POVERTY AND INEQUALITY UNDER INCOME AND PRICE DISPERSIONS*

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Abstract

Many policies simultaneously affect the distribution of prices and incomes in the economy. Moreover, a bias may occur when there is a stochastic relationship between prices and incomes and this relationship is being ignored. It is therefore important to dispose of an analytical framework for welfare analysis that could account for changes in this joint distribution.

How can the joint influences of price and income distributions on poverty and inequality indicators be analysed? We offer a method to deal with this problem by using parametric formulae of poverty and inequality measures. We propose statistical indicators for the levels, variabilities and a statistical link of price indices and nominal living standards. These indicators are consistent with an approximation of the problem based on a bivariate lognormal distribution of price indices and nominal living standards.

Our analysis provides hints about the social welfare impact of economic shocks or policies affecting levels, variabilities and correlation of prices and incomes. Intuitive insights are obtained, while using arithmetic means and variances instead does not lead to illuminating results. The role of price and income variabilities for poverty and inequality is more complex than generally expected, with the possibility of several variation regimes. Empirical investigation of these statistics would guide social policies.

Keywords**:** Strategy-proofness, public goods economies, differentiable mechanisms

Classification Numbers: D61, D78, H41.

1. Introduction

The distributions of prices and income across households play central roles for poverty and inequality analyses. The fact that income widely varies across households is at the root of social welfare analysis. The fact that the prices faced by households may vary considerably in the population is less often noticed in social welfare studies. In this paper, we propose a method to investigate the role of joint income and price distributions for poverty and inequality measurement.

Differences in prices people face may affect their capacity to consume goods even more than overall income levels[1]. Furthermore, the existence of unaccounted price differences across households may bias non-deflated poverty or inequality measures[2]. Thus, understanding the distributional interaction of prices and incomes in the calculation of poverty and inequality measures is crucial for social policies.

The issue we address is determining how to assess inequality and poverty when there is a stochastic relationship between prices and incomes and that relationship is being ignored. Indeed, if we use an aggregate price index to determine real incomes, for example in a situation of price discrimination against the poor, we underestimate poverty and inequality. To deal with such situations, we shall use a parametric model of the joint distribution of incomes and prices to help us understand the bias that would occur if the relationship between prices and incomes is not taken into account.

If one has joint data on prices and living standards, it is always possible to produce statistics with and without price effects. However, without clear statistical criteria, the interpretation of these results would be difficult. The statistics we propose provide guidance, and allow the classification of price and income situations in several regimes easy to interpret. So, the estimation of these statistics would help us to understand the approximative relationship between incomes and prices for welfare analysis. Second, if data on prices are missing or imperfectly observed, estimates of the distribution parameters from incomplete data, or conjectures about the values would help us control for the robustness of poverty and inequality analyses to price distributions.

There are several explanations of why price indices associated with distinct households may differ. First, spatial dispersion in prices is common. It is caused by imperfect markets, transport and commercialisation costs, and imperfect com-

[1] Sen (1981).

[2] Jazairy, Alamgir and Panuccio (1992), Koen and Phillips (1993).

munication. Meanwhile, spatial movements of absolute and relative prices accompany structural adjustment plans or pricing policies[3] that affect poverty and inequality[4].

Second, transaction costs specific to households may generate price differences. Rich households owning modern transport means often have lower transaction costs. The poor may face higher prices because they are subject to liquidity constraints, forcing them to buy goods at retail prices or at disadvantageous periods (Rao, 1997, 2000). The poor may also suffer higher search costs because they live far from commercial centres.

Third, households may be confronted with price discrimination by monopolists or oligopolists[5]. Sometimes, legal rules or public subsidies also implement price discrimination (e.g., cinema ticket discounts for students).

Fourth, because households' preferences and incomes vary, their individual true price indices differ according to variations in the chosen baskets of goods. Substitution effects have been well studied, and the distribution of the true price indices across households can be derived from estimates of a demand system[6].

Despite all these arguments, in the literature there does not seem to exist any explicit result showing how price and income distributions combine to affect poverty and inequality[7]. This paper fills this gap by developing a new tractable approach.

This is important for policy analysis. Structural adjustment plans or other economic policy measures are accompanied by large temporal and geographical movements of prices[8]. Adjustment plans have raised misgivings of poverty[9] rise, and their impact on poverty has been studied[10]. However, although most authors mention the importance of movements in aggregate relative prices on the wake of adjustment, they do not consider how these plans affect spatial prices. In particular, policies improving market efficiency may not only reduce price variability,

[3] Taylor and Phillips (1991), Duncan and Jones (1993).

[4] Bourguignon, De Melo and Morrisson (1991), Sahn and Sarris (1991), Koen and Phillips (1993).

[5] Basu and Bell (1991).

[6] As in Braithwait (1980) and Slesnick (1993).

[7] Muller (2002) surveys the sources of statistical links between PI and NLS.

[8] Alderman and Shively (1996), and Sahn, Dorosh and Younger (1997).

[9] The World Bank (1990, 2000), Bourguignon, De Melo, Morrisson (1991), Sahn and Sarris (1991).

[10] e.g. Kanbur (1987), Demery and Squire (1990), The World Bank (1990), Bourguignon, de Melo and Morrisson (1991), Duncan and Howell (1992), Schneider (1992), Sahn and Sarris (1991), Balisacan (1995), Sahn, Dorosh and Younger (1997).

but also alleviate possible negative statistical links between prices and living standards. Policies against price discrimination are likely to reduce price variability by favouring transactions based on uniform prices and diminish the correlation of prices and incomes. Other economic policies are directly designed to modify spatial prices[11]. Finally, one expects that policies raising supply would reduce prices and that policies that stimulate demand would augment them. However, the effect on price index variability would depend on the geographical impact of the policies in the different districts, assuming markets are not spatially integrated because of transport costs for example.

What emerges from a policy survey that we conducted is that: (1) a joint statistical framework is needed to analyse distributions of income and prices; (2) policies affect both level and dispersion of prices, with sometimes some hints about the direction of the effects; (3) more information, notably on the spatial structure of policies, would be useful for the analysis. However, since the latter information is generally missing, one can start as well with an aggregate perspective on the joint distribution of incomes and prices.

Our approach is based on a bivariate distribution model to exploit parametric formulae of poverty and inequality indicators. This approach brings intelligibility to the role of distribution parameters that is difficult to express in a general framework. It also exhibits relevant descriptive statistics for discussing poverty and inequality in terms of levels, variabilities and statistical links of prices and incomes. In this sense, we provide vocabulary in terms of simple statistics to describe, at least approximately, the interaction of price and income distributions for poverty and inequality analyses.

How can we analyse the effects induced by levels, dispersions and statistical links of prices and incomes on poverty and inequality? The aim of this article is to examine this question. What is important here is the capacity to intuitively describe the interactions of price and income distributions for poverty and inequality indicators. We clarify the relative roles of these distributions by using a parametric representation of the joint distribution of price indices and nominal living standards. We first define in Section 2 the Watts poverty measure, the Head-count index and the Gini coefficient of inequality. Then, we present their

[11] See for example Muellbauer (1974a, 1974b), Besley and Kanbur (1988), Pinstrup-Andersen (1988), Bevan, Collier and Gunning (1990), Ravallion and van de Walle (1991), Duncan and Jones (1993), Koen and Phillips (1993), Lipton and Ravallion (1993), Alderman and Shively (1996), Bardhan (1996), Binswanger and Deininger (1997), Badiane and Shively (1998).

formulae under lognormality. We proceed in Section 3 by analysing their variations with respect to the parameters of the joint distribution. We exhibit several variation regimes of poverty and inequality with respect to price and income variabilities. This leads to a more complex framework of analysis of social policies than generally used. We discuss extensions to other poverty measures in Section 4. And finally, Section 5 concludes.

2. Poverty and Inequality under Price Dispersion

The living standard indicator for an individual in household i at period t is defined as $y_{it} = \frac{c_{it}}{es_i \, P_{it}} = \frac{w_{it}}{P_{it}}$, where c_{it} is the value of the nominal consumption of household i at period t; w_{it} is the nominal living standard (NLS) of household i at period t. es_i is the equivalence scale of household i. P_{it} is the price index (PI) associated with household i and period t. We use price indices to summarise the price information. The price dispersion is therefore described by the price index dispersion across households. Similarly, the distributions of incomes and household characteristics are summarised by the distribution of the NLS. Statistical information on NLS and PI variables can be obtained from official statistics and many published articles.

Note that the analysis we conduct is also valid with other decompositions of living standards as ratios or products of correlated variables. In particular, it is often thought that equivalence scales[12] should depend on income. Thus, the role of the joint distribution of real income and equivalence scale can be studied by using the methods of this paper and systematically replacing 'NLS' with 'real income', and 'PI' with 'equivalence scale'. However, in that case the equivalence scale does not correspond to a few discrete values but rather to a continuous data generating process that can be approximated as lognormal.

What is known about theoretical effects of price and income distributions on poverty? Very little, although some rudimentary information can be mobilised from axioms used in the welfare literature. For example, poverty is generally believed to increase if all prices rise or incomes fall. Although transfer axioms suggest that an increase in a special type of income dispersion augments inequality, they provide too few insights for our purpose.

Our strategy is first to summarise the relevant price and income information

[12] Equivalence scales are deflators of living standards used to make comparisons of welfare or real income across households of different sizes and compositions.

by focusing on price indices and nominal living standards; second, to describe the joint distribution of these variables by the most parsimonious probabilistic model that would still provide a credible approximation; third, to choose a few poverty and inequality indicators. More summarising variables (e.g., equivalence scales, different sub-group indices, different income categories) could be used, as well as more complex probabilistic models.

We first base our investigation on the Head-Count Index (the most popular poverty measure), the Watts measure[13] (one of the main axiomatically sound poverty measures) and on the Gini coefficient of inequality (the most popular inequality measure). Then, we extend it in Section 4 to general classes of poverty measures. We choose these measures because they are commonly used and lead to convenient parametric expressions.

The Watts poverty measure is defined as $W = \int_0^z \ln(z/y)\, dF(y)$, where F is the cumulative density function of real living standards y, and z is the poverty line defined in real terms[14]. The Watts measure satisfies the focus, monotonicity, sub-group consistency, transfer and transfer sensitivity axioms and other attractive properties (Zheng, 1993)[15]. The Head-Count Index is $H = \int_0^z \; dF(y)$. The Gini coefficient of inequality is $G = 1 - 2\int_0^1 L(p)dp$, where $L(p)$ describes the Lorenz curve. The Lorenz curve indicates the share of total income that is received by the bottom p percent of income units. Let us define the generalised inverse of a cdf F as $F^{-1}(t) = \inf_U\{U : F(U) \geq t\}$. Then, for the cdf F of y, the mean is equal to $\lambda = \int_0^1 F^{-1}(t)dt$ and the corresponding Lorenz curve is defined as $L(p) = \frac{1}{\lambda}\int_0^p F^{-1}(t)dt$, for $p \in [0, 1]$.

Our analysis is based on approximating the joint distribution of (w, P) with a

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[13]Watts (1968).

[14]Here, the poverty line z is defined regardless of the distributions of NLS and PI. Therefore, once fixed it does not depend on prices and living standards. Nevertheless, many methods for calculating poverty lines are available (Ravallion, 1998). For z, depending on the distributions of NLS and PI, we should incorporate an explicit function $z(F)$ in the calculation, which would add terms to the formulae obtained in the paper. However, because no general result can be derived for all the methods used to calculate the poverty line, we do not pursue this approach.

[15]Focus axiom: The poverty index $P(y, z)$ is independent of the income distribution above z. Monotonicity: $P(y, z)$ is increasing if one poor has a drop in income. Transfer: $P(y, z)$ increases if income is transferred from a poor person to someone more wealthy. Transfer-sensitivity: The increase in $P(y, z)$ in the previous Transfer axiom is inversely related to the income level of the donator. Sub-group consistency: If an income distribution is partitioned in two sub-groups y' and y'', then an increase in $P(y'', z)$ with $P(y', z)$ constant, increases $P(y, z)$.

bivariate lognormal distribution. The lognormal approximation has already been used in empirical studies of living standards[16] and sometimes fits well income data[17]. Also, the parametric formulae of poverty and inequality measures under lognormality may be considered as the first term of a Taylor series expansion under the true distribution (as in Maasoumi, 1989). The lognormality of income has also been used in economic theory[18]. Similarly, the lognormality of price distributions has been used in the applied literature[19].

Other distribution models for living standards or incomes[20] or other distribution models for prices[21], although they could be used, would be less convenient for exposition purposes. We adopt a bivariate lognormal specification to simplify matters so a systematic approach can be taken with intuitive insights being generated.

Another method, when one has data on the full joint distribution, is to compute non-parametric estimates of poverty and inequality measures, with and without price effects and to examine the differences in the results. Here, there are still arguments for a parametric approach. First, joint observation of prices and living standards is rather rare in a form that allows joint non-parametric measurement, and some sort of model is necessary to fill the gap left by not observing prices and incomes for the same statistical units. Second, there are cases where individual data (out of a sample survey) are not available, while information is from which estimates of distribution parameters can be inferred (e.g. means and Gini coefficient for living standards). Last, our aim is to obtain insights that can be intuitive and qualitatively general. This is difficult with a non-parametric approach, based on an infinity of parameters to consider. It is also unclear how to reach an intuitive understanding of the role of the characteristics of the joint distribution of prices and incomes for non-parametric poverty and inequality analyses. In all these situations our parametric analysis will provide some insight into how price variability and its relationship to incomes will create a bias in poverty and inequality measurement, about which we can at least indicate the direction of the bias.

The parametric expressions of the Head-Count Index and the Gini coefficient

[16] Alaiz and Victoria-Feser (1996), Slesnick (1993).

[17] E.g. in Cramer (1980), van Praag, Hagenaars and van Eck (1983), Cowell (1993).

[18] Hildenbrand (1998).

[19] Eaton (1980), Deaton and Grimard (1992).

[20] Champernowne (1952), Salem and Mount (1974), Kloek and van Dijk (1978), Singh and Maddala (1976), Slottje (1984), Hirschberg and Slottje (1989).

[21] Creedy and Martin (1994).

of inequality under lognormality are well-known in the univariate case. Muller (2001) derives the expression of the Watts measure under lognormality[22] in the univariate case. Then, the parametric expressions of W, H and G under lognormality of (w, P) can directly be obtained by considering the distribution of real living standards y. Proofs are provided in the appendix. One must consider the joint bivariate distribution of w and P for distinguishing the role of these variables.

Let us assume that NLS and PI follow the bivariate lognormal distribution, $\mathrm{LN}\left[\begin{pmatrix} \mu_w \\ \mu_P \end{pmatrix}, \begin{bmatrix} \sigma_w^2 & \rho\sigma_w\sigma_P \\ \rho\sigma_w\sigma_P & \sigma_P^2 \end{bmatrix}\right]$. Note that μ_w and σ_w (respectively μ_P and σ_P) are the mean and the standard deviation of the *logarithms* of NLS, (respectively the *logarithms* of PI) and ρ is the correlation coefficient of the *logarithms* of NLS and PI. In this case, the Watts measure is equal to

$$\begin{aligned} W &= (\ln z - m_w + m_P)\,\Phi\left(\frac{\ln z - m_w + m_P}{\sqrt{\sigma_w^2 + \sigma_P^2 - 2\rho\sigma_w\sigma_P}}\right) \\ &\quad + \sqrt{\sigma_w^2 + \sigma_P^2 - 2\rho\sigma_w\sigma_P}\,\phi\left(\frac{\ln z - m_w + m_P}{\sqrt{\sigma_w^2 + \sigma_P^2 - 2\rho\sigma_w\sigma_P}}\right) \end{aligned} \tag{2.1}$$

where ϕ and Φ are respectively the pdf and cdf of the standard normal distribution, and the poverty line z is expressed in units of real income (see Appendix 1). Let $S = \sqrt{\sigma_w^2 + \sigma_P^2 - 2\rho\sigma_w\sigma_P}$ and $Z = \frac{\ln z - \mu_w + \mu_P}{\sqrt{\sigma_w^2 + \sigma_P^2 - 2\rho\sigma_w\sigma_P}}$. S is the standard deviation of the logarithm of the real living standards, that we denote *global variability.* S is equal to the square root of the variance of logarithms, a common inequality measure[23]. Z is the normalised difference of the logarithm of the poverty line with the mean of the logarithms of real living standards. We call $-Z$ the *standardised poverty logarithmic gap.* The knowledge of Z and S is sufficient to calculate W. Indeed, $W = S[Z\Phi(Z) + \phi(Z)]$.

[22] Due to the presence of integrals that cannot be solved explicitly, no explicit formula is known for the other axiomatically sound poverty measures such as Foster-Greer-Thorbecke poverty indices (P_a) and Clark-Hemming-Ulph-Chakravarty's measures, with poverty severity parameters consistent with the transfer axiom.

Parametric formulae for a few inequality indicators under income Gamma distribution have also been proposed (McDonald and Jensen, 1979) or under income Pareto Distribution (Chipman, 1974).

[23] Although Foster and Ok (1999) showed that it may disagree with Lorenz rankings, that is not the case under lognormality.

Under bivariate lognormality, the Head-Count Index H and the Gini coefficient G can be written as

$$H = \Phi[Z] = \Phi\left[\frac{\ln z - \mu_w + \mu_P}{\sqrt{\sigma_w^2 + \sigma_P^2 - 2\rho\sigma_w\sigma_P}}\right] \quad (2.2)$$

$$\text{and } G = 2\Phi\left[\frac{S}{\sqrt{2}}\right] - 1 = 2\Phi\left[\frac{\sqrt{\sigma_w^2 + \sigma_P^2 - 2\rho\sigma_w\sigma_P}}{\sqrt{2}}\right] - 1. \quad (2.3)$$

It is clear from (2.1), (2.2) and (2.3) that prices cannot be neglected in poverty or inequality analyses unless all PI are strictly concentrated around 1. In these formulae, the parameters associated with the respective distributions of w and P play symmetrical roles. Parameters $\mu_w, \mu_P, \sigma_w, \sigma_P$ and ρ, which stand for themselves as making sense, are the relevant notions of levels, dispersions and a statistical link to obtain intelligible results. In that sense, these parameters correspond to natural statistics describing the role of PI and income distribution for poverty and inequality analyses[24]. In contrast, translating the formulae of poverty and inequality indicators in terms of mean and variance of these variables makes them hard to analyse[25]. To be brief, we denote *variabilities* the standard deviation parameters σ_w and σ_P. The next step is to study the variations of poverty and inequality indicators.

3. Variations of poverty and inequality measures

Studying the variations in poverty and inequality measures with respect to parameters is useful on several grounds. First, one does not know a priori what to expect for the effect of income and price variabilities on poverty and inequality. Our results provide an interpretation grid of otherwise opaque interactions of living standards and prices. Second, the results of the variation analysis inform us about what are the social welfare impacts of economic shocks or policies affecting levels, variabilities and correlations of prices and incomes.

[24] These empirical analogs are sufficient statistics when the lognormal assumption is strictly satisfied.

[25] For example, even with an indicator as simple as the Head-Count Index, $\partial H/\partial \bar{P}$ has the sign of $Z\sigma_P^2(\sigma_P - \rho\sigma_w) + e^{\sigma_P}\bar{P}^2 S\sigma_P + \sigma_P^2 S\sigma_P$, where $\bar{P}$ is the mean of P, hardly a palatable formula.

3.1. Aggregate variations

In this paper, we focus on the case $Z < 0$, i.e. when the poverty line is below the median of the living standard distribution, which corresponds to most empirically relevant situations[26]. We start with the parameters describing the levels, since one expects to find a positive shift in prices reduces welfare while a positive shift in incomes increases it. Indeed, this is what happens. With the lognormal distribution, the quantile of order α of the real living standard distribution is $e^{\mu_w - \mu_P + u_\alpha S}$ (Johnston, Kotz and Balakrishna, 1994), where u_α is the quantile of order α of the standard normal distribution. Clearly, the quantile is increasing in μ_w and decreasing in μ_P. Since all living standards shift in parrallel, poverty measured with the Watts measure and the Head-Count Index decreases in the mean logarithm of NLS and increases in the mean logarithm of PI. The corresponding marginal variations of W are bounded, but not necessarily that of H. Indeed, variations in the percentage of the poor can be extremely sensitive to changes in μ_w and μ_P if real living standards are strictly concentrated around the poverty line. The variations with respect to variability parameters are more complex.

Let us examine the change in the poverty and inequality measures caused by ignoring underlying prices. This change can be interpreted as the bias resulting when price data is missing: $\triangle H = \Phi(Z) - \Phi\left(\frac{\ln z - \mu_w}{\sigma_w}\right), \triangle G = \Phi(S/\sqrt{2}) - \Phi\left(\sigma_w/\sqrt{2}\right), \triangle W = S.\triangle C + \triangle S.C$, where $C = Z\Phi(Z) + \phi(Z) > 0$. In these formulae the impact of omitting μ_P and σ_P is essentially given by the impact on the actual welfare indicators (H, G, W) since the inital values of the welfare indicators do not depend on μ_P, ρ and σ_P.

In order to obtain a clear picture of the impact of price level and price dispersion on the bias, we first focus on the simple case where $\rho = 0$. In that case, the Gini coefficient rises with σ_P while it does not change with μ_P. The Head-Count Index increases with μ_P, as expected, and drops with σ_P, which is a new result to the best of our knowledge. The Watts index increases with μ_P, while its variations with σ_P are less obvious even under $\rho = 0$. To understand them, notice that C is an increasing function of Z, and therefore varies in the same direction as H. In particular, $\triangle C$ decreases with σ_P, while S rises with σ_P, still for $\rho = 0$. The sign of the variations of the Watts index with σ_P is therefore ambiguous. However, this sign can be seen as the integral of the marginal changes $\frac{\partial W}{\partial \sigma_P}$. This justifies studying the comparative statics of the poverty and inequality measures when σ_P, σ_w and ρ vary. We proceed to this task in the next subsection. Introducing

[26] The case $Z > 0$ is worked out in the discussion paper Muller (2004).

parameter ρ further complicates the analysis since for negative ρ an augmentation of σ_P may increase or decrease G and H. Moreover, it will allow us to determine what happens when the relationship between prices and incomes is ignored.

3.2. Comparative statics

Table 1 shows the marginal variations of H, W and G (shown in column) with respect to the distribution parameters (shown in lines).

Table 1: Marginal Variations of H, W and G

	H	W	G
$\frac{\partial}{\partial \mu_w}$	$-\frac{\phi(Z)}{S} < 0$	$-\Phi(Z) < 0$	0
$\frac{\partial}{\partial \mu_p}$	$\frac{\phi(Z)}{S} > 0$	$\Phi(Z) > 0$	0
$\frac{\partial}{\partial \sigma_w}$	$-\frac{\phi(Z)}{S^2} Z(\sigma_w - \rho\sigma_P)$	$\frac{\sigma_w - \rho\sigma_P}{S}\phi(Z)$	$\frac{\phi(\frac{S}{\sqrt{2}})}{S}\sqrt{2}(\sigma_w - \rho\sigma_P)$
$\frac{\partial}{\partial \sigma_P}$	$-\frac{\phi(Z)}{S^2} Z(\sigma_P - \rho\sigma_w)$	$\frac{\sigma_P - \rho\sigma_w}{S}\phi(Z)$	$\frac{\phi(\frac{S}{\sqrt{2}})}{S}\sqrt{2}(\sigma_P - \rho\sigma_w)$
$\frac{\partial}{\partial \rho}$	$\frac{\phi(Z)}{S^2} Z\sigma_w\sigma_P$	$\frac{-\sigma_w\sigma_P}{S}\phi(Z) < 0$	$-\frac{\phi(\frac{S}{\sqrt{2}})}{S}\sqrt{2}\sigma_w\sigma_P < 0$

Proof : First, recall that $H = \Phi(Z); G = 2\Phi(S/\sqrt{2}) - 1; W = S\left[Z\Phi(Z) + \phi(Z)\right]$; $Z = \frac{\ln z - \mu_w + \mu_P}{S}$ and $S = \sqrt{\sigma_w^2 + \sigma_P^2 - 2\rho\sigma_w\sigma_P}$. Then, the marginal variations of these indicators can be expressed through that of Z and S. We have $\frac{\partial H}{\partial Z} = \phi(Z) > 0; \frac{\partial H}{\partial S} = 0; \frac{\partial G}{\partial Z} = 0; \frac{\partial G}{\partial S} = \sqrt{2}\phi(\frac{S}{\sqrt{2}}) > 0; \frac{\partial W}{\partial S} = Z\Phi(Z) + \phi(Z) = \frac{W}{S} \geq 0; \frac{\partial W}{\partial Z} = S\left[\Phi(Z) + Z\phi(Z) - Z\phi(Z)\right] = S\Phi(Z) > 0$. The results are then obtained by deriving Z and S, and applying the chain rule. $\frac{\partial S}{\partial \mu_w} = \frac{\partial S}{\partial \mu_P} = \frac{\partial S}{\partial z} = 0; \frac{\partial Z}{\partial \mu_w} = -\frac{1}{S} < 0; \frac{\partial Z}{\partial \mu_P} = \frac{1}{S} > 0$;

$\frac{\partial S}{\partial \sigma_w} = \frac{\sigma_w - \rho\sigma_P}{S}; \frac{\partial S}{\partial \sigma_P} = \frac{\sigma_P - \rho\sigma_w}{S}; \frac{\partial S}{\partial \rho} = \frac{-\sigma_w\sigma_P}{S} < 0$;

$\frac{\partial Z}{\partial \sigma_w} = -\frac{Z(\sigma_w - \rho\sigma_P)}{S^2}; \frac{\partial Z}{\partial \sigma_P} = -\frac{Z(\sigma_P - \rho\sigma_w)}{S^2}; \frac{\partial Z}{\partial \rho} = \frac{Z\sigma_w\sigma_P}{S^2}$.

Therefore: $\frac{\partial H}{\partial \sigma_w} = \frac{\partial H}{\partial Z}\frac{\partial Z}{\partial \sigma_w} = \phi(Z)\frac{-Z}{S^2}(\sigma_w - \rho\sigma_P)$ of the sign of $(\rho\sigma_P - \sigma_w)Z$;

$\frac{\partial G}{\partial \sigma_w} = \frac{\partial G}{\partial S}\frac{\partial S}{\partial \sigma_w} = \sqrt{2}\phi(\frac{S}{\sqrt{2}})\frac{(\sigma_w - \rho\sigma_P)}{S}$ of the sign of $\sigma_w - \rho\sigma_P$;

$\frac{\partial W}{\partial \sigma_w} = \frac{\partial W}{\partial S}\frac{\partial S}{\partial \sigma_w} + \frac{\partial W}{\partial Z}\frac{\partial Z}{\partial \sigma_w} = \frac{W}{S}\frac{(\sigma_w - \rho\sigma_P)}{S} + S\Phi(Z)\frac{(-Z)}{S^2}(\sigma_w - \rho\sigma_P)$

$= \phi(Z)\frac{1}{S}(\sigma_w - \rho\sigma_P)$ of the sign of $\sigma_w - \rho\sigma_P$.

The marginal variations of H, W and G with respect to σ_P correspond to analogous formulae. Finally, $\frac{\partial H}{\partial \rho} = \frac{\partial H}{\partial Z}\frac{\partial Z}{\partial \rho} = \phi(Z)\frac{Z}{S^2}\sigma_w\sigma_P$ of the sign of Z;

$\frac{\partial G}{\partial \rho} = \frac{\partial G}{\partial S}\frac{\partial S}{\partial \rho} = -\phi(\frac{S}{\sqrt{2}})\frac{\sqrt{2}}{S}\sigma_w\sigma_P < 0$;

$\frac{\partial W}{\partial \rho} = \frac{\partial W}{\partial Z}\frac{\partial Z}{\partial \rho} + \frac{\partial W}{\partial S}\frac{\partial S}{\partial \rho} = S\Phi(Z)\frac{Z}{S^2}\sigma_w\sigma_P - W\frac{\sigma_w\sigma_P}{S^2} = -\frac{\sigma w\sigma_P}{S}\phi(Z) < 0. QED.$

Since G is a relative welfare measure, it is not affected by changes in μ_w or μ_P. Indeed, changing y into λy with λ a positive scalar only changes the mean of $\ln y$ into λ plus the mean of $\ln y$ and leaves its variance constant. Thus, relative (differentiable) welfare measures are such that $\frac{\partial \ln y}{\partial \mu_w} = \frac{\partial \ln y}{\partial \mu_P} = 0$. Similar situations would arise with relative poverty measures, such as the poverty gap with a fraction of the median as the poverty line. In practice, many poverty measures are not scale invariant, as opposed to many inequality measures.

The changes in poverty and inequality with σ_w and σ_P are less elementary than the changes with μ_w and μ_P. They involve several variation regimes summarised in Table 2. Poverty and inequality indicators are shown in lines. The regimes are shown in columns and depend on the level of σ_w/σ_P relatively to ρ and $1/\rho$. Moreover, $\frac{\partial H}{\partial \rho}$ is of the sign of Z, while $\frac{\partial W}{\partial \rho} < 0$ and $\frac{\partial G}{\partial \rho} < 0$.

Table 2: Regimes of Variations of H, W and G with respect to σ_P and σ_w

	$\rho < 0$	$\frac{\sigma_w}{\sigma_P} < \rho$	$\rho < \frac{\sigma_w}{\sigma_P} < 1/\rho$	$1/\rho < \frac{\sigma_w}{\sigma_P}$
H if $Z < 0$	$+, +$	$-, +$	$+, +$	$+, -$
W and G	$+, +$	$-, +$	$+, +$	$+, -$

A $+$ or $-$ sign respectively indicates the signs of $\frac{\partial .}{\partial \sigma_w}$ and $\frac{\partial .}{\partial \sigma_P}$ for the considered welfare indicators. Four regimes appear in columns that correspond to specific directions of variations for H, W and G.

These results highlight the impact of price and income dispersions on poverty and inequality that critically depends on the relative positions of σ_w/σ_P (denoted 'relative variability'), ρ and $1/\rho$. The results are interesting because little a priori intuition is available for the effects of the variabilities. In particular, positive and negative impacts are possible for the changes in any of these variabilities, depending on the value of ρ. Figure 1 illustrates these regimes by showing the levels of W, H and G for different values of σ_w/σ_P and ρ. The variations of the welfare measures can be visualized by starting from a given point of the graph and moving along the ρ-axis and the σ_w/σ_P-axis, while mentally keeping σ_w or σ_P fixed. The graphs show that the variations of the welfare measures with price index and living standard variabilities can be positive and negative, and its sign and intensity depend on the value of ρ.

Looking at the case $\rho = 0$ and σ_P rising helps to understand what happens. In this case, $\frac{\partial H}{\partial \sigma_P} = -\phi(Z) Z \sigma_w / \sqrt{\sigma_w^2 + \sigma_P^2}$. A growing price variability would shift some people below the poverty line in real income terms and other people above the poverty line. Then at the poverty line, if the density function is increasing (our case $Z < 0$), the number of people previously below the poverty line but who are forced above the poverty line will be less than the number of people who are above the poverty line but forced below. This reasoning must be modified when $\rho \neq 0$ by considering who is affected the most by the rise in price index variability, and the direction of the effect may change. Then, to identify for whom what variability is more important two elements matter: (1) the shape of the density function, especially near the poverty line; and (2) the statistical link of PI and NLS. In that sense, the lognormal approximation could be refined to incorporate features of interest, such as the presence of several modes near the poverty line, and more complex statistical links of PI and NLS.

However, subject to this caveat, our results show several fundamental insights. First, the correlation that matters is between the logarithm of NLS and the logarithm of PI. Second, the PI variability can offset or reinforce the NLS variability. Third, there are four variation regimes defined by the relative position of σ_w/σ_P with respect to ρ and $1/\rho$ and by the sign of ρ. Four, the variation regimes are valid across broad families of poverty measures (as we shall show below). Five, provided the poverty line is below the median of the real living standards, changing the poverty line does not modify the qualitative direction of the effects of the variabilities on poverty measures. Six, the results hint at specific roles for nonparametric analysis versus parametric modelling: (1) identifying minor modes of the living standard distribution near the poverty line; and (2) refining the representation of the statistical link of PI and NLS.

Identifying what is the most important variability is central. The results show that the size of the ratio σ_w/σ_P provides the right criteria for this, as compared to ρ and $1/\rho$. The influence of both variabilities are according to the sign and magnitude of ρ: (1) compensating or reinforcing each other, and (2) positive or negative for poverty and inequality.

Finally, the variation regimes reveal that understanding the roles of prices and incomes for poverty and inequality analysis is facilitated by the knowledge of the statistics defined as the sample analogs to ρ and σ_w/σ_P. We now say a few words on how the results may help monitoring policies that affect prices and incomes.

3.3. Monitoring of policies

Many policies simultaneously affect price and income levels and variabilities, with sometimes hints about a priori direction of effects of levels and variabilities. Indeed, most policies alter the average level of prices as well as their dispersion. For example, a policy improving market efficiency may: (1) raise aggregate logincome level through the higher efficiency of production that it entails; (2) reduce price index inequality, described by its variability, because of the elimination of transaction costs, liquidity constraints or price discrimination, and (3) reduce the correlation between logarithms of PI and logarithms of NLS by making the price environment of the poor more similar to that of the rich. Therefore, the analysis of a policy should be based on the total differential of the poverty or inequality measure, denoted M here, and not only its marginal variation with respect to some parameters only: $dM = \frac{\partial M}{\partial \mu_w} d\mu_w + \frac{\partial M}{\partial \mu_P} d\mu_P + \frac{\partial M}{\partial \sigma_w} d\sigma_w + \frac{\partial M}{\partial \sigma_P} d\sigma_P + \frac{\partial M}{\partial \rho} d\rho$.

In these conditions, ten numbers need to be known: the value of five partial derivatives and the five finite changes in parameters (approximating the corresponding differentials). This suggests producing statistics describing these elements. Sometimes, theoretical information can be used to guide the analysis. In the example of the policy above, one expects $d\mu_w > 0, d\sigma_w < 0, sgn(\rho).d\rho < 0$.

To effectively monitor policies that affect variabilities in prices and incomes, it seems necessary to first ascertain what is the relevant regime to consider. A policy marginally reducing PI variability σ_P, for example through price information broadcast during radio programmes and published in newspapers, and not affecting the other parameters, would raise the Gini coefficient and the Watts poverty index in all regimes, except if $\sigma_w/\sigma_P > 1/\rho$. Coordinated surveys about household living standards and local prices could be conducted to produce estimates of σ_w, σ_P and ρ, which would indicate what is the actual variation regime and help monitoring price and income policies.

Note that the effects of the changes in the variabilities are not necessarily monotonic. Switches in variation regime may occur according to evolution of values of σ_w, σ_P, ρ. For example, with fixed $\rho > 0$ and fixed σ_w, starting with a high σ_P, a decrease in the variability of PI reduces poverty and inequality until $\sigma_P^* \equiv \rho\sigma_w$ is reached. Then, any further reduction of σ_P entails rises in poverty and inequality. Thus, for example, improvements in market infrastructures that would diminish σ_P may have a complex dynamic effect on poverty and inequality when changes in price and income distributions are large. Again, a precise monitoring of the values of σ_w, σ_P and ρ may help in assessing the consequences of such economic measures. We now turn to the last parameter, ρ.

For all variation regimes an increase in the correlation between the logarithms of NLS and PI is associated with a decrease in poverty and inequality. This is consistent with the poor facing lower prices increasingly more often than the rich. Such an increasingly favourable price discrimination for the poor could be pursued by systematic policy measures. What makes the result remarkable is that

ρ is not the correlation of NLS and PI, but the correlation of their logarithms. Then, positive discrimination schemes should be based on the logarithms. We now extend our results to a broad class of poverty measures.

4. Other poverty measures

All the previous results can be generalised to most poverty measures used in practice. These measures can be written as $P = \int_0^z k(y/z)\, dF(y)$, where k is a kernel function describing the poverty severity for a given household of real living standard y. Under lognormality of y, these measures can be rewritten as $P = \int_{-\infty}^{Z} k\left(\frac{e^{St+\mu}}{e^{SZ+\mu}}\right) \phi(t)dt = \int_{-\infty}^{Z} k\left(e^{S(t-Z)}\right) \phi(t)dt \equiv J(Z,S)$, where $\mu = \mu_w - \mu_P$. Then, the marginal variations of a differentiable welfare measure of the type $J(Z,S)$ can also be decomposed by using Z and S as intermediary statistics: $dJ = \frac{\partial J}{\partial Z}dZ + \frac{\partial J}{\partial S}dS$. What matters therefore are the properties of $\frac{\partial J}{\partial Z}$ and $\frac{\partial J}{\partial S}$, notably their signs and relative values, which can be analysed specifically for each measure. By pursuing the calculation, we obtain the following results.

(i) $\frac{\partial J}{\partial \mu_P} = \frac{\partial J}{\partial Z}\frac{\partial Z}{\partial \mu_P} + \frac{\partial J}{\partial S}\frac{\partial S}{\partial \mu_P} = \frac{\partial J}{\partial Z}\frac{1}{S} > 0$ for all measures such that $\frac{\partial J}{\partial Z} > 0$. The latter inequality is expected because $-Z$ is the standardised poverty logarithmic gap.

(ii) $\frac{\partial J}{\partial \sigma_P} = \frac{\partial J}{\partial Z}\frac{\partial Z}{\partial \sigma_P} + \frac{\partial J}{\partial S}\frac{\partial S}{\partial \sigma_P} = \frac{\sigma_P - \rho\sigma_w}{S}\left[-\frac{Z}{S}\frac{\partial J}{\partial Z} + \frac{\partial J}{\partial S}\right] \equiv \frac{\sigma_P - \rho\sigma_w}{S}Q$. This is the quantity Q that matters for linking with the results that have been obtained for H, W and G. In particular, this is simple when Q keeps the same sign, which is often the case. In that situation, the variation regimes of J with respect to the variabilities are similar to that obtained for W and G. In the cases where Q changes its sign, identifying the sign change is enough to relate qualitatively the marginal variations of $J(Z,S)$ to that of W and G, with simple modifications in the regimes according to Q's sign changes.

(iii) $\frac{\partial J}{\partial \rho} = \frac{\partial J}{\partial Z}\frac{\partial Z}{\partial \rho} + \frac{\partial J}{\partial S}\frac{\partial S}{\partial \rho} = \frac{\sigma_w\sigma_P}{S}\left[\frac{Z}{S}\frac{\partial J}{\partial Z} - \frac{\partial J}{\partial S}\right] = -\frac{\sigma_w\sigma_P}{S}Q$, leading to similar analogies.

Let us look at the Foster-Greer-Thorbecke poverty severity index:

$P_2 = \int_0^z (1 - y/z)^2\, dF(y)$. We obtain $P_2 = \int_{-\infty}^{Z} (1 - e^{S(t-Z)})^2\, \phi(t)dt$;

$\frac{\partial P_2}{\partial Z} = 2S \int_{-\infty}^{Z} (e^{S(t-Z)} - e^{2S(t-Z)})\, \phi(t)dt \geq 0$ and

$\frac{\partial P_2}{\partial S} = 2\int_{-\infty}^{Z} (e^{2S(t-Z)} - e^{S(t-Z)})(Z - t)\, \phi(t)dt \leq 0$.

Here, $Q = -2\int_{-\infty}^{Z} \left(e^{2S(t-Z)} - e^{S(t-Z)}\right) t\phi(t)dt \geq 0$, which implies as for W and G: $\frac{\partial P_2}{\partial \sigma_2}$ of the sign of $\sigma_P - \rho\sigma_w$; $\frac{\partial P_2}{\partial \sigma_w}$ of the sign of $\sigma_w - \rho\sigma_P$; $\frac{\partial P_2}{\partial \rho} < 0$. Other examples can be easily developed that show the qualitative findings for the comparative statics of H, W and G have a relatively general scope. It is also interesting to note that Sen's and SST poverty measures can be analysed by using multiplicative decompositions involving Head-Count Index and Gini coefficients to which we can apply our formulae[27].

5. Conclusion

Price index dispersion across households arises from spatial price variations, transaction costs, price discrimination or consumer substitution effects. These price differences can seriously affect poverty and inequality measurement and analyses. However, it is unclear how to reach an intuitive understanding of the role of the characteristics of the joint distribution of prices and incomes for these analyses. In this article, we offer an approach that provides an intuitive interpretation grid based on simple statistics.

For this, we use the parametric formulae of poverty and inequality measures under bivariate lognormality. These formulae provide an integrated framework exhibiting statistics well-adapted to the study: the mean and the variance of the logarithm of price indices, the mean and the variance of the logarithms of nominal living standards, and the correlation coefficient of the logarithms of the prices indices and of the nominal living standards. Such statistics can be easily produced by statistical offices.

We study the variations of poverty and inequality measures with respect to the parameters of the joint distribution of price indices and nominal living standards. While the effects of a change in the mean of logarithms of price indices or nominal living standards on these measures are monotonic, the effects of the variabilities is more complex than usually expected. Several variation regimes are exhibited. We provide suggestions for using the results for policies.

The approach of this paper could be extended in several ways. One could deal with the distribution of equivalence scales and to higher dimensional settings describing the heterogeneity of living standards. Also, other parametric distributions could be used, although at the cost of the simplicity of the formulae and therefore the intuitiveness of the result. Moreover, the bias caused by the lognormality assumption could be studied by using Edgeworth expansion methods. Finally, when enough data are available, simulation methods could be used to complement our results with non-parametric statistics.

[27] Shorrocks (1995), Xu and Osberg (2002).

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Appendix 1: Proof of the parametric formula

Let ln(y) $\sim$ N(m , σ), whose c.d.f. is denoted K. The Watts' index can be decomposed as follows

$W(z) = \int_o^z -\ln(y) + \ln(z)\, dF(y)$

which yields using the transfer theorem with $u = ln(y)$:

$W(z) = \ln(z)K(\ln(z)) - \int_{-\infty}^{\ln z} u\, dK(u),$

and again with normalisation of u with $t = \frac{u-m}{\sigma}$

$W(z) = \ln(z)\Phi\left[\frac{\ln(z)-m}{\sigma}\right] - \int_{-\infty}^{\frac{\ln z - m}{\sigma}} \sigma\, t + m\ d\Phi(t),$

where Φ is the cumulative distribution function of the standard normal law. Then,

$W(z) = (\ln(z)\ - m)\Phi\left[\frac{\ln(z)-m}{\sigma}\right] - \sigma\, J(z),$

where $J(z) = \int_{-\infty}^{\frac{\ln z - m}{\sigma}} t\, d\Phi(t).$

Integration of the latter equation yields

$J(z) = -\frac{1}{\sqrt{2\pi}}\, e^{-\left(\frac{\ln(z)-m}{\sigma}\right)^2/2}$.

Finally, $W = (\ln(z) - m)\,\Phi\left[\frac{\ln z - m}{\sigma}\right] + \frac{\sigma}{\sqrt{2\,\pi}}\,e^{-(\frac{\ln(z)-m}{\sigma})^2/2}$. Then, since $y = w/P$ with $(\ln w, \ln P)' \sim N\left[\begin{pmatrix} \mu_w \\ \mu_P \end{pmatrix}, \begin{pmatrix} \sigma_w^2 & \rho\sigma_w\sigma_P \\ \rho\sigma_w\sigma_P & \sigma_P^2 \end{pmatrix}\right]$, we have $\sigma = \sqrt{\sigma_w^2 + \sigma_P^2 - 2\rho\sigma_w\sigma_P}$ and $m = \mu_w - \mu_P$, which implies the formulae of the text. The formulae for H and G can be similarly obtained. *QED.*