f\|_q \quad (f \in C_c^\infty(\mathbb{R}^N \setminus \{0\}))$$

for $1 < q < \infty$. If $N/(N - 1) < p < N$ then $C_c^\infty(\mathbb{R}^N \setminus \{0\})$ is dense in $W_p^1(\mathbb{R}^N)$ and $W_{p'}^1(\mathbb{R}^N)$, and by (13) for $q = p, p'$, Hölder’s inequality and the boundedness of Riesz transforms we obtain for $f \in W_p^1(\mathbb{R}^N)$, $g \in W_{p'}^1(\mathbb{R}^N)$

$$\begin{aligned} \left| \left\langle \frac{1}{|x|^2} f, g \right\rangle \right| &= \left| \left\langle \frac{f}{|x|}, \frac{g}{|x|} \right\rangle \right| \leq \left\| \frac{f}{|x|} \right\|_p \left\| \frac{g}{|x|} \right\|_{p'} \\ &\leq C_p C_{p'} \|\nabla f\|_p \|\nabla g\|_{p'} \leq C \|A^{1/2} f\|_p \|(A')^{1/2} g\|_{p'}, \end{aligned}$$

which means that $|x|^{-2}$ defines a bounded multiplication operator from $W_p^1(\mathbb{R}^N)$ to $(W_{p'}^1(\mathbb{R}^N))' = W_p^{-1}(\mathbb{R}^N)$. Hence, if $|C|$ is small enough, we obtain maximal regularity for the operator $(A_{-1/2} + C|x|^{-2})|_{L_p}$, i.e. for $-\Delta + C|x|^{-2}$. Depending on the size of $C < 0$ and the dimension N the corresponding semigroup only acts in $L_p(\mathbb{R}^N)$ for p in a (small) symmetric interval around 2 (cf. [16], p. 181). If C is not real we cannot apply the result in [23, Sect. 4 d)] which requires the semigroup to be positive and contractive in $L_p(\mathbb{R}^N)$.

The situation is similar for more general second order operators in divergence form when the lower order terms satisfy form bounds and the corresponding semigroups do not act in all L_p -spaces, $1 \leq p \leq \infty$, see [15], [16], [17]. The semigroups are positive for real potentials but may not be quasi-contractive for certain p -intervals.

5.2. – Higher order elliptic operators

Elliptic operators of order $2m$ with $m > 1$ defined on \mathbb{R}^N by the form method (as studied, e.g., in [6]) do not give rise to positive semigroups nor to contraction semigroups.

If $N > 2m$ and the highest order coefficients do not satisfy further regularity properties then the corresponding semigroup does not act in all $L_p(\mathbb{R}^N)$, $1 \leq p \leq \infty$, see [7]. As pointed out for Schrödinger semigroups in the previous subsection, this may also be achieved by ‘bad’ lower order perturbations. We give a concrete example of a fourth order operator and a perturbation which cannot be treated as a form perturbation on the form domain.

EXAMPLE. Consider the operator $A := \Delta^2$ which has domain $W_p^4(\mathbb{R}^3)$ in $X := L_p(\mathbb{R}^3)$. We obtain $X_\alpha = W_p^{4\alpha}(\mathbb{R}^3)$. The form associated with A in $L_2(\mathbb{R}^3)$ is $q(f, g) := \int \Delta f \overline{\Delta g}$ with form domain $W_2^2(\mathbb{R}^3)$. Now consider the lower order perturbation $B := C|x|^{-2}(\Delta + 1)$ where C is a constant As before, the function $|x|^{-2}$ acts boundedly $W_p^1(\mathbb{R}^3) \rightarrow W_p^{-1}(\mathbb{R}^3)$ for $3/2 < p < 3$, and $\Delta + 1 : W_p^3(\mathbb{R}^N) \rightarrow W_p^1(\mathbb{R}^N)$ is an isomorphism. Hence, for $|C|$ small enough, we can apply Theorem 8 with $\alpha = 3/4$ and obtain maximal regularity for the operator $(A_{-1/4} + B)|_{L_p(\mathbb{R}^3)}$. For $p = 2$ this cannot be obtained by a perturbation on the form domain since $B : W_2^2(\mathbb{R}^3) \rightarrow W_2^{-2}(\mathbb{R}^3)$ is not bounded. Indeed, if $|x|^{-2}$ acts boundedly $W_{p'}^2(\mathbb{R}^3) \rightarrow L_{p'}(\mathbb{R}^3)$ (which by duality is equivalent to a bounded action $L_p(\mathbb{R}^3) \rightarrow W_p^{-2}(\mathbb{R}^3)$) then $p' < 3/2$ since $|x|^{-2}$ does not belong to $L_{p',loc}(\mathbb{R}^3)$ for other values of p' . Since we are now in a Hilbert space the main assertion is, of course, that the perturbed operator generates an analytic semigroup.

5.3. – Perturbation of boundary conditions

For $\alpha = 0$ in Theorem 8 we have $B : X \rightarrow X_{-1}$ and it is known that operators of this kind may be used to perturb the boundary conditions of a generator (cf. [10, Sect. III.3.a] on Desch-Schappacher perturbations). We show by a simple example that the consideration of $\alpha \in (0, 1)$ in Theorem 8 allows for more general perturbations of boundary conditions.

EXAMPLE. Let $1 < p < \infty$ and $1 > \delta > 1 - 1/p$. We want to show that a suitable translate of $-\Delta$ on $L_p(-1, 1)$ with boundary conditions $f(\pm 1) = \int_{-1}^1 \phi_\delta f dx$ where $\phi_\delta(x) := (x^2 - 1)^{-\delta}$ has maximal L_p -regularity.

To this end let $-A$ be the Laplace operator in $X = L_p(-1, 1)$ with Dirichlet boundary conditions, i.e. $D(A) = \{u \in W_p^2(-1, 1) : u(\pm 1) = 0\}$. Then A has bounded imaginary powers, the domains of the fractional powers of A can be obtained by complex interpolation, and due to a result of Seeley ([21]) we have

$$X_\alpha = \begin{cases} W_p^{2\alpha}(-1, 1) & 2\alpha - 1/p < 0 \\ \{u \in W_p^{2\alpha}(-1, 1) : u(\pm 1) = 0\} & 2\alpha - 1/p > 0 \end{cases}$$

Hence in particular $1 \in X_\alpha$ for $\alpha < (2p)^{-1}$ and $1 \notin X_\alpha$ for $\alpha > (2p)^{-1}$. Now choose β, s, r such that $1/p > 2\beta > s > 1/p + 1/r - 1 > 1/p + \delta - 1$. Then the function $\phi_\delta(x) = (x^2 - 1)^{-\delta}$ belongs to $L_r(-1, 1)$ but not to $L_{r'}(-1, 1) = X'$. By Sobolev embedding we have $W_p^s(-1, 1) \hookrightarrow L_{r'}(-1, 1)$, hence $L_r(-1, 1) \hookrightarrow (W_p^s(-1, 1))'$. This means that the operator $f \mapsto Bf := -\langle \phi_\delta, f \rangle A_{\beta-1} 1$ is bounded $W_p^s(-1, 1) \rightarrow X_{\beta-1}$. Letting $\alpha := s/2$ and recalling $s < 1/p$ we see that $B : X_\alpha \rightarrow X_{\beta-1}$ is bounded. Since $0 < \alpha < \beta$ Corollary 12 applies and we obtain maximal regularity for a suitable translate of the operator $A_B := (A_{\beta-1} + B)|_X$ which has domain

$$\begin{aligned} D(A_B) &= \{f \in X_\beta : A_{\beta-1}(f - \langle \phi_\delta, f \rangle 1) \in X\} \\ &= \{f \in W_p^{2\beta}(-1, 1) : f - \langle \phi_\delta, f \rangle 1 \in D(A)\} \\ &= \left\{ f \in W^2(-1, 1) : f(\pm 1) = \int_{-1}^1 \phi_\delta f \, dx \right\}. \end{aligned}$$

For $f \in D(A_B)$ we have

$$\begin{aligned} A_B f &= A_{\beta-1}(f - \langle \phi_\delta, f \rangle 1) = A(f - \langle \phi_\delta, f \rangle 1) \\ &= -\Delta(f - \langle \phi_\delta, f \rangle 1) = -\Delta f. \end{aligned}$$

Hence $A_B = -\Delta$ with domain $D(A_B)$. Note that the perturbation B is not bounded $X \rightarrow X_{-1}$ since $f \mapsto \langle \phi_\delta, f \rangle$ is not bounded on $L_p(-1, 1)$.

Of course, we may take any other function $\phi \in L_r(-1, 1)$ instead of ϕ_δ and argue in the same way with $1/p > 2\beta > s > 1/p + 1/r - 1$. Moreover, if $\phi, \psi \in L_r(-1, 1)$, the same argument applies to

$$Bf := -A_{\beta-1}(\langle \phi, f \rangle (1 - (\cdot)) / 2 + \langle \psi, f \rangle ((\cdot) + 1) / 2),$$

the resulting domain being $D(A_B) = \{f \in W_p^2(-1, 1) : f(1) = \langle \psi, f \rangle, f(-1) = \langle \phi, f \rangle\}$.

5.4. – Pseudodifferential operators with non-smooth symbols

Our perturbation theorem may be applied to pseudo-differential operators with non-smooth and unbounded symbols.

EXAMPLE. Again, let us take $A := -\Delta$ in $L_p(\mathbb{R}^N)$ with domain $W_p^2(\mathbb{R}^N)$ for simplicity. Then, the domains of the fractional powers A^α of A are the Bessel potential spaces $H_p^{2\alpha}(\mathbb{R}^N)$, $\alpha \in \mathbb{R}$ (cf. [1]), and $A' = -\Delta$ in $L_{p'}(\mathbb{R}^N)$.

If T is a classical pseudo-differential operator of order $m \in [0, 2]$ (e.g. with C^∞ -symbol in the class $S_{\rho, \delta}^m$) then T acts bounded $H_p^{s+m}(\mathbb{R}^N) \rightarrow H_p^s(\mathbb{R}^N)$ for all $s \in \mathbb{R}$, cf. [12]. We combine such an operator T with multiplication

operators $M_\gamma := \gamma \cdot : H_q^k(\mathbb{R}^N) \rightarrow L_q(\mathbb{R}^N)$ where $k \in [0, \infty)$ and $q \in \{p, p'\}$, i.e. we consider operators $T \circ M_\gamma : H_p^k \rightarrow H_p^{-m}$ or $M_\gamma \circ T : H_p^m \rightarrow H_p^{-k}$ (recall that, by duality, the multiplication operator M_γ is bounded $H_{p'}^k \rightarrow L_{p'}$ if and only if it is bounded $L_p \rightarrow H_p^{-k}$; for boundedness criteria for M_γ see the remark below).

Now let $S \in \{T \circ M_\gamma, M_\gamma \circ T\}$. If $m + k < 2$ then $A + S - \nu$ is R -sectorial for some $\nu \geq 0$, if $m + k = 2$ then $A + S - \nu$ is R -sectorial for some $\nu \geq 0$ if the norm of the operator S is sufficiently small $H_p^k \rightarrow H_p^{-m}$ or $H_p^m \rightarrow H_p^{-k}$, respectively.

We remark that we could in the same way consider operators of the form $S = M_{\gamma_2} \circ T \circ M_{\gamma_1} : H_p^m \rightarrow H_p^{-k}$ where T is a pseudo-differential operator of order 0 and $M_{\gamma_1} : H_p^m \rightarrow L_p, M_{\gamma_2} : H_{p'}^k \rightarrow L_{p'}$ are bounded multiplication operators.

REMARK 18. The multipliers $H_q^k \rightarrow L_q$ have been characterized in [18] in terms of certain capacities (cf. [18, p. 59]):

$$\|M_\gamma\|_{H_q^k \rightarrow L_q} \sim \sup_{E \text{ cpt}} \frac{\|\gamma\|_{L_q(E)}}{(\text{cap}(E, H_q^k))^{1/q}} \sim \sup_{E \text{ cpt}, d(E) \leq 1} \frac{\|\gamma\|_{L_q(E)}}{(\text{cap}(E, H_q^k))^{1/q}}$$

where the supremum is taken over all compact subsets E of \mathbb{R}^N with positive capacity, $d(E)$ denotes the diameter of the set E , and the capacity is defined by (cf. [18, p. 52]):

$$\text{cap}(E, H_q^k) := \inf\{\|u\|_{H_q^k}^q : u \in C_c^\infty, u \geq 1 \text{ on } E\}.$$

Recall that $\|u\|_{H_q^k} := \|\Lambda^k u\|_{L_q}$ where $\Lambda^k := (1 - \Delta)^{k/2} = \mathcal{F}^{-1}(1 + |\xi|^2)^{k/2} \mathcal{F}$ and \mathcal{F} denotes the Fourier transform.

There are several sufficient conditions implying boundedness of $M_\gamma : H_q^k \rightarrow L_q$ for a given measurable function γ : [18, Prop. 2 on p. 52] implies that we have for $kq < N$

$$\sup_{E \text{ cpt}, d(E) \leq 1} \frac{\|\gamma\|_{L_q(E)}}{(\text{cap}(E, H_q^k))^{1/q}} \leq c \sup_{E \text{ cpt}, d(E) \leq 1} |E|^{k/N-1/q} \|\gamma\|_{L_q(E)};$$

[18, Prop. 1 (ii) on p. 73] states that $\gamma \in L_{N/k, \text{unif}}$ is sufficient if $kq < N$; and [18, Thm. 2.3.3 (ii), p. 79] states that $\gamma \in B_{r, \infty, \text{unif}}^\mu$ is sufficient if $r \geq q, kq < N$ and $\mu := N/q - k \notin \mathbb{N}$. We also want to mention that, for $kq > N$, the space of multipliers $M_\gamma : H_q^k \rightarrow L_q$ coincides with $L_{q, \text{unif}}$, cf. [18, Remark 3 on p. 56].

REFERENCES

- [1] H. AMANN, “Linear and quasilinear parabolic problems”, Vol. 1: Abstract linear theory, Birkhauser, Basel, 1995.
- [2] H. AMANN – M. HIEBER – G. SIMONETT, *Bounded H_∞ -calculus for elliptic operators*, Differential Integral Equations **7** (1994), 613-653.
- [3] PH. CLÉMENT – S. LI, *Abstract parabolic quasilinear equations and application to a groundwater flow problem*, Adv. Math. Sci. Appl. **3** (1994), 17-32.
- [4] PH. CLÉMENT – B. DE PAGTER – F. A. SUKOCHEV – H. WITVLIET, *Schauder Decomposition and Multiplier Theorems*, Studia Math. **138** (2000), 135-163.
- [5] PH. CLÉMENT – J. PRÜSS, *An operator-valued transference principle and maximal regularity on vector-valued L_p -spaces*, in Proc. 6th Int. Conf. on Evolution Equations, Bad Herrenalb, 1998; Marcel Dekker 2001, 67-87.
- [6] E. B. DAVIES, *Uniformly elliptic operators with measurable coefficients*, J. Funct. Anal. **132** (1995), 141-169.
- [7] E. B. DAVIES, *Limits on L^p regularity of self-adjoint elliptic operators*, J. Differential Equations **135** (1997), 83-102.
- [8] G. DORE, *Maximal regularity in L_p -spaces for an abstract Cauchy problem*, to appear in Adv. Differential Equations.
- [9] X. T. DUONG – G. SIMONETT, *H_∞ -calculus for elliptic operators with nonsmooth coefficients*, Differential Integral Equations **10** (1997), 201-217.
- [10] K.-J. ENGEL – R. NAGEL, “One-parameter semigroups for linear evolution equations, Graduate Texts in Mathematics 194”, Springer, Berlin, 2000.
- [11] M. HIEBER – J. PRÜSS, *Heat kernels and maximal L^p - L^q estimates for parabolic evolution equations*, Commun. Partial Differential Equations **22** (1997), 1647-1669.
- [12] L. HÖRMANDER, “The Analysis of Linear Partial Differential Operators”, Vol. III, Springer, 1985.
- [13] P. C. KUNSTMANN, *Maximal L_p -regularity for second order elliptic operators with uniformly continuous coefficients on domains*, submitted.
- [14] D. LAMBERTON, *Equations d'évolution linéaires associées a des semi-groupes de contractions dans les espaces L^p* , J. Funct. Anal. **72** (1987), 252-262.
- [15] V. A. LISKEVICH, *On C_0 -semigroups generated by elliptic second order differential expressions on L^p -spaces*, Differential Integral Equations **9** (1996), 811-826.
- [16] V. A. LISKEVICH – YU. A. SEMENOV, *Some problems on Markov semigroups*, in Demuth, Michael (ed.) et al., Schrödinger operators, Markov semigroups, wavelet analysis, operator algebras, Berlin, Akademie Verlag, (1996), 163-217.
- [17] V. A. LISKEVICH – H. VOGT, *On L^p -spectra and essential spectra of second order elliptic operators*, Proc. London Math. Soc. (3) **80** (2000), 590-610.
- [18] V. G. MAZ'YA – T. O. SHAPOSHNIKOVA, “Theory of multipliers in spaces of differentiable functions”, Pitman, Boston, London, Melbourne, 1985.
- [19] J. PRÜSS – H. SOHR, *Imaginary powers of elliptic second order differential operators in L^p -spaces*, Hiroshima Math. J. **23** (1993), 161-192.
- [20] M. REED – B. SIMON, “Methods of modern mathematical physics II”, Academic Press, New York, San Francisco, London, 1975.
- [21] R. SEELEY, *Interpolation in L^p with boundary conditions*, Studia Math. **44** (1972), 47-60.

- [22] L. WEIS, *Operator-valued Fourier Multiplier Theorems and Maximal L_p -Regularity*, to appear in *Math. Ann.*
- [23] L. WEIS, *A new approach to maximal L_p -regularity*, in *Proc. 6th Int. Conf. on Evolution Equations*, Bad Herrenalb, 1998; Marcel Dekker 2001, 195–214.
- [24] A. YAGI, *Applications of the purely imaginary powers of operators in Hilbert spaces*, *J. Funct. Anal.* **73** (1987), 216–231.

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