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Interfaces in solutions of diffusion-absorption equations

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Abstract. We study the properties of interfaces in solutions of the Cauchy problem for the nonlinear degenerate parabolic equation $u_t = \Delta u^m - u^p$ in $\mathbb{R}^n \times (0,T]$ with the parameters $m > 1$, $p > 0$ satisfying the condition $m + p \geq 2$. We show that the velocity of the interface $\Gamma(t) = \partial\overline{\{\operatorname{supp} u(x,t)\}}$ is given by the formula $\mathbf{v} = \left[-\dfrac{m}{m-1}\nabla u^{m-1} + \nabla \Pi\right]\Big|_{\Gamma(t)}$ where Π is the solution of the degenerate elliptic equation $\operatorname{div}(u\nabla \Pi) + u^p = 0$, $\Pi = 0$ on $\Gamma(t)$. We give explicit formulas which represent the interface $\Gamma(t)$ as a bijection from $\Gamma(0)$. It is proved that the solution u and its interface $\Gamma(t)$ are analytic functions of time t and that they preserve the initial regularity in the spatial variables.

Interfaces en soluciones de las ecuaciones de absorción-difusión

Resumen. Se estudian las propiedades de las interfaces de las soluciones del problema de Cauchy para ecuaciones parabólicas no lineales degeneradas $u_t = \Delta u^m - u^p$ en $\mathbb{R}^n \times (0,T]$ con parámetros $m > 1$, $p > 0$ que satisfagan la condición $m + p \geq 2$. Se demuestra que la velocidad de la interface $\Gamma(t) = \partial\overline{\{\operatorname{supp} u(x,t)\}}$ viene dada por la fórmula $\mathbf{v} = \left[-\dfrac{m}{m-1}\nabla u^{m-1} + \nabla \Pi\right]\Big|_{\Gamma(t)}$, donde Π es la solución de la ecuación elíptica degenerada $\operatorname{div}(u\nabla \Pi) + u^p = 0$, $\Pi = 0$ sobre $\Gamma(t)$. Se deducen las formulas que representan explícitamente la interface $\Gamma(t)$ como una biyeción de $\Gamma(0)$. Se demuestra que la solución u y su interface $\Gamma(t)$ son analíticas como funciones del tiempo t y que conservan la regularidad inicial respecto de las variables espaciales.

1. Lagrangian coordinates

We study the Cauchy problem

$$u_t = \Delta u^m - u^p \quad \text{in } S = \mathbb{R}^n \times (0,T), \qquad u(x,0) = u_0(x) \geq 0 \qquad \text{in } \mathbb{R}^n \tag{1}$$

with the parameters $m > 1$, $p > 0$ subject to the condition $m + p \geq 2$. It is known that in the case $m > 1$, $p \in (0,1)$ the initially compact support can split into several components and that the solution vanishes in a finite time. We refer to [1, Ch.3] for the background information on the behavior of interfaces in solutions of problem (1).

We are concerned with the study of the interface dynamics and regularity. In the one-dimensional case, the study of the interface regularity was performed in [3, 4] but the method proposed and developed in these papers does not directly apply to the multidimensional case.

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Let $u(x,t)$ be a continuous solution of problem (1) and $\Omega(t) = \{x \in \mathbb{R}^n : u(x,t) > 0\}$. For the sake of presentation we assume that the set $\Omega(0)$ is one-connected in $\mathbb{R}^n$, which means that the solution only has the outer interface. Our arguments extend without any change to the case when the set $\partial\overline{\{\Omega(0)\}}$ consists of a finite number of simple-connected components.

Since $\Omega(0)$ is one-connected in $\mathbb{R}^n$ and the solution is continuous, $\Omega(t)$ is also one-connected for $t \in (0,T_+)$ with some T_+. The solution $u(x,t)$ is strictly positive inside $\bigcup_{t\in(0,T_+]}\Omega(t)$, which allows us to take a set $\mathcal{D} \subset \Omega(0)$ with the smooth boundary γ such that $\mathcal{D} \times (0,T_+] \subset \bigcup_{t\in(0,T_+]}\Omega(t)$ and $u \geq \delta$ on $\Sigma = \gamma \times [0,T_+]$ with some $\delta > 0$. Adopt the notation $D = \mathbb{R}^n \setminus \mathcal{D}$, $\omega(t) = \Omega(t) \setminus \mathcal{D}$, $\omega = \omega(0)$. The weak solution to the Cauchy problem (1) solves the initial-and-boundary value problem

$$u_t = \Delta\, u^m - u^p \text{ in } E = D \times (0,T_+], \quad u(x,0) = u_0 \text{ in } D, \quad (\nabla\, u^m, \mathbf{n})|_{\Sigma} = \phi(x,t) \tag{2}$$

with a prescribed function ϕ; $\mathbf{n}$ denotes the unit vector of outer normal to Σ. Notice that since the solution u of the Cauchy problem (1) is smooth inside its support, we have $\phi \in C^\infty(\Sigma)$.

Definition 1 *A function $u(x,t)$ is said to be a weak solution of problem (2) if u is bounded, nonnegative and continuous in $\overline{E}$, $\nabla\, u^m \in L_2(E)$, and for every test-function $\eta \in C^1\left(\overline{E}\right)$, vanishing for $t = T_+$ and all x large enough*

$$\int_E \left(u\,\eta_t - \nabla\,\eta \cdot \nabla\, u^m - \eta\, u^p\right)\, dx\, dt + \int_D u_0\, \eta(x,0)\, dx + \int_\Sigma \eta\, \phi\, dS = 0. \tag{3}$$

Let us consider the following auxiliary mechanical problem: the flow of the politropic gas with density u, pressure $p = m/(m-1)u^{m-1}$, and velocity $\mathbf{v}$ through the porous medium that occupies the region D. It is assumed that the surface Σ is immobile and that the total mass of the gas is constant. We will describe the gas flow using *Lagrangian coordinates* [6]. In this description all characteristics of the motion are considered as functions of the initial positions of the particles and time t. Let us denote by $X(\xi,t)$ the position of the particle that initially occupied the position $\xi \in \omega$, and by $U(\xi,t)$ the density at this particle. The flow is described by the following relations:

$$X_t(\xi,t) = \mathbf{v}[X(\xi,t),t] \quad \text{in } Q = \omega \times (0,T],\ T \leq T_+, \quad \text{(equation of the trajectories)} \tag{4}$$

$$U \det\left[\partial X/\partial\xi\right] = u_0 \qquad \text{in } Q \qquad \text{(the mass conservation law).} \tag{5}$$

These equations are endowed with the initial and boundary conditions: $X(\xi,0) = \xi$, $U(\xi,0) = u_0(\xi)$ in ω (the initial data), $\mathbf{v}|_\Sigma = 0$ (the surface Σ is immobile), $U(\xi,t) = 0$ on $\Gamma = \partial\omega \times [0,T]$ (the free boundary). Let us assume that for a prescribed vector-field $\mathbf{v}(X,t) \in L_2(Q)$ and a given set of the initial and boundary data we can construct a solution (X,t), $U)$ of problem (4)–(5). This solution generates the map $\xi \mapsto x = X(\xi,t)$, which we assume to be such that $|J| = \det\left[\partial X/\partial\xi\right]$ is separated away from zero and infinity. Consider the function

$$u(x,t) = \begin{cases} U(\xi,t) & \text{for } x = X(\xi,t),\ (\xi,t) \in \overline{Q}, \\ 0 & \text{elsewhere.} \end{cases} \tag{6}$$

For any test-function $\eta(x,t)$ satisfying the conditions of Definition 1 the following equality holds:

$$-\int_D \eta(x,0)u_0\, dx = \int_0^T \frac{d}{dt}\left(\int_{\omega(t)} \eta\, u\, dx\right) dt = \int_0^T \int_D \left(\eta_t u + u\, \nabla_x \eta \cdot \mathbf{v}\right)\, dxdt. \tag{7}$$

Comparing (7) with (3) we see that the function $u(x,t)$ is a weak solution of problem (2) if $\mathbf{v} = -m/(m-1)\nabla_x u^{m-1} + \nabla_x \Pi$, where Π is a weak solution of the degenerate elliptic problem

$$\operatorname{div}_x (u \nabla_x \Pi) = -u^p \quad \text{in } \omega(t), \quad \Pi|_{\partial\omega(t)} = 0, \quad u(\nabla \Pi, \mathbf{n})|_\Sigma = -\phi. \tag{8}$$

Gathering conditions (4)–(5) with (8), and writing the latter in Lagrangian coordinates, we arrive at the the Lagrangian counterpart of problem (2): it is requested to find a solution $X = I+Y$, $P = m/(m-1)U^{m-1}$, and $\pi(\xi,t) = \Pi(x,t)$ of the system of nonlinear equations

$$\left\{ \begin{array}{ll} (I + \nabla^* Y) Y_t + \nabla \, (P - \pi) = 0, \quad P|J|^{m-1} - P_0 = 0 & \text{in } Q, \\ \operatorname{div} \left(u_0 J^{-1} \left(J^{-1} \right)^* \nabla \pi \right) + u_0^p |J|^{1-p} = 0 \quad \text{in } \omega & \end{array} \right. \tag{9}$$

under the initial and boundary conditions $Y(\xi,0) = 0$, $P(\xi,0) = P_0 \equiv m/(m-1)u_0^{m-1}$ in ω, $P = 0$, $\pi = 0$ on Γ, $P^{1/(m-1)}(\nabla \pi, \mathbf{n}) = -\phi \, (m/(m-1))^{1/(m-1)}$, $Y(\xi,t) = 0$ on Σ.

Theorem 1 *Let (Y,P,π) be a solution of problem* (9) *such that ∇P, $\nabla \pi \in L_2(Q)$ and $|J|$ is separated away from zero and infinity. If for every $t \in (0,T]$ the map $\xi \mapsto I + Y(\xi,t)$ is a bijection between ω and $\omega(t)$, then formula* (6) *defines a weak solution of problem* (2)*. The interface of this solution is given by* (4).

2. The function spaces

Adopt the notation $d \equiv d_\xi = \text{dist}(\xi, \partial\omega)$, $d_{\xi,\eta} = \min(d_\xi, d_\eta)$, $\left|D^k v\right| = \sum_{|\beta|=k} \left|D^\beta v\right|$, ($\beta$ is a multi-index.) Given a set $G \subseteq Q$ and a number $\alpha \in (0,1)$, we define the seminorms and norms: $|u|_G^{(0)} = \sup_G |u|$,

$$\{u\}_G^{(\alpha)} = \sup_{G,\, \xi\neq\eta} \left\{ d_{\xi,\eta}^\alpha \frac{|u(\xi,t) - u(\eta,t)|}{|\xi-\eta|^\alpha} \right\} + \sup_{G,\, t\neq\tau} \left\{ d^{\alpha/2} \frac{|u(\xi,t) - u(\xi,\tau)|}{|t-\tau|^{\alpha/2}} \right\},$$

$$\langle u \rangle_{0,G} = |u|_G^{(0)} + \{u\}_G^{(\alpha)}, \quad \langle u \rangle_{1,G} = |u|_G^{(0)} + |Du|_G^{(0)} + \{Du\}_G^{(\alpha)}, \quad \langle u \rangle_{2k+1,G} =$$

$$= \sum_{2r+|\beta|=0}^{k} \left|D_t^r D^\beta u\right|_G^{(0)} + \sum_{2r+|\beta|=k+1}^{2k+1} \left| d^{|\beta|-k+r} D_t^r D^\beta u \right|_G^{(0)} + \sum_{2r+|\beta|=2k+1} \{d^{k+1} \, D_t^r D^\beta u\}_G^{(\alpha)}.$$

Let $P_0 \in C^1(\overline{\omega})$, $P_0 = 0$ on $\partial\omega$, and $|\nabla P_0| + P_0 \geq \kappa > 0$ in $\overline{\omega}$. Then the $(n-1)$-dimensional manifold $\partial\omega$ can be parametrized as follows: 1) given an arbitrary point $\xi_0 \in \partial\omega$ we may introduce a local coordinates in $\mathbb{R}^n$ with the origin ξ_0 so that the axis ξ_n coincides with the inner normal to $\partial\omega$ at ξ_0; 2) there exists $\rho > 0$ such that for every $\xi_0 \in \partial\omega$ the set $B_\rho(\xi_0) \cap \partial\omega$ is defined by the formulas $\xi_i = y_i$ if $i \neq n$, $y_n = P_0(y', \xi_n)$, $y' = (y_1, \ldots, y_{n-1}) \in B_\rho(\xi_0) \cap \{\xi_n = 0\}$. We set $\omega_\rho = \omega \setminus \cup_{\xi_0 \in \partial\omega} B_\rho(\xi_0)$, denote $D = B_\rho(\xi_0) \times (0,T]$, and define

$$\begin{aligned} \|u\|_{W_{2k+1}(D)} = \langle u \rangle_{2k+1,D} &+ \sum_{0\leq 2r+|\beta|\leq k-1} \; \sum_{i\neq N} \left| d^{-\alpha/2} D_t^r D_{\xi_i} \left(D^\beta u\right) \right|_D^{(0)} \\ &+ \sum_{0\leq 2r+|\beta|\leq k-2} \; \sum_{i,j\neq N} \left| d^{-\alpha} D_t^r D^2_{\xi_i\xi_j} \left(D^\beta u\right) \right|_D^{(0)} \\ &+ \sum_{k\leq 2r+|\beta|\leq 2k} \; \sum_{i\neq N} \left| d^{-\alpha/2+r+|\beta|-k+1} D_t^r D_{\xi_i} \left(D^\beta u\right) \right|_D^{(0)} \\ &+ \sum_{k-1\leq 2r+|\beta|\leq 2k-1} \; \sum_{i,j\neq N} \left| d^{-\alpha+r+|\beta|-k+2} D_t^r D^2_{\xi_i\xi_j} \left(D^\beta u\right) \right|_D^{(0)} \qquad k \geq 1. \end{aligned}$$

The Banach spaces $V(k,Q)$ are defined as completion of the space $C^{\infty}\left(\overline{Q}\right)$ in the norms $\|u\|_{V(k,Q)} = \sup_{\xi_0\in\partial\Omega}\|u\|_{W(k,B_\rho(\xi_0)\times(0,T))} + \|u\|_{H^{k+\alpha,(k+\alpha)/2}(Q\setminus\{\omega_\rho\times(0,T]\})}$, where $\|\cdot\|_{H^{k+\alpha,(k+\alpha)/2}}$ denotes the standard Hölder norm. If a function w does not depend on t, we consider the function $\widetilde{w}(\xi,t)\equiv w(\xi)$ with the dummy variable t and use the notation $\|w\|_{V(2k+1,\omega)} = \|\widetilde{w}\|_{W(2k+1,Q)}$. The Banach spaces Λ_i are defined as completion of $C^{\infty}\left(\overline{Q}\right)$ with respect to the norms $\|u\|_{\Lambda_i} = \sum_{k=0}^{\infty}\dfrac{1}{k!M^k}\|t^kD_t^ku\|_{V(i,Q)}$. In this definition M is a finite number which will be specified later. The elements of Λ_i, viewed as functions of the variable t depending on $\xi\in\overline{\Omega}$ as a parameter, are real analytic. The norms of the functions defined on the surface $\Sigma\subset Q$ are given by $\|g\|_{V(2k,\Sigma)} = \inf\left\{\|G\|_{V(2k+1,Q)}:\ (\nabla G,\mathbf{n})|_{\Sigma}=g\right\}$, $\|g\|_{\Lambda_{2k}(\Sigma)} = \inf\left\{\|G\|_{\Lambda_{2k+1}}:\ (\nabla G,\mathbf{n})|_{\Sigma}=g\right\}$.

3. Assumptions and results

Let

$$\begin{cases} P_0,\ u_0^{m+p-2}\in V(2k+1,\omega) \text{ with some } k\ge 2,\ \alpha\in(0,1), \quad P_0\in C^1(\overline{\omega}), \\ P_0+|\nabla P_0|\ge\kappa>0 \text{ in } \overline{\omega}, \quad P_0>0 \text{ in } \omega, \quad P_0=0 \text{ on } \partial\omega. \end{cases} \tag{10}$$

Theorem 2 *Let $n=1,2,3$ and conditions (10) be fulfilled. There exists $\epsilon^*<1$, M and T^* such that for every $\|P_0\|_{V(2k+1,\omega)}<\epsilon^*$ problem (9) has in the cylinder Q with $T<T^*$ a unique solution (X,P,π). The function P is strictly positive inside ω and $P=0$ on $\partial\omega$. The vector $X(\xi,t)$ is represented in the form $X=\xi+\nabla v+\mathbf{rot}\,\mathbf{s}$, ($X=\xi+v_\xi$ if $n=1$). The solution (X,P,π) satisfies the estimate*

$$\|\pi\|_{\Lambda_{2k+1}}+\|v\|_{\Lambda_{2k+3}}+\sum_{i=1}^{n}\|s_i\|_{\Lambda_{2k+3}}+\|P\|_{\Lambda_{2k+1}}\le C\left(\|P_0\|_{V(2k+1,\omega)}+\|\phi\|_{\Lambda_{2k}(\Sigma)}\right)$$

with a finite constant C independent of X, π, and P.

Theorem 3 *Under the conditions of Theorem 2, there exists T^* such that*

1. *for every $t\in[0,T^*]$ mapping*

$$X(\xi,t)=\xi-\int_0^t (J^{-1})^*\nabla_\xi(P-\pi)(\xi,\tau)\,d\tau \tag{11}$$

is a bijection of $\overline{\omega}$ onto $\overline{\omega(t)}$ and the set $X(\partial\omega,t)$ is a $(n-1)$-dimensional manifold in $\mathbb{R}^n$;

2. *the weak solution $u(x,t)$ of problem (2) is defined by formulas (6) and (11) and is continuous in $\mathbb{R}^n\times[0,T^*]$; moreover, for every $t\in[0,T^*]$ we have $\partial\left(\overline{\operatorname{supp} u(x,t)}\right)=X(\partial\omega,t)$;*

3. *the set $\operatorname{supp} u(x,t)$ is defined by formula (11), where $\nabla\Pi|_{\Gamma(t)}=0$ if $m+p>2$.*

Theorem 4 *Under the conditions of Theorem 2, the function $p(x,t)$ satisfies conditions (10) in $\omega(t)$, and $p-\Pi\in V(2k+3,\omega(t))$, p, $\Pi\in V(2k+1,\omega(t))$. Moreover, for every fixed $\xi\in\overline{\omega}$ the functions $x=X(\xi,t)$, $P(\xi,t)=p(x,t)$, $\pi(\xi,t)=\Pi(x,t)$ are real analytic function of the variable t and $\|P\|_{\Lambda_{2k+1}}+\|\pi\|_{\Lambda_{2k+1}}\le K\|P_0\|_{V(2k+1,\omega)}$.*

Remark 1 It is easy to show that the regularity results stated in Theorem 4 remain true until the moment when the surface $\partial\omega(t)$ changes the topology i.e when $\partial\overline{\{\operatorname{supp} p(x,t)\}}$ ceases to be a $(n-1)$-dimensional manifold and there appears a point of auto-intersection.

Corollary 1 *The Cauchy problem for the porous medium equation $u_t = \Delta\, u^m$ with $m > 1$ can be viewed as a partial case of problem (1). Passing to the Lagrangian coordinates we arrive at problem (9) with $\Pi \equiv 0$. It follows from Theorems 3, 4 that the interface velocity is defined by the Darcy law, $\mathbf{v} = -\nabla\, p$, and that the inclusion $p(x, 0) \in V(2k+1, \omega)$ implies the inclusion $p(x, t) \in V(2k+3, \omega(t))$. By iteration we have that the solution $p(x, t)$ and its interface are infinitely differentiable with respect to the spatial variables and analytic in t. This recovers recent results of $[2,\ 5]$.*

4. Solution of problem (9). The linear model

Problem (9) is considered as the nonlinear equation $\mathcal{F}(v, \mathbf{s}, P, \pi) = 0$, where $Y = \nabla\, v + \mathbf{rot}\, \mathbf{s}$. Denote by $\mathcal{G}$ the Fréchet derivative of $\mathcal{F}$ at the initial state $v = 0$, $\mathbf{rot}\, \mathbf{s} = 0$, P_0, and π_0, where π_0 is the solution of the degenerate elliptic problem

$$\operatorname{div}(u_0 \nabla\, \pi_0) + u_0^p = 0 \quad \text{in}\ \omega, \quad (\nabla\,(\pi_0 + P_0), \mathbf{n})|_{\Sigma \cap \{t=0\}} = 0, \quad \pi = 0 \text{ on } \Gamma \cap \{t = 0\}. \tag{12}$$

The solution of the equation $\mathcal{F}(v, \mathbf{s}, P, \pi) = 0$ is obtained as the limit of the sequence of solutions of the linear problems $x_{n+1} = x_n - \mathcal{G}^{-1}\Big\langle \mathcal{F}(x_n) \Big\rangle$, $n = 0, 1, 2, \ldots$, with $x_n = (v_n, \mathbf{s}_n, P_n, \pi_n)$, $x_0 = (0, 0, P_0, \pi_0)$ (the modified Newton method). Construction of the operator $\mathcal{G}^{-1}$ reduces to solving the following problem: given the functions Φ, Ψ, H, one has to find a solution (Y, P, π), $Y = \nabla\, v + \mathbf{rot}\, \mathbf{s}$, of the linear system

$$\left\{ \begin{array}{l} Y_t + \nabla\ (P - \pi) = \Phi, \quad P + (m-1) P_0 \operatorname{div} Y = \Psi \quad \text{in}\ Q, \\ \operatorname{div}(u_0 \nabla\, \pi - u_0\, \mathbf{D}(v) \cdot \nabla\, \pi_0) + (1-p) u_0^p \Delta\, v = H \qquad \text{in}\ Q, \quad [\mathbf{D}(v)]_{ij} = 2 D^2_{\xi_i \xi_j} v, \end{array} \right. \tag{13}$$

under the initial and boundary conditions $Y(\xi, 0) = 0$, $P(\xi, 0) = 0$, $\pi(\xi, 0) = \pi_0$ in ω, $P = 0$, $\pi = 0$ on Γ, $Y = 0$, $u_0\, [(\nabla\, \pi, \mathbf{n}) + P/((m-1)P_0)(\nabla\, \pi_0, \mathbf{n})] = \psi\, (m/m-1)^{1/(m-1)}$ on Σ. Separating the potential and divergence-free parts of the prescribed vector $\Phi = \nabla\, f + \mathbf{rot}\, \sigma$ we may split problem (13) into separate problems for defining v, π, P, and $\mathbf{s}$. The vector $\mathbf{s}$ is found from the first equation in (13) by integration in t, P is defined from the second equation in (13). The scalar functions v and π are defined as the solutions of the parabolic and elliptic equations, coupled in the right-hand sides:

$$\begin{array}{l} v_t - (m-1) P_0 \Delta\, v = \pi + f - \Psi \quad \text{in}\ Q, \qquad (\nabla\, v, \mathbf{n}) = 0 \text{ on } \Sigma, \qquad v = 0 \text{ on } \Gamma \text{ and for } t = 0, \\ \operatorname{div}(u_0 \nabla\, \pi) = \operatorname{div}(u_0\, \mathbf{D}(v) \cdot \nabla\, \pi_0) + (p-1) u_0^p \Delta\, v + H \quad \text{in}\ \omega, \\ u_0\ [(\nabla\, \pi, \partial \mathbf{n}) - (\Delta\, v - \Psi/((m-1)P_0))\, (\nabla\, \pi_0, \partial \mathbf{n})]|_{\Sigma} = \psi, \quad \pi = 0 \text{ on } \Gamma, \quad \pi = \pi_0 \text{ for } t = 0. \end{array} \tag{14}$$

Existence of a solution to the linear parabolic-elliptic system (14) is proved by application of the Contraction Mapping Principle. To this end, we separately study the degenerate elliptic and parabolic problems:

$$(1) \left\{ \begin{array}{l} \operatorname{div}(u_0 \nabla\, \pi) = h \text{ in } \omega, \\ \pi = 0 \text{ on } \Gamma,\ (\nabla\, \pi, \mathbf{n}) = g \text{ on } \Sigma, \end{array} \right. \qquad (2) \left\{ \begin{array}{l} v_t - (m-1) P_0 \Delta\, v = F \text{ in } Q, \\ (\nabla\, v, \mathbf{n}) = 0 \text{ on } \Sigma,\ v = 0 \text{ on } \Gamma,\ v(\xi, 0) = 0. \end{array} \right. \tag{15}$$

Lemma 1 1) *Let $u_0^{m-2} h \in \Lambda_{2k+1}$, $g \in \Lambda_{2k,\Sigma}$ with $k \geq 1$. Then problem $(15)_1$ has a unique classical solution that satisfies the estimate $\|\pi\|_{\Lambda_{2k+1}} \leq K\,(1 + \|P_0\|)\,\big(\|w\| + \|g\| + \big\|u_0^{m-2} h\big\|\big)$.*

2) *If $F \in \Lambda_{2k+1}$ with a sufficiently large constant M and $P_0 \in V(2k+1, \omega)$ with $k \geq 1$, then problem $(15)_2$ has a unique classical solution v satisfying the estimate $\|v\|_{\Lambda_{2k+3}} \leq L \|F\|_{\Lambda_{2k+1}}$.*

3) *Let $m + p \geq 2$ and P_0, f, $u_0^{p-1}\Psi$, $u_0^{m-2} H \in \Lambda_{2k+1}$, $\psi \in \Lambda_{2k,\Sigma}$ with $k \geq 2$. Then there exists $\overline{T} > 0$ such that for every $T \in (0, \overline{T})$ problem (14) has a unique solution $(v, \pi) \in \Lambda_{2k+3} \times \Lambda_{2k+1}$.*

Once problem (9) is solved, the regularity of the solution to problem (2) easily follows provided that the bijectivity of the mapping $\omega \mapsto \omega(t)$ is established. In the one-dimensional case, the last condition is a byproduct of the second equation in (9) (the function $X(\xi,t)$ is bounded and monotone in ξ). The situation is not that simple in the multidimensional case where the topology of the set $\omega(t)$ may change with time. To establish bijectivity of the mapping $\omega \mapsto \omega(t)$ amounts to proving that $X(\partial\,\omega,t) = \partial\,\omega(t)$ for every $t > 0$. The inclusion $X(\partial\,\omega,t) \subset \partial\,\omega(t)$ follows from the second equation in (9). To prove the inverse inclusion we take two arbitrary points $\xi,\eta \in \partial\,\omega$, $\xi \neq \eta$, the point $\xi_0 \in \mathbb{R}^n$ such that $|\eta - \xi_0| = 1$ and $\cos(\xi - \eta, \eta - \xi_0) = 0$. We consider the function $\cos\left(X(\xi,t) - X(\eta,t), X(\eta,t)\right) = \dfrac{(X(\xi,t) - X(\eta,t), X(\eta,t))}{|X(\xi,t) - X(\eta,t)||X(\eta,t)|}$. Using representation (11) and the estimates on the solution $(v,\,\pi)$ of the problem posed in Lagrangian coordinates we check that there exists T^*, independent of the choice of ξ and η, such that $\cos\left(X(\xi,t) - X(\eta,t), X(\eta,t)\right) < 1/2$ for $t < T^*$, which means that the particles initially located at the points ξ and η do not belong to the same ray and, thus, their trajectories cannot hit one another within the time interval $[0,T^*]$.

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