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A repetitive controller for discrete-time passive systems

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A Repetitive Controller for Discrete-Time Passive Systems

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Abstract

This work proposes and studies a new repetitive controller for discrete-time systems which are required to track or to attenuate periodic signals. The main characteristic of the proposed controller is its passivity. This fact implies closed-loop stable behavior when it is used with discrete-time passive plants. The work also discusses the energetic structure, the frequency response and the time response of the proposed controller structure. Some examples are included to illustrate its practical use.

CONTENTS

I	Introduction	2
II	Basic Cell	3
III	The energetic structure of the repetitive cell	8
IV	Frequency response of the repetitive cell	11
V	Time Response	13
VI	Closed-loop Numerical Example	16
	VI-A Example 1	16
	VI-B Example 2 : the odd-harmonic case	20
VII	Conclusion	23
	References	23

I. INTRODUCTION

The concept of repetitive control has been largely used in different control areas such as CD and disk arm actuators [1], robotics [2], electro-hydraulics [3], electronic rectifiers [4], pulse-width modulated (PWM) inverters [5], [6], and current harmonics active filters [7].

It is known that the usual repetitive controller cell causes a great increase in the order of the system and, then, the closed-loop stability is difficult to analyze using algebraic methods. Normally, the stability study of these systems is based on splitting the closed-loop system in three series-connected subsystems which are required to be stable. For two of them, checking the stability is a trivial matter but, for the remaining system the Small Gain Theorem is used to assure its stability [8], [9], [16], [10]. This last step takes only into account the norm so, it may be very restrictive in some cases.

Moreover, these repetitive cells have a big relative degree that implies a slow response time in front of sudden changes of the periodical references to track or disturbances to attenuate.

This work presents a new repetitive cell¹. This new cell has the same interesting properties than the traditional one: it introduces infinite (or high gain) at harmonic frequencies, it can be used as an odd-harmonic repetitive controller [10], it has a simple structure and it has a very low computational cost. However, in contrast to the traditional repetitive controllers, the proposed one has passivity structure (equivalently, it is Positive Real), so it only introduces a limited phase loss. As a consequence it can be used to reject/follow periodic signals in discrete-time passive plants, or plants that can be passivized using feedback [11], without worrying about the overall system closed-loop stability [12].

¹In this work the terms *repetitive controller* and *repetitive cell* are used as a synonyms.

II. BASIC CELL

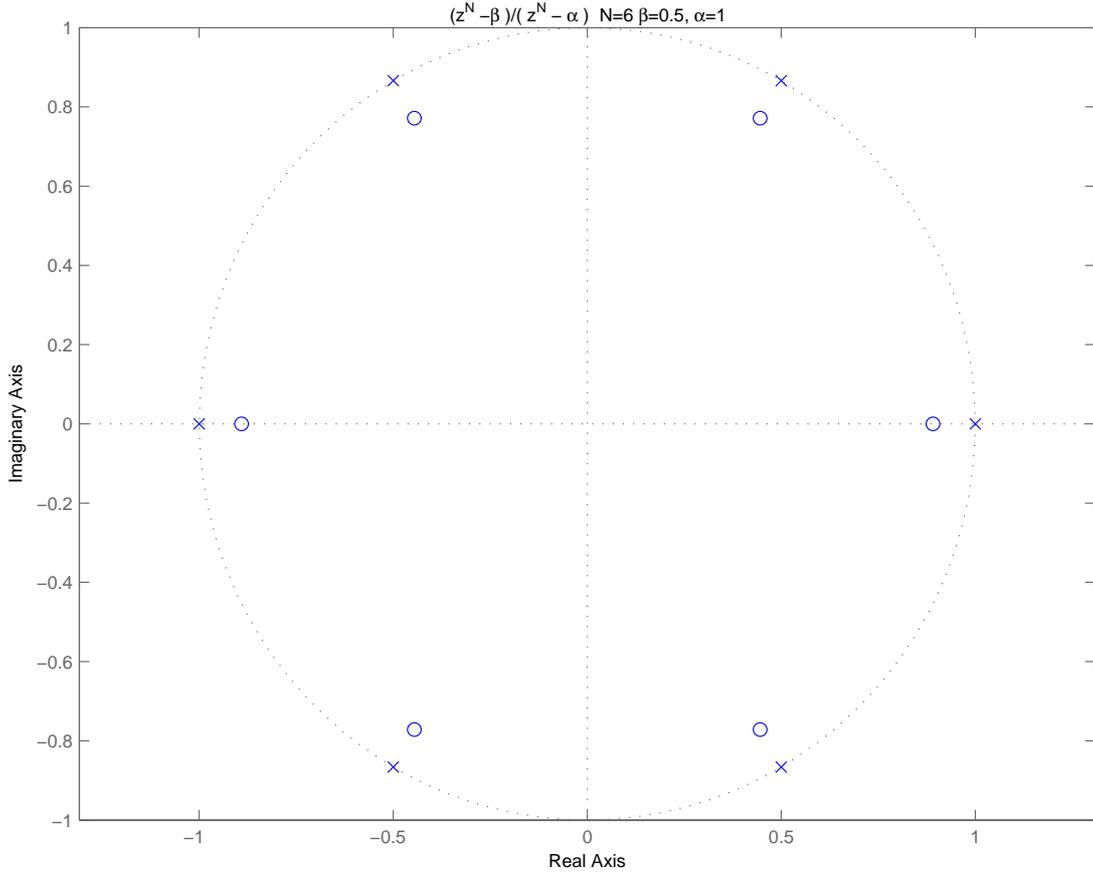


Fig. 1. Pole Zero map for α and β of the same sign ($N = 6, \beta = 0.5, \alpha = 1$)

Fig. 3 shows the block-diagram of the proposed repetitive cell. Its input-output behavior is described by the transfer function

$$C(z) = \frac{Y(z)}{U(z)} = k \frac{z^N - \beta}{z^N - \alpha} \quad (1)$$

It is not difficult to see that the poles of (1) are²:

$$p_k = \sqrt[N]{|\alpha|} e^{j\left(\frac{2\pi}{N}k + \frac{\pi(1-\text{sign}(\alpha))}{2N}\right)}, \quad k = 0, 1, \dots, N-1, \quad (2)$$

so they are uniformly distributed over a circumference of radius³ $\sqrt[N]{|\alpha|}$. The frequencies associated to the poles are $\omega_k = \frac{2\pi}{N}k + \frac{\pi(1-\text{sign}(\alpha))}{2N}$, so the poles are placed covering all the harmonic frequencies of the fundamental one, $\frac{2\pi}{N}$, in the frequency principal period $[-\pi, \pi]$. This pole placement is the same as the one obtained in the traditional repetitive cell.

In contrast to the traditional repetitive cell, the proposed one has zeroes placed in:

$$z_k = \sqrt[N]{|\beta|} e^{j\left(\frac{2\pi}{N}k + \frac{\pi(1-\text{sign}(\beta))}{2N}\right)}, \quad k = 0, 1, \dots, N-1, \quad (3)$$

so they are also uniformly distributed over a circumference of radius $\sqrt[N]{|\beta|}$ with associated frequencies $\omega_k = \frac{2\pi}{N}k + \frac{\pi(1-\text{sign}(\beta))}{2N}$.

² $\text{sign}(x)$ equals 1 for $x \geq 0$ and -1 for $x < 0$.

³ $|\alpha| < 1$ is necessary to assure the stability of the repetitive cell.

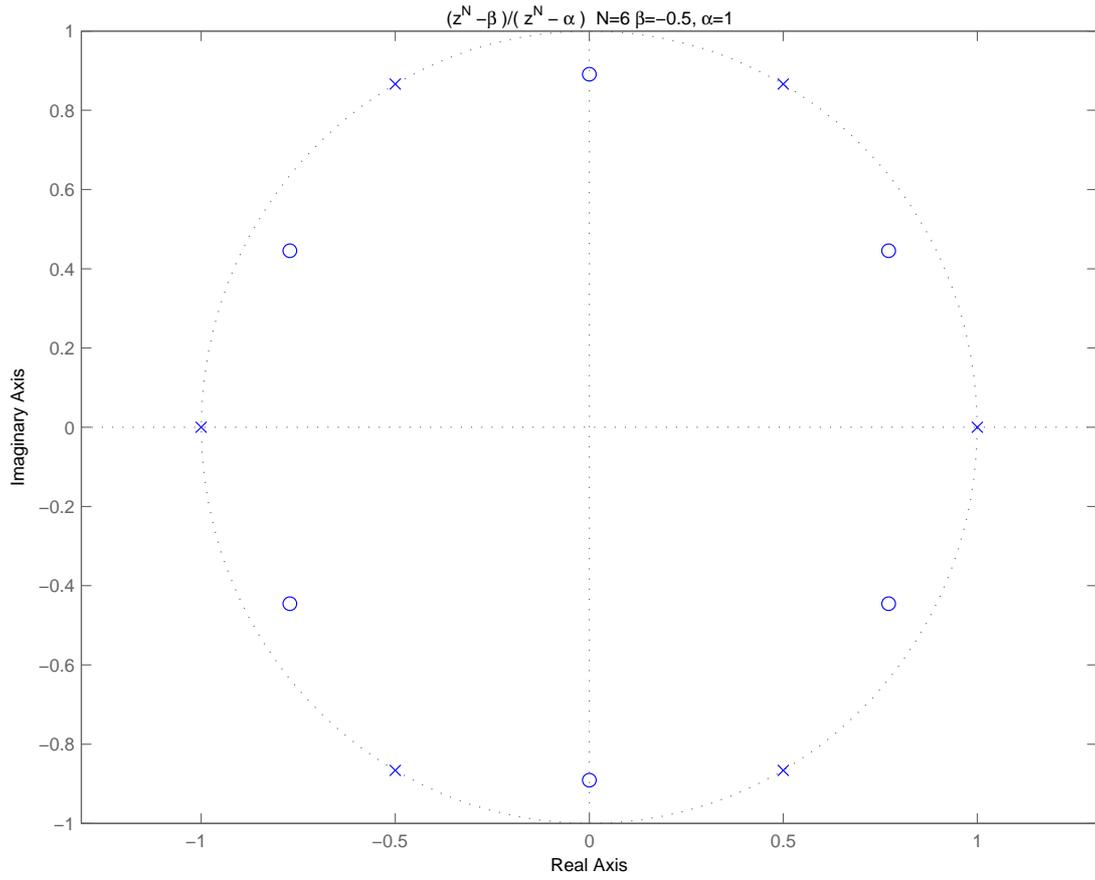


Fig. 2. Pole Zero map for α and β of different sign ($N = 6, \beta = -0.5, \alpha = 1$)

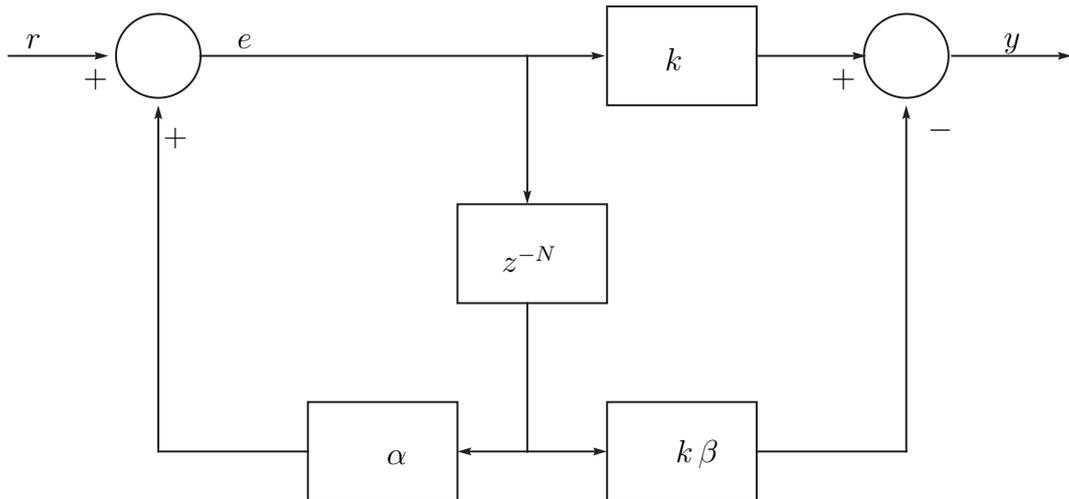


Fig. 3. Block-diagram of the proposed repetitive cell.

Depending on the signs of α and β appear the following cases: if $\text{sign}(\beta) = \text{sign}(\alpha)$ the poles and the zeroes are placed at the same frequencies [Figure 1], and if $\text{sign}(\beta) \neq \text{sign}(\alpha)$ the poles and the zeroes are placed at shifted frequencies. In this last case the frequencies associated with the zeros are exactly in the middle between the frequencies that correspond to the adjacent poles [Figure 2].

```

1  close all
2  clear all
3  disp('-----_ini_-----')
4  %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
5  %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
6  N=6; beta=0.5; alpha=1;
7  Num=zeros(1,N+1);
8  Num(N+1)=-beta;
9  Num(1)=1;
10
11 Den=zeros(1,N+1);
12 Den(N+1)=-alpha;
13 Den(1)=1;
14
15 F=tf(Num,Den,-1);
16
17 figure,pzmap(F),title('z^N-\beta)/(-z^N-\alpha) N=6,\beta=0.5,\alpha=1')
18 axis equal
19 print -depsc2 dibu_paper_1.eps
20
21 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
22 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
23 N=6; beta=-0.5; alpha=1;
24 Num=zeros(1,N+1);
25 Num(N+1)=-beta;
26 Num(1)=1;
27
28 Den=zeros(1,N+1);
29 Den(N+1)=-alpha;
30 Den(1)=1;
31
32 F=tf(Num,Den,-1);
33 figure,pzmap(F),title('z^N-\beta)/(-z^N-\alpha) N=6,\beta=-0.5,\alpha=1')
34 axis equal
35 print -depsc2 dibu_paper_2.eps
36
37
38 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
39 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
40 N=7; beta=0.5; alpha=1;
41 Num=zeros(1,N+1);
42 Num(N+1)=-beta;
43 Num(1)=1;
44 Num=Num*(1/(1-beta));
45 Den=zeros(1,N+1);
46 Den(N+1)=-alpha;
47 Den(1)=1;
48
49 F1=tf(Num,Den,-1);
50
51
52 N=7; beta=-0.5; alpha=1;
53 Num=zeros(1,N+1);
54 Num(N+1)=-beta;
55 Num(1)=1;
56 Num=Num*(1/(1-beta));
57

```

```

58 Den=zeros(1,N+1);
59 Den(N+1)=-alpha;
60 Den(1)=1;
61
62 F2=tf(Num,Den,-1);
63 Fo=tf([1,zeros(1,N)],Den,-1);
64
65 TFinal=4*N;
66 y1=impz(F1,TFinal);
67 y2=impz(F2,TFinal);
68 yo=impz(Fo,TFinal);
69
70
71 figure,
72 stem(y1,'r'); hold on
73 stem(yo,'b');
74 stem(y2,'g');
75 axis([0,TFinal,-0.2,2.2])
76 title('k(z^N-\beta)/(z^N-\alpha) N=7'), grid
77 text(2,1.9,'\alpha=1,\beta=0.5,k=2')
78 text(2,0.6667,'\alpha=1,\beta=-0.5,k=0.6667')
79 text(2,1,'\alpha=1,\beta=0,k=1')
80 print -depsc2 dibu_paper_3.eps
81
82
83 disp('_____ fi _____')

```

III. THE ENERGETIC STRUCTURE OF THE REPETITIVE CELL

This section studies the energetic structure of the proposed repetitive cell showing that it has a passivity structure. Particularly, it displays that the repetitive cell is QSR-dissipative. These properties are of great relevance when the repetitive cell is combined with other systems and when its frequency response is analyzed.

A state-space description of the transfer function in equation (1) is

$$\mathbf{x}_{n+1} = \mathbf{A} \mathbf{x}_n + \mathbf{B} u_n \quad (4)$$

$$y_n = \mathbf{C} \mathbf{x}_n + \mathbf{D} u_n \quad (5)$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ \alpha & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \quad (6)$$

$$\mathbf{B} = [0 \ 0 \ 0 \ \cdots \ 0 \ 1]^T \quad (7)$$

$$\mathbf{C} = [k(\alpha - \beta), 0, 0, \cdots, 0] \quad (8)$$

$$\mathbf{D} = [k] \quad (9)$$

Definition 1 (Discrete-Time Passivity): A discrete-time system is passive with storage function $V_n = \frac{1}{2} \mathbf{x}_n^T \mathbf{P} \mathbf{x}_n$ if :

$$\Delta V_n \triangleq V_{n+1} - V_n < \mathbf{y}_n^T \mathbf{u}_n \quad (10)$$

where \mathbf{y}_n and \mathbf{u}_n are the output and input vectors at time n .

Definition 2 (Discrete-Time Positive Real [13]): Let $\mathbf{H}(z)$ be a square matrix of real rational functions. Then $\mathbf{H}(z)$ is called Discrete-Time Positive Real (DTPR) if it has the following properties⁴:

- All the elements of $\mathbf{H}(z)$ are analytic in $|z| > 1$.
- $\mathbf{H}^*(z) + \mathbf{H}(z) \geq 0$, $\forall |z| > 1$.

Lemma 1 ([13]): Let $\mathbf{H}(z)$ be a square matrix of real rational function of z with no poles in $|z| > 1$ and simple poles only on $|z| = 1$ and let $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ be a minimal realization of $\mathbf{H}(z)$. Then necessary and sufficient conditions for $\mathbf{H}(z)$ to be DTPR are that there exist a real symmetric positive definite matrix \mathbf{P} and real matrices \mathbf{L} and \mathbf{W} such that:

$$\mathbf{A}^T \mathbf{P} \mathbf{A} - \mathbf{P} = -\mathbf{L}^T \mathbf{L} \quad (11)$$

$$\mathbf{A}^T \mathbf{P} \mathbf{B} = \mathbf{C}^T - \mathbf{L}^T \mathbf{W} \quad (12)$$

$$\mathbf{W}^T \mathbf{W} = (\mathbf{D} + \mathbf{D}^T) - \mathbf{B}^T \mathbf{P} \mathbf{B} \quad (13)$$

Using Lemma 1 and the state-space description of the repetitive cell in equations (4)-(5) gives the following proposition.

Proposition 1: The repetitive cell, $C(z) = k \frac{z^N - \beta}{z^N - \alpha}$, introduced in equation (1), for $k > 0$, $|\alpha| \leq 1$, $|\beta| \leq 1$ and $\alpha\beta \neq 1$ is DPTR.

Proof: It is not difficult to check that (6)-(9) is a minimal realization of (1). Its system matrix \mathbf{A} has all its eigenvalues on the closed unit circle and all of them are simple poles. Also, by direct inspection it is easy to check that (\mathbf{A}, \mathbf{B}) is a controllable pair and (\mathbf{C}, \mathbf{A}) is an observable pair. Due to the fact that

⁴ $\mathbf{H}^*(z)$ stands for the complex conjugate transpose of $\mathbf{H}(z)$.

$|\alpha| \leq 1$, $|\beta| \leq 1$ and $\alpha\beta \neq 1$ the inequality $1 - \alpha\beta > 0$ is always fulfilled⁵. Then, a set of matrices which fulfills the equations (11)-(13) in Lemma 1 is:

$$\mathbf{P} = \frac{k(\beta - \alpha)^2}{1 - \beta\alpha} \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \quad (14)$$

$$\mathbf{L} = \begin{bmatrix} \sqrt{\frac{k(1-\alpha^2)(\alpha-\beta)^2}{1-\beta\alpha}} & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \quad (15)$$

$$\mathbf{W} = \begin{bmatrix} \frac{(\alpha-\beta)\sqrt{k(1-\alpha^2)}}{\sqrt{(1-\beta\alpha)(\alpha-\beta)^2}} \\ -\sqrt{\frac{k(1-\beta^2)}{1-\beta\alpha}} \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (16)$$

It is important to note that for $|\alpha| \leq 1$, $|\beta| \leq 1$ and $\alpha\beta \neq 1$ \mathbf{P} is a symmetric and positive definite matrix and, also, \mathbf{L} and \mathbf{W} are matrices over the reals. \blacksquare

Remark 1: Matrix \mathbf{P} stated in equation (14) can be used to describe the energy of the system $C(z)$. So, it can be defined the following Lyapunov function:

$$V_n \triangleq \frac{1}{2} \mathbf{x}_n^T \mathbf{P} \mathbf{x}_n \quad (17)$$

By straightforward algebraic manipulations using equations (11)-(13) it can be proven that:

$$\Delta V_n = V_n - V_{n-1} \quad (18)$$

$$= -\frac{1}{2} (\mathbf{L} \mathbf{x}_n + \mathbf{W} u_n)^T (\mathbf{L} \mathbf{x}_n + \mathbf{W} u_n) + y_n u_n \quad (19)$$

This equation is composed by a dissipative term which depends on \mathbf{L} and \mathbf{W} , and an additive one as a function of the product of two power variables, y_n and u_n . From this equation it is possible to show that \mathbf{L} represents the autonomous system dissipation while \mathbf{W} represents the input dissipation.

Remark 2: Due to the discrete-time positive realness of $C(z)$ its polar plot ($C(e^{j\omega})$, $\omega \in [0, \pi]$) lies on the closed right half-plane of the Nyquist plane.

The following definition characterizes the energy structure of a dissipative system.

Definition 3 ([14]): Dissipative systems with supply functions of the form

$$\Delta V_n = \mathbf{y}_n^T \mathbf{Q} \mathbf{y}_n + 2 \mathbf{y}_n^T \mathbf{S} \mathbf{u}_n + \mathbf{u}_n^T \mathbf{R} \mathbf{u}_n \quad (20)$$

with \mathbf{Q} and \mathbf{R} being symmetric matrices and \mathbf{S} an appropriate size matrix, are regarded as $(\mathbf{Q}, \mathbf{S}, \mathbf{R})$ – *dissipative* systems.

⁵The condition $\alpha\beta \neq 1$ only precludes two trivial and uninteresting cases, namely: $\alpha = 1$, $\beta = 1$ and $\alpha = -1$, $\beta = -1$. In these two situations the repetitive cell reduces to a proportional controller $C(z) = k$ and, thus, they have no interest.

Taking into account this definition the dissipativity of the system under study is accordingly classified in the next Proposition.

Proposition 2: The repetitive cell, $C(z) = k \frac{z^N - \beta}{z^N - \alpha}$, introduced in equation (1), for $k > 0$, $|\alpha| \leq 1$, $|\beta| \leq 1$ and $\alpha\beta \neq 1$ is $(\mathbf{Q}, \mathbf{S}, \mathbf{R})$ – *dissipative*.

Proof: In Proposition 1 it is stated that system in equation (1) is DPTR, so it is passive. Matrix \mathbf{P} is used to define a storage function, equation (17), and its evolution in terms of \mathbf{L} and \mathbf{W} appears in equation (19). Due to the particular structure of \mathbf{C} , equation (8), and \mathbf{L} , equation (15), it is possible to rewrite equations (5) and (19) as:

$$\Delta V_n = -\frac{1}{2} (\mathbf{L}^{1,1} \mathbf{x}_n^1 + \mathbf{W} u_n)^T (\mathbf{L}^{1,1} \mathbf{x}_n^1 + \mathbf{W} u_n) + y_n u_n \quad (21)$$

$$y_n = \mathbf{C}^1 \mathbf{x}_n^1 + \mathbf{D} u_n \quad (22)$$

where \mathbf{x}_n^1 and \mathbf{C}^1 stands for the first component of \mathbf{x}_n and \mathbf{C} respectively, and $\mathbf{L}^{1,1}$ stands for the first diagonal element of \mathbf{L} .

From equation (22) it is possible to obtain \mathbf{x}_n^1 as a function of y_n and u_n which, then, can be replaced in (21) in order to obtain ΔV_n as function of y_n and u_n . After some algebraic manipulations it is possible to rewrite the energy evolution of the system in the $(\mathbf{Q}, \mathbf{S}, \mathbf{R})$ – *dissipative* form, see equation (20). The values of \mathbf{Q} , \mathbf{S} and \mathbf{R} are the scalars:

$$Q = \frac{1 - \alpha^2}{2k(\beta\alpha - 1)} \quad (23)$$

$$R = \frac{k(1 - \beta^2)}{2(\beta\alpha - 1)} \quad (24)$$

$$S = \frac{1}{2} \quad (25)$$

Once the values of Q , S and R have been obtained the proof is finished. ■

Remark 3: Following the classification in [14] the repetitive cell, $C(z) = k \frac{z^N - \beta}{z^N - \alpha}$, introduced in equation (1), for $k > 0$, $|\alpha| \leq 1$, $|\beta| \leq 1$ and $\alpha\beta \neq 1$ is:

- *passive* if $|\alpha| = 1$ and $|\beta| = 1$.
- *input strictly passive* if $|\alpha| = 1$ and $|\beta| \neq 1$.
- *output strictly passive* if $|\alpha| \neq 1$ and $|\beta| = 1$.
- *very strictly passive* if $|\alpha| \neq 1$ and $|\beta| \neq 1$.

The following Lemma makes possible to characterize the polar plot of the repetitive cell in the proposition below.

Lemma 2 ([14]): If a system $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ with transfer function $G(z)$ is (Q, S, R) – *dissipative* then:

- 1) If $Q < 0$ then the graph of $G(e^{j\omega})$ lies inside the circle on the complex plane with center $\frac{S}{|Q|}$ and radius $\left(\frac{1}{|Q|}\right) \sqrt{S^2 + R|Q|}$.
- 2) If $Q = 0$ then the graph of $G(e^{j\omega})$ lies to the right (if $S > 0$) or the left (if $S < 0$) of the vertical line $\text{Re}\{z\} = -\frac{R}{2S}$.

Proposition 3: The Nyquist plot of $C(z) = k \frac{z^N - \beta}{z^N - \alpha}$ where $k > 0$, $|\alpha| \leq 1$, $|\beta| \leq 1$ and $\alpha\beta \neq 1$, lies

- inside a circle of center $\frac{k(\alpha\beta - 1)}{\alpha^2 - 1}$ and radius $\frac{|k(\alpha - \beta)|}{1 - \alpha^2}$ if $|\alpha| < 1$.
- to the right of $\text{Re}\{z\} = \frac{1 - \beta^2}{2k(1 - \beta\alpha)}$ if $|\alpha| = 1$.

Proof: Straightforward by applying the values obtained for Q , S and R in Proposition 2 to the Lemma 2. ■

IV. FREQUENCY RESPONSE OF THE REPETITIVE CELL

This section details some geometric aspects of the frequency response ($C(e^{j\omega}) = k \frac{e^{j\omega} - \beta}{e^{j\omega N} - \alpha}$, $\omega \in [-\pi, \pi]$) of the repetitive cell stated in equation (1). Some of these characteristics can be useful to assess the performance of a closed-loop system including the repetitive cell as a controller by looking its contribution to the open-loop transfer function shape. In this sense, the following proposition states its geometrical shape.

Proposition 4: The frequency response of $C(z) = k \frac{z^N - \beta}{z^N - \alpha}$ with $k > 0$, $|\alpha| < 1$ and $|\beta| \leq 1$ is a circumference with center $c \triangleq k \frac{1 - \beta\alpha}{1 - \alpha^2}$ and radius $r \triangleq k \frac{|\alpha - \beta|}{1 - \alpha^2}$, i.e. located in the open right half-plane of the Nyquist plane.

Proof: Rewriting $C(e^{j\omega})$ in cartesian form

$$\begin{aligned} C(e^{j\omega}) &= \text{Re}\{C(e^{j\omega})\} + j \text{Im}\{C(e^{j\omega})\} \\ &= \frac{(\alpha + \beta) \cos(N\omega) - \beta\alpha - 1}{2 \cos(N\omega)\alpha - \alpha^2 - 1} + j \frac{(\alpha - \beta) \sin(N\omega)}{2 \cos(N\omega)\alpha - \alpha^2 - 1} \end{aligned}$$

the equation $(\text{Re}\{C(e^{j\omega})\} - c)^2 + (\text{Im}\{C(e^{j\omega})\})^2 = r^2$ is verified. \blacksquare

Remark 4: The maximum and minimum gains of the repetitive cell $C(z)$ are $|C(e^{j\omega})|_{\max} = c + r = \max\{k \frac{1 - \beta}{1 - \alpha}, k \frac{1 + \beta}{1 + \alpha}\}$ and $|C(e^{j\omega})|_{\min} = c - r = \min\{k \frac{1 - \beta}{1 - \alpha}, k \frac{1 + \beta}{1 + \alpha}\}$, respectively. The maximum phase loss of $C(z)$ is $\phi_{\max} = \min_{\omega} |\angle C(e^{j\omega})| = \arctan\left(\frac{(\beta - \alpha) \sqrt{((1 - \alpha^2)(1 - \beta^2))}}{(\alpha^3 - \alpha)\beta - \alpha^2 + 1}, \frac{1 - \beta^2}{1 - \beta\alpha}\right)$ and it is always in the $[-\frac{\pi}{2}, \frac{\pi}{2}]$ range. The phase minima occur at frequencies $\omega_l = \frac{1}{N} \arccos\left(\frac{\alpha + \beta}{\beta\alpha + 1}\right) + \frac{2\pi}{N}l$, $l = 0, 1, \dots, N - 1$ reflected to the $[-\pi, \pi]$ range. The bandwidth of the gain peaks in $|C(e^{j\omega})|$ measured in the $|C(e^{j\omega})| = 1$ level is $\Delta\omega = \frac{2}{N} \arccos\left(\frac{k^2(\beta^2 + 1) - (\alpha^2 + 1)}{2(k^2\beta - \alpha)}\right)$.

It is worth to mention that the frequency domain characteristics of $C(z) = k \frac{z^N - \beta}{z^N - \alpha}$ can be obtained in a simple way from the same characteristics of $C'(z) = k \frac{z - \beta}{z - \alpha}$. Specifically, $C(z) = C'(z^N)$ so the relation in the frequency domain is $C(e^{j\omega}) = C'(e^{jN\omega})$ which implies only a compression by N of the frequency axis.

In order to obtain the desired tracking/rejection performance in the closed-loop system the controller must contribute with high gain in the harmonic frequencies, the interesting cases among all the possible values of α and β are:

- $0 < \alpha < 1$ and $-1 \leq \beta < \alpha$
- $-1 < \alpha < 0$ and $\alpha < \beta \leq 1$

In both cases the gain offered by the repetitive cell at the harmonic frequencies is greater than one, so it contributes to reduce the closed-loop sensitivity function magnitude at these frequencies. Fig. 4 shows the bode diagrams of the repetitive controller when $\alpha > 0$. In this case gain peaks are uniformly distributed in frequency beginning with zero frequency, thus including high dc gain. When $\beta > 0$ (solid line) the zeroes are located at the same frequencies as the poles thus narrowing the gain peaks (more narrow when the zero is closer to the pole). In the other case, $\beta < 0$ (dashed line), the zeroes are located at the mean frequency between the frequencies of the two adjacent poles thus reducing the gain at their frequencies and widening the gain peaks corresponding to the poles⁶. In short, the position of the gain peaks is affected by the sign of α and its modulus and shape depends on $|\alpha|$ and β . The phase Bode diagram in Fig. 4 shows, and the equation for ϕ_{\max} in Remark 4 quantifies, that the maximum phase loss of the repetitive cell $C(z)$ falls as β get closer to α .

Fig. 6 shows the pole-zero map corresponding to the frequency response plots in Fig. 4. When α and β have the same sign poles and zeros are located at the same frequencies (angles), but when they have different sign poles and zeros appear in an alternate pattern of frequencies.

⁶This frequency response could be compared to the cascade connection of a traditional repetitive cell and a comb filter.

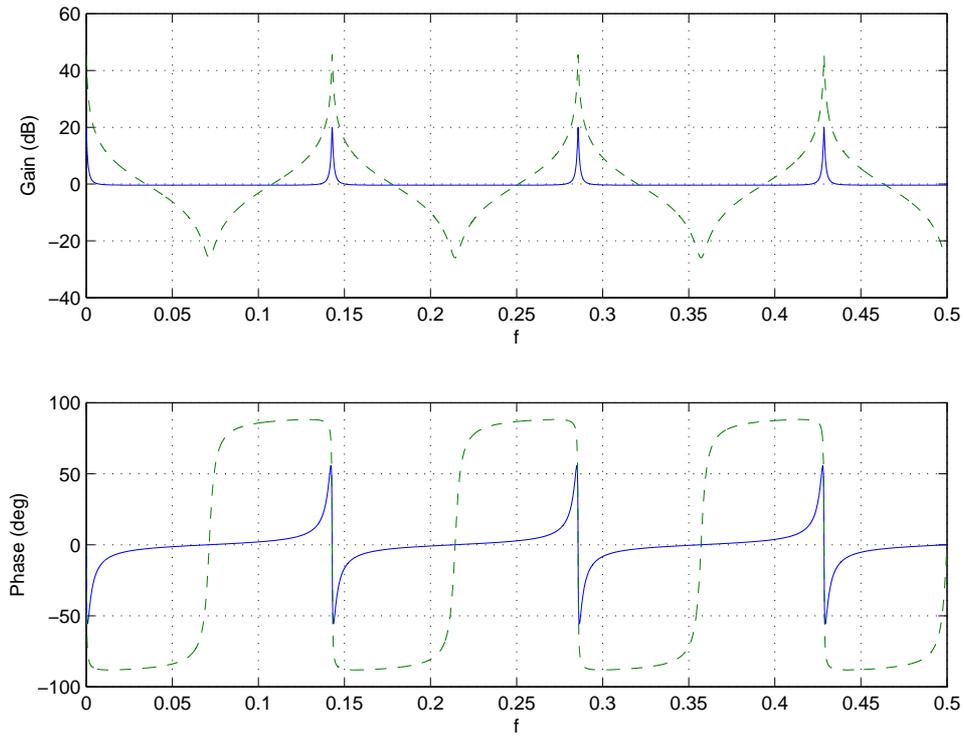


Fig. 4. Frequency response for $\alpha = 0.99, \beta = 0.90$ (solid) and $\alpha = 0.99, \beta = -0.90$ (dash), $N = 7$ and $k = 1$.

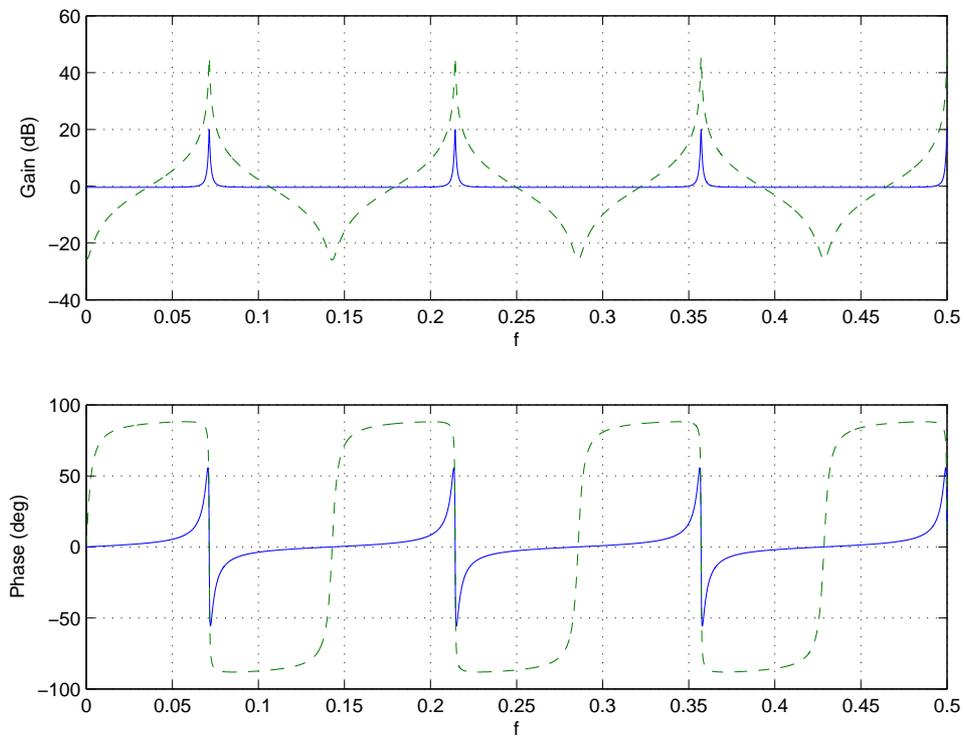


Fig. 5. Frequency response for $\alpha = -0.99, \beta = 0.90$ (solid) and $\alpha = -0.99, \beta = -0.90$ (dash), $N = 7$ and $k = 1$.

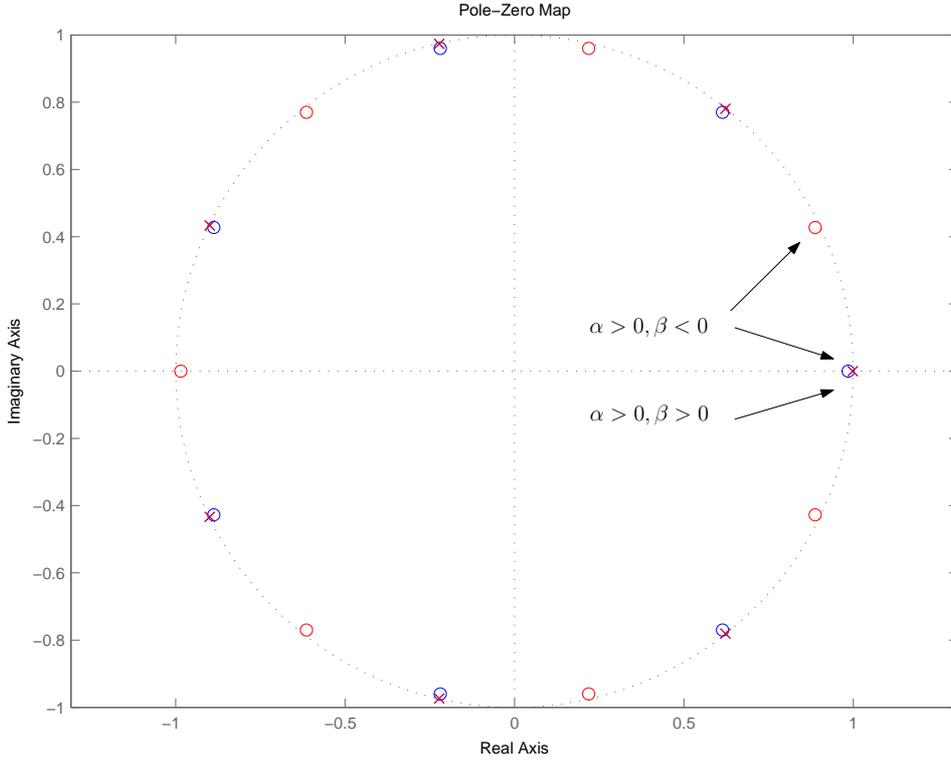


Fig. 6. Pole-zero map for $\alpha = -0.99, \beta = -0.90$ (same frequency) and $\alpha = -0.99, \beta = 0.90$ (different frequency), $N=7$.

V. TIME RESPONSE

The repetitive cell $C(z)$, like the traditional repetitive controllers [10], can be understood as a signal generator when a finite duration signal of length N is applied to its input. In this sense, the repetitive controller is interpreted, according to the Internal Model Principle [15], as the generator of the reference signals to track or the disturbance signals to attenuate that is necessary to include in the open-loop transfer function of the system.

Proposition 5: Let $x_1(n)$ a discrete-time sequence of finite duration N ($x_1(n) = 0, n < 0, n \geq N$) and $x(n) = k \sum_{m=0}^{\infty} (x_1(n - mN) - \beta x_1(n - (m + 1)N)) \alpha^m$ a discrete-time infinite-duration signal that is made up of attenuated and shifted in time copies of $x_1(n)$. Then the z -transform of $x(n)$ is

$$X(z) = \mathfrak{z}\{x(n)\} = k \frac{z^N - \beta}{z^N - \alpha} X_1(z) \quad (26)$$

where $X_1(z) \triangleq \mathfrak{z}\{x_1(n)\}$.

Proof: Straightforward applying the z -transform to $x(n)$. ■

Remark 5: When $\alpha = 1, \beta = 0$ and $k = 1$ Eq. (26) reduces to $X(z) = \frac{z^N}{z^N - 1} X_1(z)$ corresponding to $x(n) = \sum_{m=0}^{\infty} x_1(n - mN)$ that represents a periodical signal obtained by repetition of $x_1(n)$. When $\alpha = 1$ and $k = \frac{1}{1 - \beta}$, $x(n) = \sum_{m=0}^{\infty} x_1(n - mN)$, $x(n)$ is a periodic signal with a period $x_1(n)$ after the first period has elapsed, see Fig. 7.

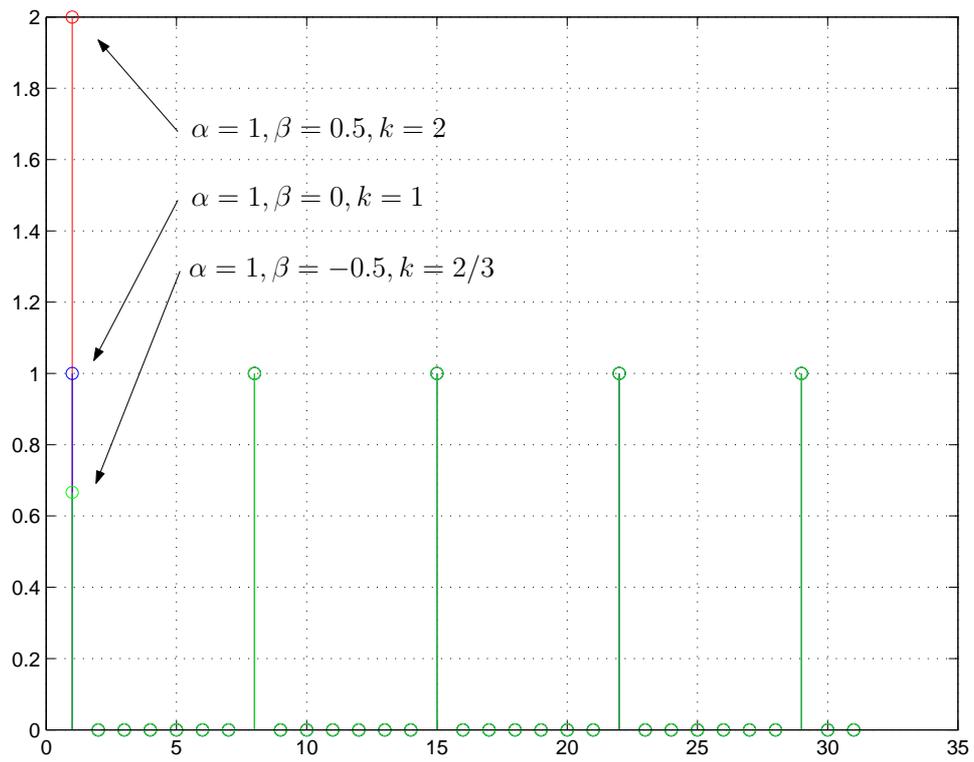


Fig. 7. Impulse response: $\alpha = 1$, $\beta \in \{-0.5, 0, 0.5\}$, $k = \frac{1}{1-\beta}$ and $N = 7$.

```

1  close all
2  clear all
3
4  T=1;
5  z=tf([1,0],[1],T);
6  planta = 0.340299846608298338026436997108e0 * ...
7          (z + 0.710216255761721015769691687030e0) * (z - 1) /...
8          (z ^ 2 - 0.785893111668710341188483317476e0 * z + ...
9          0.367879441171442321595523770163e0);
10
11 nyquist(planta)
12
13 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
14 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
15 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
16
17 beta1=1;
18 beta2=1./5;
19 N=10/2;
20
21 Num=zeros(1,N+1);
22 Num(N+1)=-beta2;
23 Num(1)=1+beta2;
24
25 Den=zeros(1,N+1);
26 Den(N+1)=beta1;
27 Den(1)=1;
28
29 F1=tf(Num,Den,T)
30
31 N=10;
32
33 Den=zeros(1,N+1);
34 Den(N+1)=-beta1;
35 Den(1)=1;
36
37 Num=zeros(1,N+1);
38 Num(N+1)=-beta2;
39 Num(1)=1+beta2;
40 F2=tf(Num,Den,T)

```

VI. CLOSED-LOOP NUMERICAL EXAMPLE

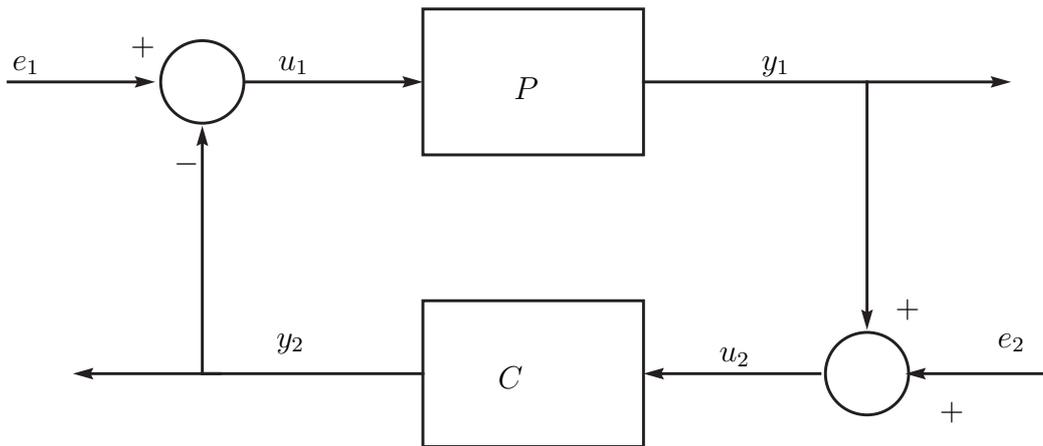


Fig. 8. Feedback connection of two passive systems.

It is well known that the closed-loop connection of two passive systems generates another passive system [12]. In this sense, the proposed repetitive cell ($C(z)$) can be connected to any passive plant ($P(z)$), see Fig. 8, giving a passive closed-loop system. Besides this, the controller structure assures the desired tracking/attenuation performance, for example, any periodic reference applied in e_1 will be followed by y_1 with low or zero steady-state error.

Traditionally, repetitive controllers are introduced in a closed-loop system in a plug-in manner [16], i.e. they are used to augment an existing controller. In this kind of connection the repetitive controller is placed in parallel with an unity gain block. Since this system is also passive this property is also preserved if the proposed repetitive cell is used as the repetitive part. So, in Fig. 8 the controller $C(z)$ could be replaced by the repetitive cell in parallel with an unity gain proportional system.

The only requirement to use the proposed repetitive cell is the discrete-time passivity of the plant. If it is not passive then an inner feedback control loop should be used to passivize it [11]. In this case, see Fig. 8, $P(z)$ should be replaced by the plant and the passivizing controller in feedback connection.

A. Example 1

The plant defined by the transfer function:

$$G(z) = \frac{4.4076(z - 0.8553)(z - 0.484)}{(z - 0.6708)(z - 0.9)} \quad (27)$$

is DTPR as it can be seen in the Nyquist plot of Fig. 9. As it is expected, it lays entirely on the right half-plane. The control performance specification is to design a controller which allows the closed-loop system to perfectly track a reference signal defined by:

$$e_2(n) = 3 + 2 \sin(\omega_1 n) + 1.5 \sin(2\omega_1 n) + 0.7 \sin(6\omega_1 n) \quad (28)$$

where $\omega_1 = \frac{2\pi}{N}$ and $N = 10$. As the reference signal is periodic with period N and nonzero mean, the controller must have high gain at frequency ω_1 and all its harmonics, including DC. Two controllers are possible, the first one with zeros and poles at the same angle:

$$C_1(z) = \frac{z^{10} - 0.5}{z^{10} - 1} \quad (29)$$

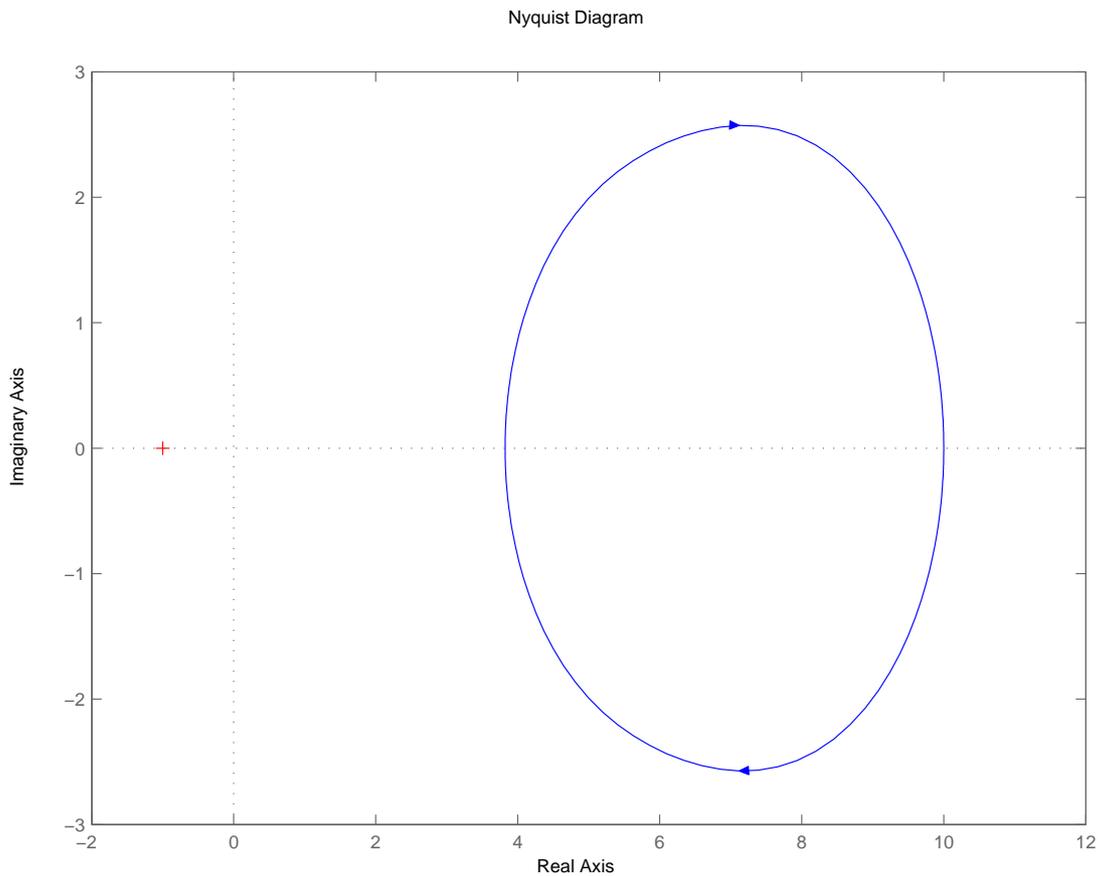


Fig. 9. Example 1: Nyquist plot of $G(z)$ in equation (27).

and the second one with zeros and poles at different angles

$$C_2(z) = \frac{z^{10} + 0.5}{z^{10} - 1} \quad (30)$$

In both cases the pole position has been fixed in order to obtain zero steady-state error. The location of the zeros is a design parameter which must be fixed to shape the frequency response in the desired way. For the purposes of comparison in both cases $k = 1$ and the modulus of the zeros has been fixed equal.

Fig. 10 shows the system output for both cases. Although the transient is slightly different both converge to the same steady state. From the performance point of view both controller are equivalent, so other criteria must be used to decide among them. As it has been presented in previous section, the equal angle controller is more selective in frequency and it almost preserves the plant frequency response between harmonic frequencies. Additionally, the introduced phase is always less than the phase introduced by the different angle controller. On the other hand, the different angle approach attenuates the frequency response magnitude of the plant between harmonic frequencies.

As a robustness measure, the minimum distance (d) from the open-loop transfer function, $L(z) = C(z)P(z)$, Nyquist plot to -1 is usually used. Although this is not a general case (it depends on the plant frequency response) in the presented example the controller (29) has $d = \|(1 + L)^{-1}\|_{\infty} = 0.2584$ while the controller (30) has $d = 0.5112$. Although this figure is of great interest in practical applications it is important to remember that, in this design, robustness is provided by the passivity structure.

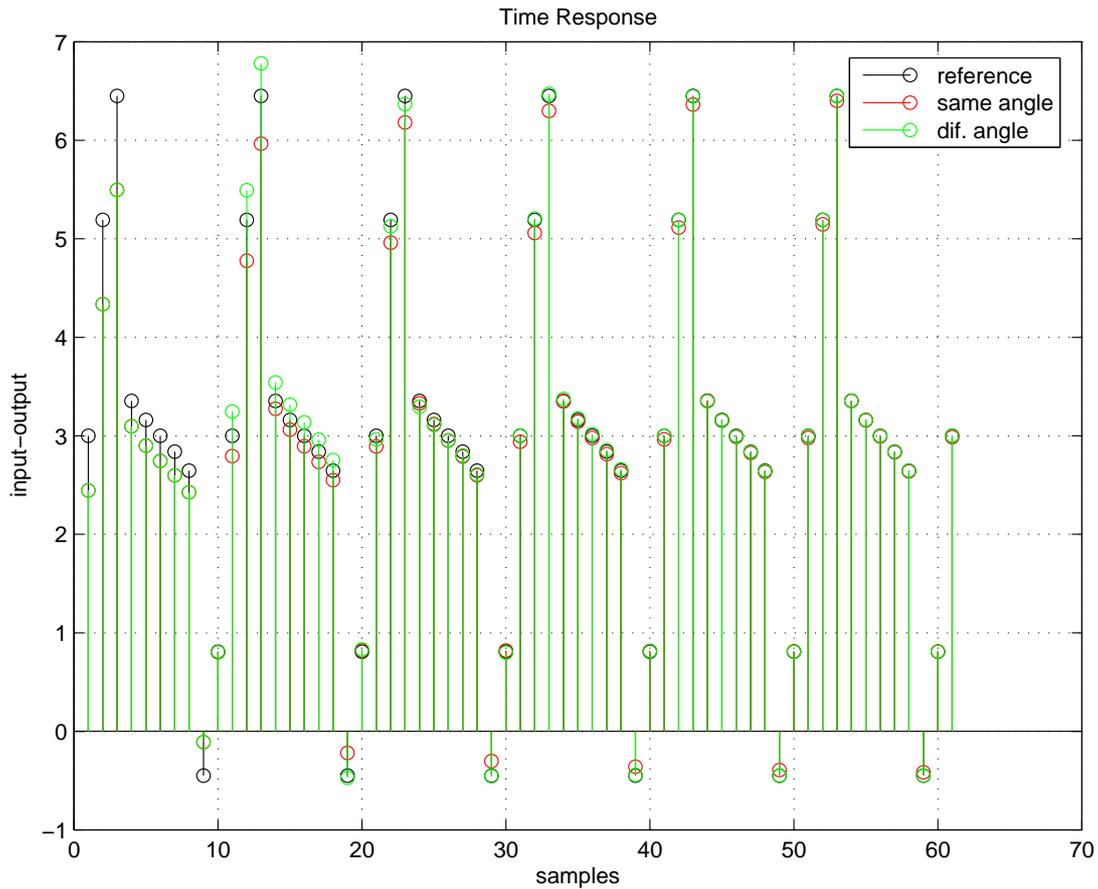


Fig. 10. Example 1 : Closed-loop time response.

```

1  close all
2  clear all
3
4  T=1;
5  z=tf([1,0],[1],T);
6  planta = 0.340299846608298338026436997108e0 * ...
7          (z + 0.710216255761721015769691687030e0) * (z - 1) /...
8          (z ^ 2 - 0.785893111668710341188483317476e0 * z + ...
9          0.367879441171442321595523770163e0);
10
11 nyquist(planta)
12
13 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%5
14 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%5
15 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%5
16
17 beta1=1;
18 beta2=1./5;
19 N=10/2;
20
21 Num=zeros(1,N+1);
22 Num(N+1)=-beta2;
23 Num(1)=1+beta2;
24

```

```
25 Den=zeros(1,N+1);
26 Den(N+1)=beta1;
27 Den(1)=1;
28
29 F1=tf(Num,Den,T)
30
31 N=10;
32
33 Den=zeros(1,N+1);
34 Den(N+1)=-beta1;
35 Den(1)=1;
36
37 Num=zeros(1,N+1);
38 Num(N+1)=-beta2;
39 Num(1)=1+beta2;
40 F2=tf(Num,Den,T)
```

B. Example 2 : the odd-harmonic case

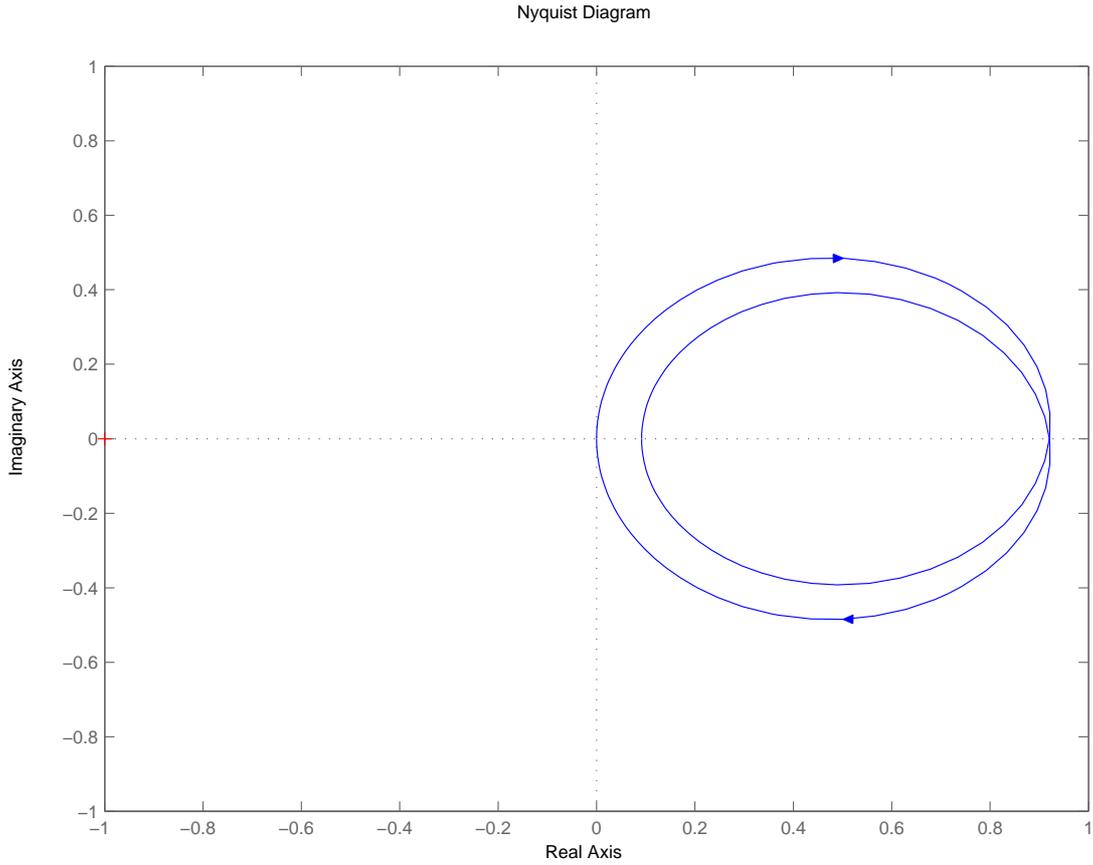


Fig. 11. Example 2: Nyquist plot of $G(z)$ in equation (31).

The plant

$$G(z) = \frac{0.3403(z + 0.7102)(z - 1)}{z^2 - 0.785z + 0.3678} \quad (31)$$

is also DTPR as it can be observed in its Nyquist plot, see Fig. 11. In this example it is desired to design a controller which allows the closed-loop system to track a reference signal defined by:

$$u(n) = 2 \sin(\omega_1 n) + 1.5 \sin(3\omega_1 n) \quad (32)$$

As this plant has a zero in $z=1$ it is compulsory to use an odd-harmonic repetitive controller [10] in order to assure internal stability. However, the use of an odd-harmonic repetitive controller reduces the tracking/rejection capabilities to the odd harmonics. It is important to note that reference signal (32) only contains this kind of harmonics.

Two possible controllers are possible, the first with zeros and poles at the same angle:

$$C_1(z) = \frac{z^5 + 0.5}{z^5 + 1} \quad (33)$$

and the second with zeros and poles at different angle

$$C_2(z) = \frac{z^5 - 0.5}{z^5 + 1} \quad (34)$$

In both cases the pole position has been fixed in order to obtain zero steady-state error. For the shake of comparison, in both cases $k = 1$ and the modulus of the zeros has been fixed equal.

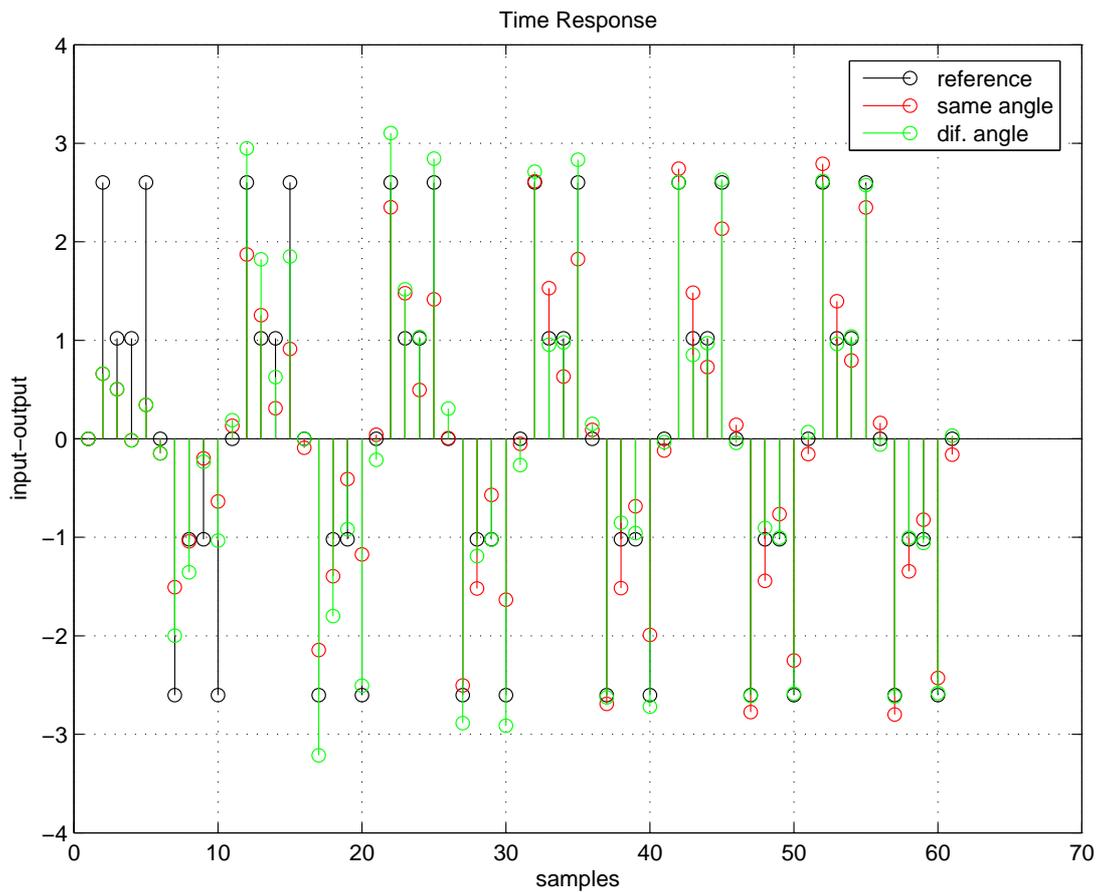


Fig. 12. Example 2: Closed-loop time response.

Fig. 12 shows the system output for both cases. Although the transient response is slightly different, as expected, both converge to the same steady state.

The minimum distances from the open-loop transfer function Nyquist plot to -1 in this example are: $d = 1.0511$ for the controller (33) and $d = 1.5876$ for the controller (34).

```

1  close all
2  clear all
3
4  z =[0.85526135458575, 0.48398374321304];
5  p =[0.67081008683825, 0.900000000000000];
6  k =[4.40756404382341];
7
8  planta=zpk(z,p,k,-1);
9
10 beta1=1;
11 beta2=1./5;
12 N=10;
13
14 Num=zeros(1,N+1);
15 Num(N+1)=beta2;
16 Num(1)=1-beta2;
17
18 Den=zeros(1,N+1);
19 Den(N+1)=-beta1;
20 Den(1)=1;
21 F1=tf(Num,Den,-1)
22
23 cl=feedback(planta*F1,1);
24 te=0:1:6*N;
25 u=3+(2*sin((2*pi/N)*te))+1.5*sin((2*2*pi/N)*te)+0.7*sin((6*2*pi/N)*te)';
26 y=lsim(cl,u,te);
27
28 figure,plot(te,[u,y]),title('Time_Response'),...
29 ylabel('input-output'),xlabel('samples'),grid

```

VII. CONCLUSION

This work has presented a new repetitive controller to track and attenuate periodic references and disturbances, respectively, for discrete-time passive plants. The energetic structure and frequency and time responses of the repetitive cell are studied proving that it has passivity structure and showing some useful characteristics for the design stage of a closed-loop system. Its main characteristic, passivity structure, assures to obtain a stable closed-loop system when the plant is also discrete-time passive. It is worth of remark that the design procedure does not include, unlike the procedures for traditional repetitive controllers, any exact or approximate inversion of the plant dynamics. From an implementation point of view, this new cell needs the same amount of memory than the traditional ones and only an additional addition and product are needed for each sampling period.

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