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Minimum cost spanning tree problems with groups*

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Abstract

We study minimum cost spanning tree problems with groups. We assume that agents are located in different villages. We introduce a rule for dividing the cost of connecting all agents to the source among the agents taking into account the group structure. We characterize this rule. We prove that the rule coincides with the Owen value of the TU game associated with the irreducible matrix.

1 Introduction

In this paper we study minimum cost spanning tree problems (*mcstp*). A group of agents (denoted by N), located at different geographical places, want a particular service which can only be provided by a common supplier, called the source (denoted by 0). Agents will be served through connections which involves some cost. Although, they do not care whether they are connected directly or indirectly to the source. This situation is described by a symmetric matrix C, where c_{ij} denotes the connection costs between i and j $(i, j \in N \cup \{0\})$.

There are many economic situations that can be modeled in this way. For instance, several towns may draw power from a common power plant, and hence have to share the cost of the distribution network (Dutta and Kar, 2004). Bergantiños and Lorenzo (2004, 2005) study a real situation where villagers had to pay the cost of constructing pipes from their respective houses to a water supplier. Other examples include communication networks, such as telephone, internet, or cable television.

We assume that agents construct a minimum cost spanning tree (*mt*). The question is how to divide the cost associated with the *mt* between the agents. Different rules give different answers to this question. One of the most important topics is the axiomatic characterization of rules. The idea is to propose desirable properties and to find out which of them characterize each rule. Properties often

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help agents to compare different rules and to decide which rule is preferred in a particular situation.

In some cases, as in Dutta and Kar (2004) or Bergantiños and Lorenzo (2004, 2005), agents are located in different villages. This means, in terms of the cost matrix, that the connection cost between two agents of the same village is not larger than the connection cost between an agent of this village and an agent from other village.

The classical model of *mcstp*, as described above, can also model these situation. Nevertheless, it ignores the fact that some group of agents are located in the same city or village. It could be interesting to include this fact in the model. We do it by considering an extra element in the model. Namely a partition $G = \left\{G^1, ..., G^m\right\}$ of the set of agents N. For each $k = 1, ..., m$, G^k represents the group of agents located in the same village, city, ...

In this paper we follow the axiomatic approach and we introduce a rule as the unique rule satisfying a set of desirable properties. Our idea is to generalize the axiomatic characterization of the rule φ given by Bergantiños and Vidal-Puga (2007d), which involves three properties. Restricted Additivity (RA) which says that the rule must be additive on the cost matrix. Population Monotonicity (PM), which says that if a new agent comes, no agent of the initial society can be worse off. Symmetry (SYM), which says that symmetric agents (with respect to the cost matrix) must pay the same.

We adapt these properties to *mcstp* with groups. The property of RA could be formulated in a similar way. Nevertheless, PM and SYM should be adapted. The main idea for adapting each of these properties is claiming both twice. Once among the groups and other among agents inside the same group.

In order to adapt these properties a question comes to our mind. The cost paid by agents of a village should depend on the internal characteristics of the other village? For instance, should this cost depends on the number of agents of the other villages? We consider that both answers, yes or no, are reasonable. In this paper we have chosen no. Then, we have adapted the properties of PM and SYM taking into account it.

We consider two properties of SYM. Symmetry among agents in the same group ($SYMA$) says that if two agents are symmetric and belongs to the same group, they must pay the same. Symmetry among groups ($SYMG$) says that if two groups are symmetric the total amount paid by the members of each group minus the cost of connecting agents inside the group among themselves must be the same. Two groups are symmetric if their connection costs to the other groups are the same.

We also consider two properties of PM. Population monotonicity over agents (PMA) says that if agent i enters in group G^k, no agent of group G^k can be worse off. Moreover, if the connection costs between group G^k and the other groups do not change, agents of the other groups must pay the same. Population monotonicity over groups (PMG) says that if a new group joins the society, no agent of the initial society can be worse off.

The main result of the paper says that there is a unique rule, we call it F, satisfying RA, $SYMA$, $SYMG$, PMA, and PMG.

We now describe the rule F characterized above. F can be considered as a two steps rule. In the first step we compute the amount that each group should pay in order to be connected to the source. We do it applying the rule φ defined in Bergantiños and Vidal-Puga (2007a). In the second step we decide the amount that each agent of each group has to pay. For each group G^k, we consider the *mcstp* inside each group $\left(G_0^k, C^{\varphi}\right)$. In $\left(G_0^k, C^{\varphi}\right)$ the connection cost between two agents of G^k is the same as in C. Nevertheless, the connection cost between any agent of G^k and the source is the amount computed for the group G^k in the first step.

Owen (1977) introduces a value for TU games with a group structure. It is assumed that agents are partitioned into different groups. Moreover, the objective is to divide the value of the grand coalition among the agents taking into account the group structure. Owen (1977) proves that his value generalizes the Shapley value.

Bird (1976) define the minimal network and the TU game (N, v_C) associated with an *mcstp* (N_0, C). Bergantiños and Vidal-Puga (2007a) define the irreducible matrix C^* associated with an *mcstp* (N_0, C) through the minimal network. The rule φ in *mcstp* is defined as the Shapley value of the TU game (N, v_{C^*}). The rule F in *mcstp* with groups generalizes the rule φ. We can ask if there is some relationship between F and the Owen value of (N, v_{C^*}, G). The answer is not trivial because (N, v_{C^*}, G) does not appear in the definition of F. Nevertheless, we will prove that F coincides with the Owen value of (N, v_{C^*}, G).

The paper is organized as follows. In Section 2 we introduce *mcstp*. In Section 3 we introduce *mcstp* with groups. In Section 4 we define the rule F and we present the axiomatic characterization. In Section 5 we prove that F coincides with the Owen value. In Appendix we present the proofs of the results obtained in the paper.

2 Minimum cost spanning tree problems

Let $\mathcal{N} = \{1, 2, ...\}$ be the set of all possible agents. Given a finite set $N \subset \mathcal{N}$, let Π_N be the set of all permutations over N. Given $\pi \in \Pi_N$, let $Pre\,(i, \pi)$ denote the set of elements of N which come before i in the order given by π, *i.e.*, $Pre\,(i, \pi) = \{j \in N : \pi\,(j) < \pi\,(i)\}$. Given $S \subset N$, let π_S denote the order induced by π among the agents in S.

We are interested in networks whose nodes are elements of a set $N_0 = N \cup \{0\}$, where $N \subset \mathcal{N}$ is finite and 0 is a special node called the *source*. Usually we take $N = \{1, ..., n\}$.

A *cost matrix* $C = (c_{ij})_{i,j \in N_0}$ on N represents the cost of direct link between any pair of nodes. We assume that $c_{ij} = c_{ji} \geq 0$ for each $i, j \in N_0$ and $c_{ii} = 0$ for each $i \in N_0$. Since $c_{ij} = c_{ji}$ we work with undirected arcs, *i.e.* $(i, j) = (j, i)$.

We denote the set of all cost matrices over N as $\mathcal{C}^N$. Given C, $C' \in \mathcal{C}^N$ we say $C \leq C'$ if $c_{ij} \leq c'_{ij}$ for all $i, j \in N_0$.

A *minimum cost spanning tree problem*, briefly an *mcstp*, is a pair (N_0, C) where $N \subset \mathcal{N}$ is a finite set of agents, 0 is the source, and $C \in \mathcal{C}^N$ is the cost matrix.

Given an *mcstp* (N_0, C), we define the *mcstp* induced by C in $S \subset N$ as (S_0, C).

A *network* g over N_0 is a subset of $\{(i, j) : i, j \in N_0\}$. The elements of g are called *arcs*. Given a network g over N_0 and $S \subset N_0$ we denote by g_S the network induced by g among the elements of S. Namely, $g_S = \{(i, j) \in g : \{i, j\} \subset S\}$.

Given a network g and a pair of nodes i and j, a *path* from i to j in g is a sequence of different arcs $\{(i_{h-1}, i_h)\}_{h=1}^{l}$ satisfying $(i_{h-1}, i_h) \in g$ for all $h \in \{1, 2, ..., l\}$, $i = i_0$, and $j = i_l$.

A *tree* is a network such that for all $i \in N$ there is a unique path from i to the source. If t is a tree, we usually write $t = \left\{\left(i^0, i\right)\right\}_{i \in N}$ where i^0 represents the first agent in the unique path in t from i to 0.

Let $\mathcal{G}^N$ denote the set of all networks over N_0. Let $\mathcal{G}_0^N$ denote the set of all networks where every agent $i \in N$ is connected to the source, *i.e.* there exists a path from i to 0 in the network.

Given an *mcstp* (N_0, C) and $g \in \mathcal{G}^N$, we define the *cost* associated with g as

$$c\left(N_0, C, g\right) = \sum_{(i,j) \in g} c_{ij}.$$

When there is no ambiguity, we write $c(g)$ or $c(C, g)$ instead of $c(N_0, C, g)$.

An *minimum cost spanning tree* for (N_0, C), briefly an *mt*, is a tree t over N_0 such that $c(t) = \min\limits_{g \in \mathcal{G}_0^N} c(g)$. It is well-known that an *mt* exists, even though it is not necessarily unique. Given an *mcstp* (N_0, C), we denote the cost associated with any *mt* as $m(N_0, C)$.

Given an *mcstp*, Prim (1957) provides an algorithm for solving the problem of connecting all agents to the source such that the total cost of creating the network is minimal. The idea of this algorithm is simple: starting from the source we construct a network by sequentially adding arcs with the lowest cost and without introducing cycles.

Formally, Prim's algorithm is defined as follows. We start with $S^0 = \{0\}$ and $g^0 = \emptyset$.

Stage 1: Take an arc $(0, i_1)$ such that $c_{0i_1} = \min\limits_{j \in N} \{c_{0j}\}$. If there are several arcs satisfying this condition, select just one. Now, $S^1 = \{0, i_1\}$ and $g^1 = \{(0, i_1)\}$.

Stage $p+1$: Assume that we have defined $S^p \subset N_0$ and $g^p \in \mathcal{G}^N$. We now define S^{p+1} and g^{p+1}. Take an arc $\left(i_{p+1}^0, i_{p+1}\right)$ with $i_{p+1}^0 \in S^p$ and $i_{p+1} \in N_0 \backslash S^p$ such that $c_{i_{p+1}^0 i_{p+1}} = \min\limits_{k \in S^p, l \in N_0 \backslash S^p} \{c_{kl}\}$. If there are several arcs satisfying this condition, select just one. Now, $S^{p+1} = S^p \cup \{i_{p+1}\}$ and $g^{p+1} = g^p \cup \left\{\left(i_{p+1}^0, i_{p+1}\right)\right\}$.

This process is completed in n stages. We say that g^n is a tree obtained following Prim's algorithm. Notice that this algorithm leads to a tree, but this is not always unique.

Given an *mcstp* (N_0, C) and an *mt* t, Bird (1976) defined the *minimal network* (N_0, C^t) associated with t as follows: $c_{ij}^t = \max\limits_{(k,l) \in g_{ij}} \{c_{kl}\}$, where g_{ij} denotes the unique path in t from i to j. Even though g_{ij} depends on the choice of t, c_{ij}^t is independent of the chosen t. Proof of this can be found, for instance, in Aarts and Driessen (1993).

The *irreducible form* of an *mcstp* (N_0, C) is defined as the minimal network (N_0, C^*) associated with a particular *mt* t. If (N_0, C^*) is an *irreducible form*, we say that C^* is an *irreducible matrix*.

One of the most important issues addressed in the literature about *mcstp* is how to divide the cost of connecting agents to the source between them. We now briefly introduce some of the rules studied in the literature.

A (*cost allocation*) *rule* is a function f such that $f(N_0, C) \in \mathbb{R}^N$ and $\sum_{i \in N} f_i(N_0, C) = m(N_0, C)$ for each *mcstp* (N_0, C). As usual, $\psi_i(N_0, C)$ represents the cost allocated to agent i.

Notice that we implicitly assume that the agents build an *mt*. As far as we know, all the rules proposed in the literature make this assumption.

A *coalitional game with transferable utility*, briefly a *TU game*, is a pair (N, v) where $v : 2^N \to \mathbb{R}$ satisfies $v(\varnothing) = 0$. $Sh(N, v)$ denotes the Shapley value (Shapley (1953)) of (N, v).

For each *mcstp* (N_0, C). Bird (1976) introduces the *TU* game (N, v_C). For each coalition $S \subset N$,

$$v_C(S) = m(S_0, C).$$

There are several rules studied in the literature. We mention, for instance, the rules studied in Bird (1976), Kar (2002), and Dutta and Kar (2004). In this paper the rule introduced by Feltkamp *et al* (1994) and called Equal Remaining Obligations rule (*ERO*) will be very important. *ERO* is called the $P-value$ in Branzei *et al* (2004).

On the other hand, in Bergantiños and Vidal-Puga (2007a) it is defined the rule φ as

$$\varphi(N_0, C) = Sh(N, v_{C^*})$$

where C^* is the irreducible matrix associated with C. Bergantiños and Vidal-Puga (2007e) prove that, surprisingly, φ coincides with *ERO*. This rule is also studied in Bergantiños and Vidal-Puga (2007b, 2007c, 2007d).

We now define several properties formally.

We say that f satisfies *Restricted Additivity* (*RA*) if for all *mcstp* (N_0, C) and (N_0, C') satisfying that there exists an *mt* $t = \{(i^0, i)\}_{i \in N}$ in (N_0, C), (N_0, C'), and $(N_0, C + C')$ and an order $\pi = (i_1, \ldots, i_{|N|}) \in \Pi_N$ such that $c_{i_1^0 i_1} \le c_{i_2^0 i_2} \le \ldots \le c_{i_{|N|}^0 i_{|N|}}$ and $c'_{i_1^0 i_1} \le c'_{i_2^0 i_2} \le \ldots \le c'_{i_{|N|}^0 i_{|N|}}$, we have that

$$f(N_0, C + C') = f(N_0, C) + f(N_0, C').$$

RA is an additivity property restricted to some subclass of problems. No rule satisfies additivity over all *mcstp*. The reason is that in the definition of a rule we are claiming that $\sum_{i \in N} f_i(N_0, C) = m(N_0, C)$, which is incompatible with additivity over all *mcstp*. See Bergantiños and Vidal-Puga (2007d) for a detailed discussion of *RA*.

We say that f satisfies *population monotonicity* (*PM*) if for all *mcstp* (N_0, C), all $S \subset N$, and all $i \in S$,

$$f_i(N_0, C) \le f_i(S_0, C).$$

PM says that, if new agents join a society, no agent of the initial society can be worse off. This is a well-known property, which has been used in many different situations.

We say that $i, j \in N$ are *symmetric* if for all $k \in N_0 \setminus \{i, j\}$, $c_{ik} = c_{jk}$.

We say that f satisfies *Symmetry* (SYM) if for all *mcstp* (N_0, C) and all pair of symmetric agents $i, j \in N$,

$$f_i (N_0, C) = f_j (N_0, C) .$$

We say that f satisfies *strong cost monotonicity* (SCM) if for all *mcstp* (N_0, C) and (N_0, C') such that $C \leq C'$ and all $i \in N$,

$$f_i(N_0, C) \leq f_i(N_0, C').$$

SCM implies that if a number of connection costs increase and the rest of connection cost (if any) remain the same, no agent can be better off. This property is called *solidarity* in Bergantiños and Vidal-Puga (2007a).

In Lemma 0 below we present some results used in the paper. The proof can be found in Bergantiños and Vidal-Puga (2007a, 2007b, 2007c, 2007d).

Lemma 0.

(a) (N_0, C) is irreducible if and only if there exists an *mt* t in (N_0, C) satisfying the two following conditions:

$(A1)$ $t = \{(i_{p-1}, i_p)\}_{p=1}^{|N|}$ where $i_0 = 0$.

$(A2)$ Given $i_p, i_q \in N_0$, $p < q$, then $c_{i_p i_q} = \max\limits_{p<r\leq q} \left\{c_{i_{r-1} i_r}\right\}$.

(b) If C is an irreducible matrix, then for all $S \subset N_0$, $i \notin S$ we have that

$$v_C (S \cup \{i\}) - v_C (S) = \min_{j \in S_0} \{c_{ij}\} .$$

(c) If C is an irreducible matrix, then v_C is a concave game. Namely, if $S \subset T \subset N$ and $i \notin T$, then

$$v_C (S \cup \{i\}) - v_C (S) \geq v_C (T \cup \{i\}) - v_C (T) .$$

(d) If C and C' are under the conditions of RA, then for all $S \subset N$,

$$v_{(C+C')^*} (S) = v_{C^*} (S) + v_{C'^*} (S) .$$

(e) φ is the unique rule on *mcstp* satisfying RA, PM, and SYM.

(f) φ satisfies SCM.

3 Minimum cost spanning tree problems with groups

There are many economic situations that can be modeled as a *mcstp*. Let us mention some examples. Several towns may draw power from a common power plant, and hence have to share the cost of the distribution network (Dutta and Kar, 2004). Bergantiños and Lorenzo (2004, 2005) study a real situation where a valley authority has to construct pipes from a dam to several houses. These houses are located in different villages of the valley.

The classical model of *mcstp*, as described in the previous section can also model this situation. Nevertheless, it ignores the fact that some group of agents are located in the same city or village. It could be interesting to include this fact in the model. That's the main issue of this section.

We do it by considering an extra element in the model. Namely a partition $G = \left\{G^1, ..., G^m\right\}$ of the set of agents N. The interpretation of G is clear. For each $k = 1, ..., m$, G^k represents a group of agents, which are located in the same city, village, ...

In many situations, for instance the examples mentioned at the beginning of this section, the cost between any pair of agents is closely related to the distance between both agents. Under these circumstances it seems reasonable that the connection cost between two agents of the city G^k is not larger than the connection cost between an agent of city G^k and an agent from other city (or the source).

We now introduce the model formally. An *mcstp* with groups is a triple (N_0, C, G) where (N_0, C) is an *mcstp*, $G = \left\{G^1, ..., G^m\right\}$ is a partition of N and for each $k = 1, ...m$

$$\max_{i,j \in G^k} \{c_{ij}\} \leq \min_{i \in G^k, j \notin G^k} \{c_{ij}\}.$$

A *rule* in *mcstp* with groups is a function f such that $f(N_0, C, G) \in \mathbb{R}^N$ and $\sum_{i \in N} f_i(N_0, C, G) = m(N_0, C)$ for each *mcstp* (N_0, C).

As in classical *mcstp*, the main objective is to divide the cost associated with an *mt* among the agents in a fair way.

Example 1. Consider the *mcstp* (N_0, C, G) with groups where $N = \{1, 2, 3\}$, $G = \left\{G^1, G^2\right\}$, $G^1 = \{1, 2\}$, $G^2 = \{3\}$, and

$$C = \begin{pmatrix} 0 & 8 & 8 & 8 \\ 8 & 0 & 2 & 4 \\ 8 & 2 & 0 & 4 \\ 8 & 4 & 4 & 0 \end{pmatrix}.$$

In this case $m(N_0, C) = 14$. Moreover, 12 units are associated with the cost of connecting cities 1 and 2 with the source and 2 units are associated with the cost of connecting agents 1 and 2 inside city 1.

Since we are looking for fair shares it seems reasonable to divide these 2 units equally between agents 1 and 2.

The 12 units comes of the construction of the network in which some of the three agents is connected with the source and some agent of G^1 is connected with agent 3. In order to divide the 12 units among the agents two approaches seems reasonable.

1. The cost paid by each city does not depend on the characteristics of the other city. Assuming it both cities are symmetric. Thus, each city should pay 6. Since agents inside city 1 are also symmetric, both pay the same. Then, agent 1 pays 1+3=4, agent 2 pays 1+3=4, agent 3 pays 6.

2. The cost paid by each city should take into account the number of agents who gets benefits of their connection. Thus, city 1 should pay twice than city 2, *i.e.*, city 1 pays 8 and city 2 pays 4. Since agents inside city 1 are also symmetric, both pay the same. Then, agent 1 pays 1+4=5, agent 2 pays 1+4=5, agent 3 pays 4.

In this paper we have decided to follow the first approach. Thus, some properties introduced later will be defined according with it.

4 The rule and the axiomatic characterization

In this section we introduce a rule for *mcstp* with groups. Moreover, we also characterize this rule. Our characterization is inspired in the characterization given in Bergantiños and Vidal-Puga (2007d).

In this paper we follow the axiomatic approach and we introduce a rule as the unique rule satisfying a set of desirable properties. Our idea is to generalize the axiomatic characterization of the rule φ given by Bergantiños and Vidal-Puga (2007d), which involves three properties: RA, PM, and SYM.

Now we adapt these properties to *mcstp* with groups. The property of RA could be formulated in a similar way. Nevertheless, PM and SYM should be adapted. The main idea for adapting each of these properties is claiming both twice. Once among the groups and other among agents inside the same group.

We say that f satisfies *Restricted Additivity* (RA) if for all *mcstp* with groups (N_0, C, G) and (N_0, C', G) satisfying that there exists an mt $t = \{(i^0, i)\}_{i \in N}$ in (N_0, C, G), (N_0, C', G), and $(N_0, C+C', G)$ and an order $\pi = (i_1, \ldots, i_{|N|}) \in \Pi_N$ such that $c_{i_1^0 i_1} \leq c_{i_2^0 i_2} \leq \ldots \leq c_{i_{|N|}^0 i_{|N|}}$ and $c'_{i_1^0 i_1} \leq c'_{i_2^0 i_2} \leq \ldots \leq c'_{i_{|N|}^0 i_{|N|}}$, we have that

$$f(N_0, C + C', G) = f(N_0, C, G) + f(N_0, C', G).$$

We say that f satisfies *symmetry among agents in the same group* ($SYMA$) if for all *mcstp* with groups (N_0, C, G) and all pair of symmetric agents $i, j \in G^k \in G$,

$$f_i (N_0, C, G) = f_j (N_0, C, G).$$

We now define symmetry among groups. We first define symmetric groups. Intuitively two groups of agents are symmetric if their connection costs to the other groups are the same. Because of the model each pair of groups G^k and $G^{k'}$ can connect in several ways. For each pair of agents $i \in G^k$, $j \in G^{k'}$ they can construct the arc (i, j). Since we are assuming that agents will construct an mt, it is reasonable to assume that they will construct an arc (i, j) with minimum cost.

We say that *two groups G^k and $G^{k'}$ are symmetric* if for all $G^l \in G_0 \setminus \left\{G^k, G^{k'}\right\}$,

$$\min_{i \in G^k, j \in G^l} \{c_{ij}\} = \min_{i \in G^{k'}, j \in G^l} \{c_{ij}\}.$$

The next step is to say that symmetric groups should pay the same. The amount paid by group G^k is $\sum_{i\in G^k} f_i(N_0,C,G)$. Thus, we can decompose this amount in two parts, the cost of connecting agents inside the group among themselves, $m\left(G^k,C\right)$, and the cost of connecting the group with the source (possible through other groups), $\sum_{i\in G^k} f_i(N_0,C,G) - m\left(G^k,C\right)$. We are assuming that the amount paid by a group should not depend on the internal characteristics of the other groups. Then, it seems reasonable to say that $m\left(G^k,C\right)$ should be paid by agents of G^k. Thus, we formulate the second symmetry property as follows.

We say that f satisfies *symmetry among groups* ($SYMG$) if for all *mcstp* with groups (N_0,C,G) and all pair symmetric groups $G^k, G^{k'} \in G$,

$$\sum_{i\in G^k} f_i(N_0,C,G) - m\left(G^k,C\right) = \sum_{i\in G^{k'}} f_i(N_0,C,G) - m\left(G^{k'},C\right).$$

We now define the two population monotonicity properties, over groups and over agents.

The idea of population monotonicity over groups is quite simple. If a new group joins the society, no agent of the initial society can be worse off. Formally,

We say that f satisfies *population monotonicity over groups* (PMG) if for all *mcstp* with groups (N_0,C,G), all $G^k \in G$, and all $i \in N\backslash G^k$,

$$f_i(N_0,C,G) \le f_i\left(\left(N\backslash G^k\right)_0, C, G\backslash G^k\right).$$

The population monotonicity over agents will say what happens when an agent enters in a group. We claim that no agent of the initial group can be worse off.

Assume that after the entrance of agent i in group G^k the minimum connection cost between group G^k and the rest of the groups did not change, *i.e.*, for each G^l, $l \neq k$, $\min_{j\in G^k, j'\in G^l}\{c_{jj'}\} = \min_{j\in G^k\cup\{i\}, j'\in G^l}\{c_{jj'}\}$. Since we are assuming that the amount paid by a group should not depend on the internal characteristics of the other groups and the entrance of agent i did not change the connection cost among groups, we claim that agents of the others groups must pay the same. Formally,

We say that f satisfies *population monotonicity over agents* (PMA) if for all *mcstp* with groups (N_0,C,G), all $G^k \in G$, and all $i \in G^k$ such that $G^k\backslash\{i\} \neq \varnothing$,

$$f_j(N_0,C,G) \le f_j\left((N\backslash\{i\})_0, C, \left(G\backslash G^k\right) \cup \left(G^k\backslash\{i\}\right)\right) \text{ if } j \in G^k\backslash\{i\}.$$

Moreover, if for each G^l with $l \neq k$, $\min_{j\in G^k, j'\in G^l}\{c_{jj'}\} = \min_{j\in G^k\backslash\{i\}, j'\in G^l}\{c_{jj'}\}$, then

$$f_j(N_0,C,G) = f_j\left((N\backslash\{i\})_0, C, \left(G\backslash G^k\right) \cup \left(G^k\backslash\{i\}\right)\right) \text{ if } j \in N\backslash G^k.$$

Remark 1. PMA can be reformulated without the condition $G^k\backslash\{i\} \neq \varnothing$. The reason is that if $G^k\backslash\{i\} = \varnothing$, then PMA will say nothing.

We now define the rule F in *mcstp* with groups. We first give the intuitive idea. This rule can be considered as a two steps rule. In the first step we compute the amount that each group should pay in order to be connected to the source. We do it applying the rule φ defined in Bergantiños and Vidal-Puga (2007a).

In the second step we decide the amount that each agent of each group has to pay. For each group G^k, we consider the *mcstp* inside each group $\left(G_0^k, C^{\varphi}\right)$. In this *mcstp*, the connection cost between two agents of G^k is the same as in C but the connection cost between any agent of G^k and the source is the amount computed by the group G^k in the first step.

We now present the definition formally. Given the *mcstp* with groups (N_0, C, G) we define the *mcstp* among groups $\left(G_0, C^G\right)$ as follows:

- $G_0 = \left\{G^0, G^1, ..., G^m\right\}$ where $G^0 = 0$.
- C^G is the cost matrix and for each $G^k, G^{k'} \in G_0$ the connection cost between G^k and $G^{k'}$ is denoted by

$$c_{kk'}^G = \min_{i \in G^k, j \in G^{k'}} \{c_{ij}\}.$$

Let (N_0, C, G) be an *mcstp* with groups and $i \in G^k$. Thus,

$$F_i(N_0, C, G) = \varphi_i\left(G_0^k, C^{\varphi}\right)$$

where

$$c_{jj'}^{\varphi} = \begin{cases} c_{jj'} & \text{if } 0 \notin \{j, j'\} \\ \varphi_k\left(G_0, C^G\right) & \text{if } 0 \in \{j, j'\}. \end{cases}$$

Before introducing the results of the paper, we present Lemma 1, which will be used often in the proofs of the main results.

Lemma 1. Given (N_0, C, G) we can find an *mt* t in (N_0, C) satisfying:
(i) For each $k = 1, ..., m$, t_{G^k} induces an *mt* in $\left(G^k, C\right)$.
(ii) $t \backslash \left(\cup_{k=1}^m t_{G^k}\right) = \left\{(k, k') : \exists i \in G^k, j \in G^{k'} \text{ with } (i, j) \in t\right\}$ is an *mt* in $\left(G_0, C^G\right)$.
(iii) For each $k = 1, ..., m$ and each $i \in G^k$, $t_{G_k} \cup \{(0, i)\}$ is an *mt* in (G_0^k, C^{φ}).

Proof. See Appendix.

In the next proposition we prove that F is a rule in *mcstp* with groups. We also prove that F generalizes the rule φ defined in Bergantiños and Vidal-Puga (2007a).

Proposition 1.
(a) For each *mcstp* with groups (N_0, C, G),

$$\sum_{i \in N} F_i(N_0, C, G) = m(N_0, C).$$

(b) Let (N_0, C, G) be an *mcstp* with groups where $G = \{N\}$. Thus, for each $i \in N$,

$$F_i(N_0, C, G) = \varphi_i(N_0, C).$$

(c) Let (N_0, C, G) be an *mcstp* with groups where $G = \{i\}_{i \in N}$. Thus, for each $i \in N$,

$$F_i (N_0, C, G) = \varphi_i (N_0, C).$$

Proof. See Appendix.

We now present the main results of the section.

Proposition 2. F satisfies RA, $SYMG$, $SYMA$, PMG, and PMA.

Proof. See Appendix.

Proposition 3. There is a unique rule satisfying RA, $SYMG$, $SYMA$, PMG, and PMA.

Proof. See Appendix.

Next theorem is a trivial consequence of propositions 2 and 3.

Theorem 1. F is the unique rule satisfying RA, $SYMG$, $SYMA$, PMG, and PMA.

Remark 2. The properties used in Theorem 1 are independent. The proof is in Appendix.

5 An approach using TU games

Owen (1977) introduces a value for TU games with a group structure. It is assumed that agents are partitioned into different groups. Moreover, the objective is to divide the value of the grand coalition among the agents taking into account the group structure. Owen (1977) proves that his value generalizes the Shapley value.

The rule φ in *mcstp* is defined as the Shapley value of the TU game (N, v_{C^*}). The rule F in *mcstp* with groups generalizes the rule φ. We can ask if there is some relationship between F and the Owen value of (N, v_{C^*}, G). The answer is not trivial because (N, v_{C^*}, G) does not appear in the definition of F. Nevertheless, we will prove that F coincides with the Owen value of (N, v_{C^*}, G).

We first introduce the Owen value formally. A TU *game with group structure* is a triple (N, v, G) where (N, v) is a TU game and $G = \left\{G^1, ..., G^m\right\}$ is a partition of N.

We say that a permutation $\pi \in \Pi_N$ is *admissible* with respect to G if given $i, i' \in G^k \in G$ and $j \in N$ with $\pi(i) < \pi(j) < \pi(i')$, then $j \in G^k$. We denote by Π^G the set of all permutations over N admissible with respect to G.

Given (N, v, G) and $i \in G^k \in G$, the Owen value is defined as

$$Ow_i (N, v, G) = \frac{1}{|\Pi^G|} \sum_{\pi \in \Pi^G} \left[v \left(Pre\left(i, \pi\right) \cup \{i\}\right) - v \left(Pre \left(i, \pi\right)\right)\right].$$

Theorem 2. For each *mcstp* with groups (N_0, C, G) and $i \in G^k \in G$,

$$F_i(N_0, C, G) = Ow_i(N, v_{C^*}, G).$$

Proof. See Appendix.

6 Appendix

6.1 Proof of Lemma 1

We prove that when we apply Prim's algorithm if at Stage p, $S^p = \left(\bigcup_{j=1}^{l} G^{k_j}\right) \cup G'$ where $G' \subset G^{k_{l+1}}, G' \neq \varnothing$, and $G' \neq G^{k_{l+1}}$, then at Stage $p+1$ we can select an arc $\left(i^0_{p+1}, i_{p+1}\right)$ where $i^0_{p+1} \in G'$ and $i_{p+1} \in G^{k_{l+1}} \backslash G'$.

Let $\left(i^0_{p+1}, i_{p+1}\right)$ be such that $i^0_{p+1} \in G'$, $i_{p+1} \in G^{k_{l+1}} \backslash G'$ and

$$c_{i^0_{p+1} i_{p+1}} = \min_{i \in G', j \in G^{k_{l+1}} \backslash G'} \{c_{ij}\}.$$

By definition of Prim's algorithm it is enough to prove that $c_{i^0_{p+1} i_{p+1}} \leq c_{ij}$ in the following cases:

1. $i \in G'$, $j \in N \backslash \left(\bigcup_{j=1}^{l+1} G^{k_j}\right)$. Thus, $c_{i^0_{p+1} i_{p+1}} \leq c_{ij}$ because $\left\{i^0_{p+1}, i_{p+1}\right\} \subset G^{k_{l+1}}$, $i \in G^{k_{l+1}}$, $j \in G^{k'}$, $k' \in \{1, ..., m\} \setminus \{k_1, ..., k_{l+1}\}$, and $k_{l+1} \neq k'$.

2. $i \in \bigcup_{j=1}^{l} G^{k_j}$, $j \in G^{k_{l+1}} \backslash G'$. Thus, $c_{i^0_{p+1} i_{p+1}} \leq c_{ij}$ because $\left\{i^0_{p+1}, i_{p+1}\right\} \subset G^{k_{l+1}}$, $i \in G^{k'}$, $k' \in \{k_1, ..., k_l\}$, $j \in G^{k_{l+1}}$, and $k_{l+1} \neq k'$.

3. $i \in \bigcup_{j=1}^{l} G^{k_j}$, $j \notin G^{k_{l+1}} \backslash G'$. Thus, $c_{i^0_{p+1} i_{p+1}} \leq c_{ij}$ because $\left\{i^0_{p+1}, i_{p+1}\right\} \subset G^{k_{l+1}}$, $i \in G^{k'}, k' \in \{k_1, ..., k_l\}$, $j \in G^{k''}$, $k'' \in \{1, ..., m\} \setminus \{k_1, ..., k_{l+1}\}$ and $k_{l+1} \neq k'$.

We now prove parts (i) and (ii).

We apply Prim's algorithm to (N_0, C). Let $(0, i_1)$ be the first arc selected according Prim's algorithm. We assume *wlog* that $i_1 \in G^1$. If $G^1 \backslash \{i_1\} \neq \varnothing$, by the previous statement, at Stage 2 of Prim's algorithm we can select an arc $\left(i^0_2, i_2\right)$ satisfying that $i^0_2 \in G^1 \cap S^1 = \{i_1\}$ and $i_2 \in G^1 \backslash \{i_1\}$.

If we repeat this argument we can prove that for each $p = 2, ..., \left|G^1\right|$ the arc $\left(i^0_p, i_p\right)$ selected at Stage p satisfies that $i^0_p \in G^1$ and $i_p \in G^1$.

At Stage $p = \left|G^1\right| + 1$ we select an arc $\left(i^0_p, i_p\right)$ where $i^0_p \in G^1 \cup \{0\}$ and $i_p \notin G^1 \cup \{0\}$. We assume *wlog* that $i_p \in G^2$. By definition of Prim's algorithm, $c_{i^0_p i_p} = c^G_{12}$ when $i^0_p \in G^1$ whereas $c_{i^0_p i_p} = c^G_{02}$ when $i^0_p = 0$. Repeating the same argument as above we can prove that for each $p = \left|G^1\right| + 2, ..., \left|G^1 \cup G^2\right|$ the arc $\left(i_p^0, i_p\right)$ selected at Stage p satisfies that $i_p^0 \in G^2$ and $i_p \in G^2$.

In general, for each $q = 1, ..., m$ and for each $p = \left|\cup_{l=1}^{q-1} G^l\right| + 2, ..., \left|\cup_{l=1}^{q} G^l\right|$ the the arc $\left(i_p^0, i_p\right)$ selected at Stage p satisfies that $i_p^0 \in G^q$ and $i_p \in G^q$. Moreover, for each $q = 1, ..., m$ and for each $p = \left|\cup_{l=1}^{q-1} G^l\right| + 1$ the the arc $\left(i_p^0, i_p\right)$ selected at Stage p satisfies that $i_p^0 \in \cup_{l=1}^{q-1} G^l \cup \{0\}$ and $i_p \in \cup_{l=q-1}^{m} G^l$.

Now it is easy to conclude that parts (i) and (ii) hold.

We now prove part (iii). Let $k \in \{1, ..., m\}$. Because of parts (i) and (ii) it is enough to prove that for each $i \in G^k$ $c_{0i}^{\varphi} \geq \max\limits_{j,j' \in G^k} \left\{c_{jj'}^{\varphi}\right\}$. Since $c_{0i}^{\varphi} = \varphi_k(G_0, C^G)$ for all $i \in G^k$ and $c_{jj'}^{\varphi} = c_{jj'}$ for all $j, j' \in G^k$, we must prove that $\varphi_k(G_0, C^G) \geq \max\limits_{j,j' \in G^k} \{c_{jj'}\}$.

Given an *mcstp* (N_0, C), for all $i \in N$, $\varphi_i(N_0, C) = Sh_i(N, v_{C^*})$. By Lemma 0 (b), for all $S \subset N$, $i \notin S$, $v_{C^*}(S \cup \{i\}) - v_{C^*}(S) = c_{ij}^*$ for some $j \in N_0 \backslash \{i\}$. So, $\varphi_i(N_0, C) \geq \min\limits_{j \in N_0 \backslash \{i\}} \left\{c_{ij}^*\right\}$.

Thus, $\varphi_k(G_0, C^G) \geq \min\limits_{k' \in G_0 \backslash \{k\}} \left\{\left(c_{kk'}^G\right)^*\right\}$. As the matrix irreducible is the minimal network associated with an *mt*,

$$\min_{k' \in G_0 \backslash \{k\}} \left\{\left(c_{kk'}^G\right)^*\right\} \geq \min_{k' \in G_0 \backslash \{k\}} \left\{c_{kk'}^G\right\}.$$

Because of the definition of (N_0, C, G), $\min\limits_{k' \in G_0 \backslash \{k\}} \left\{c_{kk'}^G\right\} \geq \max\limits_{j,j' \in G^k} \{c_{jj'}\}$, Thus, Claim 1 (iii) holds.

6.2 Proof of Proposition 1

(a) By definition of F,

$$\sum_{i \in N} F_i(N_0, C, G) = \sum_{k=1}^{m} \sum_{i \in G^k} \varphi_i\left(G_0^k, C^{\varphi}\right).$$

Since φ is a rule in *mcstp*, for each $k = 1, ..., m$, $\sum\limits_{i \in G^k} \varphi_i\left(G_0^k, C^{\varphi}\right) = m(G_0^k, C^{\varphi})$. So,

$$\sum_{i \in N} F_i(N_0, C, G) = \sum_{k=1}^{m} m(G_0^k, C^{\varphi}).$$

By Lemma 1 (iii), for any $k = 1, ..., m$, we can construct an *mt* in (G_0^k, C^{φ}) $t_{G^k} \cup \{(0, i)\}$ with $i \in G^k$. Thus, $m(G_0^k, C^{\varphi}) = m(G^k, C^{\varphi}) + \varphi_k(G_0, C^G)$. Hence,

$$\begin{aligned}
\sum_{i \in N} F_i(N_0, C, G) &= \sum_{k=1}^{m} \left[m(G^k, C^{\varphi}) + \varphi_k(G_0, C^G)\right] \\
&= \sum_{k=1}^{m} m(G^k, C^{\varphi}) + \sum_{k=1}^{m} \varphi_k(G_0, C^G).
\end{aligned}$$

Since φ is a rule in *mcstp*, $\sum_{k=1}^{m} \varphi_k(G_0, C^G) = m(G_0, C^G)$. Now,

$$\sum_{i \in N} F_i\left(N_0, C, G\right) = \sum_{k=1}^{m} m(G^k, C^{\varphi}) + m(G_0, C^G).$$

By definition of C^{φ}, $c_{ij}^{\varphi} = c_{ij}$ for all $i, j \in G^k$. Thus, $m(G^k, C^{\varphi}) = m(G^k, C)$. Hence,

$$\sum_{i \in N} F_i\left(N_0, C, G\right) = \sum_{k=1}^{m} m(G^k, C) + m(G_0, C^G).$$

By Lemma 1 (i) and (ii), $m(N_0, C) = \sum_{k=1}^{m} m(G^k, C) + m(G_0, C^G)$.

Replacing this expression in equation above, we obtain the result.

(b) Let $G = \{N\}$. Thus, $F(N_0, C, G) = \varphi(N_0, C^{\varphi})$.

By definition, $c_{ij}^{\varphi} = c_{ij}$ for all $i, j \in N$ and $c_{0i}^{\varphi} = \min_{j \in N} \{c_{0j}\}$ for all $i \in N$. Thus, $C \geq C^{\varphi}$. Since φ satisfies SCM, $\varphi(N_0, C) \geq \varphi(N_0, C^{\varphi})$. By Lemma 1 (iii), $m(N_0, C) = m(N_0, C^{\varphi})$. Now, $\varphi(N_0, C) = \varphi(N_0, C^{\varphi})$.

(c) Let $G = \{i\}_{i \in N}$. For each $i \in N$,

$$F_i(N_0, C, G) = \varphi_i(\{i\}_0, C^{\varphi}) = c_{0i}^{\varphi} = \varphi_i(G_0, C^G)$$

Moreover, $G_0 = N_0$ and $c_{ij}^G = c_{ij}$ for all $i, j \in N_0$, that is, $C^G = C$. Therefore, $F(N_0, C, G) = \varphi(N_0, C)$.

6.3 Proof of Proposition 2

We divide the proof in several claims.

Claim 1. F satisfies RA.

Proof of Claim 1. Let (N_0, C, G) and (N_0, C', G) be two *mcstp* with groups satisfying that there exists an *mt* $t = \{(i^0, i)\}_{i \in N}$ in (N_0, C, G), (N_0, C', G), and $(N_0, C + C', G)$ and an order $\pi = (i_1, \ldots, i_{|N|}) \in \Pi_N$ such that $c_{i_1^0 i_1} \leq c_{i_2^0 i_2} \leq \ldots \leq c_{i_{|N|}^0 i_{|N|}}$ and $c'_{i_1^0 i_1} \leq c'_{i_2^0 i_2} \leq \ldots \leq c'_{i_{|N|}^0 i_{|N|}}$.

We first prove that it is possible to find a tree t satisfying the conditions of RA defined above and the three conditions of Lemma 1 for the problems C, C', and $C + C'$.

Let t be the tree satisfying the conditions of RA. Assume that there exists $G^k \in G$ such that t_{G^k} is not a tree in G^k. Since t is a tree in N_0, t_{G^k} has no cycles. Let $\{X_1, ..., X_l\}$ the partition of G^k in connected components induced by t_{G^k}. Namely, if $i, j \in X_{l'}$ for some $l' = 1, ..., l$, then there is a path in t_{G^k} from i to j. Moreover, if $i \in X_{l'}$, $j \in X_{l''}$ and $l' \neq l''$, then there is no path in t_{G^k} from i to j.

Let $i \in X_{l'}$, $j \in X_{l''}$ and $l' \neq l''$. Since t is a tree there is a path g_{ij} in t from i to j. For each arc $(i', j') \in g_{ij}$ we have that $t^1 = (t \backslash \{(i', j')\}) \cup \{(i, j)\}$ is a tree in (N_0, C). Since t is an *mt* in (N_0, C) and (N_0, C'), $c_{ij} \geq c_{i'j'}$ and $c'_{ij} \geq c'_{i'j'}$ Because of the definition of (N_0, C, G) and (N_0, C', G') we deduce that if $(i',j') \in g_{ij}$, $i' \in G^{k'}$, $j' \in G^{k''}$, and $k' \neq k''$, then $c_{ij} = c_{i'j'}$ and $c'_{ij} = c'_{i'j'}$. Since $i \in X_{l'}$ and $j \in X_{l''}$ we can find $(i',j') \in g_{ij}$ such that $i' \in G^{k'}$, $j' \in G^{k''}$, and $k' \neq k''$. Now t^1 is an mt in (N_0, C, G), (N_0, C', G), and $(N_0, C + C', G)$.

The order $\pi' = (i'_1, \ldots, i'_{|N|}) \in \Pi_N$ obtained by changing in the order π the arc (i',j') by (i,j) also satisfies the conditions of the definition of RA.

Now, $t^1_{G^k}$ induces a partition of G^k in $l-1$ connected components. If $l-1 = 1$, then $t^1_{G^k}$ induces a tree in G^k. Otherwise we proceed with t^1 as with t. Finally, we find an mt t^{l-1} such that $t^{l-1}_{G^k}$ induces a tree in G^k.

Once we finish with G^k we proceed with the other groups. At the end of the procedure we find an mt t^p such that $t^p_{G^k}$ induces a tree in each $G^k \in G$. That is, t^p satisfies the conditions of Lemma 1 (i). Since t^p is a tree in N_0, we deduce that t^p also satisfies the conditions of Lemma 1 (ii). Using arguments similar to those used in the proof of Lemma 1 (iii), we can prove that t^p also satisfies the conditions of Lemma 1 (iii).

Then, we can assume that the tree t also satisfies the conditions of Lemma 1.

Let $G^k \in G$. Since t satisfies the conditions of Lemma 1 (ii), $\left(G_0, C^G\right)$ and $\left(G_0, C'^G\right)$ are under the conditions of RA. Since φ satisfies RA, $\varphi_k\left(G_0, C^G\right) + \varphi_k\left(G_0, C'^G\right) = \varphi_k\left(G_0, C^G + C'^G\right)$. Moreover, it is easy to see that $C^G + C'^G = \left(C + C'\right)^G$.

Since t satisfies the conditions of Lemma 1 (iii), we have that $t^* = t_{G^k} \cup \{(0, i_j)\}$ with $i_j \in G^k$ is an mt in (G_0^k, C^φ), (G_0^k, C'^φ), and $(G_0^k, (C+C')^\varphi)$.

Let $\pi_{G^k} = (i_1, ..., i_{|G^k|})$ be the order induced by π over the agents in G^k. We have proved above that for all $(j,j') \in t_{G^k}$, $c^\varphi_{0i} \geq c^\varphi_{jj'}$ and $c'^\varphi_{0i} \geq c'^\varphi_{jj'}$. Therefore, (G_0^k, C^φ) and (G_0^k, C'^φ) are under the conditions of RA. Since φ satisfies RA, $\varphi_i(G_0^k, C^\varphi) + \varphi_i(G_0^k, C'^\varphi) = \varphi_i(G_0^k, C^\varphi + C'^\varphi)$ for all $i \in G^k$. Moreover, it is easy to see that $C^\varphi + C'^\varphi = (C + C')^\varphi$.

Now, for all $G^k \in G$ and all $i \in G^k$

$$\begin{aligned} F_i(N_0, C, G) + F_i(N_0, C', G) &= \varphi_i(G_0^k, C^\varphi) + \varphi_i(G_0^k, C'^\varphi) \\ &= \varphi_i(G_0^k, (C+C')^\varphi) \\ &= F_i(N_0, C + C', G). \blacksquare \end{aligned}$$

Claim 2. F satisfies $SYMG$.

Proof of Claim 2. Let G^k and $G^{k'}$ be two symmetric groups. Then, for all $G^l \in G_0 \backslash \{G^k, G^{k'}\}$,

$$c^G_{kl} = \min_{i \in G^k, j \in G^l} \{c_{ij}\} = \min_{i \in G^{k'}, j \in G^l} \{c_{ij}\} = c^G_{k'l}.$$

That is, k and k' are symmetric agents in $\left(G_0, C^G\right)$. Since φ satisfies SYM, $\varphi_k\left(G_0, C^G\right) = \varphi_{k'}\left(G_0, C^G\right)$.

By Lemma 1 (iii), $m(G_0^k, C^\varphi) = m(G^k, C) + \varphi_k(G_0, C^G)$. Thus,

$$\begin{aligned} \sum_{i \in G^k} F_i(N_0, C, G) - m(G^k, C) &= \sum_{i \in G^k} \varphi_i(G_0^k, C^\varphi) - m(G^k, C) \\ &= m(G_0^k, C^\varphi) - m(G^k, C) \\ &= \varphi_k(G_0, C^G). \end{aligned}$$

Repeating the same argument with $G^{k'}$ instead of G^k, we obtain that

$$\sum_{i\in G^{k'}} F_i(N_0,C,G) - m(G^{k'},C) = \varphi_{k'}(G_0,C^G).$$

Thus, F satisfies $SYMG$. ■

Claim 3. F satisfies $SYMA$.

Proof of Claim 3. Let $i,j\in G^k\in G$ be symmetric agents in (N_0,C,G). By definition of C^φ, for all $j'\in G^k$, $c^\varphi_{ij'}=c_{ij'}$ and $c^\varphi_{jj'}=c_{jj'}$. Moreover, $c^\varphi_{0i}=c^\varphi_{0j}=\varphi_k(G_0,C^G)$. Hence, i and j are symmetric agents in $\left(G^k_0,C^\varphi\right)$.

Since φ satisfies SYM, $\varphi_i\left(G^k_0,C^\varphi\right)=\varphi_j\left(G^k_0,C^\varphi\right)$. Thus,

$$F_i(N_0,C,G)=\varphi_i\left(G^k_0,C^\varphi\right)=\varphi_j\left(G^k_0,C^\varphi\right)=F_j(N_0,C,G).$$

Hence, F satisfies $SYMA$. ■

Claim 4. F satisfies PMG.

Proof of Claim 4. Let $G^k\in G$. Since φ satisfies PM, $\varphi_l(G_0,C^G)\le\varphi_l((G\backslash G^k)_0,C^G)$ for all $l\neq k$.

Let C'^φ denote the matrix C^φ associated with the problem $(\left(N\backslash G^k\right)_0,C,G\backslash G^k)$.

Let $G^l\in G\backslash G^k$. For all $i,j\in G^l$, $c^\varphi_{ij}=c'^\varphi_{ij}$. For all $i\in G^l$,

$$c^\varphi_{0i}=\varphi_l(G_0,C^G)\le\varphi_l((G\backslash G^k)_0,C^G)=c'^\varphi_{0i}$$

That is $C^\varphi\le C'^\varphi$. Let $i\in G^l$. Since φ satisfies SCM, $\varphi_i(G^l_0,C^\varphi)\le\varphi_i(G^l_0,C'^\varphi)$. So, $F_i(N_0,C,G)\le F_i(\left(N\backslash G^k\right)_0,C,G\backslash G^k)$, *i.e.*, F satisfies PMG. ■

Claim 5. F satisfies PMA.

Proof of Claim 5. Let $G^k\in G$ and $i\in G^k$. By convenience, let us denote as C' the cost matrix C restricted to the problem $((N\backslash\{i\})_0,C,(G\backslash G^k)\cup(G^k\backslash\{i\}))$. Notice that C' coincides with C on the agents of $(N\backslash\{i\})_0$.

We consider several cases:

1. Assume that $c^G_{kl}=c'^G_{kl}$ for all $l\in\{0,1,...,m\}$. Thus, $\varphi_l\left(G_0,C^G\right)=\varphi_l\left((G\backslash G^k)_0\cup(G^k\backslash\{i\}),C'^G\right)$ for all $l=1,...,m$. Hence, $\left((G^k\backslash\{i\})_0,C^\varphi\right)=\left((G^k\backslash\{i\})_0,C'^\varphi\right)$.

   - Since φ satisfies PM, $\varphi_j\left(G^k_0,C^\varphi\right)\le\varphi_j\left((G^k\backslash\{i\})_0,C^\varphi\right)$ for all $j\in G^k\backslash\{i\}$. Then,

     $$\begin{aligned}F_j(N_0,C,G) &= \varphi_j\left(G^k_0,C^\varphi\right)\le\varphi_j\left((G^k\backslash\{i\})_0,C^\varphi\right)\\ &= \varphi_j\left((G^k\backslash\{i\})_0,C'^\varphi\right)\\ &= F_j((N\backslash\{i\})_0,C,(G\backslash G^k)\cup(G^k\backslash\{i\}))\end{aligned}$$

     for all $j\in G^k\backslash\{i\}$.

- Let $G^l \in G$ such that $l \neq k$. Then, $c^{\varphi}_{jj'} = c'^{\varphi}_{jj'}$ for all $j, j' \in G^l \cup \{0\}$. Hence, $\varphi_j(G^l_0, C^{\varphi}) = \varphi_j(G^l_0, C'^{\varphi})$ for all $j \in G^l$. So,

$$\begin{aligned} F_j(N_0, C, G) &= \varphi_j(G^l_0, C^{\varphi}) = \varphi_j(G^l_0, C'^{\varphi}) \\ &= F_j((N \backslash \{i\})_0, C', (G \backslash G^k) \cup (G^k \backslash \{i\})) \end{aligned}$$

  for all $j \in G^l$.

2. Assume that $c^G_{kk'} \neq c'^G_{kk'}$ for some $k' \in \{0, 1, ..., m\}$. Then, $c^G_{kk'} < c'^G_{kk'}$. Moreover, $c^G_{ll'} \leq c'^G_{ll'}$ for all $l, l' \in \{0, 1, ..., m\}$.

   Since φ satisfies SCM, $\varphi_k\left(G_0, C^G\right) \leq \varphi_k\left(G_0, C'^G\right)$. Now, $c^{\varphi}_{jj'} = c'^{\varphi}_{jj'}$ for all $j, j' \in G^k \backslash \{i\}$ and $c^{\varphi}_{0j} \leq c'^{\varphi}_{0j}$ for all $j \in G^k \backslash \{i\}$. Since φ satisfies SCM, $\varphi_j\left((G^k \backslash \{i\})_0, C^{\varphi}\right) \leq \varphi_j\left((G^k \backslash \{i\})_0, C'^{\varphi}\right)$ for all $j \in G^k \backslash \{i\}$.

   Since φ satisfies PM, we have that $\varphi_j\left(G^k_0, C^{\varphi}\right) \leq \varphi_j\left((G^k \backslash \{i\})_0, C^{\varphi}\right)$. Then,

$$\begin{aligned} F_j(N_0, C, G) &= \varphi_j\left(G^k_0, C^{\varphi}\right) \leq \varphi_j\left((G^k \backslash \{i\})_0, C^{\varphi}\right) \\ &\leq \varphi_j\left((G^k \backslash \{i\})_0, C'^{\varphi}\right) \\ &= F_j((N \backslash \{i\})_0, C', (G \backslash G^k) \cup (G^k \backslash \{i\})) \end{aligned}$$

   for all $j \in G^k \backslash \{i\}$. ■

6.4 Proof of Proposition 3

Let f be a rule in *mcstp* with groups satisfying RA, $SYMG$, $SYMA$, PMG, PMA. We prove that $f = F$. We proceed with several claims.

Claim 1. If $G = \left\{\{i\}_{i \in N}\right\}$ or $G = \{\{N\}\}$, then $f(N_0, C, G) = \varphi(N_0, C)$.
Proof of Claim 1.
Let $G = \left\{\{i\}_{i \in N}\right\}$, *i.e.* each agent forms a group. Given an *mcstp* (N_0, C) we define $f'(N_0, C) = f(N_0, C, G)$. Then,

$$\sum_{i \in N} f'_i(N_0, C) = \sum_{i \in N} f_i(N_0, C, G) = m(N_0, C).$$

Hence, f' is a rule in *mcstp*.

Since f satisfies $SYMG$ in *mcstp* with groups, f' satisfies SYM in *mcstp*. Since f satisfies PMG in *mcstp* with groups, f' satisfies PM in *mcstp*. Moreover, f' also satisfies RA. By Lemma 0 (e), φ is the unique rule in *mcstp* satisfying SYM, RA, and PM. Thus,

$$f(N_0, C, G) = f'(N_0, C) = \varphi(N_0, C).$$

Let $G = \{N\}$, *i.e.* all agents are in the same group. Given an *mcstp* (N_0, C) we define $f'(N_0, C) = f(N_0, C, G)$. As above, f' is a rule in *mcstp*.

Since f satisfies $SYMA$ in *mcstp* with groups, f' satisfies SYM in *mcstp*. Since f satisfies PMA in *mcstp* with groups, f' satisfies PM in *mcstp*. Moreover, f' also satisfies RA. By Lemma 0 (e), φ is the unique rule in *mcstp* satisfying SYM, RA, and PM. Thus,

$$f(N_0, C, G) = f'(N_0, C) = \varphi(N_0, C). \blacksquare$$

Claim 2. For each *mcstp* with groups (N_0, C, G) and each $G^k \in G$, let (N_0', C', G') the problem obtained from (N_0, C, G) by considering that the rest of the groups have a unique agent whose connection cost to the rest of the agents is given by $\left(G_0, C^G\right)$. Namely, $N' = G^k \cup \left(\bigcup_{l \neq k} \{i_l\}\right)$, $G' = \left\{G^k, \{i_l\}_{l \neq k}\right\}$, and C' is defined as follows: if $i, j \in G^k \cup \{0\}$, then $c'_{ij} = c_{ij}$. If $i \in G^k$ and $j = i_l$ with $l \neq k$, then $c'_{ij} = c^G_{kl}$. If $i = 0$ and $j = i_l$ with $l \neq k$, then $c'_{ij} = c^G_{0l}$. If $i = i_l$, $j = i_{l'}$ and $k \notin \{l, l'\}$, then $c'_{ij} = c^G_{ll'}$.

Thus, for each $i \in G^k$

$$f_i(N_0, C, G) = f_i(N_0', C', G').$$

Proof of Claim 2.

Let (N_0, C, G) be an *mcstp* with groups and $G^k \in G$. We assume, *wlog*, that $k = m$.

We take $(N_0'^0, C'^0, G'^0) = (N_0, C, G)$. For each $l = 1, ..., m-1$ we define $(N_0'^l, C'^l, G'^l)$ as follows.

- $N'^l = N'^{l-1} \cup \{i_l\}$.
- $\left(G'^l\right)^l = \left(G'^{l-1}\right)^l \cup \{i_l\} = G^l \cup \{i_l\}$. For any $l' \neq l$, $\left(G'^l\right)^{l'} = \left(G'^{l-1}\right)^{l'}$.
- C'^l is defined as follows:

$$c'^l_{ij} = \begin{cases} c_{ij} & \text{if } i, j \in N_0'^{l-1} \\ 0 & \text{if } i = i_l,\, j \in G^l \\ c^G_{ll'} & \text{if } i = i_l,\, j \in G'^{l'},\, l \neq l' \end{cases}$$

For each $l = 1, ..., m-1$ and for each $l' \neq l$,

$$\min_{i \in (G'^l)^l, j \in (G'^l)^{l'}} \{c'^l_{ij}\} = \min_{i \in (G'^{l-1})^l, j \in (G'^{l-1})^{l'}} \{c'^{l-1}_{ij}\}.$$

Since f satisfies PMA, for all $i \in G^m$ and all $l = 1, ..., m-1$,

$$f_i(N_0'^{l-1}, C'^{l-1}, G'^{l-1}) = f_i(N_0'^l, C'^l, G'^l).$$

Now,

$$f_i(N_0, C, G) = f_i(N_0'^0, C'^0, G'^0) = f_i(N_0'^{m-1}, C'^{m-1}, G'^{m-1}).$$

Since f satisfies PMA, for all $i \in G^k$,

$$f_i(N_0'^{m-1}, C'^{m-1}, G'^{m-1}) = f_i(N_0', C', G'). \blacksquare$$

Claim 3. For each *mcstp* with groups (N_0, C, G) and each $G^k \in G$,

$$\sum_{i \in G^k} f_i(N_0, C, G) = f_k\left(G_0, C^G, \{G^l\}_{l=1}^m\right) + m\left(G^k, C\right).$$

Proof of Claim 3.

Consider the problem among groups $\left(G_0', C'^G, \{G'^l\}_{l=1}^m\right)$ associated with (N_0', C', G') as in Claim 2. It is trivial to see that C'^G coincides with C^G.

Applying Lemma 1 it is easy to deduce

$$m\left(N_0', C', G'\right) = m\left(G_0, C^G, \{G^l\}_{l=1}^m\right) + m(G^k, C).$$

By Claim 2, $\sum_{i \in G^k} f_i\left(N_0, C, G\right) = \sum_{i \in G^k} f_i\left(N_0', C', G'\right)$. Then,

$$\begin{aligned}
\sum_{l \neq k} f_{i_l}\left(N_0', C', G'\right) + \sum_{i \in G^k} f_i\left(N_0, C, G\right) &= \sum_{l \neq k} f_{i_l}\left(N_0', C', G'\right) + \sum_{i \in G^k} f_i\left(N_0', C', G'\right) \\
&= m\left(G_0, C^G, \{G^l\}_{l=1}^m\right) + m(G^k, C) \\
&= \sum_{l=1}^m f_l\left(G_0, C^G, \{G^l\}_{l=1}^m\right) + m(G^k, C).
\end{aligned}$$

Now, it is enough to prove that for all $l \neq k$,

$$f_{i_l}\left(N_0', C', G'\right) = f_l\left(G_0, C^G, \{G^l\}_{l=1}^m\right).$$

Let $l \neq k$. Applying Claim 2 to (N_0', C', G') with G'^l instead of G^k we obtain that

$$f_{i_l}\left(N_0', C', G'\right) = f_{i_l}\left(N_0'', C'', G''\right)$$

where $N'' = \{i_1, ..., i_m\}$, $c''_{i_j i_{j'}} = c^G_{G^j G^{j'}}$ for all $j, j' = 0, 1, ..., m$, and $G'' = \left\{\{i_j\}_{j=1}^m\right\}$. Notice that (N_0'', C'', G'') is equivalent to $\left(G_0, C^G, \{G^l\}_{l=1}^m\right)$. By Claim 1 we have

$$f_{i_l}\left(N_0'', C'', G''\right) = \varphi_{i_l}\left(N_0'', C''\right) = \varphi_l\left(G_0, C^G\right) = f_l\left(G_0, C^G, \{G^l\}_{l=1}^m\right). \blacksquare$$

Claim 4. It is enough to prove that f is unique on the subclass of *mcstp* (N_0, C, G) satisfying that there exists $x \in \mathbb{R}_+$ and a network g such that $c_{ij} = x$ if $(i, j) \in g$ and $c_{ij} = 0$ otherwise.

Proof of Claim 4.

Norde et al (2004) proved that if C is a cost matrix, then there exists a family $\{C^p\}_{p=1}^a$ of cost matrices satisfying three conditions:

1. $C = \sum_{p=1}^a C^p$.
2. For each $p \in \{1, ..., a\}$ there exist $x^p \in \mathbb{R}$ and a network g^p such that $c^p_{ij} = x_p$ if $(i, j) \in g^p$ and $c^p_{ij} = 0$ otherwise.
3. There exists $\sigma : \{(i, j)\}_{i, j \in N_0, i < j} \to \left\{1, 2, ..., \frac{n(n+1)}{2}\right\}$ such that if $i, j, k, l \in N$ with $i < j$, $k < l$, and $\sigma(i, j) < \sigma(k, l)$, then $c_{ij} \leq c_{kl}$ and $c^p_{ij} \leq c^p_{kl}$ for all $p \in \{1, ..., a\}$.

By condition 3, C^1 and $\sum_{p=2}^{a} C^p$ satisfy the conditions of the definition of RA. Then,

$$f(N_0, C, G) = f\left(N_0, C^1, G\right) + f\left(N_0, \sum_{p=2}^{a} C^p, G\right).$$

By condition 3, C^2 and $\sum_{p=3}^{a} C^p$ satisfy the conditions of the definition of RA. Then,

$$f\left(N_0, \sum_{p=2}^{a} C^p, G\right) = f\left(N_0, C^2, G\right) + f\left(N_0, \sum_{p=3}^{a} C^p, G\right).$$

Repeating the same argument we obtain that

$$f(N_0, C, G) = \sum_{p=1}^{a} f(N_0, C^p, G).$$

By condition 2, Claim 4 holds. ■

Let (N_0, C, G) be an *mcstp* with groups and $G^k \in G$. By Claim 2 we can assume that (N_0, C, G) has the same structure as the problem (N_0', C', G') defined in Claim 2.

By Claim 4 we can assume that there exists $x \in \mathbb{R}_+$ and a network g such that $c_{ij} = x$ if $(i, j) \in g$ and $c_{ij} = 0$ otherwise.

Claim 5. Let $G^k \in G$. Assume that there exists $l \in \{0, 1, ..., m\}$, $l \neq k$, and $i' \in G^k$ such that $c_{i'i_l} = 0$. Thus, for each $i \in G^k$,

$$f_i(N_0, C, G) = \frac{f_k\left(G_0, C^G, \{G^l\}_{l=1}^m\right)}{|G^k|}.$$

Proof of Claim 5.

We know that $\max_{i,j \in G^k} \{c_{ij}\} \leq \min_{i \in G^k, j \in N_0 \setminus G^k} \{c_{ij}\}$. Since $\min_{i \in G^k, j \in N_0 \setminus G^k} \{c_{ij}\}$ $\leq c_{i'i_l} = 0$, we have that $c_{ij} = 0$ for all $i, j \in G^k$. Therefore, $m(G^k, C) = 0$. By Claim 3,

$$\sum_{i \in G^k} f_i(N_0, C, G) = f_k\left(G_0, C^G, \{G^l\}_{l=1}^m\right).$$

Moreover, $c_{0i} \in \{0, x\}$ for all $i \in G^k$. Three cases are possible:

1. $c_{0i} = 0$ for all $i \in G^k$. Thus, all agents in G^k are symmetric. Since f satisfies $SYMA$, for all $i \in G^k$

$$f_i(N_0, C, G) = \frac{f_k\left(G_0, C^G, \{G^l\}_{l=1}^m\right)}{|G^k|} = 0.$$

2. $c_{0i} = x$ for all $i \in G^k$. Thus, all agents in G^k are symmetric. Since f satisfies $SYMA$, for all $i \in G^k$

$$f_i(N_0, C, G) = \frac{f_k\left(G_0, C^G, \{G^l\}_{l=1}^m\right)}{|G^k|}.$$

3. There exist $i_0 \in G^k$ such that $c_{0i_0} = 0$ and $i_x \in G^k$ such that $c_{0i_x} = x$.

   Then, $c^G_{0k} = \min_{i \in G^k} \{c_{0i}\} = 0$. Now it is not difficult to prove that for all $S \subset G_0$, $v_{C^{G*}}(S \cup \{k\}) - v_{C^{G*}}(S) = 0$. Thus, $\varphi_k\left(G_0, C^G\right) = Sh_k\left(G, v_{C^{G*}}\right) = 0$.

   By Claim 1, for all $i \in G^k$, $f_i\left(G^k, C, \{G^k\}\right) = \varphi_i\left(G^k, C\right) = 0$.

   By PMG, for all $i \in G^k$, $f_i(N_0, C, G) \leq f_i\left(G^k, C, \{G^k\}\right) = 0$.

   By Claim 1, $f_k\left(G_0, C^G, \{G^l\}_{l=0}^m\right) = \varphi_k\left(G_0, C^G\right) = 0$.

   Thus, for all $i \in G^k$, $f_i(N_0, C, G) = 0$. ■

Claim 6. Assume that for all $l \in \{0, 1, ..., m\}$ with $l \neq k$ and all $i \in G^k$, $c_{ii_l} = x$. Thus, for each $i \in G^k$,

$$f_i\left(N_0, C, G\right) = f_i\left(G^k_0, C, \{G^k\}\right).$$

Proof of Claim 6.

Let t' be an mt in $\left(G^k_0, C\right)$ and let t'' be an mt in $\left(\left(N \backslash G^k\right)_0, C\right)$. Following Prim's algorithm, we can construct an mt t in (N_0, C, G) such that $t = t' \cup t''$. Therefore,

$$m(N_0, C) = m\left(G^k_0, C\right) + m\left(\left(N \backslash G^k\right)_0, C\right).$$

Since f satisfies PMG, $f_i(N_0, C, G) \leq f_i\left(G^k_0, C, \{G^k\}\right)$ for all $i \in G^k$ and $f_i(N_0, C, G) \leq f_i\left(\left(N \backslash G^k\right)_0, C, \{G_l\}_{l \neq k}\right)$ for all $i \in N \backslash G^k$. Thus,

$$\begin{aligned} m(N_0, C) &= \sum_{i \in G^k} f_i(N_0, C, G) + \sum_{i \in N \backslash G^k} f_i(N_0, C, G) \\ &\leq \sum_{i \in G^k} f_i\left(G^k_0, C, \{G^k\}\right) + \sum_{i \in N \backslash G^k} f_i\left(\left(N \backslash G^k\right)_0, C, \{G_l\}_{l \neq k}\right) \\ &= m\left(G^k_0, C\right) + m\left(\left(N \backslash G^k\right)_0, C\right). \end{aligned}$$

Thus, $f_i(N_0, C, G) = f_i\left(G^k_0, C, \{G^k\}\right)$ for all $i \in G^k$. ■

Claim 7. For all $i \in G^k$, $f_i\left(N_0, C, G\right) = F_i\left(N_0, C, G\right)$.

Proof of Claim 7.

We distinguish two cases, given by claims 5 and 6.

1. There exists $l \in \{0, 1, ..., m\}$, $l \neq k$, and $i' \in G^k$ such that $c_{i'i_l} = 0$.

   Let $i \in G^k$. By Claim 5,

$$f_i\left(N_0, C, G\right) = \frac{f_k\left(G_0, C^G, \{G^l\}_{l=0}^m\right)}{|G^k|}.$$

   By Claim 1, $f_k\left(G_0, C^G, \{G^l\}_{l=0}^m\right) = \varphi_k(G_0, C^G)$. So,

$$f_i\left(N_0, C, G\right) = \frac{\varphi_k(G_0, C^G)}{|G^k|}.$$

Consider now the problem (G_0^k, C^φ) where $c_{jj'}^\varphi = c_{jj'}$ if $0 \notin \{j, j'\}$ and $c_{0j}^\varphi = \varphi_k(G_0, C^G)$ for all $j \in G^k$.

We have seen in the proof of Claim 5 that $c_{jj'} = 0$ for all $j, j' \in G^k$. Therefore, $m(G_0^k, C^\varphi) = \varphi_k(G_0, C^G)$ and all agents in G^k are symmetric in (G_0^k, C^φ). Since φ satisfies SYM,

$$\varphi_i(G_0^k, C^\varphi) = \frac{\varphi_k(G_0, C^G)}{|G^k|}.$$

Then,

$$f_i(N_0, C, G) = \varphi_i(G_0^k, C^\varphi) = F_i(N_0, C, G).$$

2. Assume that for all $l \in \{0, 1, ..., m\}$ with $l \neq k$ and all $i \in G^k$, $c_{ii_l} = x$.

   Let $i \in G^k$. By Claim 6, $f_i(N_0, C, G) = f_i\left(G_0^k, C, \left\{G^k\right\}\right)$. By Claim 1, $f_i\left(G_0^k, C, \left\{G^k\right\}\right) = \varphi_i\left(G_0^k, C\right)$. Thus,

$$f_i(N_0, C, G) = \varphi_i\left(G_0^k, C\right).$$

   Consider now the problem (G_0^k, C^φ). We know that $c_{jj'}^\varphi = c_{jj'}$ if $0 \notin \{j, j'\}$ and $c_{0j}^\varphi = \varphi_k(G_0, C^G)$ for all $j \in G^k$.

   For all $l \neq k$, $c_{kl}^G = x$. Now it is not difficult to prove that for all $S \subset G_0$, $v_{C^{G*}}(S \cup \{k\}) - v_{C^{G*}}(S) = x$. Thus, $\varphi_k\left(G_0, C^G\right) = Sh_k\left(G, v_{C^{G*}}\right) = x$.

   Hence, $(G_0^k, C^\varphi) = \left(G_0^k, C\right)$. Then,

$$f_i(N_0, C, G) = \varphi_i\left(G_0^k, C\right) = \varphi_i(G_0^k, C^\varphi) = F_i(N_0, C, G).$$

6.5 Proof of Remark 2

We prove that if we remove some of the properties of Theorem 1, we can find more rules satisfying the other properties. We do it by considering several claims. In each claim we define a rule satisfying four properties but failing the other. We do not make the proofs rigorously in order to do not enlarge the paper. We simply give an idea of the proof.

Claim 1. There exist rules satisfying RA, $SYMG$, $SYMA$, and PMG but failing PMA.

Proof of Claim 1. We define the rule f^1 as follows. Let (N_0, C, G) be an *mcstp* with groups and $i \in G^k \in G$. Thus,

$$f_i^1(N_0, C, G) = \frac{\varphi_k\left(G_0, C^G\right) + m(G^k, C)}{|G^k|}.$$

1. f^1 satisfies RA. Using arguments similar to those used in the proof of Claim 1 of Proposition 2 we can prove that

$$\varphi_k\left(G_0,(C+C')^G\right)=\varphi_k\left(G_0,C^G\right)+\varphi_k\left(G_0,C'^G\right)$$

and
$$m(G^k,C+C')=m(G^k,C)+m(G^k,C').$$

Now it is trivial to conclude that

$$f_i^1(N_0,C+C',G)=f_i^1(N_0,C,G)+f_i^1(N_0,C',G).$$

2. f^1 satisfies $SYMG$. Let G^k and $G^{k'}$ be two symmetric groups. Then, k and k' are symmetric agents in $\left(G_0,C^G\right)$. Since φ satisfies SYM, $\varphi_k\left(G_0,C^G\right)=\varphi_{k'}\left(G_0,C^G\right)$. Now,

$$\begin{aligned}\sum_{i\in G^k} f_i^1(N_0,C,G)-m(G^k,C) &= \varphi_k\left(G_0,C^G\right)\\ &= \varphi_{k'}\left(G_0,C^G\right)\\ &= \sum_{i\in G^{k'}} f_i^1(N_0,C,G)-m(G^{k'},C).\end{aligned}$$

3. f^1 satisfies $SYMA$. It is trivial.

4. f^1 satisfies PMG. Let $G^k\in G$. Since φ satisfies PM, $\varphi_l(G_0,C^G)\leq\varphi_l((G\backslash G^k)_0,C^G)$ for all $l\neq k$.

   Thus, for all $i\in G^l$, $l\neq k$,

$$\begin{aligned}f_i^1(N_0,C,G) &= \frac{\varphi_l\left(G_0,C^G\right)+m(G^l,C)}{|G^l|}\\ &\leq \frac{\varphi_l((G\backslash G^k)_0,C^G)+m(G^l,C)}{|G^l|}\\ &= f_i^1(\left(N\backslash G^k\right)_0,C,G\backslash G^k).\end{aligned}$$

5. f^1 fails PMA. Assume that $G=\{N\}$. Thus, f^1 divides $m\left(N_0,C\right)$ equally among the agents. In this case it is trivial to see that f^1 does not satisfy PMA.

Claim 2. There exist rules satisfying RA, $SYMG$, $SYMA$, and PMA but failing PMG.

Proof of Claim 2. We define the rule f^2 as follows. Let (N_0,C,G) be an *mcstp* with groups and $i\in G^k$. Thus,

$$f_i^2\left(N_0,C,G\right)=Sh_i\left(G^k,v_C^0\right)+\frac{m\left(G_0,C^G\right)}{m\left|G^k\right|}$$

where for all $S\subset G^k$, $v_C^0\left(S\right)=m\left(S,C^*_{|S}\right)$ and $C^*_{|S}$ is the irreducible matrix associated with the problem $\left(S,C\right)$.

1. f^2 satisfies RA. By Lemma 0 (d), for all $S \subset N$, $v_{(C+C')^*}(S) = v_{C^*}(S) + v_{C'^*}(S)$. Using similar arguments we can conclude that for all $S \subset G^k$, $v^0_{C+C'}(S) = v^0_C(S) + v^0_{C'}(S)$. Since Sh is additive on v, we conclude that

$$Sh_i\left(G^k, v^0_{C+C'}\right) = Sh_i\left(G^k, v^0_C\right) + Sh_i\left(G^k, v^0_{C'}\right).$$

   We have proved in the proof of Claim 1 of Proposition 2 that C^G and C'^G are also under the conditions of RA. Thus,

$$m\left(G_0, (C+C')^G\right) = m\left(G_0, C^G\right) + m\left(G_0, C'^G\right).$$

   Now, it is obvious that f^2 satisfies RA.

2. f^2 satisfies $SYMG$. Let G^k and $G^{k'}$ be two symmetric groups.

$$\begin{aligned}\sum_{i\in G^k} f_i^2(N_0,C,G) - m(G^k,C) &= \sum_{i\in G^k} Sh_i\left(G^k, v^0_C\right) + \frac{m\left(G_0, C^G\right)}{m} - m(G^k,C)\\ &= v^0_C\left(G^k\right) + \frac{m\left(G_0, C^G\right)}{m} - m(G^k,C)\\ &= m(G^k, C^*_{|G^k}) + \frac{m\left(G_0, C^G\right)}{m} - m(G^k,C)\\ &= m(G^k, C) + \frac{m\left(G_0, C^G\right)}{m} - m(G^k,C)\\ &= \frac{m\left(G_0, C^G\right)}{m}.\end{aligned}$$

   Analogously,

$$\sum_{i\in G^k} f_i^2(N_0,C,G) - m(G^k,C) = \frac{m\left(G_0, C^G\right)}{m}.$$

   Hence, f^2 satisfies $SYMG$.

3. f^2 satisfies $SYMA$. Let $i,j \in G^k$ be a pair of symmetric agents. It is trivial to see that i and j are also symmetric in $\left(G^k, v^0_C\right)$. Then, $Sh_i\left(G^k, v^0_C\right) = Sh_j\left(G^k, v^0_C\right)$. Hence, $f_i^2(N_0,C,G) = f_j^2(N_0,C,G)$.

4. f^2 satisfies PMA. Let $i \in G^k$ be such that $G^k \setminus \{i\} \neq \varnothing$.

   We know that for all $j \in G^k \setminus \{i\}$,

$$Sh_j\left(G^k, v^0_C\right) = \frac{1}{|G^k|!} \sum_{\pi \in \Pi_{G^k}} \left[v^0_C\left(Pre\left(j,\pi\right) \cup \{j\}\right) - v^0_C\left(Pre\left(j,\pi\right)\right)\right].$$

   By Lemma 0 (c), v_{C^*} is a concave game. Using similar arguments we can prove that v^0_C is a concave game. Then, for all $\pi \in \Pi_{G^k}$,

$$v^0_C\left(Pre\left(j,\pi\right) \cup \{j\}\right) - v^0_C\left(Pre\left(j,\pi\right)\right) \le v^0_C\left(\left(Pre\left(j,\pi\right) \setminus \{i\}\right) \cup \{j\}\right) - v^0_C\left(\left(Pre\left(j,\pi\right) \setminus \{i\}\right)\right).$$

Making some computations it is possible to prove that for all $j \in G^k \backslash \{i\}$,

$$Sh_j\left(G^k, v_C^0\right) \leq Sh_j\left(G^k \backslash \{i\}, v_C^0\right).$$

Let us denote $\left(N^{-i}, C, G^{-i}\right) = \left((N \backslash \{i\})_0, C, \left(G \backslash G^k\right) \cup \left(G^k \backslash \{i\}\right)\right)$. Then, $m\left(G_0^{-i}, C^{G^{-i}}\right) \geq m\left(G_0, C^G\right)$. Now, for all $j \in G^k \backslash \{i\}$,

$$\begin{aligned} f_j^2\left(N_0, C, G\right) &= Sh_j\left(G^k, v_C^0\right) + \frac{m\left(G_0, C^G\right)}{m\left|G^k\right|} \\ &\leq Sh_j\left(G^k \backslash \{i\}, v_C^0\right) + \frac{m\left(G_0^{-i}, C^{G^{-i}}\right)}{m\left(\left|G^k\right| - 1\right)} \\ &= f_j^2\left(N^{-i}, C, G^{-i}\right). \end{aligned}$$

Assume that for each G^l with $l \neq k$, $\min\limits_{j \in G^k, j' \in G^l} \{c_{jj'}\} = \min\limits_{j \in G^k \backslash \{i\}, j' \in G^l} \{c_{jj'}\}$. Let $j \in G^l \in G \backslash G^k$. Then

$$\begin{aligned} f_j^2\left(N_0, C, G\right) &= Sh_j\left(G^l, v_C^0\right) + \frac{m\left(G_0^{-i}, C^{G^{-i}}\right)}{m\left|G^l\right|} \\ &= Sh_j\left(G^l, v_C^0\right) + \frac{m\left(G_0, C^G\right)}{m\left|G^l\right|}. \end{aligned}$$

5. f^2 fails PMG. Assume that $G = \left\{\{i\}_{i \in N}\right\}$. Thus, f^2 divides $m\left(N_0, C\right)$ equally among the agents. Now it is trivial to see that f^2 does not satisfy PMG.

Claim 3. There exist rules satisfying RA, $SYMG$, PMG, and PMA but failing $SYMA$.

Proof of Claim 3. We define the rule f^3 as follows. Given $T \subset \mathcal{N}$, let π^N denote the order induced in N by the index of the agents. Namely, given $i, j \in N$, $\pi^N(i) < \pi^N(j)$ if and only if $i < j$. For each *mcstp* $\left(N_0, C\right)$ and $i \in N$ we define

$$\psi_i\left(N_0, C\right) = v_{C^*}\left(Pre\left(i, \pi^N\right) \cup \{i\}\right) - v_{C^*}\left(Pre\left(i, \pi^N\right)\right).$$

Let $\left(N_0, C, G\right)$ be an *mcstp* with groups and $i \in G^k$. Thus,

$$f_i^3\left(N_0, C, G\right) = \psi_i\left(G_0^k, C^\varphi\right).$$

1. f^3 satisfies RA. Let C and C' as in the definition of RA. Proceeding as in the proof of Claim 1 in Proposition 2, we obtain that (G_0^k, C^φ) and (G_0^k, C'^φ) are under the conditions of RA. Bergantiños and Vidal-Puga (2007d) proved that ψ satisfies RA. Therefore, $\psi_i(G_0^k, (C + C')^\varphi) = \psi_i(G_0^k, C^\varphi) + \psi_i(G_0^k, C'^\varphi)$ for all $i \in G^k$. Thus, given $i \in G^k \in G$,

$$\begin{aligned} f_i^3(N_0, C + C', G) &= \psi_i(G_0^k, (C + C')^\varphi) \\ &= \psi_i(G_0^k, C^\varphi) + \psi_i(G_0^k, C'^\varphi) \\ &= f_i^3(N_0, C, G) + f_i^3(N_0, C', G). \end{aligned}$$

2. f^3 satisfies $SYMG$. Let G^k and $G^{k'}$ be two symmetric groups. Then, k and k' are symmetric agents in $\left(G_0, C^G\right)$. Since φ satisfies SYM, $\varphi_k\left(G_0, C^G\right) = \varphi_{k'}\left(G_0, C^G\right)$.
   By Lemma 1 (iii), $m(G_0^k, C^\varphi) = \varphi_k(G_0, C^G) + m(G^k, C)$.
   Therefore,
$$\begin{aligned}\sum_{i \in G^k} f_i^3(N_0, C, G) - m(G^k, C) &= \sum_{i \in G^k} \psi_i(G_0^k, C^\varphi) - m(G^k, C) \\ &= m(G_0^k, C^\varphi) - m(G^k, C) \\ &= \varphi_k(G_0, C^G)\end{aligned}$$
   Proceeding in the same way for $G^{k'}$ we obtain that
$$\sum_{i \in G^{k'}} f_i^3(N_0, C, G) - m(G^{k'}, C) = \varphi_{k'}(G_0, C^G).$$
   Therefore, f^3 satisfies $SYMG$.

3. f^3 satisfies PMG. It is not difficult to prove that ψ satisfies SCM. Using arguments similar to those used in the proof of Claim 4 of Proposition 2, we can prove that f^3 satisfies PMG.

4. f^3 satisfies PMA. It is not difficult to prove that ψ satisfies PM. Using arguments similar to those used in the proof of Claim 5 of Proposition 2, we can prove that f^3 satisfies PMA.

5. f^3 fails $SYMA$. Consider the *mcstp* with groups where $N = \{1, 2, 3\}$, $G = \{G^1, G^2\}$, $G^1 = \{1, 2\}$, $G^2 = \{3\}$ and
$$C = \left(\begin{array}{cccc} 0 & 4 & 4 & 4 \\ 4 & 0 & 1 & 4 \\ 4 & 1 & 0 & 4 \\ 4 & 4 & 4 & 0 \end{array}\right).$$
   Agents 1 and 2 are symmetric. $\varphi_1(G_0, C^G) = \varphi_2(G_0, C^G) = 4$. Therefore, $c_{01}^\varphi = c_{02}^\varphi = 4$ and $c_{23}^\varphi = 1$. Now, $f_1^3(N_0, C, G) = 4$ and $f_2^3(N_0, C, G) = 1$.

Claim 4. There exist rules satisfying RA, $SYMA$, PMG, and PMA but failing $SYMG$.

Proof of Claim 4. We define the rule f^4 as follows. Let (N_0, C, G) be an *mcstp* with groups and $i \in G^k \in G$.

Let π' be an order over the set of all finite subsets of $\mathcal{N}$, π' induces an order over the elements of G. We also denote this order as π'. We define the rule ϕ over $\left(G_0, C^G\right)$. For each $G^l \in G$,
$$\phi_l\left(G_0, C^G\right) = v_{(C^G)^*}\left(Pre\left(l, \pi'\right) \cup \{l\}\right) - v_{(C^G)^*}\left(Pre\left(l, \pi'\right)\right).$$
Now,
$$f_i^4\left(N_0, C, G\right) = \varphi_i\left(G_0^k, C^\phi\right)$$
where
$$c_{jj'}^\phi = \begin{cases} c_{jj'} & \text{if } 0 \notin \{j, j'\} \\ \phi_k\left(G_0, C^G\right) & \text{if } 0 \in \{j, j'\}. \end{cases}$$

1. f^4 satisfies RA.

   We have proved in Claim 1 of Proposition 2 that $\left(G_0, C^G\right)$ and $\left(G_0, C'^G\right)$ are under the conditions of RA. Moreover, $\left(C+C'\right)^G = C^G + C'^G$.

   By Lemma 0 (d), $v_{(C+C')^*}(S) = v_{C^*}(S) + v_{C'^*}(S)$ for all $S \subset N$. So, for each $G^l \in G$,

   $$\phi_l\left(G_0, \left(C+C'\right)^G\right) = \phi_l\left(G_0, C^G\right) + \phi_l\left(G_0, C^G\right).$$

   By Lemma 0 (b), for all $S \subset N$, $v_{C^*}\left(S \cup \{i\}\right) - v_{C^*}\left(S\right) = \min_{j \in S_0}\{c^*_{ij}\}$. Therefore,

   $$\phi_k(G_0, C^G) \geq \min_{k' \in G_0 \setminus \{k\}}\{(c^G_{kk'})^*\}.$$

   Since the irreducible matrix is the minimal network associated with an mt,

   $$\min_{k' \in G_0 \setminus \{k\}}\{(c^G_{kk'})^*\} \geq \min_{k' \in G_0 \setminus \{k\}}\{c^G_{kk'}\}.$$

   Because of the definition of (N_0, C, G),

   $$\min_{k' \in G_0 \setminus \{k\}}\{c^G_{kk'}\} \geq \max_{jj' \in G^k}\{c_{jj'}\}.$$

   A similar result can be obtained for C'. Now, it is easy to conclude that $t^* = t_{G^k} \cup \{(0, i_j)\}$ with $i_j \in G^k$ is an mt in (G^k_0, C^ϕ), (G^k_0, C'^ϕ). Hence, (G^k_0, C^ϕ) and (G^k_0, C'^ϕ) are under the conditions of RA. Moreover, $C^\phi + C'^\phi = \left(C + C'\right)^\phi$. Since φ satisfies RA, for all $i \in G^k$,

   $$\begin{aligned} f^4_i\left(N_0, C+C', G\right) &= \varphi_i(G^k_0, (C+C')^\phi) \\ &= \varphi_i(G^k_0, C^\phi + C'^\phi) \\ &= \varphi_i\left(G^k_0, C^\phi\right) + \varphi_i\left(G^k_0, C'^\phi\right) \\ &= f^4_i\left(N_0, C, G\right) + f^4_i\left(N_0, C', G\right). \end{aligned}$$

2. f^4 satisfies $SYMA$. Let $i, j \in G^k \in G$ be symmetric agents in (N_0, C, G). By definition of C^ϕ, i and j are symmetric agents in $\left(G^k_0, C^\phi\right)$. Since φ satisfies SYM, $\varphi_i\left(G^k_0, C^\phi\right) = \varphi_j\left(G^k_0, C^\phi\right)$. Thus,

   $$f^4_i(N_0, C, G) = \varphi_i\left(G^k_0, C^\phi\right) = \varphi_j\left(G^k_0, C^\phi\right) = f^4_j(N_0, C, G).$$

3. f^4 satisfies PMG. Let $G^k \in G$. It is easy to prove that ϕ satisfies PM. Using arguments similar to those used in the proof of Claim 4 of Proposition 2 we can prove that f^4 satisfies PMG.

4. f^4 satisfies PMA.

   We first prove that ϕ satisfies SCM. By Lemma 0 (b), for all $S \subset N$, $v_{C^*}\left(S \cup \{i\}\right) - v_{C^*}\left(S\right) = \min_{j \in S_0}\{c^*_{ij}\}$. Bergantiños and Vidal-Puga (2007a) prove that if $C \leq C'$, then $C^* \leq C'^*$. Now, it is easy to conclude that ϕ satisfies SCM.

   Using arguments similar to those used in the proof of Claim 5 of Proposition 2 we can prove that f^4 satisfies PMA.

5. f^4 fails $SYMG$. Consider the *mcstp* with groups where $N = \{1, 2\}$, $G = \{G^1, G^2\}$, $G^1 = \{1\}$, $G^2 = \{2\}$ and

$$C = \begin{pmatrix} 0 & 10 & 10 \\ 10 & 0 & 2 \\ 10 & 2 & 0 \end{pmatrix}.$$

Assume that G^1 comes before than G^2 in π'. Groups G^1 and G^2 are symmetric and $m(G^1, C) = m(G^2, C) = 0$. Nevertheless

$$\begin{aligned} f_1^4(N_0, C, G) - m(G^1, C) &= \phi_1(G_0, C^G) = 10 \text{ and} \\ f_2^4(N_0, C, G) - m(G^2, C) &= \phi_2(G_0, C^G) = 2. \end{aligned}$$

Claim 5. There exist rules satisfying $SYMG$, $SYMA$, PMG, and PMA but failing RA.

Proof of Claim 5. We define the rule f^5 as follows. Let (N_0, C, G) be an *mcstp* with groups and $i \in G^k$.

We first define the rule σ over $\left(G_0, C^G\right)$. Let Π_G^e be the subset of permutations in which the groups with the expensive cost to the source connect first, *i.e.*

$$\Pi_G^e = \left\{\pi \in \Pi_G \mid c_{0\pi(l)}^G \leq c_{0\pi(l')}^G \text{ when } \pi(l) > \pi(l')\right\}.$$

For each $G^l \in G$, let σ be the rule defined as

$$\sigma_l\left(G_0, C^G\right) = \frac{1}{|\Pi_G^e|} \sum_{\pi \in \Pi_G^e} \left[v_{(C^G)^*}\left(Pre\left(l, \pi\right) \cup \{l\}\right) - v_{(C^G)^*}\left(Pre\left(l, \pi\right)\right)\right].$$

Now,

$$f_i^5\left(N_0, C, G\right) = \varphi_i\left(G_0^k, C^\sigma\right)$$

where

$$c_{jj'}^\sigma = \begin{cases} c_{jj'} & \text{if } 0 \notin \{j, j'\} \\ \sigma_k\left(G_0, C^G\right) & \text{if } 0 \in \{j, j'\}. \end{cases}$$

1. f^5 satisfies $SYMG$. It is trivial to see that σ satisfies SYM. Using arguments similar to those used in the proof of Claim 2 of Proposition 2 we can prove that f^5 satisfies $SYMG$.

2. f^5 satisfies $SYMA$. Using arguments similar to those used in the proof of Claim 3 of Proposition 2 we can prove that f^5 satisfies $SYMA$.

3. f^5 satisfies PMG. Using arguments similar to those used in Bergantiños and Vidal-Puga (2007a), it is possible to prove that σ satisfies PM. Using arguments similar to those used in the proof of Claim 4 of Proposition 2 we can prove that f^5 satisfies PMG.

4. f^5 satisfies PMA. Let $G^k \in G$ and $i \in G^k$. Let us denote as C' the cost matrix C restricted to the problem $((N \backslash \{i\})_0, C, (G \backslash G^k) \cup (G^k \backslash \{i\}))$ and $G' = (G \backslash G^k) \cup (G^k \backslash \{i\})$. We consider two cases:

(a) Assume that $c_{kl}^{G}=c_{kl}^{\prime G}$ for all $l\in\{0,1,...,m\}$. Thus, $\sigma_l\left(G_0,C^G\right)=\sigma_l\left((G\backslash G^k)_0\cup(G^k\backslash\{i\}),C^{\prime G}\right)$ for all $l=1,...,m$. Hence, $\left((G^k\backslash\{i\})_0,C^{\sigma}\right)=\left((G^k\backslash\{i\})_0,C^{\prime\sigma}\right)$. Moreover, σ satisfies PM. Using arguments similar to those used in the proof of Claim 5 of Proposition 2 we can prove that for all $j\in G^k\backslash\{i\}$,

$$f_j^5(N_0,C,G)=f_j^5((N\backslash\{i\})_0,C,(G\backslash G^k)\cup(G^k\backslash\{i\}))$$

and for all $G^l\in G$, $l\neq k$ and all $j\in G^l$

$$f_j^5(N_0,C,G)=f_j^5((N\backslash\{i\})_0,C,(G\backslash G^k)\cup(G^k\backslash\{i\})).$$

(b) Assume that $c_{kk'}^{G}\neq c_{kk'}^{\prime G}$ for some $k'\in\{0,1,...,m\}$. Then, $c_{kk'}^{G}<c_{kk'}^{\prime G}$. Moreover, $c_{0l}^{G}=c_{0l}^{\prime G}$ for all $l\neq k$.
This means that $\Pi_{G'}^{e}\subset\Pi_{G}^{e}$. Moreover, if $\pi\in\Pi_G^e$ and $\pi\notin\Pi_{G'}^e$ there exists $\pi'\in\Pi_{G'}^e$ such that $\pi'_{G\backslash G^k}=\pi_{G\backslash G^k}$ and $\pi'(k)<\pi(k)$. Intuitively, group k comes first in the orders of $\Pi_{G'}^e$ than in the orders of Π_G^e.
By Lemma 0 (c), for each cost matrix C, v_{C^*} is a concave game. Making some computations it is possible to prove that $\sigma_k\left(G_0,C^G\right)\leq\sigma_k\left(G_0,C^{\prime G}\right)$.
Now, using arguments similar to those used in the proof of Claim 5 of Proposition 2 we can prove that for all $j\in G^k\backslash\{i\}$,

$$f_j^5(N_0,C,G)\leq f_j^5((N\backslash\{i\})_0,C,(G\backslash G^k)\cup(G^k\backslash\{i\})).$$

5. f^5 fails RA. Consider the *mcstp* with groups where $N=\{1,2\}$, $G=\{G^1,G^2\}$, $G^1=\{1\}$, $G^2=\{2\}$

$$C=\begin{pmatrix}0&10&10\\10&0&2\\10&2&0\end{pmatrix}\text{ and }C'=\begin{pmatrix}0&10&12\\10&0&2\\12&2&0\end{pmatrix}.$$

If we take $t=\{(0,1),(1,2)\}$ we realize that C and C' are under the conditions of RA.

Now $\Pi_G^e(C)=\{12,21\}$, $\Pi_G^e(C')=\{21\}$, $\Pi_G^e(C+C')=\{21\}$. Thus, $f^5(N_0,C,G)=(6,6)$, $f^5(N_0,C',G)=(2,10)$, and $f^5(N_0,C+C',G)=(4,20)$.

6.6 Proof of Theorem 2

Let (N_0,C,G) be an *mcstp* with groups and $i\in G^k\in G$.

Let (N_0',C',G') the problem obtained from (N_0,C,G) as in Claim 2 of the proof of Proposition 3. Namely, $N'=G^k\cup\left(\bigcup_{l\neq k}\{i_l\}\right)$, $G'=\left\{G^k,\{i_l\}_{l\neq k}\right\}$, and C' is defined as follows: if $i,j\in G^k\cup\{0\}$, then $c_{ij}'=c_{ij}$. If $i\in G^k$ and $j=i_l$ with $l\neq k$, then $c_{ij}'=c_{kl}^G$. If $i=0$ and $j=i_l$ with $l\neq k$, then $c_{ij}'=c_{0l}^G$. If $i=i_l$, $j=i_{l'}$ and $k\notin\{l,l'\}$, then $c_{ij}'=c_{ll'}^G$.

We know that
$$F_i(N_0, C, G) = F_i(N'_0, C', G').$$
We proceed with several claims.

Claim 1. $Ow_i(N, v_{C^*}, G) = Ow_i(N', v_{C'^*}, G')$.
Proof of Claim 1.
We know that
$$\begin{aligned} Ow_i(N, v_{C^*}, G) &= \frac{1}{|\Pi^G|} \sum_{\pi \in \Pi^G} \left[v_{C^*}(Pre(i,\pi) \cup \{i\}) - v_{C^*}(Pre(i,\pi)) \right] \text{ and} \\ Ow_i(N', v_{C'^*}, G') &= \frac{1}{|\Pi^{G'}|} \sum_{\pi' \in \Pi^{G'}} \left[v_{C'^*}(Pre(i,\pi') \cup \{i\}) - v_{C'^*}(Pre(i,\pi')) \right]. \end{aligned}$$

For each $\pi' \in \Pi^{G'}$ let $O(\pi')$ denote the set of orders of Π^G inducing the same order than π' among the agents in G^k and among the groups. Namely, $O(\pi')$ is the set of orders $\pi \in \Pi^G$ satisfying two conditions:

1. $\pi_{G^k} = \pi'_{G^k}$.
2. Given $j \in G^l$, $j' \in G^{l'}$, $k \notin \{l, l'\}$ we have that $\pi(j) < \pi(j')$ if and only if $\pi'(i_l) < \pi'(i_{l'})$.

Thus, for all $\pi' \in \Pi^{G'}$, $|O(\pi')| = \prod_{l \neq k} (|G^l|!)$.

We now prove that given $\pi' \in \Pi^{G'}$ and $\pi \in O(\pi')$, we have that
$$v_{C'^*}(Pre(i,\pi') \cup \{i\}) - v_{C'^*}(Pre(i,\pi')) = v_{C^*}(Pre(i,\pi) \cup \{i\}) - v_{C^*}(Pre(i,\pi)).$$
By Lemma 0 (b)
$$\begin{aligned} v_{C'^*}(Pre(i,\pi') \cup \{i\}) - v_{C'^*}(Pre(i,\pi')) &= \min_{j \in Pre(i,\pi')_0} \left\{ c'^*_{ij} \right\} \text{ and} \\ v_{C^*}(Pre(i,\pi) \cup \{i\}) - v_{C^*}(Pre(i,\pi)) &= \min_{j \in Pre(i,\pi)_0} \left\{ c^*_{ij} \right\}. \end{aligned}$$

We consider two cases:

- $Pre(i,\pi') \cap G^k \neq \varnothing$.
We know that for all $l = 1, ..., m$, $\max_{j,j' \in G^l} \{c_{jj'}\} \leq \min_{j \in G^l, j' \notin G^l} \{c_{jj'}\}$. Because of the definition of the irreducible matrix as the minimal network associated with the minimal tree given by Lemma 1, it is easy to deduce that for all $l = 1, ..., m$, $\max_{j,j' \in G^l} \left\{ c^*_{jj'} \right\} \leq \min_{j \in G^l, j' \notin G^l} \left\{ c^*_{jj'} \right\}$. Now,
$$v_{C^*}(Pre(i,\pi) \cup \{i\}) - v_{C^*}(Pre(i,\pi)) = \min_{j \in (Pre(i,\pi) \cap G^k)_0} \left\{ c^*_{ij} \right\}.$$
Analogously,
$$v_{C'^*}(Pre(i,\pi') \cup \{i\}) - v_{C'^*}(Pre(i,\pi')) = \min_{j \in (Pre(i,\pi') \cap G^k)_0} \left\{ c'^*_{ij} \right\}.$$
Because of the definition of C', $c^*_{ij} = c'^*_{ij}$ for all $j \in G^k$. Since $\pi \in O(\pi')$, $Pre(i,\pi) \cap G^k = Pre(i,\pi') \cap G^k$. Then, the result holds.

- $Pre\left(i,\pi'\right)\cap G^{k}=\varnothing$.

  Let t be the mt given by Lemma 1. We can compute C^{*} as the minimal network associated with t.

  By Lemma 1, we know that t_{G^k} is an mt in $\left(G^{k},C\right)$ and $t\backslash\left(\cup_{l=1}^{m}t_{G^l}\right)$ is an mt in $\left(G_0,C^{G}\right)$. Now it is easy to deduce that $t'=t_{G^k}\cup t\backslash\left(\cup_{l=1}^{m}t_{G^l}\right)$ induces an mt in $\left(N_0',C',G'\right)$. Then, we can compute C'^{*} as the minimal network associated with t'.

  Since $Pre\left(i,\pi'\right)\cap G^{k}=\varnothing$, we can assume that $\min\limits_{j\in Pre(i,\pi')_0}\left\{c'^{*}_{ij}\right\}=c'^{*}_{ii_l}$ with $l\neq k$ ($i_l=i_0=0$ is also possible). Let g_{ii_l} be the unique path in t' joining i and i_l. Then, $c'^{*}_{ii_l}=c'_{i_ai_b}$ where $(i_a,i_b)\in g_{ii_l}$. By definition of C', $c'_{i_ai_b}=c_{j_aj_b}$ where $j_a\in G^{a}$ and $j_b\in G^{b}$.

  Since $\min\limits_{j\in Pre(i,\pi')_0}\left\{c'^{*}_{ij}\right\}=c'^{*}_{ii_l}$, $G^{l}\subset Pre\left(i,\pi\right)_0$. Now, there exists $j_l\in G^{l}$ such that

  $$\min_{j\in Pre(i,\pi)_0}\left\{c^{*}_{ij}\right\}=c^{*}_{ij_l}.$$

  Because of the definition of C^{*} as the minimal network associated with t we have that (j_a,j_b) belongs to the unique path in t joining i and j_l. Thus,

  $$\min_{j\in Pre(i,\pi)_0}\left\{c^{*}_{ij}\right\}=c^{*}_{ij_l}\geq c_{j_aj_b}=\min_{j\in Pre(i,\pi')_0}\left\{c'^{*}_{ij}\right\}.$$

  Using arguments similar to those used above we can prove that $\min\limits_{j\in Pre(i,\pi)_0}\left\{c^{*}_{ij}\right\}\leq\min\limits_{j\in Pre(i,\pi')_0}\left\{c'^{*}_{ij}\right\}$.

It is easy to see that, $\left|\Pi^{G}\right|=m!\left(\prod\limits_{l=1}^{m}\left(\left|G^{l}\right|!\right)\right)$, $\left|\Pi^{G'}\right|=m!\left(\left|G^{k}\right|!\right)$, and for each $\pi'\in\Pi^{G'}$, $\left|O\left(\pi'\right)\right|=\prod\limits_{l\neq k}\left(\left|G^{l}\right|!\right)$. Thus,

$$\begin{aligned}
Ow_i\left(N,v_{C^*},G\right) &= \frac{1}{\left|\Pi^{G}\right|}\sum_{\pi'\in\Pi^{G'}}\sum_{\pi\in O(\pi')}\left[v_{C^*}\left(Pre\left(i,\pi\right)\cup\{i\}\right)-v_{C^*}\left(Pre\left(i,\pi\right)\right)\right]\\
&= \frac{1}{\left|\Pi^{G}\right|}\sum_{\pi'\in\Pi^{G'}}\sum_{\pi\in O(\pi')}\left[v_{C'^*}\left(Pre\left(i,\pi'\right)\cup\{i\}\right)-v_{C'^*}\left(Pre\left(i,\pi'\right)\right)\right]\\
&= \frac{1}{\left|\Pi^{G}\right|}\sum_{\pi'\in\Pi^{G'}}\left(\prod_{l\neq k}\left|G^{l}\right|!\right)\left[v_{C'^*}\left(Pre\left(i,\pi'\right)\cup\{i\}\right)-v_{C'^*}\left(Pre\left(i,\pi'\right)\right)\right]\\
&= \frac{1}{\left|\Pi^{G'}\right|}\sum_{\pi'\in\Pi^{G'}}\left[v_{C'^*}\left(Pre\left(i,\pi'\right)\cup\{i\}\right)-v_{C'^*}\left(Pre\left(i,\pi'\right)\right)\right]\\
&= Ow_i\left(N',v_{C'^*},G'\right).\ \blacksquare
\end{aligned}$$

Thus, we can assume that (N_0,C,G) satisfies that $\left|G^{l}\right|=1$ for all $l\neq k$.

Claim 2. Let $\pi \in \Pi^G$ such that $Pre(i,\pi) \cap G^k \neq \varnothing$. Thus,

$$v_{C^*}(Pre(i,\pi) \cup \{i\}) - v_{C^*}(Pre(i,\pi)) = v_{(C^\varphi)^*}(Pre(i,\pi_{G^k}) \cup \{i\}) - v_{(C^\varphi)^*}(Pre(i,\pi_{G^k})).$$

Proof of Claim 2.
We have seen in the proof of Claim 1 that

$$v_{C^*}(Pre(i,\pi) \cup \{i\}) - v_{C^*}(Pre(i,\pi)) = \min_{j \in (Pre(i,\pi) \cap G^k)_0} \{c^*_{ij}\}.$$

By Lemma 0 (b),

$$v_{(C^\varphi)^*}(Pre(i,\pi_{G^k}) \cup \{i\}) - v_{(C^\varphi)^*}(Pre(i,\pi_{G^k})) = \min_{j \in Pre\left(i,\pi_{G^k}\right)_0} \left\{c^{\varphi *}_{ij}\right\}.$$

Let t be an mt as in Lemma 1. Thus, $t^k = t_{G^k} \cup \{(0,i)\}$ is an mt in $\left(G^k_0, C^\varphi\right)$. Because of the proof of Lemma 1, for all $(j,j') \in t^k$, $c^\varphi_{jj'} \leq c^\varphi_{0i}$. Since $(C^\varphi)^*$ is the minimal network associated with t^k, we deduce that

$$v_{(C^\varphi)^*}(Pre(i,\pi_{G^k}) \cup \{i\}) - v_{(C^\varphi)^*}(Pre(i,\pi_{G^k})) = \min_{j \in \left(Pre\left(i,\pi_{G^k}\right) \cap G^k\right)_0} \left\{c^{\varphi *}_{ij}\right\}.$$

Since $Pre(i,\pi_{G^k}) \cap G^k = Pre(i,\pi) \cap G^k$, it is enough to prove that for all $j \in Pre(i,\pi) \cap G^k$, $c^{\varphi *}_{ij} = c^*_{ij}$. Let $j \in G^k$.

We know that $(C^\varphi)^*$ is the minimal network associated with t^k and C^* is the minimal network associated with t. Let g^φ_{ij} denote the unique path in t^k joining i and j. Let g_{ij} denote the unique path in t joining i and j. By Lemma 1, t_{G^k} is a tree in $\left(G^k, C^\varphi\right)$. Since $t^k_{G^k} = t_{G^k}$, we deduce that $g^\varphi_{ij} = g_{ij} \subset t_{G^k}$. Then,

$$c^{\varphi *}_{ij} = \max_{(a,b) \in g^\varphi_{ij}} \{c^\varphi_{ab}\} = \max_{(a,b) \in g_{ij}} \{c^\varphi_{ab}\}.$$

By definition of C^φ, $c^\varphi_{ij'} = c_{ij'}$ for all $j' \in G^k$. Now,

$$c^{\varphi *}_{ij} = \max_{(a,b) \in g_{ij}} \{c_{ab}\} = c^*_{ij}. \blacksquare$$

Claim 3. Let $\pi \in \Pi^G$ such that $Pre(i,\pi) \cap G^k = \varnothing$. Let π' denote the order induced by π among groups in G. Namely, $\pi'(l) < \pi'(l')$ if and only if there exist $j \in G^l$ and $j' \in G^{l'}$ such that $\pi(j) \leq \pi(j')$. Since $\pi \in \Pi^G$, π' is well defined.

Thus,

1. $v_{C^*}(Pre(i,\pi) \cup \{i\}) - v_{C^*}(Pre(i,\pi)) = v_{(C^G)^*}(Pre(k,\pi') \cup \{k\}) - v_{(C^G)^*}(Pre(k,\pi'))$.
2. $v_{(C^\varphi)^*}(Pre(i,\pi_{G^k}) \cup \{i\}) - v_{(C^\varphi)^*}(Pre(i,\pi_{G^k})) = \varphi_k\left(G_0, C^G\right)$.

Proof of Claim 3.

1. By Lemma 0 (b),

$$\begin{aligned} v_{C^*}(Pre(i,\pi) \cup \{i\}) - v_{C^*}(Pre(i,\pi)) &= \min_{l \in Pre(i,\pi)_0} \{c^*_{il}\} \text{ and} \\ v_{(C^G)^*}(Pre(k,\pi') \cup \{k\}) - v_{(C^G)^*}(Pre(k,\pi')) &= \min_{l \in Pre(k,\pi')_0} \left\{c^{G*}_{kl}\right\}. \end{aligned}$$

It is obvious that $Pre(i,\pi) = Pre(k,\pi')$. Let t be an mt as in Lemma 1. Thus, $t^G = t \backslash \left(\cup_{l=1}^m t_{G^l}\right)$ is an mt in $\left(G_0, C^G\right)$. Using arguments similar to those used in the proof of Claim 2, we can prove that for all $l \in Pre(i,\pi)$, $c_{il}^* = c_{kl}^{G*}$.

2. By Lemma 0 (b),

$$v_{(C^\varphi)^*}\left(Pre\left(i,\pi_{G^k}\right)\cup\{i\}\right) - v_{(C^\varphi)^*}\left(Pre\left(i,\pi_{G^k}\right)\right) = \min_{j \in Pre\left(i,\pi_{G^k}\right)_0} \left\{c_{ij}^{\varphi *}\right\}.$$

Since $Pre(i,\pi) \cap G^k = \varnothing$, $Pre\left(i,\pi_{G^k}\right)_0 = \{0\}$. Thus,

$$v_{(C^\varphi)^*}\left(Pre\left(i,\pi_{G^k}\right)\cup\{i\}\right) - v_{(C^\varphi)^*}\left(Pre\left(i,\pi_{G^k}\right)\right) = c_{0i}^{\varphi *}.$$

By Lemma 1 (iii), $t^k = t_{G^k} \cup \{(0,i)\}$ is an mt in $\left(G_0^k, C^\varphi\right)$. Since $C^{\varphi *}$ is the minimal network associated with t^k,

$$c_{0i}^{\varphi *} = c_{0i}^{\varphi} = \varphi_k\left(G_0, C^G\right). \blacksquare$$

Claim 4. $F_i(N_0, C, G) = Ow_i(N, v_{C^*}, G)$.
Proof of Claim 4.
We know that

$$\begin{aligned} F_i(N_0,C,G) &= \varphi_i\left(G_0^k, C^\varphi\right) = Sh_i\left(G^k, v_{(C^\varphi)^*}\right) \\ &= \frac{1}{\left|\Pi_{G^k}\right|}\sum_{\pi\in\Pi_{G^k}}\left[v_{(C^\varphi)^*}\left(Pre(i,\pi)\cup\{i\}\right) - v_{(C^\varphi)^*}\left(Pre(i,\pi)\right)\right]. \end{aligned}$$

Let X_1^k, X_2^k the partition of Π_{G^k} where

$$\begin{aligned} X_1^k &= \left\{\pi\in\Pi_{G^k} : Pre(i,\pi)\cap G^k \neq \varnothing\right\} \text{ and} \\ X_2^k &= \left\{\pi\in\Pi_{G^k} : Pre(i,\pi)\cap G^k = \varnothing\right\}. \end{aligned}$$

Since $\left|\Pi_{G^k}\right| = \left|G^k\right|!$,

$$\begin{aligned} F_i(N_0,C,G) &= \frac{1}{|G^k|!}\sum_{\pi\in X_1^k}\left[v_{(C^\varphi)^*}\left(Pre(i,\pi)\cup\{i\}\right) - v_{(C^\varphi)^*}\left(Pre(i,\pi)\right)\right] \\ &\quad + \frac{1}{|G^k|!}\sum_{\pi\in X_2^k}\left[v_{(C^\varphi)^*}\left(Pre(i,\pi)\cup\{i\}\right) - v_{(C^\varphi)^*}\left(Pre(i,\pi)\right)\right]. \end{aligned}$$

By Claim 3.2,

$$\begin{aligned} \frac{1}{|G^k|!}\sum_{\pi\in X_2^k}\left[v_{(C^\varphi)^*}\left(Pre(i,\pi)\cup\{i\}\right) - v_{(C^\varphi)^*}\left(Pre(i,\pi)\right)\right] &= \frac{1}{|G^k|!}\left|X_2^k\right|\varphi_k\left(G_0,C^G\right) \\ &= \frac{1}{|G^k|}\varphi_k\left(G_0,C^G\right). \end{aligned}$$

We know that

$$Ow_i(N, v_{C^*}, G) = \frac{1}{|\Pi^G|}\sum_{\pi\in\Pi^G}\left[v_{C^*}\left(Pre(i,\pi)\cup\{i\}\right) - v_{C^*}\left(Pre(i,\pi)\right)\right].$$

Let X_1, X_2 the partition of Π^G where

$$\begin{aligned} X_1 &= \left\{\pi \in \Pi^G : Pre\,(i,\pi) \cap G^k \neq \varnothing\right\} \text{ and} \\ X_2 &= \left\{\pi \in \Pi^G : Pre\,(i,\pi) \cap G^k = \varnothing\right\}. \end{aligned}$$

Since $\left|\Pi^G\right| = m! \left|G^k\right|!$,

$$\begin{aligned} Ow_i\,(N, v_{C^*}, G) &= \frac{1}{m!\,|G^k|!} \sum_{\pi \in X_1} \left[v_{C^*}\,(Pre\,(i,\pi) \cup \{i\}) - v_{C^*}\,(Pre\,(i,\pi))\right] \\ &\quad + \frac{1}{m!\,|G^k|!} \sum_{\pi \in X_2} \left[v_{C^*}\,(Pre\,(i,\pi) \cup \{i\}) - v_{C^*}\,(Pre\,(i,\pi))\right]. \end{aligned}$$

By Claim 2,

$$\frac{1}{m!\,|G^k|!} \sum_{\pi \in X_1} \left[v_{C^*}\,(Pre\,(i,\pi) \cup \{i\}) - v_{C^*}\,(Pre\,(i,\pi))\right]$$

$$= \frac{1}{m!\,|G^k|!} \sum_{\pi \in X_1} \left[v_{(C^\varphi)^*}\,(Pre\,(i,\pi_{G^k}) \cup \{i\}) - v_{(C^\varphi)^*}\,(Pre\,(i,\pi_{G^k}))\right].$$

For each $\pi^k \in \Pi_{G^k}$, $\left|\left\{\pi \in X_1 : \pi_{G^k} = \pi^k\right\}\right| = m!$. Thus, the last expression coincides with

$$= \frac{1}{|G^k|!} \sum_{\pi^k \in X_1^k} \left[v_{(C^\varphi)^*}\left(Pre\left(i,\pi^k\right) \cup \{i\}\right) - v_{(C^\varphi)^*}\left(Pre\left(i,\pi^k\right)\right)\right].$$

Let Π_G denote the set of all orders of the m groups $\{G_1, ..., G_m\}$. Given $\pi \in \Pi_N$, let π' denote the order induced by π among the groups (as in Claim 3). For each $\pi_G \in \Pi_G$,

$$|\{\pi \in X_2 : \pi' = \pi_G\}| = \left(\left|G^k\right| - 1\right)!$$

By Claim 3.1,

$$\frac{1}{m!\,|G^k|!} \sum_{\pi \in X^2} \left[v_{C^*}\,(Pre\,(i,\pi) \cup \{i\}) - v_{C^*}\,(Pre\,(i,\pi))\right]$$

$$\begin{aligned} &= \frac{1}{m!\,|G^k|!} \sum_{\pi \in X^2} \left[v_{(C^G)^*}\,(Pre\,(k,\pi') \cup \{k\}) - v_{(C^G)^*}\,(Pre\,(k,\pi'))\right] \\ &= \frac{1}{m!\,|G^k|} \sum_{\pi' \in \Pi_G} \left[v_{(C^G)^*}\,(Pre\,(k,\pi') \cup \{k\}) - v_{(C^G)^*}\,(Pre\,(k,\pi'))\right] \\ &= \frac{1}{|G^k|} \varphi_k\left(G_0, C^G\right). \end{aligned}$$

Then, $F_i\,(N_0, C, G) = Ow_i\,(N, v_{C^*}, G)$. ■

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