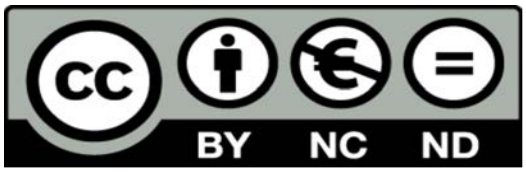
UNIVERSIDAD DE LA RIOJA

TESIS DOCTORAL

Título
Algebraic Reliability. Monomial ideals applied to multi-state system reliability
Autor/es
Patricia Pascual Ortigosa
Director/es
Eduardo Sáenz de Cabezón Irigaray
Facultad
Facultad de Ciencia y Tecnología
Titulación
Departamento
Matemáticas y Computación
Curso Académico



publicaciones.unirioja.es
E-mail: publicaciones@unirioja.es

PhD. thesis

Algebraic Reliability.
Monomial ideals applied to multi-state system reliability

Patricia Pascual-Ortigosa

Supervisor: Prof. Dr. Eduardo Sáenz de Cabezón

UNIVERSIDAD DE LA RIOJA

Logroño, 2022
PhD. Program on Mathematics and Computations

This work has been partially supported by grant *MTM2017-88804-P* from Ministerio de Economía, Industria y Competitividad (Spain), grant *PID-2020-116641GB-100* from Ministerio de Economía, Industria y Competitividad (Spain), grant of a *pre-doctoral contract for the formation of personal researcher* in the announcement supported by University of La Rioja, and grants ATUR 19/36, ATUR 20/31, ATUR 21/30 from the same University.

PhD. tesis

Fiabilidad algebraica. Ideales monomiales aplicados a la fiabilidad de sistemas multi-estado

Patricia Pascual-Ortigosa

Director: Prof. Dr. Eduardo Sáenz de Cabezón

Logroño, 2022
Programa de Doctorado en Matemáticas y Computación

Este trabajo ha sido financiado por el proyecto *MTM2017-88804-P* del Ministerio de Economía, Industria y Competitividad (España), el proyecto *PID-2020-116641GB-100* del Ministerio de Economía, Industria y Competitividad (España), el contrato predoctoral para formación de personal investigador financiado por la Universidad de La Rioja (convocatoria año 2018) y las ayudas ATUR 19/36, ATUR 20/31, ATUR 21/30 subvencionadas por la misma universidad.

Acknowledgments

A mis padres, que siempre me han apoyado
en todas las decisiones que he ido tomando.
A la memoria de mi abuelo, Daniel.
A Félix, lo mejor que me ha dado la vida.
A Félix Júnior & cía, lo mejor que le he dado a la vida.

There are lot of people to whom I am grateful, and the very first person on this list is my director, Eduardo Sáenz-de-Cabezón. Not only is he a great mathematician, but also a wonderful person. He has taught me so many things on the years we have been working together and supported me in the bad times, guiding my path when it was a bit dark.

I also want to thank to all persons that made me love mathematics. There have been a lot of: Judit, Guti, Chema, Óscar, LJ, Luis Español, Fillat, Tere, Manuel Bello, Javi, Ansorena, Arregui, and Ignacio. I would like to emphasize María Pilar Benito Clavijo (who is and has always been a mother-professor since I started the University until nowadays); and Jesús Laliena, who directed my Final Degree Work and showed me the little I knew (and I know).

I would like to thank the ones who taught me maths while I was not noticing it: Laure, Bea, Gadea, Jónathan, the Rubio's brothers (Ángel Luis and Julio), Clara, Dani, JuanMi and Jesús Mari. I would like to give and special space to Ana Romero and Jose Divasón, who has taught me a lot of things about research-life and listened to me when I needed, giving me the best pieces of advice.

I am grateful too to my PhD. mates: Adrián Inés, Ángela Casado, Manu García y Jorge Roldán. With all of you, doing administrative work is much more fun! Of course, thank you, Rodrigo Iglesias, my little math-brother, for helping me every time (and I mean every time) I needed.

All the persons -I apologize for the ones I forget- I knew along my research period and from whom I learnt so many things: Adrián Latorre, Edu Mota, Nacho, Míchel, Alberto, Edgar, Ujué, Anna Bigatti, Moji, Fatemmeh Mohammadi and Henry P. Wynn.

Natalia, Itziar, Maca and Judit I am sure I couldn't do it without you. You listen to me and let me disconnect from my work when I needed to. Thank you, my friends.

To my family, Yoli, Javi and Dani. Only you know how difficult this path has been. To the rest of my family, Andrea, Joanito, Ana Luci and Conrado; Ángel Luis, Esther, Aitana and Naiara; Luci and Ángel; María, Alberto, Javi and Sergio; Nicolás and Vicenta; thank you for supporting me.

And last, but the most important: thank you Félix for being with me on my way (and what's left!). You are my lobster.

Thank you, to all of you.

Contents

Introduction

In this thesis we study the reliability of multi-state systems using an algebraic approach based on monomial ideals.

We consider a *system* to be a set of components together with a *structure function*. Both the components and the system are said to be *binary* if they can reach only two levels of performance: 0 for failure and 1 for working; whereas we call them *multi-state* if they can take more than two states of performance. The *state* or *level of performance* of the system is determined by the state of the components by means of the structure function of the system.

The *reliability* (respectively *unreliability*) of a system is defined as the probability that the system is in a working (respectively failing) state, where each component of the system has an associated probability for each level that it can reach. For binary systems, it represents the probability that the system is in state 1 (0) while in multi-state systems we have different *reliability levels*, depending on the number of states of the system.

The study of system reliability is a branch of engineering that has turned into a very important topic in the last decades. Indeed, the first studies about multi-state systems were published in the late 70s [40, 70] and, since then, the methods for computing system reliability have advanced. This increase on system reliability interest is due to the fact that knowing the probability of a system failing or working has a huge importance in engineering, networks, biology, etc. Some examples of its applications are the followings:

- *Network Reliability*. Let us have a network in which we select one or more vertices as *source vertices* and one or more vertices as *target*

vertices. Each of the connections between the vertices of the network has a probability to be working. The problem consists in computing the probability that there exists at least a working path between the source and the target vertices. Some references that include algorithms developed for computing network reliability are [9, 54, 56, 146].

- *Power generation*. The safety and reliability of power systems is an essential component of energy security and it is becoming more important due to global warming. Considering that a national electricity net has many sources of supply and different components are in different states, this represents a multi-component multi-state network reliability problem. Some references are [87, 108, 128].

- *k-out-of-n systems* are systems that work (respectively fail) whenever at least k of its n components work (respectively fail). These systems and their variants are some of the most relevant types of systems studied in reliability theory due to their theoretical interest and wide range of applications. There exist efficient and particular methods to compute their reliability. Some references on k-out-of-n systems are [69, 75, 43].

This thesis is focused in studying multi-state system reliability from an algebraic approach based on monomial ideals. The main advantages of this approach are the following:

- The algebraic method provides useful *information about the structure of the system* such as the importance of the components, *i.e.* which is the component whose deterioration implies the higher devaluation of the system reliability, or the minimal working/cut paths.

- The generality respect to the *structure of the system* and the *probability distribution of the probability of failing or working of the components of the system*. The applicability of the algebraic method for computing system reliability is not affected by the structure of the system, *i.e.* it works for every kind of system, although it might not be

the most efficient one for some particular systems, if compared to other ad-hoc methods. Furthermore, it is able to compute the reliability either if the probabilities of the components are equal or different, independent or dependent, taking into account the differences between dependent and independent ones.

- The algebraic method presented in this thesis is an enumerative one which means that the list of all minimal working or failing states of the system needs to be computed. This in some cases represents a disadvantage. However, there exist implemented algorithms on algebraic computational systems such as `Macaulay2` or `CoCoA` which make the computations efficient.

The relation between algebra and reliability theory is provided by monomial ideals: square-free monomial ideals in the binary case and monomial ideals with exponents in the multi-state case. Each level of performance of a system has associated a *path ideal*, which allows to compute the reliability of the system, and a *cut ideal*, which gives the *unreliability* of the system.

With respect to *algebraic reliability*, in [116, 118, 119, 120] the authors study the reliability of binary systems. The results of these investigations are the basis for this thesis.

In this thesis we are going to go forward on the algebraic approach based on monomial ideals to compute system reliability. In particular, our research is focused on multi-state systems. We need to investigate the relationship between squarefree monomial ideals and monomial ideals with exponents. To look into this relationship, we are going to study the operations polarization and depolarization, and support posets. Once the relation of monomial ideals is treated, we are going to be able to start working on the analysis of multi-state system reliability. There exist different methods to compute the reliability of a system and there are some that are really efficient for specific systems. The algebraic method that we are going to investigate for multi-state system is general (such as *Markov's chains* or *Inclusion-Exclusion method*, cf. [69, 75]) and provides a good performance, although it is not as fast as some of the specific ones.

We are going to check, in some situations, how the algebraic method based on monomial ideals behaves in computational terms. To do that, we developed a `C++` class within the computer algebra library `CoCoALib`. Not only is this class going to allow us to perform some computational experiments, but it is also available for everyone who needs to use it.

These goals can be summarized as:

- Research on polarization and depolarization of monomial ideals.
- Investigate the application of polarization and depolarization operations for the analysis of multi-state system reliability.
- Study different kinds of multi-state systems and its associated algebraic structures.
- Development of algorithms to implement algebraic reliability.

Taking into account the objectives that we have just presented, we can summarise the contributions of this thesis as:

- *Polarization and depolarization operations*. Polarization is an operation that transforms a monomial ideal into a squarefree one. For each monomial ideal there is a unique polarization. Depolarization is the inverse operation but the resulting monomial ideal with exponents is not unique. In order to find all possible depolarizations of a monomial ideal (Theorem 2.3.8), we develop a combinatoric tool: *support posets*. Polarization and depolarization are interesting because the original ideal and its polarization or depolarization share some important properties such as the Betti numbers and the Hilbert series in the sense that the Hilbert series of the polarization is the polarization of the Hilbert series of the monomial ideal. However, it is easier to compute them for the square-free monomial ideal. In this thesis we investigate more properties shared between both ideals (Lemma 2.3.12 shows that the lcm-lattices of two ideals sharing the same polarization are isomorphic and Theorem 2.3.13 says that the width of the support poset of a monomial ideal is an upper bound of the projective dimension of the ideal). Furthermore, we

investigate the conditions needed by a poset for being a support poset of a monomial ideal. Although we have not characterized which posets are support posets of a monomial ideal, Proposition 2.3.1 provides a sufficient condition for building a monomial ideal with a given poset (under certain conditions), and in Proposition 2.3.14, a sufficient condition needed by a poset for being the support poset of a 0-dimensional monomial ideal is given. Finally, we have shown some families of posets (lines, diamonds, trees and forest) which are support poset for some monomial ideal (Proposition 2.3.18, Proposition 2.3.23, Proposition 2.3.27 y Theorem 2.3.30). For those ideals, Betti numbers have been computed.

- *Algebraic Reliability of Multi-state systems*. In this area we made a generalization of the work started by Eduardo Sáenz-de-Cabezón and Henry P. Wynn for binary systems and show how the algebraic method works for multi-state systems. Moreover, we thoroughly study multi-state k-out-of-n systems for which we have reviewed the different definitions given in literature. For all of them, we gave an algebraic definition based on monomial ideals (Proposition 3.2.4, Definition 3.2.13 and Proposition 3.2.13). For some variants of k-out-of-n systems (binary k-out-of-n systems with multi-state components, multi-state consecutive k-out-of-n, sparsely connected homogeneous multi-state k-out-of-n and weighted multi-state k-out-of-n systems) we give algebraic structures associated to them and explicit formulas for computing their Betti numbers.

- *Algorithms to compute reliability of multi-state systems*. We develop a `C++` class within the open source computer algebra library `CoCoALib` which allows to compute the reliability of a multi-state systems. The class is presented in
  `http://www.dima.unige.it/~bigatti/data/AlgebraicReliability/`

The structure of this thesis is the following: in Chapter 1 we explain the algebraic and reliability theory background required for the next chapters. Chapter 2 shows the relationship between squarefree monomial ideals and monomial ideals with exponents by analyzing thoroughly polarization

and depolarization operations and support posets. In Chapter 3 we study multi-state systems and compute their reliability using the algebraic approach based on monomial ideals. Chapter 4 is focused on computacional algorithms for the reliability of multi-state systems.

Introducción

Esta tesis se centra en el estudio de la fiabilidad de sistemas multi-estado con un acercamiento algebraico basado en el uso de ideales monomiales.

Consideramos un *sistema* como un conjunto de componentes junto con una *función de estructura*. Tanto las componentes como el sistema se dicen *binarios* si únicamente pueden alcanzar dos niveles o estados de funcionamiento diferentes: 0 es fallo y 1 es funcionamiento; mientras que se dicen *multi-estado* si pueden alcanzar más de dos estados diferentes. El *estado* o *nivel de funcionamiento* de un sistema está determinado por el estado de sus componentes por medio de su función de estructura.

La *fiabilidad* (respectivamente *no fiabilidad*) de un sistema se define como la probabilidad de que el sistema se encuentre en un estado de funcionamiento (o en un estado de fallo, respectivamente), donde cada componente del sistema tiene asociada una probabilidad para cada uno de los estados que puede alcanzar. Para sistemas binarios, la fiabilidad representa la probabilidad de que el sistema se encuentre en estado 1, mientras que para los sistemas multi-estado tenemos diferentes *niveles de fiabilidad* dependiendo del número de estados del sistema.

El estudio de fiabilidad de sistemas es una rama de la ingeniería que ha cobrado gran relevancia en las últimas décadas. De hecho, los primeros artículos relacionados con sistemas multi-estado fueron publicados a finales de los años 70 [40, 70] y, desde entonces, los métodos para calcular la fiabilidad de un sistema han avanzado notablemente. Este interés creciente en la fiabilidad de sistemas es debido a que conocer la probabilidad de que un sistema funcione o falle es de gran relevancia en campos como la

ingeniería, redes, biología, etc. Algunos ejemplos de sus aplicaciones son los siguientes:

- *Fiabilidad de redes*. Imaginemos que tenemos una red en la que seleccionamos uno o más vértices como *vértices fuente* y uno o más vértices como *vértices objetivo*. Cada una de las conexiones entre los vértices de la red tiene un probabilidad de funcionar. El problema de la fiabilidad de redes consiste en calcular la probabilidad de que exista al menos un camino entre los vértices fuente y los vértices objetivo. Algunas referencias con algoritmos actuales desarrollados para calcular la fiabilidad de redes son [9, 54, 56, 146].

- *Generación de energía*. La seguridad y la fiabilidad de sistemas de energía es una componente esencial de la seguridad energética y se está volviendo más y más importante debido al calentamiento global. Considerando que una red eléctrica nacional tiene muchas fuentes de suministro y que diferentes componentes están en diferentes estados de funcionamiento, esto es un problema de fiabilidad de redes multi-estado. Algunas referencias en las que se aborda este problema son [87, 108, 128].

- Los *sistemas k-entre-n* son sistemas que funcionan (o fallan) cuando al menos k de sus n componentes funcionan (o fallan). Estos sistemas y sus variantes son uno de los tipos de sistemas más relevantes y más estudiado en el campo de la fiabilidad algebraica debido a su interés teórico y su amplio rango de aplicaciones. Existen métodos particulares y eficientes para calcular la fiabilidad de este tipo de sistemas. Algunas referencias de sistemas k-entre-n son [69, 75, 43].

Esta tesis está centrada en estudiar la fiabilidad de sistemas multi-estado desde un punto de vista algebraico basado en ideales monomiales. Las ventajas principales de este acercamiento algebraico son las siguientes:

- El método algebraico proporciona información relevante *sobre la estructura del sistema* como puede ser la importancia de las componentes (cual es la componente cuyo deterioro implica una mayor

devaluación en la fiabilidad del sistema) o los caminos y cortes mínimos.

- La generalidad respecto a la *estructura del sistema* y a la *distribución de probabilidad de las probabilidades de fallo o funcionamiento de las componentes del sistema*. La aplicabilidad del método algebraico para calcular la fiabilidad de un sistema no se ve afectado por la estructura del sistema, es decir, funciona para cualquier sistema, aunque puede no ser el más eficiente. Además, es capaz de calcular la fiabilidad de un sistema en el que las probabilidades de las componentes son iguales o diferentes o incluso si las probabilidades de las componentes son dependientes, teniendo en cuenta alguna diferencia con el modo de calcular la fiabilidad cuando son independientes.

- El método algebraico presentado para calcular la fiabilidad de un sistema es un método enumerativo, es decir, es necesario calcular todos los caminos o cortes mínimos. Este hecho suele ser una desventaja cuando hablamos de cálculos computacionales. Sin embargo, existen algoritmos implementados en sistemas computacionales algebraicos tales como `Macaulay2` o `CoCoA` que hacen estos *cálculos eficientes*.

La relación entre algebra y teoría de la fiabilidad viene dada por los ideales monomiales: ideales libres de cuadrados en el caso binario e ideales monomiales con exponentes en el caso multi-estado. Cada nivel de funcionamiento del sistema tiene asociado un *ideal de caminos*, el cual nos permite calcular la fiabilidad del sistema, y un *ideal de cortes*, con el que podemos calcular la no fiabilidad del sistema.

Con respecto a la fiabilidad algebraica basada en ideales monomiales, en [116, 118, 119, 120] los autores realizan un estudio de fiabilidad de sistemas binarios. Los resultados de esas investigaciones son la base de esta tesis.

En esta tesis vamos a avanzar en el acercamiento algebraico basado en ideales monomiales para calcular la fiabilidad de un sistema. En particular, nuestro estudio está centrado en sistemas multi-estado. Por ello, necesitamos estudiar con detenimiento la relación existente entre los ideales monomiales libres de cuadrados y los monomiales con exponentes.

Para investigar esta relación, vamos a explorar las operaciones polarización, depolarización y los *support posets*.

Una vez tratada la relaciones entre los ideales monomiales citados, podremos comenzar a analizar la fiabilidad de un sistema multi-estado. Existen diferentes métodos para calcular la fiabilidad de un sistema, incluso los hay que son muy eficientes para sistemas específicos y bajo ciertas condiciones. El método algebraico que vamos a investigar para calcular la fiabilidad es general (como pueden serlo las *cadenas de Markov* o el método de *Inclusión-Exclusión* cf. [69, 75]) y ofrece un buen rendimiento, aunque no es tan rápido como algunos de los métodos específicos. Además, vamos a comprobar cómo se comporta, en algunos casos, el método algebraico basado en ideales monomiales en términos computaciones. Para ello, hemos desarrollado una clase en `C++` con la librería de álgebra computacional `CoCoALib`. La clase no solo nos va a permitir realizar experimentos computacionales con ella, si no que, además, está disponible para cualquier persona que la necesite.

Estos resultados quedan resumidos en los siguientes puntos:

- Estudio de la polarización y depolarización de ideales monomiales.
- Definir la aplicabilidad de las operaciones polarización y depolarización al análisis de la fiabilidad de un sistema.
- Estudio de los distintos tipos de sistemas multi-estado y sus estructuras algebraicas asociadas.
- Implementación de algoritmos.

Teniendo en cuanta los objetivos que acabamos de presentar, el resumen de las contribuciones de esta tesis es:

- *Operaciones polarización y depolarización*. La polarización es una operación que transforma un ideal monomial con exponentes en un ideal monomial libre de cuadrados. Para cada ideal monomial existe una única polarización. La depolarización es la operación inversa, pero el resultado no es único: al depolarizar un ideal monomial libre de cuadrados podemos obtener más de un ideal monomial

con exponentes. Con el objetivo de encontrar todas las posibles depolarizaciones de un ideal monomial (Teorema 2.3.8), hemos desarrollado una herramienta combinatoria llamada *support posets*. Las operaciones polarización y depolarización son interesantes porque los ideales originales y su polarización o depolarización comparten algunas propiedades importantes como pueden ser los números de Betti o la serie de Hilbert (no es exactamente la misma, pero es sencillo calcular una a partir de la otra) y, en muchas ocasiones, es más sencillo calcular estas propiedades en un caso determinado, es decir, a veces es más sencillo calcularlas en el caso libre de cuadrados y, otras, en un caso monomial con exponentes. En esta tesis hemos visto qué otras propiedades se comparten entre ambos ideales (en el Lema 2.3.12 mostramos que los lcm-lattice de dos ideales copolares son isomorfas y en el Teorema 2.3.13 que la anchura del *support poset* de la polarización de un ideal es una cota superior de la dimensión proyectiva del ideal monomial original). Además, hemos investigado bajo qué condiciones un *poset* es *support poset* de un ideal monomial. Aunque no hemos podido caracterizar cuáles lo son y cuales no, la Proposición 2.3.1 nos proporciona una condición suficiente para construir un ideal monomial dado un *poset* que cumpla ciertas condiciones y la Proposición 2.3.14 se da una condiciones suficiente para que un *poset* sea *support poset* de un ideal monomial 0-dimensional. Finalmente, hemos mostrado ciertas familias de *posets* (líneas, diamantes, árboles y bosques) que son el *support poset* de un ideal monomial (Proposición 2.3.18, Proposición 2.3.23, Proposición 2.3.27 y Teorema 2.3.30). Para todos los ideales que tienen este tipo de *support poset* hemos encontrado sus números de Betti.

- *Fiabilidad algebraica de sistemas multi-estado*. En este área hemos realizado una generalización del trabajo iniciado por Eduardo Sáenz-de-Cabezón y Henry P. Wynn para sistemas binarios y mostramos cómo funciona el método algebraico para sistemas multi-estado. Además, estudiamos con especial detalle los sistemas multi-estado k-entre-n, para los que revisamos las diferentes definiciones que se han dado en la literatura y las interpretamos en términos de ide-

ales monomiales (Proposición 3.2.4, Definición 3.2.13 y Proposición 3.2.13). Para ciertas variantes de los sistemas k-entre-n multi-estado (sistemas binarios k-entre-n con componentes multi-estado, sistemas multi-estado k-entre-n consecutivos, sistemas homogéneos *sparsely* conectados k-entre-n y sistemas multi-estado k-entre-n con pesos) damos sus estructuras algebraicas asociadas y fórmulas explícitas para calcular sus números de Betti.

- *Algoritmos para calcular la fiabilidad de sistemas multi-estado*. Se ha desarrollado una clase de `C++` con la librería libre de álgebra computacional `CoCoALib` que permite calcular la fiabilidad de sistemas multi-estado. La clase está disponible en
  `http://www.dima.unige.it/~bigatti/data/AlgebraicReliability/`

La estructura de esta tesis es la siguiente: en el Capítulo 1 se presentan los conceptos preeliminares algebraicos y de teoría de la fiabilidad necesarios en los capítulos posteriores. El Capítulo 2 se centra en la relación existente entre ideales monomiales libres de cuadrados e ideales monomiales con exponentes, analizando en profundidad las operaciones polarización y depolarización y el *support poset*. En el Capítulo 3 se estudian los sistemas multi-estado y se calcula su fiabilidad mediante el método algebraico basado en ideales monomiales. Terminamos con el Capítulo 4, donde mostramos los algoritmos computacionales desarrollados para calcular la fiabilidad de sistemas multi-estado.

Chapter 1

Background

In this thesis concepts from two different areas are going to take part: Commutative Algebra (in particular, monomial ideals) and Reliability Theory. This chapter is devoted to collect the notions and results that are going to be used in the following chapters.

For readers interested in the algebraic background we recommend [62, 31, 114]. For the ones interested in a deeper study of Reliability Theory than the presented here, we recommend [27, 75] and [116, 117, 118, 119, 120, 99], for an algebraic point of view.

1.1 Monomial ideals

Monomial ideals have been deeply studied due to their relevance in Commutative Algebra or in the intersection between Algebra and Combinatorics. See for example [23, 62, 94] and references therein. Monomial ideals can be used to encode the information of simplicial complexes, to reduce problems from polynomial ideals to monomial ideal ones or to connect them to combinatorial objects due to their combinatorial structure. Some computer algebra systems such as `CoCoA` or `Macaulay2` have developed fast algorithms to work with monomial ideals.

Monomial ideals are the ones who relate algebra and reliability theory: a coherent system has one (or more) monomial ideal associated. This topic

is extensively covered in Section 1.2.2 for binary systems, and in Chapter 3 for multi-state ones.

Let $\Bbbk$ be a field and $R = \Bbbk[x_1, \ldots, x_n]$ the polynomial ring in n variables over $\Bbbk$. Let $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}_0^n$. A *monomial* in R is a product $\mathbf{u} = x_1^{a_1} \ldots x_n^{a_n}$, denoted by $\mathbf{u} = \mathbf{x}^{\mathbf{a}}$, and we say that $\mathbf{a}$ is the *multidegree* of the monomial $\mathbf{u}$. If all the components of $\mathbf{a}$ are 0 or 1, we say that $\mathbf{u}$ is a *squarefree monomial*.

The set of monomials in R, $\mathrm{Mon}(R)$, is a $\Bbbk$-basis of R and the monomials $\mathbf{u} \in \mathrm{Mon}(R)$ correspond biyectively to the lattice points in $\mathbb{N}_0^n$. They satisfy that $\mathbf{x}^{\mathbf{a}} \cdot \mathbf{x}^{\mathbf{b}} = \mathbf{x}^{\mathbf{a}+\mathbf{b}}$, where the sum is componentwise.

Definition 1.1.1. An ideal $I \subseteq R$ generated by (squarefree) monomials is called *(squarefree) monomial ideal*.

Let $I = \langle g_1, \ldots, g_m \rangle$ be a monomial ideal. The set $\{g_1, \ldots, g_m\}$ is a set of generators of I if every $g \in I$ can be written as $g = \sum_{i=1}^{m} a_i g_i$, where $a_i \in R$. An interesting and important question is how can one know when a polynomial $f \in R$ belongs to I. In the case of working with polynomial ideals, this question is not easy to answer. However, in the case of monomial ideals it is. Firstly, we start answering an easier question: when does a monomial belong to a monomial ideal?

Proposition 1.1.2. *Let $I = \langle u_1, \ldots, u_k \rangle$ be a monomial ideal. Then the monomial $\mathbf{v} \in I$ if and only if there exists a monomial $\mathbf{w}$ such that $\mathbf{v} = \mathbf{w}u_i$, for some $i \in \{1, \cdots, k\}$.*

Proof. $\Rightarrow$) Let $\mathbf{v} \in I$. We know, by the definition of set of generators of I that $\mathbf{v} = a_1 u_1 + \cdots + a_k u_k$, where $a_i \in R$. Then, we have that $\mathbf{v} \in \bigcup_{i=1}^{k} \mathrm{supp}(a_i u_i)$, where $\mathrm{supp}(\mathbf{f}) = \{g_i \in \mathrm{Mon}(R) \; s.t. \; b_i \neq 0\}$, for $\mathbf{f} = b_1 g_1 + \cdots + b_m g_m \in R$ and $b_i \in \Bbbk$. $v \in \bigcup_{i=1}^{k} \mathrm{supp}(a_i u_i)$ implies that $\mathbf{v} \in \mathrm{supp}(a_i u_i)$ for some $i \in \{1, \ldots, k\}$. Hence, $\mathbf{v} = u_i \mathbf{w}$ for some $\mathbf{w} \in \mathrm{supp}(a_i)$.
$\Leftarrow$) We have that $\mathbf{v} = \mathbf{w}u_i$. By definition, $\mathbf{v} \in I$. □

Remark 1.1.3. A monomial ideal is uniquely determined by its monomials. Therefore, two monomial ideals I and J are equal if and only if both contain the same monomials.

As a Corollary, we have the answer to our question.

Corollary 1.1.4. *If f is a polynomial in $R = \Bbbk[x_1, \ldots, x_n]$ then $f \in I$ if and only if every monomial in f lies in I, i.e., if and only if f is a $\Bbbk$-linear combination of the generators of I.*

We know that ideals always have a set of generators. A natural question is if there exists a generating set better than the others and if this one is unique. The following proposition gives us an answer for the monomial ideal case:

Proposition 1.1.5. *Each monomial ideal has a unique minimal monomial set of generators. More precisely, let $G(I)$ denote the set of monomials in I which are minimal with respect to divisibility. Then $G(I)$ is the unique minimal set of monomial generators.*

From this point, we denote by $G(I)$ (or just G, when there is no possibility of mistake), the unique minimal set of monomial generators of I.

For monomial ideals, Dickson's Lemma assures that a monomial ideal is always finitely generated.

Lemma 1.1.6 (Dickson's Lemma.)**.** *Let $I = \langle \mathbf{x}^{\mathbf{a}} \text{ s.t. } \mathbf{a} \in A \rangle \subseteq \Bbbk[x_1, \ldots, x_n]$ be a monomial ideal. Then I can be written in the form $I = \langle \mathbf{x}^{\mathbf{a}(1)}, \ldots, \mathbf{x}^{\mathbf{a}(s)} \rangle$, where $\mathbf{a}(1), \ldots, \mathbf{a}(s) \in A$. In particular, I has a finite set of generators.*

For any monomial ideal I, we saw that it is possible to know which monomials are in the ideal and which not. An important concept related to monomial ideals are maximal standard pairs, which tells us how is the boundary of the monomials belonging to the ideal and the ones that not:

Definition 1.1.7. Let I be a monomial ideal in $R = \Bbbk[x_1, \ldots, x_n]$ and $\sigma \subseteq \{1, \ldots, n\}$. The pair (x^μ, σ) is a *standard pair* for I if it satisfies:

- $\operatorname{supp}(x^\mu) \cap \sigma = \emptyset$, where $\operatorname{supp}(x^\mu)$ is the set of indices $i \in \{1, \ldots, n\}$ such that x_i divides x^μ.
- for all monomials x^ν such that $\operatorname{supp}(x^\nu) \subseteq \sigma$ we have that $x^\mu x^\nu \notin I$.

- $(x^\mu, \sigma) \not\subseteq (x^\nu, \tau)$ for any other (x^ν, τ) satisfying the two previous conditions.

We say that (x^μ, σ) is a *maximal standard pair* if there is no other standard pair (x^ν, σ) such that x^μ divides x^ν.

Another advantage about monomial ideals is that some relevant operations between them behave well in the sense that the property of being a monomial ideal is preserved.

Let I and J be two monomial ideals whose unique minimal sets of monomial generators are $G(I) = \{g_1, \ldots, g_m\}$ and $G(J) = \{f_1, \ldots, f_t\}$ respectively. We have that $G(I+J) = G(I) \cup G(J)$ and $G(IJ) = G(I)G(J) = \{g_i f_j \text{ s.t. } g_i \in G(I) \text{ and } f_j \in G(J)\}$.

The following proposition shows how to compute the set of generators of the intersection of two monomial ideals:

Proposition 1.1.8. *Given two monomial ideals I and J in the polynomial ring $R = \Bbbk[x_1, \ldots, x_n]$, then $I \cap J$ is a monomial ideal which is generated by $\{\operatorname{lcm}(u, v) \text{ s.t. } u \in G(I), v \in G(J)\}$.*

Proposition 1.1.9. *Let I and J be two monomial ideals in the polynomial ring $R = \Bbbk[x_1, \ldots, x_n]$. The* colon ideal, *$I : J = \{f \in R \text{ s.t. } fg \in I, \forall g \in J\}$, is a monomial ideal and*

$$I : J = \bigcap_{v \in G(J)} I : \langle v \rangle.$$

Moreover, a set of generators of $I : \langle v \rangle$ is $\{u / \gcd(u, v) \text{ s.t. } u \in G(I)\}$.

Now we define the radical and the saturation of a monomial ideal, which are, in particular, homogenoeous.

Let $I \subset R = \Bbbk[x_1, \ldots, x_n]$ be a graded ideal and $\mathfrak{m} = \langle x_1, \ldots, x_n \rangle$ the graded maximal ideal of R.

The *saturation* $\widetilde{I}$ of I is the ideal $I : \mathfrak{m}^\infty = \bigcup_{k=1}^{\infty} I : \mathfrak{m}^k$.

The *radical* ideal of I is the ideal $\sqrt{I} = \left\{ f \in R \text{ s.t. } f^k \in I \text{ for some } k \right\}$. We say that an ideal I is radical if $I = \sqrt{I}$.

The proposition below says that the radical and the saturation of a monomial ideal are, again, monomial ideals.

Proposition 1.1.10. *The saturation and the radical of a monomial ideal are monomial ideals.*

Let $\mathbf{u} = \mathbf{x}^{\mathbf{a}}$ be a monomial. We have $\sqrt{u} = \prod_{i,a_i \neq 0} x_i$. The following proposition gives us a generating set of the radical of a monomial ideal.

Proposition 1.1.11. *Let I be a monomial ideal. Then, a set of generators of $\sqrt{I}$ is $\left\{\sqrt{u} \text{ s.t. } u \in G(I)\right\}$.*

When working with polynomial ideals, it is relevant to know if the ideal is decomposable into prime, primary or irreducible ideals. Although this task is difficult for polynomial ideals, it is not in the monomial ideal case.

Proposition 1.1.12. *A monomial ideal I is a* prime ideal *if and only if it is generated by some set of variables, that is $I = \langle x_{i_1}, \ldots, x_{i_m} \rangle$, for some subset $\{i_1, \ldots, i_m\}$ of $\{1, \ldots, n\}$.*

Definition 1.1.13. A monomial ideal I is called *irreducible* if it cannot be written as proper intersection of two other monomial ideals *i.e.* there not exists monomial ideals Q_1 and Q_2 such that $I = Q_1 \cap Q_2$.

A characterization of an irreducible monomial ideal is the following:

Proposition 1.1.14. *A monomial ideal I is* irreducible *if and only if I is generated by pure powers of the variables, i.e., I can be written as $I = \langle x_1^{a_1}, \ldots, x_m^{a_m} \rangle$, for $a_i \geq 1, \forall i$ and $m \leq n$.*

Definition 1.1.15. An ideal I is a *primary ideal* if, whenever $xy \in I$ one has $x \in I$ or $y^n \in I$ for some $n \in \mathbb{N}$.

Proposition 1.1.16. *A monomial ideal I is a* primary ideal *if and only if every variable appearing in I has some power in I,i.e.,*

$$I = \langle x_1^{a_1}, \ldots, x_m^{a_m}, \mathbf{x}^{\mathbf{b}_1}, \ldots, \mathbf{x}^{\mathbf{b}_r} \rangle,$$

for $a_i \geq 1, \forall i$ and $\bigcup_{i=1}^{r} \operatorname{supp} \mathbf{x}^{\mathbf{b}_i} \subset \{1, \ldots, m\}$, $m \leq n$.

Example 1.1.17. (A) The monomial ideal $I_1 = \langle x_1, x_2 \rangle \subset \Bbbk[x_1, x_2]$ is prime, primary and irreducible.

(B) The monomial ideal $I_2 = \langle x_1^2, x_1x_2, x_2^2 \rangle \subset \mathbb{k}[x_1, x_2]$ is primary but it is not prime or irreducible.

□

Proposition 1.1.18. *A monomial ideal $I \subset R$ is squarefree if and only if any of the following conditions hold:*

1. $\sqrt{I} = I$
2. *I is an intersection of prime ideals.*
3. *A monomial u is in I if and only if $x_1 \cdots x_r \in I$, where* $\operatorname{supp} u = \{x_i\}_{i=1}^r$

Once we know how to characterize an squarefree monomial ideal, we are going to work with minimal prime ideals and presentations.

Definition 1.1.19. Let $I \subset R$ be an ideal. A prime ideal P is called *minimal prime ideal* of I, if $I \subset P$ and there is no prime ideal containing I which is contained in P. We denote by $\operatorname{Min}(I)$ the set of minimal prime ideals of I.

Definition 1.1.20. Let $\bigcap_{i=1}^m Q_i$ be a primary decomposition of an ideal I. This descomposition is called *irredundant primary* if none of the Q_i can be omitted.

The following theorem shows that a monomial ideal always has an irredundant presentation in terms of irreducible ideals.

Theorem 1.1.21. *Let $I \subset R$ be a monomial ideal. Then $I = \bigcap_{i=1}^m Q_i$, where each Q_i is of the form $\langle x_1^{a_1}, \ldots, x_m^{a_m} \rangle$. Moreover, an irredundant presentation of this form is unique.*

Lemma 1.1.22. *Suppose that I has irredundant presentation $I = P_1 \cap \cdots \cap P_m$ as an intersection of prime ideals. Then $Min(I) = \{P_1, \ldots, P_m\}$.*

The following corollary describes an squarefree monomial ideal as an intersection of minimal prime ideals:

Corollary 1.1.23. *Let $I \subset R$ be a squarefree monomial ideal. Then*

$$I = \bigcap_{P \in Min(I)} P,$$

and each $P \in Min(I)$ is a monomial prime ideal.

Example 1.1.24. (A) Let $I = \langle x_1^2 x_2^2, x_2^2 x_3, x_3^2 \rangle$ be a monomial ideal.Then, one has

$$\begin{aligned} I &= \langle x_1^2, x_2^2 x_3, x_3^2 \rangle \cap \langle x_2^2, x_3^2 \rangle \\ &= \langle x_1^2, x_2^2, x_3^2 \rangle \cap \langle x_1^2, x_3 \rangle \cap \langle x_2^2, x_3^2 \rangle \end{aligned}$$

By Theorem 1.1.21 we have that the presentation above is unique.

(B) Let I be the squarefree monomial ideal $I = \langle x_1 x_2 x_3, x_2 x_3 x_4, x_4 x_5 \rangle$. Then, we have the following presentation in terms of minimal prime ideals:

$$\begin{aligned} I &= \langle x_1 x_2 x_3, x_4 \rangle \cap \langle x_1 x_2 x_3, x_2 x_3 x_4, x_5 \rangle \\ &= \langle x_1, x_4 \rangle \cap \langle x_2, x_4 \rangle \cap \langle x_3, x_4 \rangle \cap \langle x_1 x_2 x_3, x_2 x_3 x_4, x_5 \rangle \\ &= \langle x_1, x_4 \rangle \cap \langle x_2, x_4 \rangle \cap \langle x_3, x_4 \rangle \cap \langle x_1, x_2 x_3 x_4, x_5 \rangle \cap \langle x_2, x_5 \rangle \cap \langle x_3, x_5 \rangle \\ &= \langle x_1, x_4 \rangle \cap \langle x_2, x_4 \rangle \cap \langle x_3, x_4 \rangle \cap \langle x_2, x_5 \rangle \cap \langle x_3, x_5 \rangle \end{aligned}$$

Furthermore, by Lemma 1.1.22 we have that

$$\text{Min}(I) = \{\langle x_1, x_4 \rangle, \langle x_2, x_4 \rangle, \langle x_2, x_5 \rangle, \langle x_3, x_4 \rangle, \langle x_3, x_5 \rangle\}.$$

□

1.1.1 Resolutions and Hilbert series

1.1.1.1 Resolutions

Every ideal $I \in R$, where R is a ring, has a natural structure as a module with the product of R. There are modules which are not free *i.e.* there exist relations between their generators. The relations satisfied by their generators are quite interesting. The set of relations of the generators of

a module $\mathcal{M} = \langle f_1, \ldots, f_m \rangle \subseteq R = \Bbbk[x_1, \ldots, x_n]$ is called *first syzygies* and is denoted by $\mathrm{Syz}(f_1, \ldots, f_m)$. $\mathrm{Syz}(f_1, \ldots, f_m)$, in turn, is a module over R^m itself. If this module is not free, one is able to find relations among its generators. As a result, the *second syzygy* module is obtained. Iterating the process, we can obtain the *i-syzygy modules*, for $i \in \mathbb{N}$. The set of the modules $\mathcal{M}$ and its *Syzygies* are the key objets to describe an exact sequence called *free resolution of* $\mathcal{M}$.

Once the main idea behind resolutions is known, we give the tools needed to define it firmly.

Definition 1.1.25. Let $\mathcal{M} = \langle f_1, \ldots, f_m \rangle$ be an R-module. A *presentation* for $\mathcal{M}$ is a set of generators $\{f_1, \ldots, f_m\}$ together with a set of generators for the syzygy module $\mathrm{Syz}(f_1, \ldots, f_m)$.

In terms of exact sequences, we have the surjective homomorphism $\varphi : R^m \to \mathcal{M}$. As the homomorphism is surjective, we obtain the exact sequence

$$R^m \xrightarrow{\varphi} \mathcal{M} \to 0,$$

where $\varphi(g_1, \ldots, g_m) = \sum_{i=1}^{m} g_i f_i \in \mathcal{M}$. It follows that the syzygies on the set of generators are the elements of $\ker(\varphi)$, *i.e.*

$$\mathrm{Syz}(f_1, \ldots, f_m) = \ker(\varphi).$$

Choosing a set of generators of $\mathrm{Syz}(f_1, \ldots, f_m)$ corresponds to choosing a homomorphism θ of R^t surjective (over $\mathrm{Syz}(f_1, \ldots, f_m)$) $\theta : R^t \xrightarrow{\theta} R^m$ and that means $\mathrm{im}(\theta) = \ker(\varphi)$. Then, we have the exact sequence

$$R^t \xrightarrow{\theta} R^m \xrightarrow{\varphi} \mathcal{M} \to 0$$

The homomorphism θ is defined by the presentation matrix for $\mathcal{M}$, *i.e.*, the matrix whose columns are the generators of $\mathrm{Syz}(f_1, \ldots, f_t)$.

Definition 1.1.26. The exact sequence

$$R^t \xrightarrow[A]{\theta} R^m \xrightarrow[(f_1 \ldots f_m)]{\varphi} \mathcal{M} \to 0$$

is the *presentation* for the R-module $\mathcal{M}$, where θ is the homomorphism defined by the presentation matrix A for $\mathcal{M}$, and φ is defined by the matrix whose columns are the generators of $\mathcal{M}$.

The presentation for the module $\mathcal{M}$ can be extended. If we continue with the process of finding relations between generators of $\mathrm{Syz}(f_1, \ldots, f_m)$ - second syzygy module-, the exact sequence turns into

$$R^l \xrightarrow[B]{\phi} R^t \xrightarrow{\theta} R^m \xrightarrow{\varphi} \mathcal{M} \to 0,$$

where B is the presentation matrix for the first syzygy module $\mathrm{Syz}(f_1, \ldots, f_m)$. This process can be continued -with the third, fourth and higher syzygy modules- leading to a free resolution:

Definition 1.1.27. Let $\mathcal{M}$ be an R-module. A *free resolution* of $\mathcal{M}$ is an exact sequence of the form

$$\cdots \to \mathcal{F}_2 \xrightarrow{\varphi_2} \mathcal{F}_1 \xrightarrow{\varphi_1} \mathcal{F}_0 \xrightarrow{\varphi_0} \mathcal{M} \to 0,$$

where $\mathcal{F}_i \cong R^{m_i}$ is a free R-module $\forall i$. If there exist s such that $\mathcal{F}_s \neq 0$ but $\mathcal{F}_{s+1} = \mathcal{F}_{s+2} = \cdots = 0$, then we call the resolution *finite of length* s.

When we have a finite resolution of length s we write

$$0 \to \mathcal{F}_s \to \mathcal{F}_{s-1} \cdots \to \mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F}_0 \to \mathcal{M} \to 0.$$

Theorem 1.1.28 (Hilbert Syzygy Theorem)**.** *Let $R = \Bbbk[x_1, \ldots, x_n]$. Then every finitely generated R-module has a finite free resolution of length at most n.*

1.1.1.2 Graded resolutions

When we add a grading to a polynomial ring, the notions we have defined in Section 1.1.1.1 vary and some new ones appear, like the notion of minimality.

Definition 1.1.29. A *graded module* over R is a module $\mathcal{M}$ with a family of subgroups $\{\mathcal{M}_t : t \in \mathbb{Z}\}$ of the additive group of $\mathcal{M}$. The elements of $\mathcal{M}_t$ are called *homogeneous elements* of degree t in the grading. The $\mathcal{M}_t$ must satisfy

1. $\mathcal{M} = \bigoplus_{t\in\mathbb{Z}} \mathcal{M}_t$.
2. $R_m\mathcal{M}_t \subseteq \mathcal{M}_{m+t}$.

If one has a module $\mathcal{M}$ with two different gradings, the following proposition allows to produce modules isomorphic matching their gradings:

Proposition 1.1.30. *Let $\mathcal{M}$ be a graded R-module, and let d be an integer. Let $\mathcal{M}(d)$ be the direct sum*

$$\mathcal{M}(d) = \bigoplus_{t\in\mathbb{Z}} \mathcal{M}(d)_t,$$

where $\mathcal{M}(d)_t = \mathcal{M}_{d+t}$. Then $\mathcal{M}(d)$ is also a graded R-module.

Remark 1.1.31. The modules satisfying $(R^m)(d) = R(d)^m$ are called *twisted graded free modules over R*. The standard basis $\{e_1, \ldots, e_m\}$ is still a module basis for $(R^m)(d)$. The basis vectors e_i are homogeneous elements of degree $-d$ in the grading.

Equivalent to module homomorphisms, graded homomorphisms are needed for working with graded resolutions:

Definition 1.1.32. Let $\mathcal{M}, \mathcal{N}$ be graded modules over R. A homomorphism $\varphi : \mathcal{M} \to \mathcal{N}$ is said to be a *graded homomorphism of degree d* if $\varphi(\mathcal{M}_t) \subseteq \mathcal{N}_{t+d}$ for all $t \in \mathbb{Z}$.

Example 1.1.33. Suppose that $\mathcal{M}$ is a graded R-module generated by the homogeneous elements $f_1, \ldots, f_m$ of degrees $d_1, \ldots, d_m$. Then we get a graded homomorphism

$$\varphi : R(-d_1) \oplus \cdots \oplus R(-d_m) \to \mathcal{M},$$

which sends the standard basis element e_i to $f_i \in \mathcal{M}$.

□

Remark 1.1.34. A graded homomorphism of degree zero

$$R(-d_1) \oplus \cdots \oplus R(-d_p) \to R(-c_1) \oplus \cdots \oplus R(-c_m)$$

is defined by and $m \times p$ matrix A where the entry $a_{ij} \in R$ is homogeneous of degree $d_j - c_i$ for all i, j. The matrix A is called *graded matrix* over R.

Definition 1.1.35. If $\mathcal{M}$ is a graded R-module, then a *graded resolution* of $\mathcal{M}$ is a resolution of the form

$$\cdots \to \mathcal{F}_2 \xrightarrow{\varphi_2} \mathcal{F}_1 \xrightarrow{\varphi_1} \mathcal{F}_0 \xrightarrow{\varphi_0} \mathcal{M} \to 0,$$

where each $\mathcal{F}_i$ is a twisted free graded module $R(-d_1) \oplus \cdots \oplus R(-d_p)$ and each homomorphism φ_i is a graded homomorphism of degree zero.

Example 1.1.36. Let I be the monomial ideal which minimal generating set is $G(I) = \{x^2, xy, y^2\}$. A resolution for this ideal is

$$0 \to R^1 \xrightarrow{\varphi_2} R^3 \xrightarrow{\varphi_1} R^3 \xrightarrow{\varphi_0} I \to 0,$$

where the homomorphisms φ_i for $i = 0, 1, 2$ are defined as

$$\varphi_0 = \begin{pmatrix} x^2 & xy & y^2 \end{pmatrix}, \qquad \varphi_1 = \begin{pmatrix} y & y^2 & 0 \\ -x & 0 & y \\ 0 & -x^2 & -x \end{pmatrix}, \qquad \varphi_2 = \begin{pmatrix} y \\ -1 \\ x \end{pmatrix}.$$

One can easily check that $\varphi_0\varphi_1 = 0$ and $\varphi_1\varphi_2 = 0$.

The resolution can be written as Example 1.1.36 is

$$0 \to R(-4) \xrightarrow{\varphi_2} R(-3) \oplus R(-4) \oplus R(-3) \xrightarrow{\varphi_1} R(-2)^3 \xrightarrow{\varphi_0} I \to 0.$$

□

Theorem 1.1.37 (Graded Hilbert Syzygy Theorem)**.** *Let $R = \Bbbk[x_1, \ldots, x_n]$. Then every finitely generated R-module has a finite graded resolution of length at most n.*

Definition 1.1.38. Suppose that

$$0 \to \mathcal{F}_s \xrightarrow{\varphi_s} \mathcal{F}_{s-1} \cdots \xrightarrow{\varphi_3} \mathcal{F}_2 \xrightarrow{\varphi_2} \mathcal{F}_1 \xrightarrow{\varphi_1} \mathcal{F}_0 \xrightarrow{\varphi_0} \mathcal{M} \to 0$$

is a graded resolution of $\mathcal{M}$. Then the resolution is *minimal* if for every $s \geq i \geq 1$, all the nonzero entries of the graded matrix of φ_i have positive degree (*i.e.* there are no constants in the graded matrix).

Example 1.1.39. As one can check in Example 1.1.36 we have that φ_2 has a non zero element of degree zero. So that, the resolution we have computed is not minimal. If one looks at the differentials carefully, then can check that φ_1 has a column which is a linear combination of the other two (2^{nd}col=$y \cdot 1^{st}$ col $+ \ x \cdot 3^{rd}$ col). It means that second column can disappear. Then, a minimal graded resolution for the ideal $I = \langle x^2, xy, y^2 \rangle$ is the following

$$0 \to R(-3)^2 \xrightarrow{\varphi_1} R(-3)^3 \xrightarrow{\varphi_0} I \to 0,$$

where the homomorphisms φ_i for $i = 0, 1$ are the ones defined below.

$$\varphi_0 = \begin{pmatrix} x^2 & xy & y^2 \end{pmatrix}, \qquad \varphi_1 = \begin{pmatrix} y & 0 \\ -x & y \\ 0 & -x \end{pmatrix}.$$

□

A set of generators of a module is *minimal* if no proper subset generates the module. Minimal resolutions can be characterized as follows:

Proposition 1.1.40. *The resolution*

$$\cdots \to \mathcal{F}_s \xrightarrow{\varphi_s} \mathcal{F}_{s-1} \cdots \xrightarrow{\varphi_1} \mathcal{F}_0 \xrightarrow{\varphi_0} \mathcal{M} \to 0$$

is minimal if and only if for each $s \geq 0$, φ_s takes the standard basis of $\mathcal{F}_s$ to a minimal generating set of $im(\varphi_s)$.

When we work with minimal resolutions, we have that

$$\mathcal{F}_s = \bigoplus_{\mathbf{a} \in \mathbb{N}^n} R^{\beta_{i,\mathbf{a}}}(-\mathbf{a}).$$

Each rank appearing in a minimal resolution, $\beta_{i,\mathbf{a}} = \beta_{i,\mathbf{a}}(\mathcal{M})$, is an invariant called the *i-th Betti number of the multigraded module $\mathcal{M}$ in multidegree* $\mathbf{a}$ or *multigraded Betti number*, where a R-module $\mathcal{M}$ is called *multigraded* if $\mathcal{M} \cong \bigoplus \mathcal{M}_{\mathbf{j}}$, with $\mathcal{M}_{\mathbf{j}} = \mathcal{M}_{j_1,\ldots,j_d}$, and $R_{\mathbf{i}}\mathcal{M}_{\mathbf{j}} \subset \mathcal{M}_{\mathbf{i}+\mathbf{j}}$.

Remark 1.1.41. When $\mathbf{a}$ is not a multidegree but just a degree, $\beta_{i,\mathbf{a}}$ are called *graded Betti numbers*.

Betti numbers always satisfy that $\beta_{i,\mathbf{a}} \leq \gamma_{i,\mathbf{a}}$, where $\gamma_{i\mathbf{a}}$ are the ranks of $R^{\gamma_{i\mathbf{a}}}$ in a non minimal resolution. When $\mathcal{M}$ is a graded module, in particular, if $\mathcal{M}$ is a monomial ideal, then $\beta_{i,\mathbf{a}}(I)$ measures the number of minimal generators required in multidegree $\mathbf{a}$ for the i-th syzygy module of I.

Other important invariants related to a graded ideal and defined in terms of its minimal free resolution are

- The *projective dimension* of I, defined as

$$\text{projdim}(I) = \max\left\{i \text{ s.t. } \beta_{i,j}(I) \neq 0 \text{ for some } j\right\},$$

  where $i, j \in \mathbb{N}$.

- The *Castelnuovo-Mumford regularity* of I, defined as

$$\text{reg}(I) = \max\left\{j \text{ s.t. } \beta_{i,i+j}(I) \neq 0 \text{ for some } i\right\},$$

  where $i, j \in \mathbb{N}$.

We finish this section with the result below

Theorem 1.1.42. *Any two minimal resolutions of $\mathcal{M}$ are isomorphic.*

1.1.1.3 Some free resolutions for monomial ideals

Resolutions are essential objects in commutative algebra. One of the main problems related to resolutions is to find the minimal one, even in the case of monomial ideals which, as we have seen in this chapter, are easy to treat. In the literature, there have been two different strategies for studying resolutions. The first one is to study the explicit minimal free resolution for particular families of monomial ideals. The result obtained by Eliahou and Kervaire [41], an explicit description of the minimal free resolution of stable ideals, is the main result of this strategy. The second approach consists on the development of general procedures to obtain

free resolutions, even if they are not minimal. The well-known Taylor [131] resolution is the main result from this approach, together with the Lyubeznik resolution.

Taylor resolution. One of the best known resolutions of monomial ideals is the Taylor resolution [131] which, in general, is nonminimal.

Let I be a monomial ideal and $\{m_1, \ldots, m_r\}$ a generating set of I. For any subset $J = \{j_1, \ldots, j_s\} \subseteq \{1, \ldots, r\}$, let us denote $m_J = \text{lcm}\left\{m_{j_1}, \ldots, m_{j_s}\right\}$ and $J^i = \{j_1, \ldots, j_{i-1}, j_{i+1}, \ldots, j_s\}$. A resolution for R/I can be constructed in the following way:

- Let $\mathcal{T}_s$, $s \geq 0$ be a free R-module generated by

$$\left\{e_J \text{ s.t. } |J| = s, \forall J \subseteq \{1, \ldots, r\}\right\},$$

  where J are all the subsets of $\{1, \ldots, r\}$.

- Let consider the R-linear differential

$$d(e_J) = \sum_{i \in J} (-1)^{i-1} \frac{m_J}{m_{J^i}} e_{J^i}.$$

Observe that $d^2 = 0$ and it is exact at each module, so it is a resolution of R/I. This resolution is due to Taylor and is denoted by $\mathbb{T}$. The length of Taylor's resolution is given by the number of elements in the given generator set and the rank of the i-th free module $\mathcal{T}_i$ is $\binom{r}{i+1}$, with $0 \leq i < r$.

Example 1.1.43. Let I be the monomial ideal which generator set is $G(I) = \left\{x^3, x^2y, xy^2, y^4\right\}$. The Taylor resolution for this ideal is

$$0 \to R^1 \xrightarrow{\varphi_3} R^4 \xrightarrow{\varphi_2} R^6 \xrightarrow{\varphi_1} R^4 \xrightarrow{\varphi_0} I \to 0,$$

where the homomorphisms φ_i for $i = 0, 1, 2, 3$ are defined as

$$\varphi_0 = \begin{pmatrix} x^3 & x^2y & xy^2 & y^4 \end{pmatrix}, \qquad \varphi_1 = \begin{pmatrix} y & y^2 & y^4 & 0 & 0 & 0 \\ -x & 0 & 0 & y & 0 & y^3 \\ 0 & -x^2 & 0 & -x & y^2 & 0 \\ 0 & 0 & -x^3 & 0 & -x & -x^2 \end{pmatrix},$$

$$\varphi_2 = \begin{pmatrix} y & 0 & y^3 & 0 \\ -1 & y^2 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ x & 0 & 0 & y^2 \\ 0 & x^2 & 0 & x \\ 0 & 0 & x & -1 \end{pmatrix}, \qquad \varphi_3 = \begin{pmatrix} y^2 \\ 1 \\ -1 \\ -x \end{pmatrix}.$$

□

Some characterizations of monomial ideals for which the Taylor's resolution is minimal are given in [15, 48, 63].

Lyubeznik resolution. A subresolution $\mathbb{L}$ of $\mathbb{T}$ was given in [89] and it is known as Lyubeznik resolution. It is defined as

- For a given subset $J \subseteq \{1, \ldots, r\}$ and an integer $1 \leq s \leq r$, define $J_{>s} = \{j \in J : j > s\}$.
- $\mathbb{L}$ generated by those u_J such that for all $1 \leq s \leq r$ one has that $m_s \not| m_{J_{>s}}$.

Lyubeznik resolution is, in general, also not minimal but smaller than the Taylor resolution. It depends on the order of the generators.

Eliahou-Kervaire resolution. The Eliahou-Kervaire resolution is a minimal free resolution in the case of stable monomial ideals [41].

Definition 1.1.44. A monomial ideal I is *stable* if for every monomial $\mathbf{x^a} \in I$

$$\mathbf{x^a} \frac{x_i}{x_{\max(\mathbf{x^a})}} \in I \text{ for all index } i < \max(\mathbf{x^a}),$$

where $\max(\mathbf{x^a})$ denotes the index of the last variable that divides $\mathbf{x^a}$.

Examples 1.1.45.

A) The monomial ideal $I = \langle x_1^2, x_1x_2 \rangle \subseteq \Bbbk[x_1, x_2]$ is a stable monomial ideal.

B) Let $I = \langle x_1^3, x_1x_2 \rangle \subseteq \Bbbk[x_1, x_2]$ be a monomial ideal. Observe that $\max(x_1x_2) = 2$. We have that

$$x_1x_2\frac{x_1}{x_2} = x_1^2 \notin I.$$

Therefore, the ideal I is not stable.

□

For describing the Eliahou-Kervaire resolution, we need the following fact:

Proposition 1.1.46. *Let I be a monomial ideal and $\mathbf{x^a}$ a monomial belonging to I. Then there exists a unique generator g and monomial $\mathbf{x^b}$ such that $\mathbf{x^a} = g\mathbf{x^b}$ and, for every x_i dividing $\mathbf{x^b}$ we have that $i \geq \max(g)$.*

We say that g is the beginning *of $\mathbf{x^a}$ and we denote it as $beg(\mathbf{x^a})$. We represent as $end(\mathbf{x^a})$ the* end *of $\mathbf{x^a}$ which is the monomial $\mathbf{x^b}$.*

Definition 1.1.47. Let I be a stable monomial ideal. An *EK*-symbol for I is a pair of the form $[f, u]$ where f is a minimal generator of I and u is a square-free monomial satisfying $\max(u) < \max(f)$.

The *Eliahou-Kervaire resolution of I* is of the form

$$0 \to \cdots \to E_l \to E_{l-1} \to \cdots \to E_0 \to I \to 0,$$

where each of the modules E_j is a free module generated by the set of *EK*-symbols $[f, u]$ such that $\deg(u) = j$. The differential of the resolution is given by

$$\varphi([f, u]) = \sum_{x_i|u} \operatorname{sgn}(x_i, u)x_i \left[f, \frac{u}{x_i}\right] - \sum_{x_i|u} \operatorname{sgn}(x_i, u)\operatorname{end}(x_if) \left[\operatorname{beg}(x_if), \frac{u}{x_i}\right],$$

where $\operatorname{sgn}(x_i, u) = 1$ if the cardinality of $\{x_j \text{ s.t. } x_j|u \text{ and } j \leq i\}$ is odd, and -1 otherwise.

Example 1.1.48. Let $I = \langle x, y^2, yz \rangle \subseteq \Bbbk[x, y, z]$ be a stable ideal. The Eliahou-Kervaire resolution for this ideal is

$$0 \to R^1 \xrightarrow{\varphi_2} R^3 \xrightarrow{\varphi_1} R^3 \xrightarrow{\varphi_0} I \to 0.$$

We use the formula from Definition 1.1.47 for computing the explicit form of the differentials:

$$\varphi_0([x]) = x, \varphi_0([y^2]) = y^2, \varphi_0([yz]) = yz,$$
$$\varphi_1([y^2; x]) = x[y^2] - y^2[x],$$
$$\varphi_1([yz; x]) = x[yz] - yz[x],$$
$$\varphi_1([yz; y]) = y[yz] - z[y^2],$$
$$\varphi_2([yz; xy]) = x[yz; y] - y[yz; x] + z[y^2; x].$$

Organizing the data computed in matrix form we have that φ_i for $i = 0, 1, 2$ are:

$$\varphi_0 = \begin{pmatrix} x & y^2 & yz \end{pmatrix}, \qquad \varphi_1 = \begin{pmatrix} -y^2 & -yz & 0 \\ x & 0 & -z \\ 0 & x & y \end{pmatrix}, \qquad \varphi_2 = \begin{pmatrix} z \\ -y \\ x \end{pmatrix}.$$

□

1.1.2 Hilbert series

In this section we present the definition and some useful results about *Hilbert series*, which allows us to enumerate the monomials that are in a monomial ideal.

Definition 1.1.49. Let $\mathcal{M}$ be a finitely generated graded R-module. The numerical function

$$HF_{\mathcal{M}}(\cdot) : \mathbb{Z} \longrightarrow \mathbb{Z}, \qquad a \mapsto HF_{\mathcal{M}}(a) := \dim_{\Bbbk} \mathcal{M}_a$$

is called the *Hilbert function of* $\mathcal{M}$.

The formal series

$$HS_{\mathcal{M}}(x) = \sum_{a \in \mathbb{Z}} HF_{\mathcal{M}}(a) x^a$$

is the *Hilbert series of* $\mathcal{M}$.

Let

$$0 \to \mathcal{M}_k \to \cdots \to \mathcal{M}_0 \to 0$$

be an exact sequence of graded R-modules. Then, we have that

$$HS_{\mathcal{M}_k}(x) = \sum_{i=0}^{k-1} (-1)^i HS_{\mathcal{M}_i}(x).$$

We are working with multigraded modules. For this reason, we need to define the multigraded version of the Hilbert series.

Definition 1.1.50. The *multigraded Hilbert series* of a finitely generated multigraded R-module $\mathcal{M}$ is the formal series

$$HS_{\mathcal{M}}(\mathbf{x}) = \sum_{\mathbf{a} \in \mathbb{N}_0^n} \dim_{\Bbbk}(\mathcal{M}_{\mathbf{a}}) \mathbf{x}^{\mathbf{a}},$$

where $\dim_{\Bbbk} \mathcal{M}_{\mathbf{a}}$ is the number of elements of degree $\mathbf{a}$ in $\mathcal{M}_{\mathbf{a}}$ and, in our particular case, $\dim_{\Bbbk} \mathcal{M}_{\mathbf{a}} < \infty$.

Remark 1.1.51. Remember that monomial ideals are, in particular, finitely generated multigraded R-modules.

Remark 1.1.52. In the ring $\mathbb{Z}[[x_1 \ldots, x_n]]$ we have that the element $1 - x_i$ has an inverse and it is the series $\frac{1}{1-x_i} = 1 + x_i + x_i^2 + \cdots$. The multigraded Hilbert series of $R = \Bbbk[x_1, \ldots, x_n]$ is an element of $\mathbb{Z}[[x_1 \ldots, x_n]]$, which is the formal sum of all monomials in R. Then, we have that $HS_R = \prod_{i=1}^n \frac{1}{1-x_i}$.

Let denote by $R(-\mathbf{a})$ the free module generated in multidegree $\mathbf{a}$, so $R(-\mathbf{a}) \cong \langle \mathbf{x}^{\mathbf{a}} \rangle$ as multigraded modules. The multigraded Hilbert series of that $\mathbb{N}_0^n$-graded translation of R is just

$$HS_{R(-\mathbf{a})}(\mathbf{x}) = \mathbf{x}^{\mathbf{a}} \cdot HS_R(\mathbf{x}) = \frac{\mathbf{x}^{\mathbf{a}}}{\prod_{i=1}^n 1 - x_i}.$$

Remark 1.1.53. If I is a monomial ideal, then the multigraded Hilbert series of R/I is the sum of all monomials not in I.

Remark 1.1.54. The multigraded Hilbert series of monomial ideals and R-modules of the form I can be expressed as

$$HS_{\mathcal{M}}(\mathbf{x}) = \frac{\mathcal{K}_{\mathcal{M}}(\mathbf{x})}{\prod_{i=1}^{n} 1 - x_i},$$

where $\mathcal{K}_{\mathcal{M}}(\mathbf{x})$ is known as the $\mathcal{K}$-polynomial of $\mathcal{M}$ or numerator of the Hilbert series. To reach our goal of computing the reliability of a system, the numerator of the Hilbert series of a monomial ideal I is going to be essential.

One can check that if I is a monomial ideal, then $\mathcal{K}_{R/I}(\mathbf{x}) = 1 - \mathcal{K}_I(\mathbf{x})$.

Example 1.1.55. Let $I = \langle x^2, xy, y^2z \rangle \subseteq R = \mathbb{k}[x, y, z]$. We have that

$$HS_I(\mathbf{x}) = \frac{\mathcal{K}_I(\mathbf{x})}{(1-x)(1-y)(1-z)},$$

where $\mathcal{K}_I(\mathbf{x}) = x^2 + xy + y^2z - x^2y^2z - xy^2z - x^2y + x^2y^2z$.

□

A good way to compute multigraded Hilbert series of ideals is using a free resolution of the ideal. We have seen that a multigraded free resolution of an ideal I is a collection of modules $\mathcal{F}_i, i > 0$ together with morphisms $\partial_i : \mathcal{F}_i \to \mathcal{F}_{i-1}$ that express the structure of the ideal. For any resolution of a monomial ideal I, we have

$$HS_I(\mathbf{x}) = \sum_{i=0}^{d} (-1)^i HS(\mathcal{F}_i; \mathbf{x}),$$

where $\mathcal{F}_i$, for $i = 0, \ldots, d$ are the modules in the resolution of I.

The resolutions we are working with are multigraded, so each $\mathcal{F}_i = \oplus_{\mathbf{a} \in \mathbb{N}^n} R(-\mathbf{a})^{\gamma_{i,\mathbf{a}}}$ for some scalars $\gamma_{i,\mathbf{a}}$. Then, we have

$$HS_I(\mathbf{x}) = \frac{\sum_{i=0}^{d} (-1)^i \left(\sum_{\mathbf{a} \in \mathbb{N}_0^n} \gamma_{i,\mathbf{a}} \mathbf{x}^{\mathbf{a}} \right)}{\prod_{j=1}^{n} (1 - x_j)}.$$

Remark 1.1.56. From Section 1.1.1.1 we know that, if the resolution is minimal, then the Betti numbers, $\beta_{i,\mathbf{a}}$ always satisfy that $\beta_{i,\mathbf{a}} \leq \gamma_{i,\mathbf{a}}$, $\forall i$. The multigraded Hilbert series then has the form

$$H_I(\mathbf{x}) = \frac{\sum_{i=0}^{d}(-1)^i \left(\sum_{\mathbf{a}\in\mathbb{N}_0^n} \beta_{i,\mathbf{a}}\mathbf{x}^{\mathbf{a}}\right)}{\prod_{j=1}^{n}(1-x_j)}.$$

Betti numbers are really important in our work due to the fact that they provide the tighest bounds for reliability by suppressing redundant terms in $\mathcal{K}_I(\mathbf{x})$ as we will see in Section 1.2.2.

1.1.3 Mayer-Vietoris Trees

Mayer-Vietoris trees (*MVT*) are a tool that can be considered as a description of an ideal together with the relevant part of its lcm-lattice or as an algorithm that helps us to compute the homological invariants of the ideals. In this section we present how to use the *MVT* of a monomial ideal to compute bounds for its Betti numbers without computing the (minimal) free resolution of such ideal. Monomial ideals for which the bounds provided by the *MVT* for the Betti numbers are sharp are called *Mayer-Viertories ideals*.

Each *MVT* of a monomial ideal provides a resolution for the ideal (see Section 3.1.2.1 of [114] for more details). In particular, it provides the multigraded Hilbert series.

Let $I \subseteq R = \Bbbk[x_1, \ldots, x_n]$ be a monomial ideal whose minimal generating set is $G(I) = \{m_1, \ldots, m_r\}$. Let us define $I' = \langle m_1, \ldots, m_{r-1}\rangle$ and $\widetilde{I} = I' \cap \langle m_r\rangle = \langle m_{1,r}, m_{2,r}, \ldots, m_{r-1,r}\rangle$, where $m_{i,j} = \operatorname{lcm}(m_i, m_j)$.

From this point, we use the following notation: for each $1 \leq s \leq r$, we denote by $I_s = \langle m_1, \ldots, m_s\rangle$ and $\widetilde{I}_s = I_{s-1} \cap \langle m_s\rangle$.

Definition 1.1.57. Given a monomial ideal $I \subseteq R = \Bbbk[x_1, \ldots, x_n]$ minimally generated by $G(I) = \{m_1, \ldots, m_r\}$, we define the *(recursive) Mayer-Vietoris exact sequence of* I in the following way:
For each $1 \leq s \leq r$ we have the following exact sequence of ideals:

$$0 \to \widetilde{I}_s \to I_{s-1} \oplus \langle m_s\rangle \to I_s \to 0.$$

The Mayer-Vietoris sequence relates the homology of two spaces, their intersection and their union by means of a short exact sequence of complexes. These sequences were used in [114] to develop an algorithm for homological computations on monomial ideals.

Remarks 1.1.58.

1. The Mayer-Vietoris sequence associated to a monomial ideal is not uniquely defined: it depends on how the minimal generators have been sorted.

2. The MVT of a monomial ideal is related with the so called cone resolution [66] in the sense that the MVT provides the ranks and the generators of the iterations of the cone resolution.

We have said that *MVTs* can be used to compute homological invariants of a monomial ideal. To compute the *MVT* of a given monomial ideal I, we have that the initial ideal I is the root of the tree. Each node J has a child $\widetilde{J}$ on the left and a child J' on the right (if J is generated by r monomials, then $\widetilde{J}$ represents $\widetilde{J_r}$ and J' denotes J_{r-1}). This is what we call the *Mayer-Vietoris tree* of the ideal I and is denoted by $MVT(I)$.

Remark 1.1.59. The construction of the *MVT* depends on how we choose the distinguished monomial used to split the ideal, which is called *(monomial) pivot*.

For a given ideal I and an ordering of the generators we have several trees depending on the way we choose the pivot.

Some strategies to choose the pivot are

1. Select the pivot according to some term order τ, taking the pivot as the biggest term. We call this kind of *MVT* as $\tau - MVT$. This trees are unique given τ and I.

2. Keep or reach some property of the ideals of the nodes in the tree. For example, choose a monomial in which appears one unique variable to obtain an ideal with one less variable.

3. Select as pivot the one having the biggest exponent in some of the variables. Then, the generators of $\widetilde{J}$ will have the same exponent

on this variable. This pivot provides smaller trees in which the left branch has, at most, length $n+1$, where n is the number of variables.

The strategy to choose the pivot can change at each node.

By definition, every father node in an $MVT(I)$ has two children. We can assign position indices to every node, which is useful for reading hidden information in the MVT, in the following way:

- I has position 1.
- If J has position p, then $\widetilde{J}$ has position $2p$ and J' has position $2p+1$.

We denote that as $MVT_1(I) = I$, $MVT_p(I) = J$, $MVT_{2p}(I) = \widetilde{J}$ and $MVT_{2p+1}(I) = J'$.

Given a monomial ideal I, all the multidegrees $\mathbf{a} \in \mathbb{N}^n$ such that $\beta_{i,\mathbf{a}}(I) \neq 0$, are present as exponents of generators in some of the nodes of any $MVT(I)$. The only multidegrees appearing on an $MVT(I)$ which are relevant to the homology computation of I are those in position 1 (for which we obtain H_0) or even. The degree of the homology to which they contribute can be read from their position in the tree. To do that, let us assign a dimension to every node in the $MVT(I)$ in the following way:

- $\dim\left(MVT_1(I)\right) = 0$.
- If $\dim\left(MVT_p(I)\right) = d$, then

$$\dim\left(MVT_{2p}(I)\right) = d+1 \text{ and } \dim\left(MVT_{2p+1}(I)\right) = d.$$

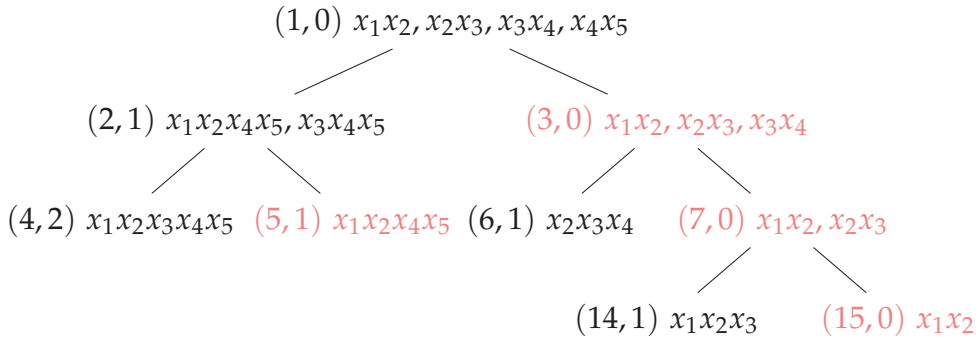
The dimension d of a node in position p is just the number of zeros in the binary expansion of p. The generators of each relevant node contribute to the homology modules in the homological degree given by the dimension of the node.

For reading Betti numbers from an MVT, we have that $\beta_{i,\mathbf{a}}$ is the number of monomials in dimension i and position 1 or even on the tree and with (multi)degree $\mathbf{a}$.

Example 1.1.60. Let us consider Mayer-Vietoris trees of the ideal

$$I = \langle x_1x_2, x_2x_3, x_3x_4, x_4x_5 \rangle \subseteq \mathbb{k}[x_1, \ldots, x_5].$$

The Mayer-Vietoris tree of I is the one displayed below, where the relevant nodes are the ones in black color, and each node is of the form `(position, dimension) generators`.



From this tree we obtain that the graded Betti numbers are $\beta_{0,2}(I) = 4$, $\beta_{1,3}(I) = 3$, $\beta_{1,4}(I) = 1$ and $\beta_{2,5}(I) = 1$. Moreover, the numerator of the Hilbert series of this ideal is

$$\begin{aligned} HN_{I_{2,5}} = &(x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5) \\ &- (x_1x_2x_4x_5 + x_3x_4x_5 + x_2x_3x_4 + x_1x_2x_3) \\ &+ x_1x_2x_3x_4x_5 \end{aligned}$$

We can also read the multigraded Betti numbers which are the followings

$$\begin{aligned} &\beta_{0,(1,1,0,0,0)}(I) = \beta_{0,(0,1,1,0,0)}(I) = \beta_{0,(0,0,1,1,0)}(I) = \beta_{0,(0,0,0,1,1)}(I) = 1, \\ &\beta_{1,(1,1,1,0,0)}(I) = \beta_{1,(0,1,1,1,0)}(I) = \beta_{1,(0,0,1,1,1)}(I) = \beta_{1,(1,1,0,1,1)}(I) = 1, \\ &\beta_{2,(1,1,1,1,1)}(I) = 1. \end{aligned}$$

□

We know that the main object expressing the homological structure of a monomial ideal is its minimal free resolution but it is not trivial to compute it.

Let I be a monomial ideal and $MVT(I)$ a Mayer-Vietoris tree of I. Let $\mathbf{a} \in \mathbb{N}^n$ and let

- $\overline{\beta_{i,\mathbf{a}}}(I) = 1$, if $\mathbf{a}$ is a multidegree of a generator of a relevant node of dimension i in $MVT(I)$ which does not appear as a generator of any relevant node.
- $\overline{\beta_{i,\mathbf{a}}(I)} = 0$, in other case.

Let $\widehat{\beta}_{i,\mathbf{a}}(I)$ be the number of times that $\mathbf{a}$ appears as the multidegree of some generator of dimension i in some relevant node in $MVT(I)$.

The generators of the relevant nodes of $\mathrm{MVT}(I)$ provide upper and lower bounds for the Betti numbers of the ideal without actually computing the resolution. These bounds can be improved using several criteria and are sharp in several families of ideals, see [115] for details. A simple useful criterion is the following:

Proposition 1.1.61. *Let μ be a multidegree such that there are generators of multidegree μ in relevant nodes of $\mathrm{MVT}(I)$ of dimensions $d_1 \dots d_k$ such that no two of them are consecutive, then*

$$\beta_{d_i,\mu}(I) = \#\mathrm{MVT}(I)_{d_i,\mu} \, \forall i = 1, \dots, k$$

Then, for all $\mathbf{a} \in \mathbb{N}^n$ we have that

$$\overline{\beta_{i,\mathbf{a}}}(I) \leq \beta_{i,\mathbf{a}}(I) \leq \widehat{\beta}_{i,\mathbf{a}}(I).$$

We have that the equality holds in the cases above

Definition 1.1.62. Let I be a monomial ideal.

- We say that I is a *Mayer-Vietoris ideal of type A* if there exists a Mayer-Vietoris tree of I such that there is no repeated generators in the ideals of the relevant nodes. In this case we have

$$\overline{\beta_{i,\mathbf{a}}}(I) = \beta_{i,\mathbf{a}}(I) = \widehat{\beta}_{i,\mathbf{a}}(I).$$

- If $\overline{\beta_{i,\mathbf{a}}}(I) = \beta_{i,\mathbf{a}}(I)$ for all $i \in \mathbb{N}, \mathbf{a} \in \mathbb{N}^n$ then I is a *Mayer-Vietoris ideal of type B1*.
- If $\beta_{i,\mathbf{a}}(I) = \widehat{\beta}_{i,\mathbf{a}}(I)$ for all $i \in \mathbb{N}, \mathbf{a} \in \mathbb{N}^n$ then I is a *Mayer-Vietoris ideal of type B2*.

Examples 1.1.63. 1. Every ideal in two variables is an MVT_A.

2. Generic monomial ideals are MVT_{B1}.
3. Ideals minimally resolved by the Taylor resolution are MVT_{B2}.

□

1.2 Reliability Theory

Reliability Theory has been a source of interest since the World War II although it is not until 1952 in a series of lectures given by R. S. Pierce, where the topic started to be developed. The main ideas of these lectures were published in [138] by J. von Neumann. The author considered the problem of constructing reliable binary circuits using redundancy of non-reliable components. Furthermore, thanks to the previous work of A. M. Turing [134] and W. S. McCulloch and W. Pitts [92], the author showed that logical propositions could be represented as networks or (idealized) nervous systems which means that logical propositions can be treated as a box that receives a finite number of inputs and returns a finite number of outputs that are determined by some specific rules.

E. F. Moore and C. E. Shannon, developed different methods in [101] that did not require to assume good reliability in the components of the circuit neither that much redundancy as was required in [138] by von Neumann. Even though there were some examples of series and parallel systems, this research was not focused on the application but in introducing the problem of reliability from a theoretical and mathematical point of view and provided bounds for reliability.

The concepts and results presented by Moore and Shannon were generalized in [26] by Birnbaum, Esary, and Saunders. In that research, the

authors continued studying reliability of binary systems, which was referred to as *dichotomic reliability*. The authors gave proper definitions of structure function, (minimal) paths and cuts, essential and not essential components, and (semi-)coherent systems. Furthermore, parallel and series systems (and the mix of both) and k-out-of-n systems were covered. The authors realized that binary reliability was not enough to study the systems that appeared in practice and, thereby they proposed a new branch of investigation: reliability of systems that were not binary.

The authors of [45] studied the case of binary independent components with different probabilities i.e. independent but non-identical components: they obtained convenient bounds for the reliability of the structures, generalized statistical properties obtained in a previous research ([26]) and then they generalized a differential equation (introduced in [101]) that allowed to relate the reliability of the components and the structure. The research was quite theoretical and had no illustrative examples.

To see a review of some different models for computing reliability we refer to [14, 75]. In [144] there is a systematic review of the different approaches that have been used to compute reliability of a system, which is summarized below:

- *An extension of binary models to the multi-state case*. Some binary methods have been employed to compute the reliability of multi-state systems, such as the reliability classical block diagram method which was extended to a repairable multi-state system [83] or the Fault Trees, employed by Caldarola in [29] to study multi-state systems after turning them into binary one using Boolean variables.

- *The stochastic process approach*. The first time this approach was used to compute the reliability of a system was by Natvig and Streller [103]. Nowadays, the stochastic approach allows to compute more properties of the system other than the reliability.

- *The Universal Generating Function (UGF) approach*. Since the introduction of the UGF in 1986 [135], this approach has been greatly studied [86, 79, 77].

- *Monte-Carlo simulation*. This method has been used in combination with other ones such as enumerative techniques, to compute the reliability of multi-state systems in different ways [25, 149, 150].

- *Recursive algorithms*. The use of recursive algorithms for computing the reliability of a system has been developed for different kind of systems such as the k-out-of-n or the series-parallel [151, 82, 126, 78].

1.2.1 Binary systems and Reliability. Definitions

This section is an introduction to reliability theory. We explain the basic definitions needed along this thesis and give several examples to help the reader understand the concepts introduced.

1.2.1.1 Binary systems

Definition 1.2.1. A *binary system* $S = (\mathcal{C}, \phi)$ is a set of components $\mathcal{C} = \{c_1, \ldots, c_n\}$ so that each component c_i can be in two different states $\mathcal{S}_i = \{0, 1\}$, together with a *structure function* $\phi : \mathcal{S}_1 \times \cdots \times \mathcal{S}_n \to \mathcal{S}$, where $\mathcal{S} = \{0, 1\}$ is the set of states of the system. The set

$$\mathfrak{D} = \{\mathbf{a} = (a_1, \ldots, a_n) \; s.t. \; a_i \in \mathcal{S}_i, \forall i \in \{1, \ldots, n\}\}$$

is call the *state space of* S and each $\mathbf{a}$ is a *component's state*.

When there is no opportunity of misunderstanding, we simply refer to the system $S = (\mathcal{C}, \phi)$ as S.

The structure function receives as input an n-tuple of the states of the components and outputs the global state of the system. That means that the states of the components determine the overall performance of the system. Along this thesis, we consider that a system (respectively a component) in state j represents *better performance* than the system (respectively a component) in state i, whenever $j > i$. Furthermore, we say that a tuple $\mathbf{a}$ is greater (lower) than the tuple $\mathbf{b}$, and denoted by $\mathbf{a} > \mathbf{b}$ ($\mathbf{a} < \mathbf{b}$), whether $a_i \geq b_i$ ($a_i \leq b_i$) for every i and there is at least one index i such that $a_i > b_i$ ($a_i < b_i$).

Definition 1.2.2. We say that he system $S = (\mathcal{C}, \phi)$ is *coherent* if $\phi(\mathbf{a}) \geq \phi(\mathbf{b})$ whenever $\mathbf{a} > \mathbf{b}$. Conversely, $\phi(\mathbf{a}) \leq \phi(\mathbf{b})$ whenever $\mathbf{a} < \mathbf{b}$.

The coherence property tells us that given a state for which the system works, if one component improves its level of performance, then the system cannot fail. Coherency can also be defined in failure states: let S be a system that is failing for a given components' state, if one component decreases its level of performance, then the system continues failing.

Examples of coherent systems include electrical and transport networks, biological and industrial systems among many others [75]. During this work, unless explicitly stated otherwise, every system is assumed to be coherent.

Example 1.2.3. (A) Let $S_1 = (\mathcal{C}, \phi)$ be a binary system with binary components such that $\mathcal{C} = \{c_1, c_2\}$ and $\phi(x_1, x_2) = \max\{x_1, x_2\}$. Given the components' state $\mathbf{x} = (0, 1)$ we obtain that $\phi(\mathbf{x}) = 1$ and, if we improve the level of performance of component 2, *i.e.*, $\mathbf{x} = (1, 1)$ we still have $\phi(\mathbf{x}) = 1$.

(B) Let $S_2 = (\mathcal{C}, \phi)$ be a binary system with binary components such that $\mathcal{C} = \{c_1, c_2, c_3\}$ and $\phi(x_1, x_2, x_3) = \min\{x_1, x_2, x_3\}$. Given the components' state $\mathbf{x} = (0, 1, 1)$ we obtain that $\phi(\mathbf{x}) = 0$ and, if we decrease the level of performance of component 2, *i.e.*, $\mathbf{x} = (0, 0, 1)$ we still have $\phi(\mathbf{x}) = 0$.

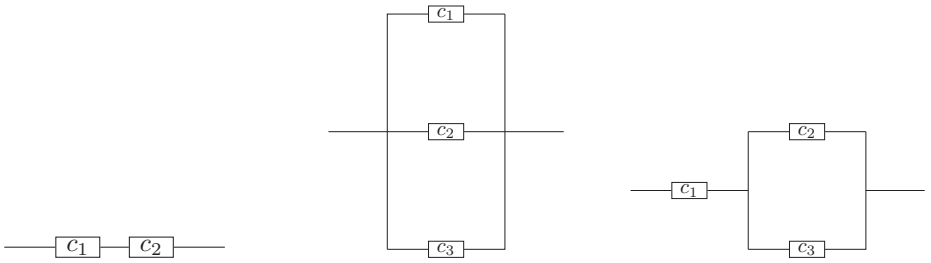
□

Definition 1.2.4. Let $S = (\mathcal{C}, \phi)$ be a binary system with n components and $\mathbf{a}$ an n-tuple of component's states. We say that $\mathbf{a}$ is a *path of S (or working state)* if $\phi(\mathbf{a}) = 1$. We say that $\mathbf{a}$ is a *minimal path of S (or minimal working state)* if any decrease on the level of any of the components implies that the level of performance of the system decreases. We denote by $\mathcal{F}_S^P$ and $\overline{\mathcal{F}}_S^P$ the set of paths and minimal paths of the system S, respectively.

Definition 1.2.5. Let $S = (\mathcal{C}, \phi)$ be a binary system with n components and $\mathbf{a}$ an n-tuple of component's states. We say that $\mathbf{a}$ is a *cut of S (or*

failure state) if $\phi(\mathbf{a}) = 0$. We say that $\mathbf{a}$ is a *minimal cut of S (or minimal failure state)* if an increase in one of the components implies that the level of performance of the system improves. We denote by $\mathcal{F}_S^C$ and $\overline{\mathcal{F}}_S^C$ the set of cuts and minimal cuts of the system S, respectively.

We use the notation :G for systems defined by their paths and :F for systems defined by their cuts.



(a) Series system (b) Parallel system (c) Series-parallel system

Figure 1.1: Systems of Example 1.2.6 (A), Example 1.2.6 (B) and Example 1.2.6 (C)

Example 1.2.6. (A) Let $S = (\mathcal{C}, \phi)$ be the system given by $\mathcal{C} = \{c_1, c_2\}$, $\mathcal{S}_i = \{0, 1\}, \forall i \in \{1, 2\}$, $\mathcal{S} = \{0, 1\}$ and

$$\phi(\mathbf{a}) = \min\{(a_1, a_2) \; s.t. \; a_i \in \mathcal{S}_i, \forall i \in \{1, 2\}\}.$$

The paths, minimal paths, cuts and minimal cuts of this example are shown in Table 1.1.

Paths	Minimal paths	Cuts	Minimal cuts
(1,1)	(1,1)	(0,0) (1,0) (0,1)	(1,0) (0,1)

Table 1.1: Paths, minimal paths, cuts and minimal cuts of Example 1.2.6 (A)

(B) Let $S = (\mathcal{C}, \phi)$ be the system given by $\mathcal{C} = \{c_1, c_2, c_3\}$, $\mathcal{S}_i = \{0, 1\}, \forall i \in \{1, 2, 3\}$, $\mathcal{S} = \{0, 1\}$ and

$$\phi(\mathbf{a}) = \max\{(a_1, a_2, a_3) \; s.t. \; a_i \in \mathcal{S}_i, \forall i \in \{1, 2, 3\}\}.$$

The paths, minimal paths, cuts and minimal cuts of this example are shown in Table 1.2

Paths	Minimal paths	Cuts	Minimal cuts
(1,0,0)			
(1,1,0)			
(1,0,1)	(1,0,0)		
(0,1,0)	(0,1,0)	(0,0,0)	(0,0,0)
(0,1,1)	(0,0,1)		
(0,0,1)			
(1,1,1)			

Table 1.2: Paths, minimal paths, cuts and minimal cuts of Example 1.2.6 (B)

(C) Let $S = (\mathcal{C}, \phi)$ be the series-parallel system given by $\mathcal{C} = \{c_1, c_2, c_3\}$, $\mathcal{S}_i = \{0, 1\}, \forall i \in \{1, 2, 3\}$, $\mathcal{S} = \{0, 1\}$ and

$$\phi(\mathbf{a}) = \min\{(a_1, \max\{(a_2, a_3)\}) \; s.t. \; a_i \in \mathcal{S}_i, \forall i \in \{1, 2, 3\}\}.$$

The paths minimal paths, cuts and minimal cuts of this example are summarized in Table 1.3.

□

Remark 1.2.7. When there is no possibility of misunderstanding between paths or cuts, we just denote by $\mathcal{F}_S$ and $\overline{\mathcal{F}}_S$ the set of paths or cuts and minimal paths or minimal cuts of the system S, respectively.

Paths	Minimal paths	Cuts	Minimal cuts
(1,1,0) (1,0,1) (1,1,1)	(1,1,0) (1,0,1)	(0,0,0) (1,0,0) (0,1,0) (0,0,1) (0,1,1)	(1,0,0) (0,1,1)

Table 1.3: Paths, minimal paths, cuts and minimal cuts of Example 1.2.6 (C)

1.2.1.2 Reliability

In this section we provide the definition of reliability for a binary system. All definitions related with multi-state systems can be found in Chapter 3.

Let us $S = (\mathcal{C}, \phi)$ be a binary system given by $\mathcal{C} = \{c_1, \ldots, c_n\}$, with $\mathcal{S}_i = \{0, 1\}$ for all $i \in \{1, \ldots, n\}$. Each of the components has a probability of failing and working. We will denote by $p_{i,0}$ and $p_{i,1}$ the probability of component i of being in state 0 and 1, respectively. The probabilities associated to each components are the ones which will allow us to compute the reliability of the system:

Definition 1.2.8. Let S be a binary system. The *reliability of system* S, denoted by $\mathbf{R}_S$, is the probability that the system works *i.e.* the probability of being in state 1.

Definition 1.2.9. Let S be a binary system. The *unreliability of system* S, denoted by $\mathbf{U}_S$, is the probability that the system fails *i.e.*the probability of being in state 0.

Remark 1.2.10. The reliability and the unreliability of a binary system S are related by the following formula

$$\mathbf{R}_S = 1 - \mathbf{U}_S. \tag{1.2.1}$$

To compute the (un)reliability of a system S is necessary to know the probability of working or fail of each component of the system. We say that the components of a systems S are

- *Independent and identical distributed (i.i.d.)* when all of them have the same probability of failing/working and their states are not dependent on the state of the other components.

- *Independent and non identical distributed* when the components have different probabilities of failing/working and their state are not dependent on other component.

- *Dependent and identical distributed* when all of them have the same probability of failing/working and their state are dependent on the state of one or more components.

- *Dependent and non identical distributed* when the components have different probabilities of failing/working and their state are dependent on the state of one or more components.

The reliability of any system can be computed with the algebraic method based on monomial ideals by making few assumptions on the given system.

Bonferroni bounds. Sometimes, rather than the exact (un)reliability of a system S, bounds are preferred. For computing the exact reliability of the system one needs to have the probabilities of working/failing of all the components of the system which is a not an easy engineering process. The difficulty of this process becomes even more evident in the case of dependent components. When there is not chance of having all the information needed, the exact reliability cannot be computed. Yet, one is able to get bounds although it is clear that the more information one has, the better the bounds will be.

The *Inclusion-Exclusion method* is an enumerative method which allows to obtain the reliability of a system. Given a binary coherent system $S = (\mathcal{C}, \phi)$ and its set of paths $\mathcal{F}_S^P = \{\mathbf{a}_1, \ldots, \mathbf{a}_s\}$, one is able to compute the reliability of S using the Inclusion-Exclusion method as follows:

$$\begin{aligned}\mathbf{R}_S =& \mathbb{P}(\mathbf{a}_1 \cup \cdots \cup \mathbf{a}_s) \\ =& [\mathbb{P}(\mathbf{a}_1) + \mathbb{P}(\mathbf{a}_2) + \cdots + \mathbb{P}(\mathbf{a}_s)] + \\ & - [\mathbb{P}(\mathbf{a}_1 \cap \mathbf{a}_2) + \mathbb{P}(\mathbf{a}_1 \cap \mathbf{a}_3) + \cdots + \mathbb{P}(\mathbf{a}_{s-1} \cap \mathbf{a}_s)] + \ldots \\ =& \sum_{i=1}^{s} (-1)^{i-1} \sum_{|\sigma|=i} \bigcap_{j \in \sigma} \mathbf{a}_j.\end{aligned}$$

The so called Bonferroni-Fréchet bounds [37] are obtained by truncating at each summand of the Inclusion-Exclusion equation as follows:

$$\mathbf{R}_S \leq \sum_{i=1}^{k} (-1)^{i-1} \sum_{|\sigma|=i} \bigcap_{j \in \sigma} \mathbf{a}_j \text{ for } k \text{ odd},$$

$$\mathbf{R}_S \geq \sum_{i=1}^{k} (-1)^{i-1} \sum_{|\sigma|=i} \bigcap_{j \in \sigma} \mathbf{a}_j \text{ for } k \text{ even}.$$

In the next section we show how the algebraic method allows to compute tight bounds for (un)reliability for binary systems in a more efficient way because it avoids some redundancy (the multi-state case is explored in Chapter 3).

1.2.2 The algebraic method

The relation between Algebra and System Reliability, which is given by a correspondence of a monomial for each state of the coherent system, was established in [55]. Then, a monomial ideal with the monomials representing the working (or failing) states of the system is associated to it. In order to obtain the reliability (unreliability) of the system, we need to compute the Hilbert series of the monomial ideal. For computing the Hilbert series, there are different methods, such as the one presented in [22]. However, these methods do not allow us to compute bounds, which are really interesting in reliability. As a result of this fact, we work with free resolutions. The main problem is to find a free resolution for the

monomial ideal that does not make the Hilbert series too redundant, *i.e.* we need to compute the closest free resolution to the minimal one. By truncating Hilbert series we obtain upper and lower bounds for reliability. We know that minimal free resolutions allow us to write the Hilbert series in the most compact form between the ones coming from resolutions (and it also make the bounds as tight as for any other resolution), but it is not easy, in general, to find it. In this article the authors presented a way to compute a resolution for a monomial ideal, with not guarantee of being minimal, nevertheless it is quite good (it improves results obtained with Taylor resolution). To obtain that resolution, they used the so called *Scarf Complex*.

Some years later, Sáenz-de-Cabezón and Wynn ([114, 116, 117, 118], all of them reviewed on [119]) showed that (minimal free) resolution is not necessary to be computed to obtain the Hilbert series, it is enough to calculate the Betti numbers which can be computed using Mayer-Vietoris trees. Some special and important kind of system such as binary k-out-of-n systems and some variants or networks are thoroughly analysed during their research.

When it comes to system reliability theory, the design of optimal systems is a topic of high importance. An advantage of the algebraic method is that it provides information about the structure of the system and it can be useful in designing optimal systems due to it provides information about the structure of the system[120]. Furthermore, the improvement of computational algebra systems (such as `CoCoA`, `Macaulay2` or `Singular`) make the algebraic method an useful tool for designing optimal systems (and for computing its reliability too).

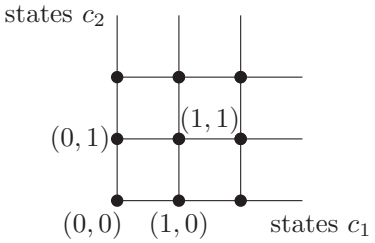
We have said before that the link between the algebraic approach and reliability theory are monomial ideals.

The relation is the following: given a system S and its state space $\mathfrak{D}$, we can represent each tuple $\mathbf{a} \in \mathfrak{D}$ in a grid of non-negative and integer coordinates. Due to the coherent property of the systems and the fact that monomials satisfy that

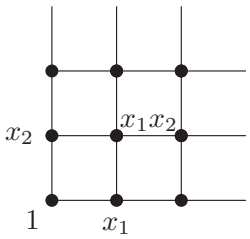
$$\mathbf{x}^{\mathbf{a}} | \mathbf{x}^{\mathbf{b}} \text{ whenever } \mathbf{a} \leq \mathbf{b} \tag{1.2.2}$$

for each $\mathbf{a} \in \mathfrak{D}$ we can construct a monomial $\mathbf{x^a} = x_1^{a_1} \cdots x_n^{a_n}$ that represents a state of the system.

The visual relation between the state space of Example 1.2.6 and our algebraic approach to reliability theory is given in Figure 1.2.



(a) State space representation



(b) Monomial representation

Figure 1.2: Monomial representation of the states of the system in Example 1.2.6

Remember from Remark 1.2.7 that $\mathcal{F}$ and $\overline{\mathcal{F}}$ are the set of paths and minimal paths, respectively. In Figure 1.3 we can notice that if we take the monomials generated by the minimal paths, all the paths of the system are included in the shadow area and satisfy the coherence property translated to monomials (if $\mathbf{x^a}$ belongs to a monomial ideal, then all the monomials $\mathbf{x^b}$ such that $\mathbf{x^a}|\mathbf{x^b}$ belong to the monomial ideal too). This area represents the monomials in a monomial ideal whose minimal generators are exactly the minimal paths of the system.

Remark 1.2.11. Observe that there exist monomials in the ideal which are not representing any state of the system. Indeed, in the binary case, every monomial with an exponent greater than 1 does not represent any state of the system. These monomials are not going to influence in the computations of the reliability because we have into account that their probability is 0 *i.e.* as the components only reach states 0 and 1, we have that $p_{i,j} = 0$ for all i and for all $j \geq 2$.

Definition 1.2.12. The ideal generated by the set of minimal paths of the system S, denoted by $I_{\mathcal{F}_S^p}$, is called *path ideal of system S*.

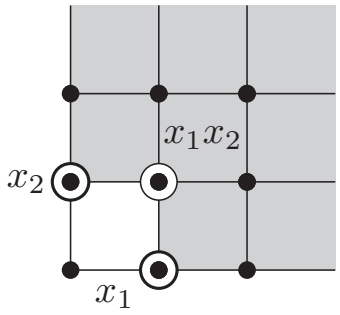


Figure 1.3: Monomials belonging to the the path ideal of the system from Example 1.2.6 (A)

Following the same idea of the minimal paths but with the minimal cuts of a system, one can define:

Definition 1.2.13. The ideal generated by the set of minimal cuts of the system S, denoted by $I_{\mathcal{F}_S^C}$, is called *cut ideal of system S*.

Remark 1.2.14. When working with cut ideals, we are focused on the failure of the system. In this case the levels of the system indicates growing levels of failure *i.e.* 0 means that the system works and 1 that the system is failing. Being in the cut ideal means failure.

Example 1.2.15 (Continuation of Example 1.2.6)**.** (A) The path ideal of the system of Example 1.2.6 (A) is $I_{\mathcal{F}_S^P} = \langle x_1x_2 \rangle$ and the cut ideal is $I_{\mathcal{F}_S^C} = \langle x_1, x_2 \rangle$.

(B) The path ideal of the system of Example 1.2.6 (B) is $I_{\mathcal{F}_S^P} = \langle x_1, x_2, x_3 \rangle$ and the cut ideal is $I_{\mathcal{F}_S^C} = \langle x_1x_2x_3 \rangle$.

(C) The path ideal of the system of Example 1.2.6 (C) is $I_{\mathcal{F}_S^P} = \langle x_1x_2, x_1x_3 \rangle$ and the cut ideal is $I_{\mathcal{F}_S^C} = \langle x_1, x_2x_3 \rangle$.

□

Once we have computed the path or cut ideal, we need to compute the numerator of the Hilbert series, $NHS(I)$ (see Section 1.1.1), which counts

how many monomials are in the ideal, and substitute the probabilities of the components: each x_i appearing in the numerator of the Hilbert series is replaced by $p_{i,j}$.

Remark 1.2.16. If we compute the path ideal, in the numerator of the Hilbert series one has to substitute each x_i for $p_{i,1}$. However, if we work with the cut ideal one has to take into account that the higher the exponent of the monomial, the higher the failing i.e. the monomials inside the cut ideal are failing states which are not minimal. Then, when substituting the probabilities we must be careful and now substitute each x_i for $1 - p_{i,1}$.

Sometimes, rather than the exact probability of working or failing, bounds are preferred. The algebraic method provides bounds by truncating the summands of the numerator of the Hilbert series:

- If one truncates the numerator of the Hilbert series in an odd step, then one get upper bounds, denoted by u.
- If, otherwise, the truncation is done in an even summand, lower bounds are obtained. Lower bounds are denoted by l.

Depending on the number of summands of the numerator of the Hilbert series, a different number of bounds can be obtained. The more summands one use, the tightest the bounds are.

Example 1.2.17 (Continuation of Example 1.2.6)**.**

(A) The numerator of the Hilbert series of the path ideal of the system of Example 1.2.6 (A) is $NHS(I_{\mathcal{F}_S^P}) = x_1x_2$ and the numerator of the Hilbert series of the cut ideal is $NHS(I_{\mathcal{F}_S^C}) = x_1 + x_2 - x_1x_2$.

(B) The numerator of the Hilbert series of the path ideal of the system of Example 1.2.6 (B) is $NHS(I_{\mathcal{F}_S^P}) = x_1 + x_2 + x_3 - (x_1x_2 + x_1x_3 + x_2x_3) + x_1x_2x_3$ and the numerator of the Hilbert series of the cut ideal is $NHS(I_{\mathcal{F}_S^C}) = x_1x_2x_3$.

(C) The numerator of the Hilbert series of the path ideal of the system of Example 1.2.6 (C) is $NHS(I_{\mathcal{F}_S^P}) = x_1x_2 + x_1x_3 - x_1x_2x_3$ and the

numerator of the Hilbert series of the cut ideal is $NHS(I_{\mathcal{F}_S^C}) = x_1 + x_2x_3 - x_1x_2x_3$.

□

Example 1.2.18 (Double bridge structure)**.** In this example we are going to work with the system represented in Figure 1.4, which is a network called *double bridge structure*.

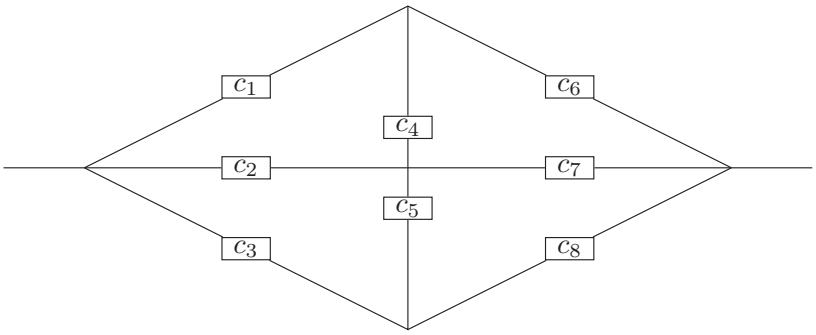


Figure 1.4: Double bridge network from Example 1.2 in [118]

The probabilities of each component for failing are

$$p_{1,0} = 0.01, \quad p_{2,0} = 0.03, \quad p_{3,0} = 0.05, \quad p_{4,0} = 0.05$$
$$p_{5,0} = 0.05, \quad p_{6,0} = 0.01, \quad p_{7,0} = 0.03, \quad p_{8,0} = 0.02$$

If we observe Figure 1.4, one can check that the minimal cuts are the ones in Table 1.4. Remember that once we know the minimal cuts of a system, one can compute the ideal associated to it. In this case, we have that the ideal is

$$I = \langle x_1x_2x_3, x_1x_2x_5x_8, x_1x_3x_4x_5x_7, x_1x_4x_7x_8, \\ x_2x_3x_4x_6, x_2x_4x_5x_6x_8, x_3x_5x_6x_7, x_6x_7x_8 \rangle.$$

To compute the numerator of the Hilbert series associated to I, we use the *MVT*, although this process is not included here due to its large size. We obtain that the numerator of Hilbert series is

Minimal cuts
(1,1,1,0,0,0,0,0)
(1,1,0,0,1,0,0,1)
(1,0,1,1,1,0,1,0)
(1,0,0,1,0,0,1,1)
(0,1,1,1,0,1,1,1)
(0,1,0,1,1,1,0,1)
(0,0,1,0,1,1,1,0)
(0,0,0,0,0,1,1,1)

Table 1.4: Minimal cuts of the system from Example 1.2.18

$$
\begin{aligned}
NHS(I) =& x_1x_2x_3 + x_1x_2x_5x_8 + x_1x_3x_4x_5x_7 + x_1x_4x_7x_8 + \\
&+x_2x_3x_4x_6 + x_2x_4x_5x_6x_8 + x_3x_5x_6x_7 + x_6x_7x_8 + \\
&-(x_1x_2x_3x_4x_6 + x_1x_2x_3x_4x_5x_7 + x_1x_2x_3x_4x_7x_8 + \\
&+x_1x_2x_3x_5x_6x_7 + x_1x_2x_3x_5x_8 + x_1x_2x_3x_6x_7x_8 + \\
&+x_1x_2x_4x_5x_6x_8 + x_1x_2x_4x_5x_7x_8 + x_1x_2x_5x_6x_7x_8 + \\
&+x_1x_3x_4x_5x_6x_7 + x_1x_3x_4x_5x_7x_8 + x_2x_3x_4x_5x_6x_7 + \\
&+x_2x_3x_4x_5x_6x_8 + x_2x_3x_4x_6x_6x_7x_8 + x_2x_4x_5x_6x_7x_8 + \\
&+x_3x_5x_6x_7x_8) + \\
&+(2 \cdot x_1x_2x_3x_4x_5x_6x_7 + x_1x_2x_3x_4x_5x_6x_8 + \\
&+x_1x_2x_3x_4x_5x_7x_8x_1x_2x_4x_5x_6x_7x_8 + \\
&+2 \cdot x_2x_3x_4x_5x_6x_7x_8) - 2 \cdot (x_1x_2x_3x_4x_5x_6x_7x_8).
\end{aligned}
$$

Observe that in this example we are working with the ideal generated by the minimal cuts i.e. the cut ideal. Because of that, instead of substituting the working probabilities, we have to substitute the fail probabilities of each components i.e. each variable x_i has to be replace with $p_{i,0}$ and we will obtain (bounds of) unreliability of the system, instead of the reliability. We have

$$\begin{aligned}
\mathbf{U} &= 2,31525 \cdot 10^{-5} - 4,398 \cdot 10^{-8} + 1,02 \cdot 10^{-10} - 2,25 \cdot 10^{-13} \\
&= 2,310862178 \cdot 10^{-5} \\
u_2 &= 2,31525 \cdot 10^{-5} - 4,398 \cdot 10^{-8} + 1,02 \cdot 10^{-10} = 2,3108622 \cdot 10^{-5} \\
l_1 &= 2,31525 \cdot 10^{-5} - 4,398 \cdot 10^{-8} = 2,310852 \cdot 10^{-5} \\
u_1 &= 2,31525 \cdot 10^{-5}
\end{aligned}$$

where $\mathbf{U}$ is the unreliability of the system, u_i are upper bounds for unreliability in which the higher the i, the tighter the bound, and l_1 is a lower bound.

□

Example 1.2.19 (Binary k-out-of-n systems)**.** A *binary k-out-of-n:G(:F) system* is a system with n components that works (fails) if and only if k components work (fail). This kind of systems are really important due to its wide range of applications (see, for instance, [75]). We will work with the multi-state version of those systems in Section 3.2.3 of Chapter 3.

Let us work with a 2-out-of-4 system in which we have the following probabilities for the components: $p_{1,0} = 0.1, p_{1,1} = 0.9, p_{2,0} = 0.15, p_{2,1} = 0.85, p_{3,0} = 0.05, p_{3,1} = 0.95, p_{4,0} = 0.17, p_{4,1} = 0.83$.

We know from [116] that the ideal associated to a binary k-out-of-n system is of the form

$$I_{k,n} = \langle \mathbf{x}^\mu \ s.t. \ \mathbf{x}^\mu \text{ is a squarefree monomial ideal of degree } k \text{ in } n \text{ variables} \rangle .$$

For this ideal, the authors showed how to compute the Betti numbers associated to these ideals (and, in consequence, how to compute the Hilbert series) but we compute the Betti numbers using Mayer-Vietoris Trees.

In our example, we have that

$$I_{2,4} = \langle x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_2x_4, x_3x_4 \rangle ,$$

which *MVT* is showed in Figure 1.5 with no `(position, dimension)` written.

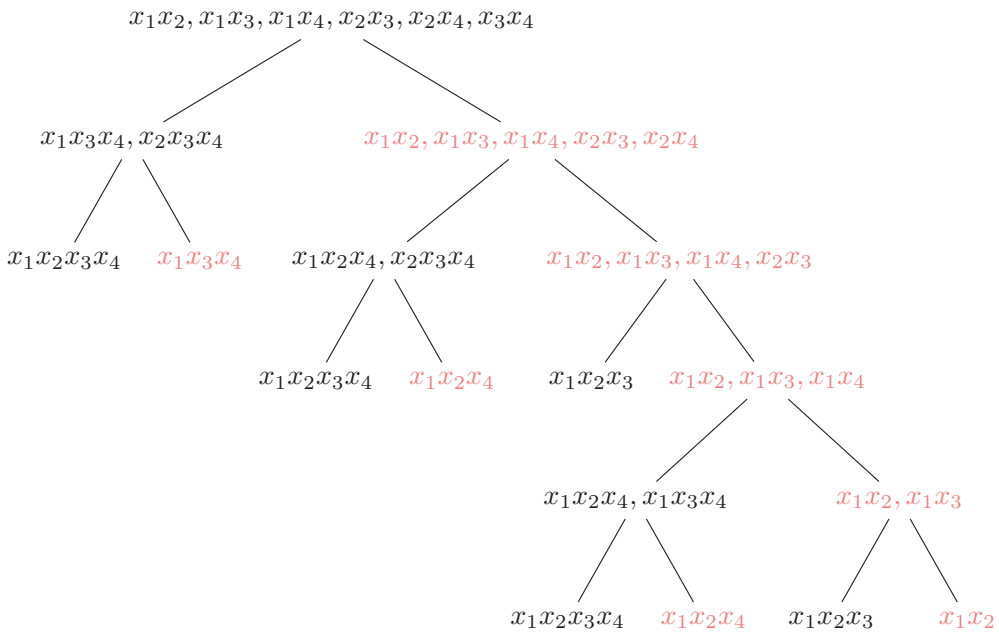


Figure 1.5: *MVT* associated to the ideal $I_{2,4}$ from Example 1.2.19

From the *MVT* we can read that the numerator of the Hilbert series associated to the ideal is

$$\begin{aligned} NHS(I_{2,4}) = {} & x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4 \\ & - 2(x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4) \\ & + 3x_1x_2x_3x_4 \end{aligned}$$

Substituting the probabilities (where it appears x_i we have to substitute the variable for $p_{i,1}$) we have that

$$\begin{aligned} \mathbf{R} &= 4.6685 - 2 \cdot 2.78195 + 3 \cdot 0.6032025 = 0.9142075, \\ l_1 &= 4.6685 - 2 \cdot 2.78195 = -0.8954, \\ u_1 &= 4.6685, \end{aligned}$$

where l_1 is a lower bound and u_1 is an upper bound for reliability. In this example, bounds are useless because the ideal does not have enough variables and generators.

□

Chapter 2

Polarization and depolarization of monomial ideals. The support poset

2.1 Introduction

Polarization is an operation that transforms a monomial ideal into a squarefree monomial ideal in a larger polynomial ring, preserving several important features of the original ideal such as the graded Betti numbers. The main idea behind polarization is the possibility of using the combinatorial properties of squarefree monomial ideals when studying problems about general monomial ideals. Polarization is used in a wide variety of applications in the theory of monomial ideals. For example it was used by Hartshorne to prove the connectedness of the Hilbert scheme by showing that distractions of ideals can be described as specializations of polarizations of monomial ideals [60]. One of the main applications is its use to study the Cohen-Macaulay[1] property of monomial ideals by passing to squarefree monomial ideals and applying Reisner's criterion on their associated simplicial complex [93, 103]. It is also used to study associated primes of monomial ideals and their powers [64, 58, 90].

[1] An R-module $\mathcal{M}$ is Cohen-Macaulay if $\operatorname{depth}(\mathcal{M}) = \operatorname{Krull}\dim(\mathcal{M})$.

Even though polarization has been used as a powerful tool in algebraic geometry and in applications, the inverse operation, depolarization, has been less investigated. Depolarization can be used to study the algebraic invariants of squarefree monomial ideals using general monomial ideals in less variables [106]. We note that depolarization is not unique, in the sense that a given squarefree monomial ideal might have different depolarizations.

One of the main goals of this chapter is to find all depolarizations of a given squarefree monomial ideal and describe their structure combinatorially. Furthermore, as we have said before, a given squarefree monomial ideal I shares several important features with its depolarizations, the aforementioned combinatorial structure can be then used to select a convenient depolarization of I in order to study the properties of either I or any of its depolarizations. For example, one immediately obtains the Hilbert function of these ideals by studying only one of them, as they are closely related. The properties and features that are not shared within the family of depolarizations of I are also interesting to study in order to identify a particular depolarization whose invariants are easier to compute and provide information about all depolarizations of I.

2.2 Polarization and depolarization

2.2.1 Polarization

Let $R = \Bbbk[x_1, \ldots, x_n]$ be a polynomial ring in n indeterminates over a field $\Bbbk$ on which we make no explicit assumptions. For any monomial ideal $I \subseteq R$, we let $G(I) = \{m_1, \ldots, m_r\}$ be the unique minimal monomial generating set of I.

Definition 2.2.1. Let $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mu = (b_1, \ldots, b_n)$ be two elements in $\mathbb{N}^n$ with $b_i \leq a_i$ for all i. The polarization of μ in $\mathbb{N}^{a_1+\cdots+a_n}$ is the multi-index

$$\overline{\mu} = (\underbrace{1, \ldots, 1}_{b_1}, \underbrace{0, \ldots, 0}_{a_1-b_1}, \ldots, \underbrace{1, \ldots, 1}_{b_n}, \underbrace{0, \ldots, 0}_{a_n-b_n}).$$

The *polarization of* $\mathbf{x}^{\mu} = x_1^{b_1} \cdots x_n^{b_n} \in R$ with respect to $\mathbf{a}$ is the squarefree monomial

$$\mathbf{x}^{\overline{\mu}} = x_{1,1} \cdots x_{1,b_1} \cdots x_{n,1} \cdots x_{n,b_n} \in \Bbbk[x_{1,1}, \ldots, x_{1,a_1}, \ldots, x_{n,1}, \ldots, x_{n,a_n}].$$

Note that for ease of notation we used $\mathbf{x}$ with two different meanings in this definition.

Example 2.2.2. Let $\mathbf{x}^{\mu} \in R = \Bbbk[x_1, x_2, x_3]$, with $\mu = (2, 1, 3)$. The polarization of $\mathbf{x}^{\mu}$ with respect to $a = (3, 2, 3)$ is

$$\mathbf{x}^{\overline{\mu}} = \mathbf{x}^{(1,1,0,1,0,1,1,1)} = x_{1,1}x_{1,2}x_{2,1}x_{3,1}x_{3,2}x_{3,3},$$

with $S = \Bbbk[x_{1,1}, x_{1,2}, x_{1,3}, x_{2,1}, x_{2,2}, x_{3,1}, x_{3,2}, x_{3,3}]$.

□

Definition 2.2.3. Let $I = \langle m_1, \ldots, m_r \rangle \subseteq R$ be a monomial ideal and let a_i be the maximum exponent to which indeterminate x_i appears among the generators of I. The *polarization of* I, denoted by I^P, is the monomial ideal in S given by $I^P = \langle \overline{m_1}, \ldots, \overline{m_r} \rangle$, where $\overline{m_i}$ is the polarization of m_i with respect to $\mathbf{a}$.

Example 2.2.4. Let Let $I = \langle x^2y^2, z^2 \rangle \subseteq \Bbbk[x, y, z]$. The polarization of I is the squarefree monomial ideal

$$I^P = \langle x_1x_2y_1y_2, z_1z_2 \rangle \subseteq \Bbbk[x_1, x_2, y_1, y_2, z_1, z_2].$$

□

Observe that Definition 2.2.3 is a combinatorial expression of the following result of Fröberg [49] as given in [137] in which I' is a polarization of I.

Proposition 2.2.5. *For any monomial ideal $I \subset R$, there is a squarefree monomial ideal $I' \subset R'$ such that $R/I = R'/(I' + (\underline{h}))$, where $\underline{h}$ is a regular sequence[2] on R'/I' of forms of degree one.*

[2]Let $\mathcal{M}$ be a graded module on $R = \Bbbk[x_1, \ldots, x_n]$. A sequence $\mathbf{f} = f_1, \ldots, f_r \in R$ is a regular sequence on $\mathcal{M}$ or $\mathcal{M}$-sequence if $\mathcal{M}/\mathbf{f}\mathcal{M} \neq 0$ and f_i is a nonzerodivisor on $\mathcal{M}/\langle f_1, \ldots, f_{i-1} \rangle \mathcal{M}$ for each $i = 1, \ldots, r$.

2.2.1.1 Polarization and free resolutions

Let

$$\mathbb{F} : \cdots \to F_i \xrightarrow{\delta_i} F_{i-1} \xrightarrow{\delta_{i-1}} F_{i-2} \to \dots$$

be a multigraded chain complex of R-modules, i.e., $F_i = \bigoplus_{j=1}^{r_i} R(-\mu_{i,j})$ where $\mu_{ij} \in \mathbb{N}^n$, and the differentials δ_i have multidegree 0. The differentials δ_i are given by matrices A_i whose entries are monomials in $\mathbb{N}^n$. We denote by $e_{i,j}$ the standard generator of the j-th summand of F_i whose multidegree is $\mu_{i,j}$. Then the j-th column of A_i is given by $(a^i_{1,j}, a^i_{2,j}, \dots, a^i_{r_{i-1},j})$ where $\delta(e_{i,j}) = \sum_{k=1}^{r_{i-1}} a^i_{k,j} e_{i-1,k}$ and the $e_{i-1,k}$ are the standard generators of F_{i-1}. The nonzero entries $a^i_{k,j}$ are given by $\mu_{i,j}/\mu_{i-1,k}$.

Definition 2.2.6. We define $\overline{\mathbb{F}}$, the *polarization* of $\mathbb{F}$, as the chain complex given by

$$\overline{\mathbb{F}} : \cdots \to \overline{F}_i \xrightarrow{\overline{\delta}_i} \overline{F}_{i-1} \xrightarrow{\overline{\delta}_{i-1}} \overline{F}_{i-2} \to \dots$$

where $\overline{F}_i = \bigoplus_{j=1}^{r_i} R(-\overline{\mu}_{ij})$ if $F_i = \bigoplus_{j=1}^{r_i} R(-\mu_{ij})$ and the matrices $\overline{A}_i$ of the differentials $\overline{\delta}_i$ are given by

$$\overline{a}^i_{j,k} = \begin{cases} 0 & \text{if } a^i_{j,k} = 0 \\ x_{1,c_1+1} \cdots x_{1,b_1} \cdots x_{n,c_n+1} \cdots x_{n,b_n} & \text{if } a^i_{j,k} \neq 0, \mu_{i,j} = \mathbf{x}^b, \mu_{i-1,k} = \mathbf{x}^c, \end{cases}$$

where $b = (b_1, \dots, b_n)$ and $c = (c_1, \dots, c_n)$. Note that if $0 \neq a^i_{j,k} \in \Bbbk$ then $\overline{a}^i_{j,k} = a^i_{j,k}$.

By polarizing a resolution of a monomial ideal we obtain a resolution of its polarization. This is a consequence of Theorem 3.3 in [139]. Here, we present our own proof to keep track of the explicit changes in each multidegree. The use of depolarization to compute resolutions of a monomial ideal from the resolutions of its polarization is also a well-known result, see Examples 3.4 in [139] and [49].

Proposition 2.2.7. *Let $\mathbb{F}$ be a multigraded free resolution of a monomial ideal I. The polarization $\overline{\mathbb{F}}$ of $\mathbb{F}$ is a multigraded free resolution of the polarization of I. Moreover, the ranks and the graded ranks of $\overline{\mathbb{F}}$ are equal to those of $\mathbb{F}$. In the*

case of multigraded ranks, we have that the (i,μ)*-rank of* $\mathbb{F}$ *equals the* $(i,\overline{\mu})$*-rank of* $\overline{\mathbb{F}}$*.*

Proof. Given the fact that $\mathbb{F}$ is a multigraded free resolution of I and the construction of $\overline{\mathbb{F}}$ by polarization, we only need to prove that $\mathrm{Im}(\overline{\delta}_i) = \mathrm{Ker}(\overline{\delta}_{i-1})$.

We know that $\delta^2 = 0$. Explicitly, we have that

$$\delta_{i-1}\delta_i(e_{i,j}) = \sum_{k=1}^{r_{i-1}}\sum_{l=1}^{r_{i-2}} a^i_{k,j}a^{i-1}_{l,k}e_{i-2,l} = 0, \quad \forall\, i,j.$$

This implies that

$$\sum_{k=1}^{r_{i-1}}\sum_{l=1}^{r_{i-2}} a^i_{k,j}a^{i-1}_{l,k} = \sum_{k=1}^{r_{i-1}}\sum_{l=1}^{r_{i-2}} \frac{\mu(e_{i,j})}{\mu(e_{i-1,k})}\frac{\mu(e_{i-1,k})}{\mu(e_{i-2,l})} = 0, \quad \forall\, i,j. \tag{2.2.1}$$

On the other hand, from the definitions of the maps in $\overline{\mathbb{F}}$ we have that for any i,j

$$\overline{\delta}_{i-1}\overline{\delta}_i(\overline{e}_{i,j}) = \sum_{k=1}^{r_{i-1}}\sum_{l=1}^{r_{i-2}} \overline{a}^i_{k,j}\overline{a}^{i-1}_{l,k}\overline{e}_{i-2,l}.$$

Now by polarizing (2.2.1), we obtain

$$\sum_{k=1}^{r_{i-1}}\sum_{l=1}^{r_{i-2}} \frac{\overline{\mu}(e_{i,j})}{\overline{\mu}(e_{i-1,k})}\frac{\overline{\mu}(e_{i-1,k})}{\overline{\mu}(e_{i-2,l})} = \sum_{k=1}^{r_{i-1}}\sum_{l=1}^{r_{i-2}} \overline{a}^i_{k,j}\overline{a}^{i-1}_{l,k} = 0,$$

and hence $\overline{\delta}_{i-1}\overline{\delta}_i(\overline{e}_{i,j}) = 0$ for all i,j. Since polarization induces a multigraded isomorphism, the result follows. □

Proposition 2.2.7 is important in our context since it allows us to use polarization in the algebraic analysis of system reliability and obtain formulas and bounds for the reliability of the system corresponding to the polarization of a given ideal. In particular, we can use the Mayer-Vietoris trees as one of the main tools applied.

Corollary 2.2.8. *Let* $\mathbb{F}$ *be a cone resolution of a monomial ideal* I*, then* $\overline{\mathbb{F}}$ *is a cone resolution of the polarization of* I*. In particular, if* $\mathbb{T}$ *is a Mayer-Vietoris tree of* I*, then the polarized tree* $\overline{\mathbb{T}}$ *is a Mayer-Vietoris tree of the polarization of* I*.*

2.2.2 Depolarization

In the Section 2.1 we have said that polarization has been extensively studied whereas the reverse process, depolarization, has not. Depolarization transforms a square-free monomial ideal into a monomial ideal with exponents, providing ideals belonging to rings with less variables than the original monomial ideal. Although polarization of a monomial ideals is unique, depolarization is not, *i.e.* there exist different monomial ideals sharing the same polarization.

We claimed before that there are some properties and features which are shared between the square-free monomial ideals and its depolarizations such as Betti numbers or the Hilbert series (it is not exactly the same, but it is close-related). Most of the times making computations with square-free monomial ideals is (much) more efficient even though, as we mentioned earlier, one has to pay the price of dealing with more variables. Nonetheless, there are times when having all these extra variables does not pay off. This usually happens when we have to deal with a monomial ideal that falls under certain category for which some invariants are explicitly defined (such is the case of the stable ideals for which Eliahou and Kervaire [41] give an explicit definition of the minimal resolution). This is mainly the reason why depolarization is so important: it allows us to perform computations with square-free or monomial ideals with exponents indistinctly. Then, one can explore which features and properties are and which are not shared between a square-free ideal and its polarization.

When it comes to reliability, depolarization allows to reduce the number of components of the system, which is sometimes profitable. This process will be explored in Chapter 3.

Definition 2.2.9. Let R, S and T be polynomial rings over a field $\Bbbk$. Let $I \subseteq R$ be a squarefree monomial ideal. A *depolarization of* I is a monomial ideal $J \subseteq S$ such that I is isomorphic to $J^P \subseteq T$, that is: there is a bijective map φ from the set of variables of R to the set of variables of T such that $\varphi(G(I)) = G(J^P)$, where $G(J^P)$ is the unique minimal monomial generating set of J^P.

Observe that the ring R and T in the definition above should have the same number of variables.

Example 2.2.10. (A) Let $I = \langle x_1x_2x_3x_4, x_5x_6\rangle \subseteq \Bbbk[x_1, x_2, x_3, x_4, x_5, x_6]$. The monomial ideals $J_1 = \langle x^2y^2, z^2\rangle \subseteq \Bbbk[x, y, z]$ and $J_2 = \langle x^4, y^2\rangle \subseteq \Bbbk[x, y]$ are two depolarizations of the squarefree monomial ideal I. Observe that

- $J_1^P = \langle a_1a_2b_1b_2, c_1c_2\rangle \subseteq \Bbbk[a_1, a_2, b_1, b_2, c_1, c_2]$ is the polarization of J_1. Furthermore, I and J_1^P are isomorphic via the correspondence $a_1 \mapsto x_1, a_2 \mapsto x_2, b_1 \mapsto x_3, b_2 \mapsto x_4, c_1 \mapsto x_5, c_2 \mapsto x_6$.
- $J_1^P = \langle a_1a_2a_3a_4, b_1b_2\rangle \subseteq \Bbbk[a_1, a_2, a_3, a_4, b_1, b_2]$ is the polarization of J_1. Furthermore, I and J_1^P are isomorphic via the correspondence $a_1 \mapsto x_1, a_2 \mapsto x_2, a_3 \mapsto x_3, a_4 \mapsto x_4, b_1 \mapsto x_5, b_2 \mapsto x_6$.

(B) Consider the squarefree monomial ideal $I = \langle xyz, xyt, yzt, ytu\rangle \subseteq R = \Bbbk[x, y, z, t, u]$. The ideals

$$J_1 = \langle ab^2, a^2b, abc, a^2c\rangle \text{ and } J_2 = \langle ab^2, abc, b^3, b^2c\rangle$$

in $S = \Bbbk[a, b, c]$ are two different depolarizations of I. To check this observe that

- $J_1^P = \langle a_1b_1b_2, a_1a_2b_1, a_1b_1c_1, a_1a_2c_1\rangle \subseteq \Bbbk[a_1, a_2, b_1, b_2, c_1]$ and we have an isomorphism between I and J_1^P via the correspondence $a_1 \mapsto y, a_2 \mapsto x, b_1 \mapsto t, b_2 \mapsto u, c_1 \mapsto z$.
- $J_2^P = \langle a_1b_1b_2, a_1b_1c_1, b_1b_2b_3, b_1b_2c_1\rangle \subseteq \Bbbk[a_1, b_1, b_2, b_3, c_1]$ is isomorphic to I by $a_1 \mapsto x, b_1 \mapsto y, b_2 \mapsto t, b_3 \mapsto u, c_1 \mapsto z$.

□

Remark 2.2.11. As seen in Proposition 2.2.5, depolarization is a combinatorial way to perform identification of variables arosen from a regular sequence of linear forms. A natural question would be whether every such identification of variables can be read as a depolarization of the original ideal. In the following example we show that this is not true in general. Consider the following three ideals from [100], Example 9.5:

$$
\begin{aligned}
M &= \langle x_1^3, x_2^2, x_3^2, x_1^2x_2, x_1^2x_3, x_1x_2x_3 \subseteq \Bbbk[x_1, x_2, x_3] \\
\mathcal{M} &= \langle x_{12}x_{13}x_{14}, x_{21}x_{24}, x_{31}x_{34}, x_{13}x_{14}x_{24}, x_{12}x_{14}x_{34}, x_{14}x_{24}x_{34}\rangle \\
&\subseteq \Bbbk[x_{12}, x_{13}, x_{14}, x_{21}, x_{24}, x_{31}, x_{31}] \\
\mathcal{O} &= \langle x_{12}x_{13}x_{14}, x_{12}x_{24}, x_{13}x_{34}, x_{13}x_{14}x_{24}, x_{12}x_{14}x_{34}, x_{14}x_{24}x_{34}\rangle \\
&\subseteq \Bbbk[x_{12}, x_{13}, x_{14}, x_{24}, x_{34}].
\end{aligned}
$$

Both ideals M and $\mathcal{O}$ can be obtained from $\mathcal{M}$ by identifying a set of variables together. More precisely, in M we relabel every variable x_{ij} with the variable x_i, and in $\mathcal{O}$ we identify the following sets of variables with each other $\{x_{13}, x_{31}\}$ and $\{x_{12}, x_{21}\}$. Lemma 10.4 from [100] implies that these identifications of variables arise from a regular sequence of linear forms. However, we note that neither of them is a depolarization of $\mathcal{M}$.

2.3 The support poset

In Section 2.2 we showed that polarization and depolarization are inverse operations but, although polarization is unique, depolarization is not. Then, some natural questions arise:

- How can one obtain all monomial ideals that have the same polarization?
- Have the monomial ideals sharing the same polarization ideal any property in common?

The support poset is a combinatoric tool which helps us to answer these questions.

2.3.1 The support poset

Let $R = \Bbbk[x_1, \ldots, x_n]$ be a polynomial ring in n variables. For any monomial m of R the *support* of m, denoted by $\operatorname{supp}(m)$, is defined as the set of indices of variables which divide m. The support of a monomial ideal

$I \subseteq R$ is $\text{supp}(I) = \bigcup_{m \in G(I)} \text{supp}(m)$, where $G(I)$ is the unique minimal monomial generating set of I. We say that an ideal I has *full support* if $\text{supp}(I) = \{1, \ldots, n\} = [n]$. For ease of notation we assume that ideals have full support, unless otherwise stated.

Let I be a squarefree monomial ideal with $G(I) = \{m_1, \ldots, m_r\}$. For each i in $\text{supp}(I)$ we define the set $C_i \subseteq \text{supp}(I)$ as,

$$C_i = \bigcap_{\substack{m \in G(I) \\ x_i \text{ divides } m}} \text{supp}(m).$$

In other words, C_i is given by the indices of all the variables that appear in every minimal generator of I in which x_i is present. Let $C_I = \{C_1, \ldots, C_n\}$. The poset on the elements of C_I ordered by inclusion is called the *support poset* of I and is denoted $\text{suppPos}(I)$. We define the *support poset* of a general monomial ideal as the support poset of its polarization obtained from Definition 2.2.1.

Given n subsets C_i of $[n]$, we form the poset $(\mathcal{C} = \{C_1, \ldots, C_n\}, \subseteq)$ on elements C_i which are ordered by inclusion. Note that some C_i can possibly be equal to C_j for $i \neq j$. A natural question is whether for such $(\mathcal{C}, \subseteq)$ we can construct a monomial ideal $I_{\mathcal{C}}$ whose support poset is $(\mathcal{C}, \subseteq)$. This question is not easy in general. See Example 2.3.2 (2). In the following proposition we provide a sufficient condition to construct such ideals. Another sufficient condition will be given in Proposition 2.3.14.

Proposition 2.3.1. *Let $(\mathcal{C} = \{C_1, \ldots, C_n\}, \subseteq)$ be a poset such that $\{i\} \subseteq C_i \subseteq [n]$ for each i, and if $k \in C_i$ and $i \in C_j$ then $k \in C_j$ for all i, j, k. Let $R = \Bbbk[x_1, \ldots, x_n]$ and let $m_i = \prod_{j \in C_i} x_j$ for each i. For any $\sigma \subseteq [n]$ let $m_\sigma = \text{lcm}(m_i | i \in \sigma)$, and for any collection Σ of subsets of $[n]$, consider the monomial ideal $I_\Sigma = \langle m_\sigma | \sigma \in \Sigma \rangle$. Then $(\mathcal{C}, \subseteq)$ is the support poset of I_Σ if the following properties hold:*

1. *$\forall i \in [n]$ there is some $\sigma \in \Sigma$ such that $x_i | m_\sigma$.*
2. *If $\{\sigma : x_i | m_\sigma\} \subseteq \{\sigma : x_j | m_\sigma\}$, then $C_j \subseteq C_i$.*

Proof. Let $(\mathcal{D} = \{D_1, \ldots, D_n\}, \subseteq)$ be the support poset of the ideal I_Σ. We want to show that $D_j = C_j$ for all $j \in \{1, \ldots, n\}$.

We have that $D_j = \bigcap_{\substack{x_j|m_\sigma \\ \sigma\in\Sigma}} \operatorname{supp}(m_\sigma)$. If x_j divides m_σ, then there is some $\ell \in \sigma$ such that x_j divides m_ℓ. This implies that m_j divides m_σ for all σ with $x_j|m_\sigma$. Hence, $\operatorname{supp}(m_j) \subset \bigcap_{\substack{x_j|m_\sigma \\ \sigma\in\Sigma}} \operatorname{supp}(m_\sigma)$ which means $C_j \subseteq D_j$.

On the other hand, $k \in D_j$ implies that x_k divides all m_σ, where $x_j|m_\sigma$. This together with condition (2) imply that $C_k \subseteq C_j$ and $k \in C_j$ which means $D_j \subseteq C_j$. □

In the following example, we show that some posets might not appear as support poset of any ideal, and on the other hand, several ideals might have the same support poset.

Example 2.3.2. 1. Let $C_1 = \{1,2\}$, $C_2 = \{2\}$, $C_3 = \{3\}, C_4 = \{4\}$ and $C_5 = \{4,5\}$. Let

$$\begin{aligned}\Sigma_1 &= \{\{1\},\{2,4\},\{3\},\{5\}\},\\ \Sigma_2 &= \{\{1\},\{2,3\},\{3,4\},\{5\}\},\\ \Sigma_3 &= \{\{1,3\},\{3,5\},\{1,4\},\{2,5\}\}.\end{aligned}$$

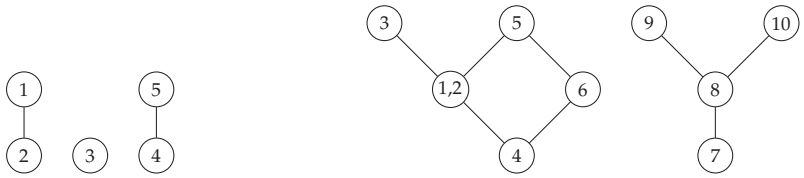
These three collections satisfy the conditions in Proposition 2.3.1 and hence $(\mathcal{C} = \{C_1,\dots,C_5\},\subseteq)$ is the support poset of the ideals $I_{\Sigma_1} = \langle x_1x_2, x_2x_4, x_3, x_4x_5\rangle$, $I_{\Sigma_2} = \langle x_1x_2, x_2x_3, x_3x_4, x_4x_5\rangle$ and $I_{\Sigma_3} = \langle x_1x_2x_3, x_3x_4x_5, x_1x_2x_4, x_2x_4x_5\rangle$.

2. Let $\mathcal{C}$ be given by $C_1 = \{1\}$, $C_2 = \{1,2\}$ and $C_3 = \{1,2,3\}$, then there is no monomial ideal $I \subseteq R[x_1,x_2,x_3]$ such that $(\mathcal{C},\subseteq)$ is the support poset of I. To see this, observe that $x_1x_2x_3$ must be one of the minimal generators of I, hence the only one, but $\mathcal{C}$ is not the support poset of $I = \langle x_1x_2x_3\rangle$.

3. Let $C_1 = \{1,2,4\}$, $C_2 = \{1,2,4\}$, $C_3 = \{1,2,3,4\}$, $C_4 = \{4\}$, $C_5 = \{1,2,4,5,6\}$, $C_6 = \{4,6\}$, $C_7 = \{7\}$, $C_8 = \{7,8\}$, $C_9 = \{7,8,9\}$, $C_{10} = \{7,8,10\}$. Then for $\Sigma = \{\{3\},\{6,7\},\{5\},\{9\},\{10\}\}$, the ideal $I_\Sigma = \langle x_1x_2x_3x_4, x_4x_6x_7, x_1x_2x_4x_5x_6, x_7x_8x_9, x_7x_8x_{10}\rangle \subseteq \Bbbk[x_1,\dots,x_{10}]$. has $(\mathcal{C} = \{C_1,\dots,C_{10}\},\subseteq)$ as its support poset.

□

Remark 2.3.3. Note that in any support poset, $k \in C_i$ and $i \in C_j$ imply that $k \in C_j$. We can use this fact to visualize support posets using their Hasse diagrams, where each node is labelled by their elements which are not in any of the nodes below it.



(a) Support poset for Example 2.3.2 (1) (b) Support poset for Example 2.3.2 (3)

The support poset of any monomial ideal $I \subseteq R = \Bbbk[x_1, \ldots, x_n]$, together with a given ordering $<$ on the variables $x_1, \ldots, x_n$ induces a partial order $\prec$ on the set of variables as follows: $x_i \prec x_j$ if $C_i \subset C_j$ or if $C_i = C_j$ and $x_i < x_j$. We call this partial order the *$<$-support poset* of I and denote it by $\text{suppPos}_<(I)$. If $C_i \neq C_j$ for every pair of indices, then $\text{suppPos}(I)$ is equal to the $<$-support poset of I for any order $<$. See Figure 1 (A) for Example 2.3.2 (1).

Note that, the Hasse diagram of $\text{suppPos}_<(I)$ can be obtained from the Hasse diagram of $\text{suppPos}(I)$ in which every node C labelled with more than one index is substituted by a vertical line of nodes labelled by distinct elements of C, ordered by $<$. In other words, every $<$-support poset of I is a refinement of $\text{suppPos}(I)$. See Figure 2.2 as a $<$-support poset of I_Σ in Example 2.3.2 (3) for any order on the variables which is compatible with $x_1 < x_2$.

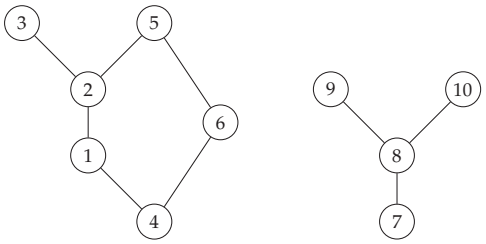


Figure 2.2: $\text{suppPos}_<(I)$ for Example 2.3.2 (3) for any order with $x_1 < x_2$

2.3.2 Support Poset applied to depolarization

2.3.2.1 Depolarization orders

Recall that a subset C of a poset $(\mathcal{P}, \prec)$ is a *chain* if any two elements of C are comparable. An antichain is a set of pairwise incomparable elements in $(\mathcal{P}, \prec)$.

Definition 2.3.4. Given an order $<$ on the variables of R, a *depolarization order* of a squarefree monomial ideal $I \subseteq R$ is a partition of $\operatorname{suppPos}_<(I)$ into disjoint chains.

We now show that depolarization orders characterize all depolarizations of I. Namely, every depolarization order gives rise to a depolarization of I, and every depolarization of I can be realized as a depolarization obtained by such an order.

Proposition 2.3.5. *Using any depolarization order of a squarefree monomial ideal I, we can construct a depolarization of I.*

Proof. Let $(\mathcal{P}, \prec)$ be a depolarization order for a squarefree monomial ideal $I \subseteq R = \Bbbk[x_1, \ldots, x_n]$, where $\mathcal{P} = \{\sigma_1, \ldots, \sigma_k\}$ and each σ_i is a chain in $\operatorname{suppPos}_<(I)$ for a given order $<$ on the variables of R. We construct a depolarization J of I in a polynomial ring $S = \Bbbk[y_1, \ldots, y_k]$ as follows: for each monomial m in $G(I)$ consider the monomial m' given by the image of m under the correspondence $x_i \mapsto y_j$ for each $i \in \sigma_j$. The monomials m' generate an ideal J whose polarization J^P is clearly equivalent to I via the map sending $y_{j,\ell} \mapsto x_{\sigma_{j_\ell}}$ where σ_{j_ℓ} is the ℓ-th element of σ_j under the order $\prec$. □

Example 2.3.6. The partition given by $\mathcal{P} = \{\{4,2,1,3\}, \{6,5\}, \{7,8,9\}, \{10\}\}$ is a depolarization order for the ideal I in Example 2.3.2 (3) for any ordering in which $x_2 < x_1$. Figure 2.4 shows this partition.

The depolarization order $(\mathcal{P}, \prec)$ depicted in Figure 2.4 gives the depolarization $J = \langle y_1^4, y_1y_2y_3, y_1^3y_2^2, y_3^3, y_3^2y_4 \rangle \subseteq \Bbbk[y_1, y_2, y_3, y_4]$ of I. The equivalence between J^P and I is given by $y_{1,1} \mapsto x_4$, $y_{1,2} \mapsto x_2$, $y_{1,3} \mapsto x_1$, $y_{1,4} \mapsto x_3$, $y_{2,1} \mapsto x_6$, $y_{2,2} \mapsto x_5$, $y_{3,1} \mapsto x_7$, $y_{3,2} \mapsto x_8$, $y_{3,3} \mapsto x_9$, $y_{4,1} \mapsto x_{10}$.

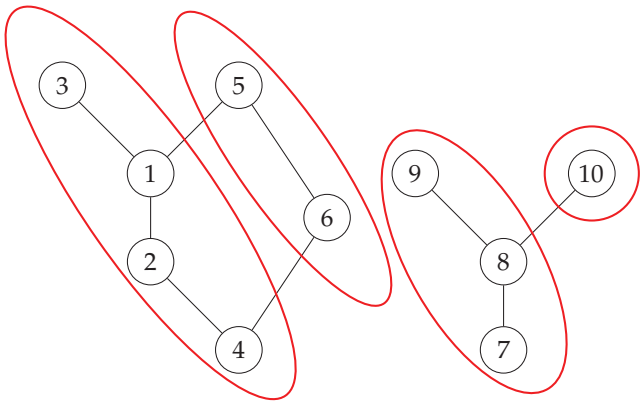


Figure 2.3: A chain partition of $\text{suppPos}_{<}(I)$ in Example 2.3.2 (3) gives a depolarization order $(\mathcal{P}, \prec)$ for I.

□

Remark 2.3.7. Observe that the order selected in the support poset for variables x_1, x_2 in Example 2.3.6 is not relevant for depolarizing the ideal, *i.e.* if we choose the order $x_1 > x_2$ the depolarizations obtained are the same. A node labeled with indexes i, j (or more than one) is a consequence of $C_i = C_j$ which means that indexes i and j appears exactly in the same monomials.

We have just seen that every depolarization order of a squarefree monomial ideal I gives a depolarization of I. Now, we study the reverse of this process and we show that given any depolarization J of I we can explicitly find a depolarization order from which we can reconstruct J.

Theorem 2.3.8. *Let $I = \langle m_1, \ldots, m_r \rangle \subseteq R = \Bbbk[x_1, \ldots, x_n]$ be a squarefree monomial ideal. Every depolarization of I can be obtained from a depolarization order of I.*

Proof. Let $J \subseteq S = \Bbbk[y_1, \ldots, y_k]$ be a depolarization of the ideal I and let $J^P \subseteq T = \Bbbk[y_{1,1}, \ldots, y_{1,j_1}, \ldots, y_{k,1}, \ldots, y_{k,j_k}]$ be the polarization of J. Since J is a depolarization of I, we know that R and T have the same number of variables and that I and J^P are equivalent under a map sending x_i to $y_{a,b}$

for some $a \in \{1, \ldots, k\}$ and $b \in \{1, \ldots, j_a\}$. Now consider in $\{1, \ldots, n\}$ the partition $\mathcal{P}$ with k subsets in which σ_i contains all j such that x_j corresponds to some $y_{i,b}$ with the total order given by $j < j'$ if $b < b'$, where $y_{i,b} \mapsto x_j$ and $y_{i,b'} \mapsto x_{j'}$. Then $(\mathcal{P}, <)$ is a depolarization order for I that produces the depolarization J. □

Example 2.3.9. Consider the depolarization $J = \langle ab^2, a^2b, abc, a^2c \rangle$ of the ideal $I = \langle xyz, xyt, yzt, ytu \rangle$ in Example 2.2.10 (B). We have that $J^P \subseteq \mathbb{k}[a_1, a_2, b_1, b_2, c_1]$ is equivalent to $I \subseteq \mathbb{k}[x, y, z, t, u]$ through the correspondence $a_1 \mapsto y$, $a_2 \mapsto x$, $b_1 \mapsto t$, $b_2 \mapsto u$, $c_1 \mapsto z$. The corresponding depolarization order is $\mathcal{P} = \{\{y, x\}, \{t, u\}, \{z\}\}$ where the elements in the sets are given in increasing order.

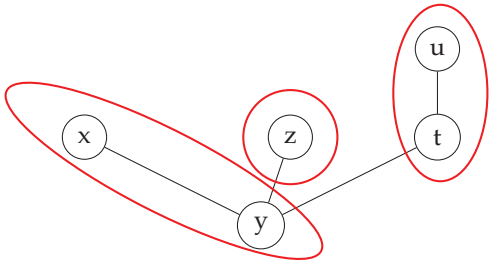


Figure 2.4: The chain partition proposed associated to the depolarization order given in Example 2.3.9.

□

2.3.2.2 Depolarization posets

Let P and P' be two chain partitions of a given poset. We say that P is a *refinement* of P' if for every chain C in P there is a chain C' in P' such that $C' \subseteq C$. The set of all chain partitions of a given poset are sorted by refinement and using this ordering they form themselves a poset. Let I be a squarefree monomial ideal and let J, J' be two depolarizations of I. We say that $J \leq J'$ if the chain partition giving rise to J is a refinement of the one corresponding to J'. Using this ordering, a collection of ideals that are depolarizations of a given squarefree monomial ideal I forms a poset

in which I is the unique minimal element. We call this the *depolarization poset* of I, denoted $\mathcal{DP}(I)$. Given any monomial ideal J (not necessarily squarefree), we define its *depolarization poset* to be the depolarization poset of its polarization J^P. In other words, $\mathcal{DP}(J) := \mathcal{DP}(J^P)$.

Every depolarization poset has a unique minimal element which is a squarefree monomial ideal, hence $\mathcal{DP}(J)$ is a meet-semilattice for every monomial ideal J, that is for every pair K and K' in $\mathcal{DP}(J)$ there is an element in $\mathcal{DP}(J)$, denoted by $K \wedge K'$, which is smaller than both of them. On the other hand, $\mathcal{DP}(J)$ might have several maximal elements and therefore it is not a lattice in general. We say that an ideal $J \subseteq \Bbbk[x_1, \ldots, x_n]$ is a *maximum* element in its depolarization poset if there is no other ideal $J' \subseteq \Bbbk[x_1, \ldots, x_m]$ in $\mathcal{DP}(J)$ such that $m < n$. That means the ambient ring of J has the minimal number of variables among the ambient rings of all ideals in $\mathcal{DP}(J)$.

2.3.2.3 Copolar ideals

Here, we study which algebraic invariants, such as their Betti numbers, are preserved with the operations polarization and depolarization. A motivation for studying such family of ideals is to find some particular ideal in the family that can provide information about the rest of the ideals in the family.

Definition 2.3.10. Two monomial ideals I and J are called *copolar* if their polarizations are equivalent, i.e., they are in the same depolarization poset.

Copolarity is an equivalence relation in the set of monomial ideals. We say that a property or numerical invariant of an ideal is *copolar* if it is shared by all ideals in the same polarity class. The following proposition gives a list of copolar properties. For more details, we refer the reader to [62], Corollary 1.6.3.

Proposition 2.3.11. *Let $I \subseteq S$ be a monomial ideal and let $J \subseteq T$ be its polarization. Then*

(1) $\beta_{i,j}(I) = \beta_{i,j}(J)$ for all i and j

(2) $HS_I(t) = (1-t)^{\delta} HS_J(t)$ *where* $\delta = \dim T - \dim S$

(3) $\text{height}(I) = \text{height}(J)$

(4) $\text{projdim}(S/I) = \text{projdim}(T/J)$ *and* $\text{reg}(S/I) = \text{reg}(T/J)$

(5) S/I *is Cohen-Macaulay if and only if* T/J *is Cohen-Macaulay.*

A reason behind the items in Proposition 2.3.11 is that the *lcm-lattice* [139, 94] of I, which is the lattice of all least common multiples of subsets of the (minimal) generators of I, ordered by divisibility, and is denoted by $\text{lcm}(I)$, is isomorphic to the *lcm-lattice* of J under the map taking $\text{lcm}(m, m')$ to $\text{lcm}(\overline{m}, \overline{m'})$ for every pair of monomials in $G(I)$.

Lemma 2.3.12. *Let* I *and* J *be two copolar ideals. Then* $\text{lcm}(I) \cong \text{lcm}(J)$.

The lcm-lattice of a monomial ideal encodes the structure of its minimal free resolution and thus its Betti numbers [139]. In fact polarization is a particular tool to generate ideals with isomorphic lcm-lattice. Some other important invariants are also fixed under polarization. For example, in [71] where the authors proved that the Stanley conjecture can be reduced to the squarefree case via polarization, and that the Stanley projective dimension is invariant under polarization (in particular, two ideals with isomorphic lcm-lattice have the same Stanley projective dimension). One recent remarkable result using polarization is given in [91], where the authors used polarization and combinatorial optimization to study the depth and regularity of powers of edge ideals on graphs and clutters.

We first take advantage of the fact that the number of variables of the ambient ring is not constant within the same polarity class, but the projective dimension is. Therefore, for any monomial ideal we can construct its depolarization poset and find the maximum elements whose ambient rings has the minimum number of variables. Since the number of variables of a polynomial ring is an upper bound for the projective dimension of its ideals, this procedure provides us with an upper bound for the projective dimension of the ideals in terms of their depolarization posets. Recall that the *width* of a poset is the maximum size of its antichains.

Theorem 2.3.13. *The width of* $\text{suppPos}(I^P)$ *is an upper bound for* $\text{projdim}(I)$.

Proof. The projective dimension of I is equal to the projective dimension of its polarization I^P which is in turn the same as that of any of its depolarizations. Let J be a depolarization of I^P whose ambient ring has the smallest number of variables, say r. By Hilbert Syzygy Theorem we know that $\text{projdim}(J) \leq r$. By Theorem 2.3.8 we know that r is given by the minimal number of chains in which we can partition the support poset of I^P (observe that this number is the same for any $\text{suppPos}_<(I)$ and $\text{suppPos}(I)$). By Dilworth's Theorem [34], this number is smaller than the size of the maximal antichain of the support poset of I^P which is the width of $\text{suppPos}(I^P)$. □

An interesting question, relegated to future work (see Chapter 5) because it is out of the scope of this thesis, is to compare this bound with other bounds for the projective dimension of monomial ideals, like the ones in [32, 33].

2.3.2.4 Quasi-stable ideals

We usually study depolarization posets to find an ideal with a particularly nice property that can be transferred to its copolar ideals. For instance, here we study *quasi-stable* ideals [123] to compute the algebraic invariants of their copolar ideals.

Quasi-stable ideals are also called ideals of *nested type* [20] or ideals of *Borel type* [65]. A monomial ideal $I \subseteq \Bbbk[x_1, \ldots, x_n]$ is of nested type if each of its associated prime ideals is of form $\mathfrak{p} = (x_1, \ldots, x_i)$ for some i. The equivalence between these families of ideals is not immediate and it has been proven by Seiler in [123, Proposition 4.4].

The invariants of such ideals have been extensively studied in [20, 123], and it is shown that their Castelnuovo-Mumford regularity and their projective dimension can be obtained in terms of their irreducible decompositions or in terms of their Pommaret bases. Moreover, a minimal free resolution of these ideals is explicitly computed in [123, Theorem 8.6].

Therefore, if a polarity class contains a quasi-stable ideal, then we can use the aforementioned results to compute the Castelnuovo-Mumford regularity and the projective dimension of each ideal in such class. Since

zero-dimensional ideals are quasi-stable we can make use of these considerations in the polarity classes that contain at least one zero-dimensional ideal.

In the same spirit as Proposition 2.3.1 we provide a sufficient condition for a poset to be a support poset of a zero-dimensional monomial ideal.

Proposition 2.3.14. *Let $n, m_1, \ldots, m_n$ be some positive integers with $1 \leq m_i \leq n$ for all i and let $m = \sum_i m_i$. Consider a poset $(\mathcal{P}, \subseteq)$ on subsets of $\{1, \ldots, m\}$ formed by n disjoint paths each of length m_i. Then there is a squarefree monomial ideal I whose support poset is $\mathcal{P}$ except if $n = 2$ and $m_1 \neq m_2$. Moreover, if $m_i > 1$ for all i, then there is a zero-dimensional monomial ideal copolar to I.*

Proof. Let $\mathcal{P} = A_1 \sqcup \cdots \sqcup A_n$ where

$$A_i = \{\{a_{i,1}\}, \{a_{i,1}, a_{i,2}\}, \ldots, \{a_{i,1}, \ldots, a_{i,m_i}\}\}.$$

We assume that $n > 2$ or $m_1 = m_2$. The remaining case is studied in Example 2.3.15.

For ease of notation, we identify each variable $x_{a_{i,j}}$ with its subindex $a_{i,j}$. We can assume without loss of generality that $m_1 \geq \cdots \geq m_n$.

We first construct a monomial ideal generated by the following sets of monomials:

1. G_1 consists of the monomials $\mu_i = a_{i,1} \cdots a_{i,m_i}$ for all i with $m_i > 1$.

2. G_2 consists of the monomials $\mu_{i,j} = a_{i,1} \cdots a_{i,j} b_{i,j}$ for all i, j with $1 < j < m_i$. Here, $b_{i,j} = a_{\lceil i+j-1 \rceil, 1}$ where $\lceil i + j - 1 \rceil$ denotes $i + j - 1$ modulo n. Note that the indices $b_{i,j}$ are pairwise distinct for each i, as $m_i \leq n$.

3. $G_3 = \{a_{i,1} a_{i',1} : \ a_{i,1}, a_{i',1} \in G_3' \text{ and } a_{i,1} a_{i',1} \nmid m \text{ for any } m \in G_1 \cup G_2\}$, where G_3' consists of all indices $a_{i,1}$ for $m_i = 1$ that appeared at most once as $b_{j,k}$ in G_2, and indices $a_{i,1}$ for $m_i > 1$ that never appeared as $b_{j,k}$ in G_2.

We now prove that the support poset of the ideal $I_G \subseteq \Bbbk[x_{a_{1,1}}, \ldots, x_{a_{n,mn}}]$ generated by the above sets of monomials is $(\mathcal{P}, \subseteq)$.

To see this, first observe that, by construction, the monomials in $G = G_1 \cup G_2 \cup G_3$ do not divide each other. Now we show that for each pair of variables $a_{i,j}$, $a_{i',j'}$ with $i \neq i'$ there is at least one monomial in G which only contains one of these variables. If $j > 1$ or $j' > 1$, then it is easy to find such a monomial in G_1. Now assume that $j = j' = 1$. If $m_i > 1$ or $m_{i'} > 1$, then such a monomial can be found in G_1. Otherwise, they appear in separate monomials in G_2 or G_3.

Now, we show that $C_{a_{i,j}} = \{a_{i,1}, \ldots, a_{i,j}\}$ for every variable a_{ij}. First note that every variable $a_{i,1}$ appears at least once in G_2 or G_3 without the rest of the variables $a_{i,j}$ for $j > 1$. Thus $C_{a_{i,1}} = \{a_{i,1}\}$. For $m_i > 1$, a_{i,m_i} appears only in the monomial $\mu_i = a_{i,1} \cdots a_{i,m_i}$ in G_1, hence $C_{a_{i,m_i}} = \{a_{i,1}, \ldots, a_{i,m_i}\}$. Now assume that $1 < j < m_i$. The variable $a_{i,j}$ appears always together with all the variables $a_{i,j'}$ for $j' < j$ since they all divide the monomials μ_i in G_1 and $\mu_{i,j} \in G_2$. On the other hand, if $m_i > \ell > j$, then by the construction of G_2, there is at last one monomial $\mu_{i,j}$ in which $a_{i,\ell}$ is not present. Hence, $C_{a_{i,j}} = \{a_{i,1}, \ldots, a_{i,j}\}$ which completes the proof.

Moreover, if $m_i > 1$ for each i, then using the chain partition given by the disjoint paths themselves, the corresponding depolarization of I has one variable for each i whose pure power appears in G_1, which implies that I is zero-dimensional. □

Example 2.3.15. Let $n = 2$, $m_1 = 2$ and $m_2 = 1$. Then the minimal generating set of any monomial ideal I with the support poset $\mathcal{P} = \{\{1\}, \{1,2\}, \{3\}\}$ must include a monomial divisible by x_1x_2. Therefore, the only candidates for such an ideal are $\langle x_1x_2x_3 \rangle$, $\langle x_1x_2, x_3 \rangle$, $\langle x_1x_2, x_1x_3 \rangle$, $\langle x_1x_2, x_2x_3 \rangle$ and $\langle x_1x_2, x_1x_3, x_2x_3 \rangle$. However, none of them has $\mathcal{P}$ as it support poset.

□

Example 2.3.16. Consider the poset $(\mathcal{P}, \subseteq)$ on the following subsets of $\{1, \ldots, 14\}$. Let

$$A_1 = \{\{1\}, \ldots, \{1,2,3,4,5,6\}\},\ A_2 = \{\{7\}, \ldots, \{7,8,9,10\}\},$$
$$A_3 = \{\{11\}\},\ A_4 = \{\{12\}\},\ A_5 = \{\{13\}\},\ A_6 = \{\{14\}\}.$$

Then following the notation of the proof of Proposition 2.3.14 we have that $(\mathcal{P}, \subseteq)$ is the support poset of the ideal generated by the monomials in $G_1 \cup G_2 \cup G_3$, where

$$\begin{aligned} G_1 =& \{x_1x_2x_3x_4x_5x_6, x_7x_8x_9x_{10}\} \\ G_2 =& \{x_1x_2x_7, x_1x_2x_3x_{11}, x_1x_2x_3x_4x_{12}, x_1x_2x_3x_4x_5x_{13}, x_7x_8x_{11}, x_7x_8x_9x_{12}\} \\ G_3 =& \{x_1x_{14}, x_{13}x_{14}\} \text{ and } G_3' = \{x_1, x_{13}, x_{14}\}. \end{aligned}$$

□

Example 2.3.17. Consider the following monomial ideal in 9 variables:

$$I = \langle x_1x_2x_3x_4, x_5x_6x_7, x_8x_9, x_1x_2x_3x_5, x_1x_2x_8, x_5x_6x_8, x_1x_5x_8\rangle.$$

One can check that $\mathfrak{p}_1 = \langle 1, 5, 8\rangle$ and $\mathfrak{p}_2 = \langle 2, 7, 8\rangle$ are associated primes of I and they are not nested. Then, I is not quasi-stable. However, by determining the depolarization poset of I we found that the ideal,

$$J = \langle y_1^4, y_2^3, y_3^2, y_1^3y_2, y_1^2y_3, y_2^2y_3, y_1y_2y_3\rangle \subseteq \mathbb{k}[y_1, y_2, y_3]$$

is one of its maximum depolarizations. As J is a zero-dimensional ideal, and hence quasi-stable, by applying Theorems 8.11 and 9.2 in [123] we obtain $\mathrm{pd}(J) = 2$ and $\mathrm{reg}(J) = 5$, thus obtaining these invariants for I.

□

2.3.3 Support poset of some monomial ideals

We now know that the support poset of a monomial ideal $I \subseteq \mathbb{k}[x_1, \ldots, x_n]$ encodes the relation between the variables $x_1, \ldots, x_n$ and the minimal monomial generators of I.

As it is shown in previous sections not every poset is realizable as the support poset of a monomial ideal. A natural problem is therefore *to find posets that can be realized as support posets of monomial ideals and provide explicit descriptions of those ideals*, so that we can describe properties of the ideal based on properties of the support poset and viceversa. In Proposition 2.3.14 we saw that a poset formed by n disjoint chains of

different lengths are always a support poset of a squarefree monomial ideal I. In fact, there if the subsets have length greater than 1, the exist a zero-dimensional monomial ideal copolar to I. In Section 2.3.3.1 we go further in the characterization of posets that are realizables as a support poset of monomial ideals: we give some families of posets for which we can always find at least one monomial ideal supported by them (collections of lines or diamonds, and forests) and provide a full explicit description of the main features of these ideals such as their Betti numbers and free resolutions, see Propositions 2.3.21 and 2.3.24 and in particular Theorem 2.3.30.

Another natural question related to support posets is *to find a natural way to describe the support poset of some families of monomial ideals*. We describe in Section 2.3.3.2 the support poset of k-out-of-n and series-parallel ideals, studied in Chapter 3, which correspond to relevant systems in reliability theory [75, 117, 118]. We find a particular relation between forests and series-parallel ideals, see Theorem 2.3.35 and Proposition 2.3.37. It is known that a given poset can be the support poset of several different monomial ideals. We see that this holds even within the classes of forests and series-parallel ideals, i.e. a given forest can be the support poset of several different series-parallel ideals.

Some open questions related to support posets can be found in Chapter 5.

2.3.3.1 Ideals with a given support poset

In this section we give some posets that are realizables as a support poset of some monomial ideals.

Proposition 2.3.18. *Let n and m be two positive integers and let $(\mathcal{P}, \subseteq)$ be a poset of subsets of the set $[nm] = \{1, \ldots, nm\}$ formed by n disjoint lines each of length m. Then there is at least one squarefree monomial ideal $I_{n,m}$ such that $\mathcal{P}$ is its support poset and there is a zero-dimensional monomial ideal $J_{n,m}$ copolar to $I_{n,m}$.*

In particular, the ideal $J_{n,m} \subseteq \mathbb{k}[y_1, \ldots, y_n]$ given by

$$J_{n,m} = \langle y_1^m, \ldots, y_n^m, y_1^{m-1}y_2, \ldots, y_1y_2^{m-1}, \ldots, y_1^{m-1}y_n, \ldots, y_1y_n^{m-1} \rangle.$$

is a zero dimensional ideal having $\mathcal{P}$ as its support poset.

Proof. We describe the construction of I stepwise as n increases. The base case is $n = 2$. Let $\mathcal{P} = A_1 \sqcup A_2$ where $A_1 = \{1, \ldots, m\}$ and $A_2 = \{m+1, \ldots, 2m\}$. The ideal $I_{2,m} \subset \mathbb{k}[x_1, \ldots, x_{2m}]$ generated by the monomials

$$x_1 \cdots x_m, x_{m+1} \cdots x_{2m}, x_1 \cdots x_{m-1} x_{m+1}, \\ x_1 \cdots x_{m-2} x_{m+1} x_{m+2}, \ldots, x_1 x_{m+1} \cdots x_{2m-1}$$

satisfies that $\operatorname{suppPos}(I_{2,m}) = \mathcal{P}$.

To see this, let us consider first the indices in A_1. Observe that x_m appears only in the first generator, $x_1 \ldots x_m$, hence $C_m = \{1, \ldots, m\}$. If $1 \leq j < m$ then every generator that contains x_j also contains $x_1 \ldots, x_{j-1}$ and if $j < k \leq m$ then there is at least one generator which contains x_j but not x_k, for instance $x_1 \cdots x_j x_{m+1} \cdots x_{2m-j}$. Finally, if $k \geq m+1$ we have that x_k is not present in $x_1 \cdots x_m$ in which x_j is, hence $C_j = \{1, \ldots, j\}$. By simmetry, the same applies to the generators in A_2.

Considering in $\mathcal{P}$ the chain partition given by the A_i's we have that the corresponding depolarization is $J_{2,m} \subset \mathbb{k}[y_1, y_2]$ given by

$$J_{2,m} = \langle y_1^m, y_2^m, y_1^{m-1} y_2, \ldots, y_1 y_2^{m-1} \rangle$$

which is zero-dimensional and $J_{2,m}^{\mathcal{P}} = I_{2,m}$. Let now $n = 3$. Then $I_{3,m} \subseteq \mathbb{k}[x_1, \ldots x_{3m}]$ is given by the same set of generators of $I_{2,m}$ plus the following ones

$$\{x_{2m+1} \cdots x_{3m}, x_1 \cdots x_{m-1} x_{2m+1}, \\ x_1 \cdots x_{m-2} x_{2m+1} x_{2m+2}, \ldots, x_1 x_{m+1} \cdots x_{3m-1}\}.$$

Using the same argument as for $n = 2$ we have that $\operatorname{suppPos}(I_{3,m}) = A_1 \sqcup A_2 \sqcup A_3$. The ideal $J_{3,m} \subseteq [y_1, y_2, y_3]$ is given by

$$J_3 = \langle y_1^m, y_2^m, y_3^m, y_1^{m-1} y_2, \ldots, y_1 y_2^{m-1}, y_1^{m-1} y_3, \ldots, y_1 y_3^{m-1} \rangle.$$

Now, proceeding in the same way adding at each step the new generators

$$x_{(n-1)m+1} \cdots x_{nm}, x_1 \cdots x_{m-1} x_{(n-1)m+1}, \ldots, x_1 x_{(n-1)m+1} \cdots x_{nm-1}$$

we obtain the ideal $I_{n,m}$ whose support poset is formed by a disjoint set of n paths of size m.

The ideal $J_{n,m} \subseteq \mathbb{k}[y_1, \ldots, y_n]$ is given by

$$J_{n,m} = \langle y_1^m, \ldots, y_n^m, y_1^{m-1}y_2, \ldots, y_1y_2^{m-1}, \ldots, y_1^{m-1}y_n, \ldots, y_1y_n^{m-1} \rangle.$$

Observe that $J_{n,m}^P = I_{n,m}$ and $J_{n,m}$ is zero-dimensional for all n, since it contains a pure power of each of the variables. □

Remark 2.3.19. If we take the same value for all m_i, $i = 1, \ldots, n$ in Proposition 2.3.14, we obtain the particular case presented on Proposition 2.3.18. The contribution of Proposition 2.3.18 is the explicit computation of the zero-dimensional ideal when all the subsets have the same length.

Example 2.3.20. Let us take $n = 2$ and $m = 3$. Then, we have the support poset showed on Figure 2.5

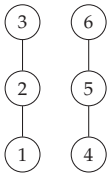


Figure 2.5: Support poset for Example 2.3.20

We can compute the monomial ideal associated to this poset by using both Proposition 2.3.14 and .2.3.18.

Using Proposition 2.3.14 (and following the constructive proof) we have that

$$A_1 = \{\{1\}, \{1,2\}, \{1,2,3\}\} \text{ and } A_2 = \{\{4\}, \{4,5\}, \{4,5,6\}\}.$$

Now, we can construct the G sets:

$$G_1 = \{x_1x_2x_3, x_4x_5x_6\} \text{ and } G_2 = \{x_1x_2x_4, x_1x_3x_4\}.$$

We have that

$$I_G = G_1 \cup G_2 = \langle x_1x_2x_3, x_4x_5x_6, x_1x_2x_4, x_1x_3x_4 \rangle$$

is a monomial ideal with the support poset given on Figure 2.5.

By using Proposition 2.3.18 we obtain that

$$J_{2,3} = y_1^3, y_2^3, y_1^2 y_2, y_1 y_2^3.$$

Polarizing we obtain

$$J_{n,m}^P = \left\{ y_{11}y_{12}y_{13}, y_{21}y_{22}y_{23}, y_{11}y_{12}y_{21}, y_{11}y_{21}y_{22} \right\},$$

which is isomorphic to I_G via the correspondence $x_1 \mapsto y_{11} x_2 \mapsto y_{12}, x_3 \mapsto y_{13}, x_4 \mapsto y_{21}, x_5 \mapsto y_{22}, x_6 \mapsto y_{23}$.

□

Observe that the ideal constructed in Proposition 2.3.18 can be obtained by taking the following collection σ in $(\mathcal{P}, \subseteq)$

$$\Sigma = \left(\bigcup_{i=1}^{n} \{im\} \right) \bigcup \left(\bigcup_{i=1}^{n-1} \bigcup_{j=1}^{m-1} \{m-j, im+j\} \right),$$

which satisfies Proposition 2.3.1. The ideals $J_{n,m}$ are generated in degree m and their Betti numbers are computed by the following result.

Proposition 2.3.21. *The Betti numbers of the ideal*

$$J_{n,m} = \langle y_1^m, \ldots, y_n^m, y_1^{m-1} y_2, \ldots, y_1 y_2^{m-1}, \ldots, y_1^{m-1} y_n, \ldots, y_1 y_n^{m-1} \rangle$$

are given by

$$\beta_0(J_{n,m}) = n + (n-1)(m-1)$$

$$\beta_i(J_{n,m}) = \binom{n-1}{i} + \sum_{j=2}^{n} (m-1) \binom{1+n-j}{i} + \binom{n-1}{i+1} \; \forall\, 1 \leq i \leq n-1$$

In particular, $\operatorname{projdim}(J_{n,m}) = n-1$ *and* $\operatorname{reg}(J_{n,m}) = (n-1)(m-1)$. *The minimal free resolution of* $J_{n,m}$ *can be obtained as an iterated mapping cone.*

Proof. We shall use Mayer-Vietoris trees. First we sort the generators of $J_{n,m}$ in the following way:

$$y_1^m, \ldots, y_n^m,$$
$$y_1 y_2^{m-1}, \ldots, y_1 y_n^{m-1},$$
$$\ldots$$
$$y_1^{m-1} y_2, \ldots, y_1^{m-1} y_n.$$

Now, we use them in turn to construct a Mayer-Vietoris tree of $J_{n,m}$ i.e. an iterated mapping cone resolution. Let us denote this resolution by $\mathbb{F}$ and let $\gamma_i(J_{n,m})$ denote the rank of the i'th module of $\mathbb{F}$. We proceed row by row with the pivots.

The first pivot, y_1^m produces the ideal $\langle y_1^m \rangle \cap \langle y_2^m, \ldots, y_1^{m-1} y_n \rangle$, minimally generated by $\langle y_1^m y_2, \ldots, y_1^m y_n \rangle$. Observe that the Taylor complex of this ideal is its minimal free resolution, and hence the contribution of this ideal to $\gamma_i(J_{n,m})$ is $\binom{n-1}{i}$ for $1 \leq i \leq n-1$. Each of the next $n-1$ pivots in the first row, namely $y_2^m, \ldots y_n^m$ produce the ideals $\langle y_1 y_j^m, y_j^m y_{j+1}^m, \ldots, y_j^m y_n^m \rangle$, $2 \leq j \leq n$. Each of these ideals is again minimally resolved by its Taylor complex and hence their contribution to $\gamma_i(J_{n,m})$ is $\binom{1+n-j}{i}$ for $1 \leq i \leq n-1$ and $2 \leq j \leq n$.

For the next $m-2$ rows of pivots, from

$$y_1 y_2^{m-1}, \ldots, y_1 y_n^{m-1}$$

to

$$y_1^{m-2} y_2^2, \ldots, y_1^{m-2} y_n^2$$

we have that each pivot $y_1^{m-k} y_j^k$ with $j = 2, \ldots, n$ and $k = 2, \ldots, m-1$ produces the ideal

$$\langle y_1^{m-k+1} y_j^k, y_1^{m-k} y_j^k y_{j+1}^k, \ldots, y_1^{m-k} y_j^k y_n^k \rangle.$$

All these ideals are again minimally resolved by their Taylor complexes and hence their contribution to $\gamma_i(J_{n,m})$ is $\binom{1+n-j}{i}$ for $1 \leq i \leq n-1$, and this is for $2 \leq j \leq n$ and $2 \leq k \leq m-1$ hence the contribution of these rows to $\gamma_i(J_{n,m})$ is $\sum_{j=2}^{n} (m-2) \binom{1+n-j}{i}$ for $1 \leq i \leq n-1$.

Finally, the last row forms a monomial ideal whose Taylor complex is its minimal resolution and is generated by $n-1$ monomials , hence its contribution to $\gamma_i(J_{n,m})$ is $\binom{n-1}{i+1}$ for $1 \leq i \leq n-1$.

Putting all these contributions together we have that

$$\gamma_i(J_{n,m}) = \binom{n-1}{i} + \sum_{j=2}^{n}(m-1)\binom{1+n-j}{i} + \binom{n-1}{i+1} \; \forall 1 \leq i \leq n-1.$$

Now it is easy to observe, by the sorting of our pivots, that the Mayer-Vitoris tree that we have built has no repeated generators, i.e. the generators of the modules in $\mathbb{F}$ all have different multidegrees, hence $\mathbb{F}$ is minimal. And we obtain that $\operatorname{projdim}(J_{n,m}) = n-1$.

Finally, observe that the ideal produced by pivot y_2^m is

$$J = \langle y_1 y_2^m, y_2^m y_3^m, \dots y_2^m y_n^m \rangle \text{ and } \operatorname{reg}(J) = (n-1)(m-1)+1.$$

By easy inspection of the degrees of the rest of ideals involved, we can see that $\operatorname{reg}(J_{n,m}) = \operatorname{reg}(J) - 1$. □

Remark 2.3.22. By keeping track of the (multi-)degrees of the generators of the ideals in the Mayer-Vietoris tree built in Proposition 2.3.21 we obtain the (multi-)graded Betti numbers of $J_{n,m}$.

Proposition 2.3.23. *Let m be a positive integer, let $(\mathcal{P}, \subseteq)$ be a poset of subsets of the set $[4m]$ formed by $m > 1$ disjoint diamonds $D_1, \dots, D_m$, $D_i = \{a_{i1}, \dots, a_{i4}\}$ with $a_{i1} < a_{i2}$, $a_{i1} < a_{i3}$, $a_{i2} < a_{i4}$, $a_{i3} < a_{i4}$. Then there is at least one squarefree monomial ideal I_m such that $\mathcal{P}$ is its support poset.*

Proof. Consider the following two sets of monomials:

$$A = \{x_{11}x_{12}x_{13}x_{14}, \dots, x_{m1}x_{m2}x_{m3}x_{m4}\}$$

$$B = \{x_{11}x_{12}x_{21}x_{23}, \dots, x_{(m-1)1}x_{(m-1)2}x_{m1}x_{m3}, x_{m1}x_{m2}x_{11}x_{13}\}$$

Let $\bar{I}_m = \langle A \cup B \rangle \subseteq \mathbb{k}[x_{11}, \dots, x_{14}, \dots, x_{m1}, \dots, x_{m4}]$, then $\mathcal{P}$ is in fact the support poset of $\bar{I}_m$. Just observe that for every i we have that x_{i4} is only present in the monomial $x_{i1}x_{i2}x_{i3}x_{i4}$ hence $C_{i4} = \{i1, i2, i3, i4\}$, x_{i3} is present in the monomials $x_{i1}x_{i2}x_{i3}x_{i4}$ and $x_{i-1,1}x_{i-1,2}x_{i1}x_{i3}$ hence

$C_{i3} = \{i1, i3\}$. The variable x_{i2} is present in the monomials $x_{i1}x_{i2}x_{i3}x_{i4}$ and $x_{i1}x_{i2}x_{i+1,1}x_{i+1,3}$ hence $C_{i2} = \{i1, i2\}$. Finally x_{i1} is present in the monomials $x_{i1}x_{i2}x_{i3}x_{i4}$, $x_{i1}x_{i2}x_{(i+1)1}x_{(i+1)3}$ and $x_{(i-1)1}x_{(i-1)2}x_{i1}x_{i3}$ hence $C_{i1} = \{i1\}$ [3]. □

One possible partition of $(\mathcal{P}, \subseteq)$ is to consider, for each i the paths $\{a_{i1}, a_{i2}, a_{i4}\}$ and $\{a_{i3}\}$, the resulting deporalization is an ideal for which we can explicitly compute the Betti numbers, hence obtaining the Betti numbers of all the ideals in its polarity class.

Proposition 2.3.24. *Let $I_m \subseteq \mathbb{k}[x_{11}, x_{12}, \ldots, x_{m1}, x_{m2}]$ the ideal given by*

$$I_m = \langle x_{11}^3 x_{12}, \ldots, x_{m1}^3 x_{m2}, x_{11}^2 x_{2,1} x_{2,2}, \ldots, x_{(m-1)1}^2 x_{m1} x_{m2}, x_{m1}^2 x_{11} x_{12} \rangle.$$

The Betti numbers of I_m are given by

$$\beta_0(I_m) = 2m$$

$$\beta_i(I_m) = 2K^m_{m-3,i-1} + K^m_{m-2,i}$$

where the numbers $K^m_{a,b}$ are given by the recurrence relation

$$K^m_{a,b} = K^m_{a-2,b-1} + K^m_{a-1,b}$$

with base cases

$$K^m_{a,0} = m + a,\ K^m_{0,i} = \binom{m}{i+1},\ K^m_{1,i} = \binom{m}{i} + \binom{m}{i+1}.$$

Proof. We divide the generators of I_m in two groups A and B. Group A consists on the following m generators: $x_{11}^3 x_{12}, \ldots, x_{m1}^3 x_{m2}$. Group B consists on the following m generators:

$$x_{11}^2 x_{21} x_{22}, \ldots, x_{(m-1)1}^2 x_{m1} x_{m2}, x_{m1}^2 x_{11} x_{12}.$$

Since I_m has m generators in each of the groups we say that it is of the form $\langle m|m \rangle$.

[3] If $i = 1$ then take m instead of $i - 1$, and if $i = m$ take 1 instead of $i + 1$.

To build the Mayer-Vietoris tree of I_m we will first use the generators of group A in the given order. When using the first generator, the ideal produced is given by the monomial $x_{11}^3 x_{12}$ multiplied by each of the following $m-3$ generators from group A: $x_{3,1}^3 x_{3,2}, \ldots, x_{m-1,1}^3 x_{m-1,2}$, and the following m monomials, one for each generator of group B: $x_{2,1}x_{2,2}, x_{2,1}^2 x_{3,1}x_{3,2}, \ldots, x_{m-1,1}^2 x_{m1}x_{m2}, x_{m1}^2$. I.e. the obtained ideal $\widetilde{I}_m$ is of the form $\langle m-3|m\rangle$. The ideal $I'_m = \langle x_{21}^3 x_{22}, \ldots, x_{m1}^2 x_{11}x_{12}\rangle$ is of the form $\langle m-1|m\rangle$.

We continue the construction of the Mayer-Vietoris tree by using as pivots the monomials in group A in their given order. If we take a pivot from an ideal of the form $\langle a|m\rangle$ then its left child is of the form $\langle a-2|m\rangle$ (or $\langle 0|m\rangle$ if $a \leq 2$) and the right child is of the form $\langle a-1|m\rangle$. Each time, when using the pivot $x_{i1}^3 x_{(i+1)1}x_{(i+1)2}$ we delete generators $x_{(i+1)1}^3 x_{(i+1)2}$ from group A and transform the generators $x_{i1}^2 x_{(i+1)1}x_{(i+1)2}$ and $x_{i-1}^2 x_{i1}x_{i2}$ into $x_{i1}^3 x_{i2}x_{(i+1)1}x_{(i+1)2}$ and $x_{i1}^3 x_{i2}x_{i-1}^2$ respectively (observe that when we use $x_{m1}^3 x_{m2}$ we transform $x_{m1}x_{11}x_{12}$ into $x_{m1}^3 x_{m2}x_{11}x_{12}$).

We continue this procedure until we reach an ideal of the form $\langle 0|m\rangle$. Ideals of this form are minimally resolved by their Taylor complex, no matter how we choose pivots, since they consist of the list of generators $x_{11}^2 x_{21}x_{22}, \ldots, x_{m1}^2 x_{11}x_{12}$ where some of them have been substituted by their corresponding $x_{i1}^3 x_{i2}x_{(i+1)1}x_{(i+1)2}$ or by $x_{(i-1)1}^3 x_{(i-1)2}x_{i1}^2$. No lcm of any set of i of these monomials is divisible by the lcm of any other set of i of them.

The ideals of the form $\langle a|m\rangle$ for $a > 1$ which are in an even position of dimension i of the tree contribute with $a+m$ generators to $\beta_i(I_m)$. The nodes of the form $\langle 0|m\rangle$ in an even position of dimension i of the tree contribute with $\binom{m}{j-i+1}$ generators to $\beta_j(I_m)$ for $j \geq i$. Finally, the nodes $\langle 0|m\rangle$ in an odd position of dimension i of the tree contribute to $\beta_j(I_m)$ with $\binom{m}{j-i+1}$ generators.

Now we add up all the contributions. The number $K_{a,i}^m$ for $a > 0$ represents the contribution of a node of the form $\langle a|m\rangle$ to $\beta_i(I_m)$. From the above considerations we have that $K_{a,i}^m = K_{a-1,i}^m + K_{a-2,i-1}^m$ and the base cases of this recursion are $K_{a,0}^m = m + a$ for $a > 0$, $K_{0,i}^m = \binom{m}{i+1}$ and $K_{1,i}^m =$

$\binom{m}{i} + \binom{m}{i+1}$ for $i > 0$. Finally, from the first step in the construction of the tree, we have that $\beta_i(I_m) = K^m_{m-3,i-1} + K^m_{m-1,i} = 2K^m_{m-3,i-1} + K^m_{m-2,i}$. □

We can use of the following binomial identity [18], to describe the Betti numbers of I_m in a direct non-recursive way.

Proposition 2.3.25.

$$K^m_{a,b} = \binom{a+1}{0}\binom{m}{b+1} + \binom{a}{1}\binom{m}{b} + \binom{a-1}{2}\binom{m}{b-1} + \cdots$$

By direct inspection of the Mayer-Vietoris tree constructed in Proposition 2.3.24 we have that

Corollary 2.3.26. *For every ideal J_m in the polarity class of I_m we have* $\operatorname{reg}(J_m) = 2m$, $\operatorname{projdim}(J_m) = \lfloor \frac{m}{2} \rfloor + m - 1$ *and its minimal free resolution is given as an iterated cone resolution.*

A more general class of posets than lines are trees and forests. Observe that a line is a tree that has only one leaf and a set of lines is a forest formed by 1-leaf trees. For trees and forest we can identify supported monomial ideals for which we can compute the main invariants.

Proposition 2.3.27. *Let $\mathcal{P}$ be a tree with nodes $\{1, \ldots, n\}$ and let $\{l_1, \ldots, l_k\} \subseteq \{1, \ldots, n\}$ be the set of leaves of the tree. There exists a squarefree monomial ideal $I_L(\mathcal{P}) \subseteq \Bbbk[x_1, \ldots, x_n]$ with k generators such that $\mathcal{P}$ is it support poset. The Taylor resolution of $I_L(\mathcal{P})$ minimally resolves it and therefore $\beta_i(I_L(\mathcal{P})) = \binom{k}{i+1}$ for all $i \geq 0$.*

Proof. Consider the ideal $I_L(\mathcal{P}) = \langle m_{l_1}, \ldots, m_{l_k} \rangle$ where $m_{l_i} = \prod_{i<l_i} x_i$; here $i < j$ means that i is an ancestor of j. We have that $\mathcal{P}$ is the support poset of $I_L(\mathcal{P})$. To see this, observe that given a variable x_i, the set C_i of variables that appear in every generator in which x_i appears is formed by the variables x_j such that $j < i$ in $\mathcal{P}$.

To see that the Taylor complex of $I_L(\mathcal{P})$ minimally resolves it and therefore $\beta_i(I_L(\mathcal{P})) = \binom{k}{i+1}$ for all i, consider the following process:

First, for each node a such that it is the unique child of node b we identify both in a new node $\overline{a}$ and in the corresponding ideal we substitute

$x_a x_b$ by $x_{\overline{a}}$. We proceed in the same way until we obtain a reduced tree $\mathcal{P}'$ such that each node is either a leaf or has more than one child. Observe that $I_L(\mathcal{P}) \simeq I_L(\mathcal{P}')$ and hence $\beta_i(I_L(\mathcal{P})) = \beta_i(I_L(\mathcal{P}'))$ for all i.

Now, take the root $\overline{a}$ of the reduced tree $\mathcal{P}'$ whose children are $\overline{b}_1, \ldots, \overline{b}_m$. Delete $\overline{a}$ and we are left with a set of m disjoint trees $\mathcal{P}'_1, \ldots, \mathcal{P}'_m$ whose nodes are either leaves or have more than one children. Observe that $I_L(\mathcal{P}') = x_{\overline{a}} \cdot \sum_{i=1}^{m} I_L(\mathcal{P}'_i)$, where multiplication by $x_{\overline{a}}$ means that we multiply each generator of each of the ideals $I_L(\mathcal{P}'_i)$ by $x_{\overline{a}}$ and the ideals $I_L(\mathcal{P}'_i)$ are supported on mutually disjoint sets of variables, hence the sum is direct and $\beta_i(I_L(\mathcal{P}')) = \sum_{j=1}^{m} \beta_i(I_L(\mathcal{P}'_j))$.

By repeated us of this process we obtain that $I_L(\mathcal{P})$ has the same total Betti numbers than a prime monomial ideal generated by one variable for each of its leaves, and hence the result. □

Remark 2.3.28. Observe that the squarefree monomial ideal I constructed in the proof of Proposition 2.3.18 is different with the one presented in Proposition 2.3.27. That is because in Proposition 2.3.18 we are taking into account the order of the variables *i.e.* the poset given in Figure 2.6 (a) is not the same that in Figure 2.6 (b), whereas the support poset of the ideal obtained with Proposition 2.3.27 can be indistinctly the one given in 2.6 (a) or in (b).

(a) (b)

Figure 2.6

Then, when we have a poset formed by lines, there exist different monomial ideals with it as a support poset.

Example 2.3.29. Consider the tree depicted in Figure 2.7 together with its reducing process as described in Proposition 2.3.27. We have that the leaf

ideal of $\mathcal{P}$ is given by

$$I_L(\mathcal{P}) = \langle x_1x_2x_3x_4, x_1x_2x_3x_5x_6, x_1x_2x_3x_7x_8x_9, \\ x_1x_2x_3x_7x_8x_{10}x_{11}, x_1x_2x_3x_7x_8x_{10}x_{12}\rangle,$$

which has the same total Betti numbers than the prime ideal generated by the variables corresponding to the leaves. i.e. $\langle x_4, x_6, x_9, x_{11}, x_{12}\rangle$.

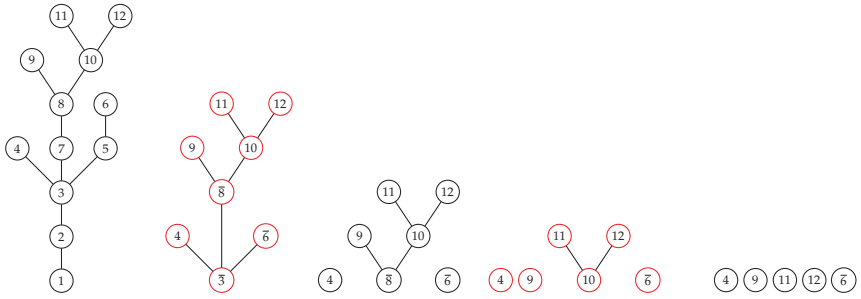


Figure 2.7: Reduction process of tree $\mathcal{P}$ in Example 2.3.29

□

We call $I_L(\mathcal{P})$ the *leaf ideal* of $\mathcal{P}$. Generalizing Proposition 2.3.27 we obtain the following result.

Theorem 2.3.30. *Let $\mathcal{P}$ be a forest whose trees $\mathcal{P}_1, \ldots, \mathcal{P}_m$ have n_i nodes and l_i leaves each, for $i = 1, \ldots, m$. Then there is a squarefree monomial ideal $I_L(\mathcal{P}) \subseteq \Bbbk[x_1, \ldots, x_n]$, $n = \sum_{i=1}^m n_i$ whose support poset is $\mathcal{P}$. The ideal $I_L(\mathcal{P})$ has $g = \sum_{i=1}^m l_i$ minimal monomial generators, its Taylor complex minimally resolves it, and $\beta_i(I_L(\mathcal{P})) = \sum_{j=1}^m \binom{l_j}{i+1}$ for all i.*

2.3.3.2 Support posets of some families of ideals

We turn now to the second question that we address in this section: given a certain class of monomial ideals, how can we describe their support posets? We will treat consecutive k-out-of-n ideals (equivalently path ideals of line graphs cf. [61]) and series-parallel ideals, i.e. path ideals of series-parallel systems cf. [117].

Consecutive linear k-out-of-n ideals. A k-out-of-n ideal $I_{k,n} \subseteq \mathbb{k}[x_1, \dots, x_n]$ is an ideal generated by all possible products of k variables. One can easily see that the support poset of such ideals is a collection of n isolated points, hence it is the only ideal in its polarity class. Consecutive k-out-of-n ideals are generated by the products of any k consecutive variables, $J_{k,n} = \langle x_1 \cdots x_k, x_2 \cdots x_{k+1}, \dots, x_{n-k+1} \cdots x_n \rangle$. These ideals are the path ideals of the line graph, and their main characteristics are well known [61, 118]. Here we describe their support poset.

Proposition 2.3.31. *Let $J_{k,n} = \langle x_1 \cdots x_k, x_2 \cdots x_{k+1}, \dots, x_{n-k+1} \cdots x_n \rangle$ a consecutive k-out-of-n ideal. Then its support poset $\mathcal{P}_{k,n}$ is given by*

- *If $k < n - k + 1$:*

$$C_i = \begin{cases} \{i, \dots, k\} & i = 1, \dots, k \\ \{i\} & i = k+1, \dots, n-k \\ \{n-k+1, \dots, i\} & i = n-k+1, \dots, n \end{cases}$$

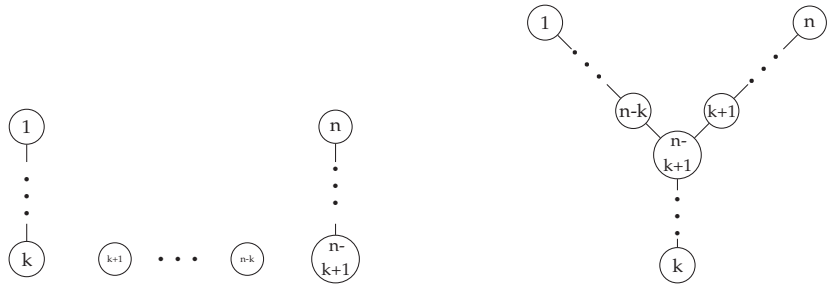
- *If $k \geq n - k + 1$:*

$$C_i = \begin{cases} \{i, \dots, k\} & i = 1, \dots, n-k \\ \{n-k+1, \dots, k\} & i = n-k+1, \dots, k \\ \{n-k+1, \dots, i\} & i = k+1, \dots, n \end{cases}$$

The form of $\mathcal{P}_{k,n}$ is given in Figure 2.8

Proof. Observe that x_1 is only present in the generator $x_1 \cdots x_k$ hence $C_1 = \{1, \dots, k\}$. x_2 is present in generators $x_1 \cdots x_k$ and $x_2 \cdots x_{k+1}$ hence $C_2 = C_1 - \{1\}$, then $C_3 = C_2 - \{2\}$ and so on. By symmetry, x_n is only present in generator $x_{n-k+1} \cdots x_n$ so $C_n = \{x_{n-k+1}, \dots, x_n\}$ and $C_{n-1} = C_n - \{n\}$ etc.

If $k < n - k + 1$ then we get in this way $C_1, \dots, C_k$ and $C_{n-k+1}, \dots, C_n$. For all the C_i with $k < i < n - k + 1$ observe that i appears in generators $x_{i-k+1} \cdots x_i$ to $x_i \dots x_{i+k-1}$ and these generators have only one variable in common, namely x_i, hence $C_i = \{i\}$.



(a) Support poset of $J_{k,n}$ for $k < n-k+1$.

(b) Support poset of $J_{k,n}$ for $k \geq n-k+1$.

Figure 2.8: Form of the support posets of consecutive k-out-of-n ideals.

If $k \geq n-k+1$ observe that $C_{n-k+1} = \cdots = C_k = \{n-k+1, \ldots, k\}$ since the monomial $x_{n-k+1} \cdots x_k$ divides every generator of $J_{k,n}$, and for each $j < n-k+1$ the variable x_j is only present in generators $x_i \cdots x_k$ for $i < j$ (and by symmetry, for every $j > k$ variable x_j is only present in generators $x_i \cdots x_n$ for $i > j$). □

Remark 2.3.32. Using the depolarization poset as described in the sections above we see that if $k \geq n-k+1$ there is a monomial ideal $J'_{k,n}$ copolar to $J_{k,n}$ in only two variables, namely

$$J'_{k,n} = \langle a^k, a^{k-1}b, \ldots, a^{2k-n}b^{n-k} \rangle,$$

which is isomorphic to the zero-dimensional ideal in two variables

$$J''_{k,n} = \langle a^{n-k}, a^{n-k-1}b, \ldots, b^{n-k} \rangle.$$

If $k < n-k+1$ then we have that there is an ideal in $2+n-2k$ variables copolar to $J_{k,n}$, namely

$$J'_{k.n} = \langle a^k, a^{k-1}b_1, \ldots, a^{3k-n}b_1 \cdots b_{n-2k}, a^{3k-n-1}b_1 \cdots b_{n-2k}c, \ldots$$
$$\ldots, ab_1 \cdots b_{n-2k}c^{3k-n-1}, b_1 b_{n-2k}c^{3k-n}, \ldots, b_{n-2k}c^{k-1}, c^k \rangle.$$

These reductions in the number of variables improve drastically the computation times of the Betti numbers and other invariants for this kind of ideals.

Remark 2.3.33. The support poset $\mathcal{P}$ of a $J_{k,n}$ ideal is always a tree or forest. We could then use Proposition 2.3.27 to construct the leaf ideal $I_L(\mathcal{P})$ of $\mathcal{P}$. Observe that $I_L(\mathcal{P}) \neq J_{k,n}$. This is an example that a given poset $\mathcal{P}$ may be the support poset of different squarefree ideals.

Series-parallel ideals Series-parallel ideals are defined as the cut ideals of series-parallel networks, a prominent class of coherent systems, cf. [117]. We can define these ideals in the following way

Definition 2.3.34. The ideal $I = \langle x_1 \rangle \subseteq \Bbbk[x_1]$ is called a *basic series-parallel* ideal. If $I_1 \subseteq \Bbbk[x_1, \ldots, x_n]$ and $I_2 \subseteq \Bbbk[x_{n+1}, \ldots, x_{n+m}]$ are series-parallel ideals then $I_1' + I_2'$ and $I_1' \cap I_2'$ in $\Bbbk[x_1, \ldots, x_{n+m}]$ are series-parallel ideals, where I_1' is the image of I_1 under the inclusion

$$\Bbbk[x_1, \ldots, x_n] \subseteq \Bbbk[x_1, \ldots, x_{n+m}]$$

and I_2' is the image of I_2 under the inclusion

$$\Bbbk[x_{n+1}, \ldots, x_{n+m}] \subseteq \Bbbk[x_1, \ldots, x_{n+m}].$$

Theorem 2.3.35. *The support poset of any series-parallel ideal is a forest.*

Proof. Let I be a series-parallel ideal. We will give a constructive proof following a building process of I.

First, the support poset of a basic series-parallel ideal $I = \langle x_1 \rangle$ has a single element 1, which is a basic forest.

Now, we start constructing I by joining basic series-parallel ideals, i.e. ideals of the form $\langle x_i \rangle$ one at a time. If we join $\langle x_i \rangle$ and $\langle x_j \rangle$ by union, the resulting ideal is $\langle x_i, x_j \rangle$ whose poset is the disjoint union of two points. If we join them by intersection, we obtain $\langle x_i x_j \rangle$ whose poset is a line with two points. Whenever we join a new basic series-parallel ideal $\langle x_i \rangle$ we either join it by addition, in which case we have a new disjoint point in the support poset of the new ideal, or we add it by intersection, in which case we obtain the new poset by setting i as its unique minimal element and joining the minimal element of each connected component of the previous support poset to i. Hence, whenever our series-parallel ideal is built by joining on new basic series-parallel ideal at a time its support poset is a tree plus zero or more disjoint points.

The next step is joining two of these ideals $I_1 \subseteq \Bbbk[x_1, \ldots, x_n]$ and $I_2 \subseteq \Bbbk[x_{n+1}, \ldots, x_{n+m}]$ whose support posets we denote by T_1 and T_2, and the support poset of the resulting ideal by T. If $I = I_1 + I_2$ then T is the disjoint union of T_1 and T_2 since the two ideals have separate sets of variables.

If $I = I_1 \cap I_2$ then the minimal monomial generating set of I is given by all the products $\{m_i m_j' | m_i$ is a generator of I_1, m_j' is a generator of $I_2\}$ and we can be in one of the following three cases:

i) T_1 and T_2 have more than one connected component each. In this case, T is the disjoint union of T_1 and T_2. To see this, observe that there are no indices $i \in \{1, \ldots, n\}$ and $j \in \{n+1, \ldots, n+m\}$ such that $i < j$ or $j < i$. If this was not the case, assume we have $i < j$, then whenever x_j appears in a generator of I so does x_i. But we have that for every generator g of I_1 there is a generator in I of the form $\mu x_j g$ with $\mu \in \Bbbk[x_{n+1}, \ldots, x_{n+m}]$, hence x_i is in every generator of I_1 and hence T_1 has one connected component which contradicts our asumption.

ii) Either T_1 or T_2 have one connected component. Let T_1 have a single connected component, then T_1 is a tree which has a set of elements $b = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$ such that $C_{i_1} = \cdots = C_{i_k}$ and $j > i_a$ for every $i_a \in b$, $j \in \{1, \ldots, n\}$ not in b, i.e. b is the *trunk* of the tree T_1. Then T is formed by the union of T_1 and T_2 plus a connection from $\max(b)$ to every minimal element in T_2. This is because in I there are no new relations among the variables $\{x_1, \ldots, x_n\}$ or among the variables in $\{x_{n+1}, \ldots, x_{n+m}\}$. Observe that all variables in b appear in every generator of I.

iii) Both T_1 and T_2 have only one connected component each. Then let b and b' be the maximal elements of their respective trunks. The support poset of T is the result of identifying both trunks.

□

Remark 2.3.36. Observe that Theorem 2.3.35 provides a criterion to detect ideals that cannot be obtained as a series-parallel ideal. For instance, the

ideal I in Example 2.3.2 (3) cannot be obtained as a series-parallel ideal since its support poset is not a forest.

We have just seen that for every series-parallel ideal I we have a tree $\mathcal{P}_I$ such that $\mathcal{P}$ is the support poset of I. Our next results states that the converse is also true.

Proposition 2.3.37. *Let $\mathcal{P}$ be a forest. There is a series-parallel ideal $I_{\mathcal{P}}$ such that $\mathcal{P}$ is its support poset.*

Proof. Let $\mathcal{P}$ be a tree whose root is r. For every leaf i of $\mathcal{P}$ let $I_i = \langle x_i \rangle$. For every inner node j whose children are $j_1, \ldots, j_k$ let $I_j = \langle x_j \rangle \cap \sum_{i=1}^{i=k} I_{j_k}$. At each stage we have that the support poset of I_j is the upper set of j, $\mathcal{P}_{\geq j}$, hence we have that $I_{\mathcal{P}} = I_r$. □

Observe that the ideal we construct in Proposition 2.3.37 is in fact the *leaf* ideal of Proposition 2.3.27. The proof is an easy inspection of each of the generators. While we always have that $\mathcal{P}_{I(\mathcal{P})} = \mathcal{P}$ it is not always the case that $I = I(\mathcal{P}_I)$ as the following example shows.

Example 2.3.38. Consider the system S_1 in Figure 2.9. It is a series-parallel system whose construction procedure following Theorem 2.3.35 yields the cut ideal

$$\begin{aligned} I_{S_1} &= (\langle x_1 \rangle \cap (\langle x_2 \rangle \cap \langle x_3 \rangle + \langle x_4 \rangle)) \cap (\langle x_5 \rangle \cap (\langle x_6 \rangle \cap \langle x_7 \rangle + \langle x_8 \rangle)) \\ &\subseteq \Bbbk[x_1, \ldots, x_8]. \end{aligned}$$

We have that $I_{S_1} = \langle x_1x_2x_3x_5x_6x_7, x_1x_2x_3x_5x_8, x_1x_4x_5x_6x_7, x_1x_4x_5x_8 \rangle$ and its support poset $\mathcal{P}_{I_{S_1}}$ is given by the sets

$$C_1 = \{1,5\},\ C_2 = \{1,2,3,5\},\ C_3 = \{1,2,3,5\},\ C_4 = \{1,4,5\},$$

$$C_5 = \{1,5\},\ C_6 = \{1,5,6,7\},\ C_7 = \{1,5,6,7\},\ C_8 = \{1,5,8\}$$

whose Hasse diagram (for $x_1 < x_5, x_2 < x_3$ and $x_6 < x_7$) is depicted in Figure 2.10

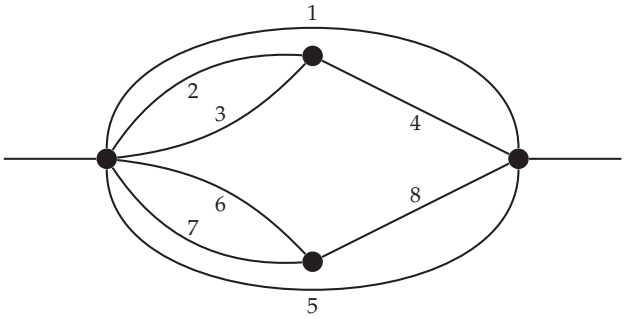


Figure 2.9: S_1, a series-parallel system.

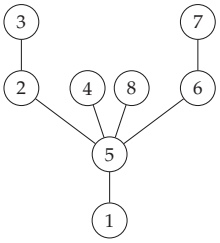


Figure 2.10: Support poset of I_{S_1}

Observe that following the procedure in Proposition 2.3.37 on $\mathcal{P}_{I_{S_1}}$ we obtain the series-parallel ideal

$$I(\mathcal{P}_{I_{S_1}}) = \langle x_1x_2x_3x_5, x_1x_4x_5, x_1x_5x_8, x_1x_5x_6x_7 \rangle$$

which is also the *leaf ideal* of $\mathcal{P}_{I_{S_1}}$. This is the cut ideal of the series-parallel system S_2 in Figure 2.11. Observe that S_1 and S_2 are two different series-parallel systems, yet their respective cut ideals have the same support poset.

□

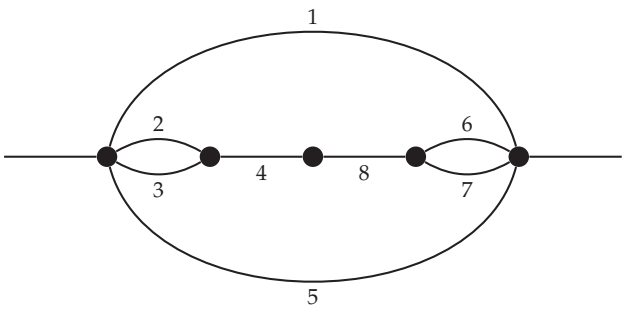


Figure 2.11: S_2, a series-parallel system.

Chapter 3

Algebraic Reliability of multi-state systems

This chapter is devoted to the application of monomial ideals to multi-state system analysis. In previous works [55, 99, 116, 117, 118, 119, 120, 97] the authors have studied the ideals associated to coherent systems for which the performing probabilities of different components are independent, and used their algebraic invariants such as Hilbert function and Betti numbers to compute the reliability of such systems. However, most of the work is devoted to binary systems whose associated ideals are squarefree. But, in practice many systems are non-binary, i.e., their components have multiple possible states, and hence their associated ideals are not squarefree. Moreover, a single ideal is not enough to model multi-state systems, we will need a collection (in fact, a filtration) of reliability ideals for each multi-state system. This chapter extends the algebraic analysis of system reliability from binary to multi-state systems.

In Chapter 1 we introduced definitions related to reliability in the particular case of binary systems. However, as studying multi-state systems is one of the aims of this thesis, Section 3.1 is devoted to generalize these definitions to the multi-state case. In this section we also explain how the algebraic method works for computing reliability of multi-state systems. Section 3.2 shows how the structure of multi-state systems can be analyzed by algebraic means and that this analysis can be transferred

between binary and multi-state systems using polarization and depolarization. Multi-state k-out-of-n systems are deeply studied in Section 3.2.3, including variants such as weighted or consecutive systems.

3.1 Multi-state coherent systems

In reliability theory [28, 13, 75, 102], a *system* S is defined as follows

Definition 3.1.1. A *system* $S = (\mathcal{C}, \phi)$ is a set of components $\mathcal{C} = \{c_1, \ldots, c_n\}$ so that each c_i can be in a discrete number of ordered states $\mathcal{S}_i = (0, \ldots, m_i)$, together with a *structure function* $\phi : \mathcal{S}_1 \times \cdots \times \mathcal{S}_n \to \mathcal{S}$, where $\mathcal{S} = \{0, 1, \ldots, m\}$ is the set of possible states of the system. The structure function receives an n-tuple of component states, and outputs a state of the system.

The set

$$\mathfrak{D} = \{\mathbf{a} = (a_1, \ldots, a_n) \text{ s.t. } a_i \in \mathcal{S}_i, \forall i \in \{1, \ldots, n\}\}$$

is call the *state space of* S and each $\mathbf{a}$ is a *component's state*.

When there is no opportunity of misunderstanding, we simply refer to the system $S = (\mathcal{C}, \phi)$ as S.

We say that the structure function ϕ is *non-decreasing* if $\phi(\mathbf{x}) \geq \phi(\mathbf{y})$ whenever $\mathbf{x} > \mathbf{y}$. The system $\mathcal{S}$ is said to be *coherent* if ϕ is non-decreasing and each component is relevant to the system, *i.e.* for each component c_i there exist a system state $\mathbf{a} = (a_1, \ldots, a_n)$ and two different levels $j, k \in S_i$ such that $\phi(\mathbf{a}_{i,j}) \neq \phi(\mathbf{a}_{i,k})$, where $\mathbf{a}_{i,\ell} = (a_1, \ldots, a_{i-1}, \ell, a_{i+1}, \ldots, a_n)$.

One can notice that Definition 3.1.1 is a generalization of Definition 1.2.1. If we choose $m_i = 1$ and $m = 1$ we have exactly the definition of binary system given in Section 1.2.1. Taking into account the general definition we have just introduced, one can classify systems with respect to their number of states:

- If $m = 1$ and $m_i = 1$ for all i, we have a binary system with binary components. These are usually simply referred to as binary systems.

- If $m > 1$ and $m_i = 1$ for all i, we have a multi-state system with binary components.

- If $m = 1$ and there is at least one i with $m_i > 1$, we have a binary system with multi-state components.

- If $m > 1$ and there is at least one i with $m_i > 1$, we have a multi-state system with multi-state components.

We follow the notation in [53, 102]. However, we allow a more general kind of system by not restricting to the case $\max(\mathcal{S}) \leq \max(\mathcal{S}_i)$ for all i. For other definitions of multi-state system and a review of multi-state reliability analysis, see [85, 144] and the references therein.

Example 3.1.2. Let $S = (\mathcal{C}, \phi)$ be the system represented in Figure 3.1. The set of components is $\mathcal{C} = \{c_1, c_2, c_3, c_4\}$. For each component c_i we define its sets of states as $\mathcal{S}_1, \mathcal{S}_3, \mathcal{S}_4 = \{0, 1, 2\}$, $\mathcal{S}_2 = \{0, 1\}$, *i.e.* we have that components c_1, c_3 and c_4 are multi-state whereas c_2 is a binary component. The set of states of the system is $\mathcal{S} = \{0, 1, 2\}$ and the structure function is

$$\phi(\mathbf{x}) = \max\{c_4, \min\{c_1, \max\{c_2, c_3\}\}\}.$$

Some examples of components' states with the output of the structure

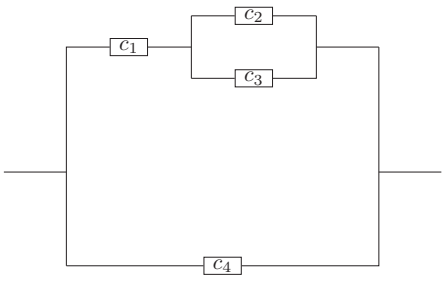


Figure 3.1: A series-parallel system

function are

$$\begin{array}{llll} \phi(0,0,0,0)=0 & \phi(0,1,1,0)=0 & \phi(1,1,2,0)=0 & \phi(2,0,0,0)=0 \\ \phi(0,0,0,1)=1 & \phi(1,1,1,0)=1 & \phi(1,1,2,0)=1 & \phi(2,1,0,1)=1 \\ \phi(0,0,0,2)=2 & \phi(2,1,2,0)=2 & \phi(2,0,2,0)=2 & \phi(2,1,1,2)=2. \end{array}$$

□

3.1.1 The algebraic method in reliability analysis

We saw in Section 1.2 of Chapter 1 the definitions related to binary systems and how one can compute their reliability by using squarefree monomial ideals. In this section we generalize the definitions and the algebraic method presented for binary systems to multi-state ones.

Definition 3.1.3. Let S be a coherent system with n components $\mathcal{C} = \{c_1, \ldots, c_n\}$ so that each c_i can be in a discrete number of ordered states $\mathcal{S}_i = (0, \ldots, m_i)$ and $\mathcal{S} = \{0, 1, \ldots, m\}$ is the set of possible states of the system. Let $0 < j \leq m$:

- The set of tuples of components' states $\mathbf{x}$ such that $\phi(\mathbf{x}) \geq j$ is denoted by $\mathcal{F}^P_{S,j}$. Its elements are called *j-working states* or *j-paths* of S.
- We call *minimal j-working states* or *minimal j-paths* to the tuples in $\mathcal{F}^P_{S,j}$ for which the degradation of the performance of any component provokes that the overall performance of the system is degraded to $j' < j$. The set of minimal j-working states is denoted by $\overline{\mathcal{F}^P}_{S,j}$.
- The tuples of components' states $\mathbf{x}$ with $\phi(\mathbf{x}) < j$ are called *j-failure states* or *j-cuts*. The set of *j-failure states* is denoted by $\mathcal{F}^C_{S,j}$.
- The *minimal j-failure states* or *minimal j-cuts* are states in $\mathcal{F}^C_{S,j}$ such that the improvement of the performance of any component provokes that the overall performance of the system is increased to $j' \geq j$. The set of minimal j-failure states is denoted by $\overline{\mathcal{F}^C}_{S,j}$.

Remark 3.1.4. Definition 3.1.3 is a generalization of Definition 1.2.4 and Definition 1.2.5. If $m = 1$ in Definition 3.1.3, then we simply speak of *working states* or *paths* and *failure states* of *cuts*.

Now, let $R = \mathbb{k}[x_1, \ldots, x_n]$ be a polynomial ring over a field $\mathbb{k}$. Each tuple of components' states $(s_1, \ldots, s_n) \in \mathcal{S}_1 \times \cdots \times \mathcal{S}_n$ corresponds to the monomial $x_1^{s_1} \cdots x_n^{s_n}$ in R. The *coherence property* of the system is equivalent to saying that the elements of $\mathcal{F}_{S,j}$ correspond to a monomial ideal in R, denoted by $I_{S,j}$ The unique minimal monomial generating set of $I_{S,j}$ is formed by the monomials corresponding to the elements of $\overline{\mathcal{F}}_{S,j}$ (see [116, §2] for more details). Hence, obtaining the set of minimal j-paths of S is equivalent to computing the minimal generating set of $I_{S,j}$.

Definition 3.1.5. Let S be a multi-state system such that it can reach $m+1$ different states. For each $0 \leq j \leq m$, the monomial ideal $I_{S,j}$ defined above is called the *j-reliability ideal* of S.

There exist some differences between the binary and the multi-state cases: for computing the reliability of binary system, we only need to assign a squarefree monomial ideal to the system and, using the numerator of the Hilbert series, compute its reliability. However, we have just showed that, for each level j of a multi-state system, we have one monomial ideal, *i.e.* for a multi-state system with $m+1$ states, we have m monomial ideals assigned to it. Due to this fact, the definition of reliability given for binary systems is not enough when treating multi-state systems: we need to generalize this definition.

Let us $S = (\mathcal{C}, \phi)$ be a multi-state system given by $\mathcal{C} = \{c_1, \ldots, c_n\}$, with $\mathcal{S}_i = \{0, 1, \ldots, m_i\}$ for all $i \in \{1, \ldots, n\}$. Each component c_i has a probability of being in one of its m_i states. We will denote it by $p_{i,j}$, where $i \in \{1, \ldots, n\}$ and $j \in \{0, \ldots, m_i\}$. The probabilities associated to each components are the ones which will allow us to compute the j-reliability of the system, which is defined as follows:

Definition 3.1.6. Let S be a multi-state system such that it can reach $m+1$ different states. For each $0 \leq j \leq m$, the *j-reliability* of S is the probability that the system is performing at least at level j.

In order to compute the j-reliability of S, we can use the numerator of the Hilbert series of $I_{S,j}$, denoted by $H_{I_{S,j}}$. The polynomial $H_{I_{S,j}}$ gives a formula, in terms of $x_1, \ldots, x_n$ that enumerates all the monomials in $I_{S,j}$, i.e., the monomials corresponding to the states in $\mathcal{F}_{S,j}$. Hence, computing the (numerator of the) Hilbert series of $I_{S,j}$ provides a method to compute the j-reliability of S by substituting x_i^a by $p_{i,a}$, the probability that the component i is at least performing at level a, as explored in Chapter 1.

In summary, the algebraic method for computing the j-reliability of a coherent system S works as follows:

1. Associate to the system S its j-reliability ideals $I_{S,j}$.

2. Obtain the minimal generating set of $I_{S,j}$ to get the set $\overline{\mathcal{F}}_{S,j}$.

3. Compute the Hilbert series of $I_{S,j}$ to have the j-reliability of S.

3′ Compute any free resolution of $I_{S,j}$. The alternating sum of the ranks of this resolution gives a formula for the Hilbert series of $I_{S,j}$ i.e., the unreliability of S, which provides bounds by truncation at each summand.

The choice between steps (3) or (3′) depends on our needs. If we are only interested in computing the full reliability formula, then we can use any algorithm that computes Hilbert series in step (3). However, if we need bounds for our system reliability, then we can compute any free resolution of $I_{S,j}$ and thus perform step (3′). If the performing probabilities of different components are independent and identically distributed (i.i.d), then in points (3) and (3′) of this procedure we only need the graded version of Hilbert series and free resolutions. Otherwise, we need their multigraded version. For more details and the proofs of the results described here, we refer to [116, 119]. To see more applications of this method in reliability analysis we refer to [117, 118, 120].

3.1.1.1 Bounds

As we have explained in Chapter 1, in practice it is often more useful to have *bounds* on the j-reliability of S rather than the complete precise formula.

Bonferroni-Fréchet bounds. In Section 1.2.1.2 we presented the Bonferroni-Fréchet bounds for binary systems. This bounds can be generalized for the case of multi-state systems as follows: Let $S = (\mathcal{C}, \phi)$ be a multi-state coherent system such that $\mathcal{S} = \{0, \dots, m\}$ and let $\mathcal{F}^P_{S,j} = \left\{ \mathbf{a}^j_1, \dots, \mathbf{a}^j_{s_j} \right\}$ be its set of paths for each level $j \in \mathcal{S}$, where $\mathbf{a}^j_i$ means the first path that satisfies $\phi(\mathbf{a}^j_i) \geq j$ with $i \in \{1, \dots, s_j\}$. The Inclusion-Exclusion method allows to compute the reliability for S as follows:

$$\begin{aligned}
\mathbf{R}_{S,j} =& \mathbb{P}(\mathbf{a}^j_1 \cup \cdots \cup \mathbf{a}^j_{s_j}) \\
&= \left[\mathbb{P}(\mathbf{a}^j_1) + \mathbb{P}(\mathbf{a}^j_2) + \cdots + \mathbb{P}(\mathbf{a}^j_{s_j}) \right] + \\
&\quad - \left[\mathbb{P}(\mathbf{a}^j_1 \cap \mathbf{a}^j_2) + \mathbb{P}(\mathbf{a}^j_1 \cap \mathbf{a}^j_3) + \cdots + \mathbb{P}(\mathbf{a}^j_{s_j-1} \cap \mathbf{a}^j_{s_j}) \right] + \dots \\
&\quad = \sum_{i=1}^{s_j} (-1)^{i-1} \sum_{|\sigma|=i} \bigcap_{l \in \sigma} \mathbf{a}^j_l,
\end{aligned}$$

where, for $i, t, m = 1, \dots, s_j$, $\mathbb{P}(\mathbf{a}^j_i)$ is the probability of the components' state being greater or equal to $\mathbf{a}^j_i$, $\mathbb{P}(\mathbf{a}^j_1 \cup \cdots \cup \mathbf{a}^j_{s_j})$ is the probability that the components' state is greater or equal to at least one of the $\mathbf{a}^j_i$, and $\mathbb{P}(\mathbf{a}^j_t \cap \mathbf{a}^j_m)$ is the probability that the components' state is greater or equal to both of $\mathbf{a}^j_t$ and $\mathbf{a}^j_m$.

As in the binary case, one can truncate the summands in order to obtain the Bonferroni-Fréchet bounds for the j-reliability of the multi-state system as follows:

$$\mathbf{R}_{S,j} \leq \sum_{i=1}^{k}(-1)^{i-1} \sum_{|\sigma|=i} \bigcap_{l\in\sigma} \mathbf{a}_l^j \text{ for } k \text{ odd},$$
$$\mathbf{R}_{S,j} \geq \sum_{i=1}^{k}(-1)^{i-1} \sum_{|\sigma|=i} \bigcap_{l\in\sigma} \mathbf{a}_l^j \text{ for } k \text{ even},$$

with $k \in \{1, \ldots, s_j\}$.

Gasemyr-Natvig bounds. In [51], Funnemark and Natvig presented the following lower bounds based on minimal paths and minimal cuts:

- Let $\mathbf{y}^{\mathbf{m}}$, $m = 1, \ldots, M_p$ be the minimal paths of the multi-state system S for level j. Then,

$$l'^j(\mathbf{p}) = \max_{1\leq m\leq M_p} \left(\prod_{i=1}^{n} p_i^{y_i^m} \right)$$

  is a lower bound for the $\mathbf{R}_{S,j}$ reliability.

- Let $\mathbf{z}^{\mathbf{m}}$, $m = 1, \ldots, M_c$ be the set of minimal cut vectors of the multi-state system S for level j, then we have the following lower minimal bound for $R_j(\mathcal{S})$

$$l^{**j}(\mathbf{p}) = \prod_{m=1}^{M_c} \coprod_{i=1}^{n} p_i^{z_i^m+1}$$

  where for $p_i \in [0,1]$ we define $\coprod_{i=1}^{n} = 1 - \prod_{i=1}^{n}(1-p_i)$.

Gasemyr and Natvig presented in [53] a lower bound which is an improvement with respect to $l'^j(\mathbf{p})$ and $l^{**j}(\mathbf{p})$. It is defined as follows: given a multi-state system $S = (\mathcal{C}, \phi)$, whose components are independent, we have that

$$\tilde{l}_\phi^j(\mathbf{p}) = \sum_{\mathbf{x}\in\mathcal{S}_1\times\cdots\times\mathcal{S}_n} \mathbf{1}(\phi(\mathbf{x}) \geq j) \prod_{i=1}^{n}(p_i^{x_i} - p_i^{x_i+1})$$

is a lower bound for $\mathbf{R}_{S,j}$. In [Theorem 2, [53]] it is proved that, if the components processes are independent, then

$$\tilde{l}^j_\phi(\mathbf{p}) \geq \max\{l'^j(\mathbf{p}), l^{**j}(\mathbf{p})\}.$$

Gasemyr and Natvig point out that in complex systems, the computation of the reliability bounds $l'^j_\phi(\mathbf{p})$, $l^{**j}_\phi(\mathbf{p})$ and $\tilde{l}^j_\phi(\mathbf{p})$, can become quite difficult and computationally expensive. A wide class of those more complicated systems are those in which we can by some means obtain the minimal cuts or paths but their structure is complicated. In these cases the algebraic approach can be a very useful tool.

Algebraic bounds. In order to have a formula that can be truncated at different summands to obtain bounds for the j-reliability, we need a special way to write the numerator of the Hilbert series of $I_{S,j}$. This convenient form is given by the alternating sum of the ranks in any free resolution of the ideal $I_{S,j}$, *i.e.*

$$HN_{I_{S,j}}(\mathbf{x}) = \sum_{i=0}^{d}(-1)^i \left(\sum_{\mathbf{a}\in\mathbb{N}_0^n} \gamma_{i,\mathbf{a}}\mathbf{x}^{\mathbf{a}} \right),$$

where $\gamma_{i,\mathbf{a}}$ is the rank of the i-th module in multi-degree $\mathbf{a}$ and d is the length of the resolution. Given the numerator of the Hilbert Series written this way, we can obtain bounds by truncating as follows

$$\mathbf{R}_{S,j} \leq \sum_{i=0}^{k}(-1)^i \left(\sum_{\mathbf{a}\in\mathbb{N}_0^n} \gamma_{i,\mathbf{a}}\mathbf{x}^{\mathbf{a}} \right) \text{ for } k = 0 \text{ or } k \text{ even},$$

$$\mathbf{R}_{S,j} \geq \sum_{i=0}^{k}(-1)^i \left(\sum_{\mathbf{a}\in\mathbb{N}_0^n} \gamma_{i,\mathbf{a}}\mathbf{x}^{\mathbf{a}} \right) \text{ for } k \text{ odd},$$

Observe that the bounds improve when k approaches to d.

As we saw in Chapter 1, every monomial ideal has a *minimal* free resolution, which provides the tightest bounds among the aforementioned ones. Remember that the ranks of the minimal free resolution

are called Betti numbers. In general, the closer the resolution is to the minimal one, the tighter the bounds obtained, for full details see, e.g., [116, §3].

3.1.2 Duality

Given a structure function ϕ, its dual ϕ^D with respect to $\mathbf{t} \in \mathbb{N}_0^n$ is given by (cf. [40])

$$\phi^D(s_1, \ldots, s_n) = m - \phi((t_1 - s_1, \ldots, t_n - s_n)). \tag{3.1.1}$$

Example 3.1.7. Consider a binary series:G system $\mathcal{S}$ with three components where $\phi(s_1, s_2, s_3) = \min\{s_1, s_2, s_3\}$. We have $\phi^D(s_1, s_2, s_3) = 0$ if and only if $(s_1, s_2, s_3) = (0, 0, 0)$ hence the minimal working states of the dual system are $(1, 0, 0), (0, 1, 0)$ and $(0, 0, 1)$, which correspond to a parallel system. The dual of a series system is always a parallel system and vice-versa.

□

There is a notion of duality in monomial ideals, called *Alexander duality* [94]. To describe it we use the following notation. Given a vector $\mu \in \mathbb{N}^n$, we denote by $\mathfrak{m}^\mu$ the monomial ideal

$$\mathfrak{m}^\mu = \langle x_i^{\mu_i} \mid \mu_i \geq 1 \rangle.$$

Given two vectors μ and ν in $\mathbb{N}^n$ let $\mu \setminus \nu$ the vector whose i'th coordinate is $\mu_i + 1 - \nu_i$ if $\nu_i \geq 1$ and 0 otherwise.

Definition 3.1.8. Let $I \subset \mathbf{k}[x_1, \ldots, x_n]$ be a monomial ideal, $\mathrm{MinGens}(I)$ its minimal set of monomial generators, and $x^\nu = \mathrm{lcm}(\mathrm{MinGens}(I))$. The Alexander dual of I is the intersection

$$I^D = \bigcap_{x^\mu \in \mathrm{MinGens}(I)} \mathfrak{m}^{\nu \setminus \mu},$$

where $\mathfrak{m}^{(s_1, \ldots, s_n)}$ denotes the monomial ideal $\langle x_i^{s_i} \mid s_i \geq 1 \rangle$

We can use the dual ideal of a system to compute its reliability in the following way. Let $\overline{pr}(x^\mu) = \prod_{i=1}^{n}(1 - p_{i,\mu_i+1})$ *i.e.* the product of the probabilities that each component i is in a state less than or equal to μ_i. We denote by $\nu = (m_1, \ldots, m_n)$ the vector of maximal possible levels of the components. Let $\overline{I}_j(\mathcal{S})$ be the ideal generated by the monomials $\{x^{\nu\setminus\mu} \mid x^\mu \text{ is a generator of } I_j(\mathcal{S})\}$. We consider the ideal $\overline{I}_j(\mathcal{S})^D$ and compute $HN_{\overline{I}_j(\mathcal{S})^D}(x_1, \ldots, x_n)$. We obtain $U_j(\mathcal{S}) = 1 - R_j(\mathcal{S})$ by formally substituting each monomial x^μ in $HN_{\overline{I}_j(\mathcal{S})^D}(x_1, \ldots, x_n)$ by $\overline{pr}(\frac{x^\nu}{x^\mu})$.

Example 3.1.9. Consider the system in Example 3.1.7. We have that $I_1(\mathcal{S}) = \langle x_1x_2x_3\rangle$, then $I_1(\mathcal{S})^D = \langle x_1, x_2, x_3\rangle$ and $\overline{I}_1(\mathcal{S})^D = \langle x_1, x_2, x_3\rangle$. By using the minimal free resolution of $\overline{I}_1(\mathcal{S})^D = \langle x_1, x_2, x_3\rangle$ we have that $HN_{\overline{I}_1(\mathcal{S})^D}(x_1, \ldots, x_n) = (x_1 + x_2 + x_3) - (x_2x_3 + x_1x_3 + x_1x_2) + x_1x_2x_3$. Hence, if we set the probabilities $p_{1,1} = 0.8$, $p_{2,1} = 0.9$ and $p_{3,1} = 0.75$, we get $U_1(\mathcal{S}) = \overline{pr}(x_1x_2) + \overline{pr}(x_1x_3) + \overline{pr}(x_2x_3) - \overline{pr}(x_1) - \overline{pr}(x_2) - \overline{pr}(x_3) + \overline{pr}(1) = 0.25 + 0.1 + 0.2 - (0.025 + 0.05 + 0.02) + 0.005 = 0.46$, and we obtain $R_1(\mathcal{S}) = 0.54$. Observe that in the equality above, $\overline{pr}(1) = \overline{pr}(x_1^0x_2^0x_3^0) = pr(x_1 \leq 0)pr(x_2 \leq 0)pr(x_3 \leq 0) = 0.005$.

□

3.2 Analysis of certain multi-state systems using monomial ideals

In applications it is sometimes convenient to work with squarefree monomial ideals, i.e, with polarizations of monomial ideals, and use all their features as combinatorial objects, as seen in [93, 98, 97]. However, on many occasions it makes sense to work on depolarizations of squarefree monomial ideals and reduce the number of variables of their corresponding rings. See, e.g., [19, 100] for similar considerations in different contexts. We propose to explore both directions in the context of algebraic analysis of the reliability of systems using polarization and depolarization.

3.2.1 Multi-state systems examples

The examples presented in this section show how to apply the algebraic method to the analysis of the reliability of multi-state coherent systems. We show that our approach, using the algebraic method, can be used to analyze the reliability of such systems in an efficient and clear way. Throughout this section, we assume that the performing probabilities of different components of each system are independent.

3.2.1.1 Multi-state system given by its minimal paths

This example is the series-parallel system presented in Example 3.1.2 and is showed in Figure 3.2.

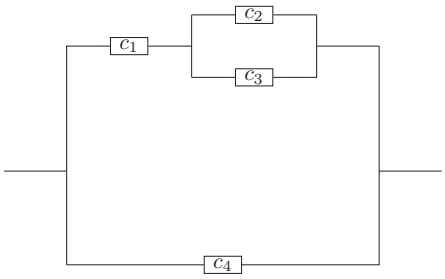


Figure 3.2: Series-parallel system

Components c_1, c_3 and c_4 are multi-state components that can reach 3 different states, while c_2 is a binary component. The set of states of the system is $\mathcal{S} = \{0, 1, 2\}$ and the minimal paths are showed in Table 3.1. We have that the probabilities of working for each component are

$$\begin{array}{lll} p_{1,0} = 1 & p_{1,1} = 0.95 & p_{1,2} = 0.9 \\ p_{2,0} = 1 & p_{2,1} = 0.9 & \\ p_{3,0} = 1 & p_{3,1} = 0.9 & p_{3,2} = 0.8 \\ p_{4,0} = 1 & p_{4,1} = 0.9 & p_{4,2} = 0.85 \end{array}$$

The first step is computing the j-reliability ideals. As the system has 3 different levels of performance $(0, 1, 2)$, we need to compute the j-reliability

Minimal 1-paths	Minimal 2-paths
(1,1,0,0)	(0,0,0,2)
(0,0,0,1)	(2,0,2,0)
(1,0,1,0)	

Table 3.1: Minimal 1-paths and minimal 2-paths for multi-state system of example presented in Section 3.2.1.1

ideal for $j = 1, 2$. We have the 1-paths and 2-paths, so obtain the j-reliability ideals is a direct computation: $I_1 = \langle x_1x_2, x_1x_3, x_4\rangle$ and $I_2 = \langle x_1^2x_3^2, x_4^2\rangle$.

Now, we compute the numerator of the Hilbert Series for both ideals and we obtain

$$NH_{I_{S,1}} = x_1x_2 + x_1x_3 + x_4 - (x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4) + x_1x_2x_3x_4,$$

$$NH_{I_{S,2}} = x_1^2x_3^2 + x_4^2 - x_1^2x_3^2x_4^2.$$

Let us start computing the 2-reliability. By substituting the components' probabilities in $NH_{I_{S,2}}$ we obtain R_{S_2}, *i.e.* the probability that the system works at level equal or greater than 2:

$$R_{S_2} = 0.72 + 0.85 - 0.612 = 0.9558.$$

Repeating the process in $NH_{I_{S,1}}$ we obtain $R_{S,1}$:

$$\begin{aligned} NH_{I_{S,1}} &= 0.855 + 0.855 + 0.9 - (0.7695 + 0.7695 + 0.7695) + 0.69255 \\ &= 2.61 - 2.3085 + 0.69255 = 0.99405 \end{aligned}$$

Observe that $R_{S,1}$ and $R_{S,2}$ are the probabilities that the system is in state greater than or equal to 1 and 2, respectively. However, one can compute the probability of the system of being in state exactly j, for $0 \geq j \geq 2$, as

$$\begin{aligned} r_{S,0} &= 1 - R_{S,1} = 1 - 0.99405 = 0.00595, \\ r_{S,1} &= R_{S,1} - R_{S,2} = 0.99405 - 0.9558 = 0.03825, \\ r_{S,2} &= R_{S,2} = 0.9558 \end{aligned}$$

For an arbitrary coherent system S that can reaches $M+1$ different level of performance, one can compute the exact probability of being in level j, denoted by $r_{S,j}$, $0 \geq j \geq M$, as

$$\begin{cases} r_{S,j} = R_{S,j} - R_{S,j+1} & \text{for } 0 \geq j \geq M-1 \\ r_{S,j} = R_{S,j} & \text{for } j = M \end{cases}$$

Furthermore, if bounds are needed for j-reliability, one can obtain them by truncating the numerator of Hilbert Series. We are going to denote the upper bounds as u_i^j, and the lower bounds as l_i^j for level j, where the greater the i, the tighter the bound. In particular, in this example we have

$$\begin{aligned} l_1^1 &= 2.61 - 2.3085 = 0.3015, \\ u_1^1 &= 2.61, \\ u_1^2 &= 1.57 \end{aligned}$$

The bounds for this example are not very good: the upper bounds are both useless and the lower bound is not accurate.

3.2.1.2 Flow network

A flow network S has n components c_i, $i = 1, \ldots, n$, where each of them can be in $m_i + 1$ states $\mathcal{S}_i = \{0, 1, \ldots, m_i\}$ and the set of states of the system is $\mathcal{S} = \{0, \ldots, m_1 + \cdots + m_n\}$. The structure function of S is $\phi = x_1 + \cdots + x_n$. The j-reliability ideal $I_j \subseteq R = \Bbbk[x_1, \ldots, x_n]$ of S is generated by all monomials in R of degree j. These ideals are stable and therefore, the resolution given in [41] is minimal and provides a formula for the j-reliability of S which can be truncated to obtain bounds.

We consider now an example of a flow network with different levels of performance, see Example 2 in [53]. The system S has two components and each of these can be in three states, $\mathcal{S}_1 = \mathcal{S}_2 = \{0, 1, 2\}$. The system itself can be in five states, $\mathcal{S} = \{0, 1, 2, 3, 4\}$. The structure function of this system is $\phi(\mathbf{x}) = x_1 + x_2$, i.e., the state of the system is the sum of the states of each of its components. The probability that each component is

at least in state 1 is $p^{(1)} = 0.9$ and the probability that each component is in state 2 is $p^{(2)} = 0.8$.

We use now the algebraic method to compute the j-reliabilities of this system:

- For performance at level 1 the minimal paths are $(1,0)$ and $(0,1)$, the corresponding ideal is $I_1 = \langle x, y \rangle$ whose Hilbert function is $H_{I_1} = x + y - xy$. By substituting the corresponding probabilities we have that $R_{S,1} = 0.9 + 0.9 - 0.9 \cdot 0.9 = 0.99$.

- For performance at level 2 the minimal paths are $(2,0)$, $(1,1)$ and $(0,2)$, the corresponding ideal is $I_2 = \langle x^2, xy, y^2 \rangle$ whose Hilbert function is $H_{I_2} = x^2 + xy + y^2 - (x^2y + xy^2)$. By substituting the corresponding probabilities we have that $R_{S,2} = 0.8 + 0.9 \cdot 0.9 + 0.8 - (0.8 \cdot 0.9 + 0.9 \cdot 0.8) = 0.97$.

- For performance at level 3 the minimal paths are $(1,2)$ and $(2,1)$, the corresponding ideal is $I_2 = \langle x^2y, xy^2 \rangle$ whose Hilbert function is $H_{I_2} = x^2y + xy^2 - x^2y^2$. By substituting the corresponding probabilities we have that $R_{S,3} = 0.8 \cdot 0.9 + 0.9 \cdot 0.8 - 0.8 \cdot 0.8 = 0.80$.

- Finally, for performance at level 4 the only minimal path is $(2,2)$, $I_4 = \langle x^2y^2 \rangle$, $H_{I_4} = x^2y^2$ and we have that $R_{S,4} = 0.8 \cdot 0.8 = 0.64$.

3.2.2 Multi-state systems via binary systems and viceversa

We can study multi-state systems via binary systems and vice versa by means of polarization and depolarization of their j-reliability ideals. The main reason behind this approach is that the Hilbert series and free resolutions of monomial ideals and their polarizations are related, see Proposition 2.3.11. For a complete application of the polarization and depolarization operations in the algebraic method, we also need the statement that the ranks of the modules in any resolution of a monomial ideal and its polarization are the same, see Proposition 2.2.7. When using the polarization of a j-reliability ideal to study the system's reliability, we have to carefully adapt the probability associated to the monomials in

the new ideal. Under independence assumption, the term $x_1^{a_1}x_2^{a_2}$ contributes $\text{prob}(c_1 \geq a_1) \cdot \text{prob}(c_2 \geq a_2)$ to the reliability of the system. If independence is not assumed, then we need to individually study the probability evaluation of each monomial. In general, one needs to know the full distribution on the failure set, although the structure of the sets are distribution-free. In the case of polarization of a system reliability ideal, we have to take care of monomials that include products of the type $x_{i,1} \cdots x_{i,k}$ which must be evaluated as $\text{prob}(c_i \geq k)$.

3.2.2.1 Coherent system given by structure function

There are different ways in which a system can be presented such as giving its minimal paths (or cuts), as in Example 3.2.1.1. Another way to present a system is giving its structure function: given a system S with n components, we only have it structure function $\phi(\mathbf{x})$. It might be given the explicit formula of the function so it is necessary to compute the paths and cuts (as in Example 3.2.1.2) or maybe just the results are given, with no more information (as in this example, the structure function of which is given in a table).

Let S be the coherent system with 4 components c_1, c_2, c_3, c_4 such that c_1, c_3, c_4 have two possible states 0 and 1 meaning failure and working, while c_2 has three possible states $0, 1, 2$. The system S itself can be in two possible states, working (1) or failure (0). The probabilities $p_{i,j}$ that component i is in state j are: $p_{1,0} = 0.2$, $p_{1,1} = 0.8$, $p_{2,0} = 0.3$, $p_{2,1} = 0.2$, $p_{2,2} = 0.5$, $p_{3,0} = 0.1$, $p_{3,1} = 0.9$, $p_{4,0} = 0.1$, $p_{4,1} = 0.9$.

c_1	0	1	0	0	0	0	1	1	1	1	0	0	0	0	0	1	1	1	1	1	0	1	1	1
c_2	0	0	1	2	0	0	1	2	0	0	1	1	2	2	0	1	2	1	2	0	1	2	1	2
c_3	0	0	0	0	1	0	0	0	1	0	1	0	1	0	1	1	1	0	0	1	1	1	1	1
c_4	0	0	0	0	0	1	0	0	0	1	0	1	0	1	1	0	0	1	1	1	1	1	1	1
$\phi(\mathbf{x})$	0	0	0	1	0	0	1	1	1	0	1	0	1	1	1	1	1	1	1	1	1	1	1	1

Table 3.2: Structure function $\phi(\mathbf{x})$ for system S.

We want to study the reliability of S whose structure function ϕ is given in Table 3.2. One can see from the table that

$$\overline{\mathcal{F}_S} = \{(1,1,0,0),(1,0,1,0),(0,2,0,0),(0,1,1,0),(0,0,1,1)\}.$$

Hence, the reliability ideal of S is $I_S = \langle xy, xz, y^2, yz, zt\rangle$. The numerator of the Hilbert series of I_S given by the alternating sum of its Betti numbers is

$$H_{I_S} = xy + xz + y^2 + yz + zt - (2xyz + xy^2 + xzt + y^2z + yzt) + xy^2z + xyzt.$$

Substituting the probabilities in H_{I_S}, we obtain that $R_S = 0.9606$. On the other hand, the polarization of I_S is

$$I_S^P = \langle x_1y_1, x_1z_1, y_1y_2, y_1z_1, z_1t_1\rangle \subset \mathbb{k}[x_1, y_1, y_2, z_1, t_1].$$

We can see that the numerator of the Hilbert series of I_S^P is

$$\begin{aligned} H_{I_S^P} \;=\; & x_1y_1 + x_1z_1 + y_1y_2 + y_1z_1 + z_1t_1 \\ & -(2x_1y_1z_1 + x_1y_1y_2 + x_1z_1t_1 + y_1y_2z_1 + y_1z_1t_1) \\ & +x_1y_1y_2z_1 + x_1y_1z_1t_1. \end{aligned}$$

Now, we substitute the probabilities, taking into account that y_1y_2 corresponds to $\text{prob}(c_2 \geq 2)$. We obtain $R_S = 0.9606$.

Studying the depolarization operation on I_S^P we find that we can use the following sets for depolarizing ideal I_S^P

$$\sigma_{x_1} = \{x_1\},\ \sigma_{y_1} = \{y_1\},\ \sigma_{y_2} = \{y_1, y_2\},\ \sigma_{z_1} = \{z_1\} \text{ and } \sigma_{t_1} = \{z_1, t_1\}.$$

Hence, using the partition $\{x_1\}, \{y_1, y_2\}, \{z_1, t_1\}$ we obtain a depolarization of I_S^P in only three indeterminates, as $J = \langle ab, ac, b^2, bc, c^2\rangle \subset \mathbb{k}[a, b, c]$. The numerator of the Hilbert series of this ideal is

$$H_J = ab + ac + b^2 + bc + c^2 - (2abc + ab^2 + ac^2 + b^2c + bc^2) + ab^2c + abc^2.$$

In order to use this expression to evaluate the reliability of S we must keep track of the meaning of the new variables in terms of the ones in I_S^P, i.e., the monomial b^2 corresponds to y_1y_2 which corresponds to $\text{prob}(c_2 \geq 2)$ but c^2 corresponds to z_1t_1 which is evaluated as $\text{prob}(c_3 \geq 1) \cdot \text{prob}(c_4 \geq 1)$. Using these evaluations we obtain the same result that $R_S = 0.9606$.

3.2.2.2 Depolarization of consecutive k-out-of-n systems and storehouse systems:

A *consecutive k-out-of-n:G system* [75], as we saw in Chapter 2, is a binary system with n components that works whenever k *consecutive* components work. The reliability ideal of such a system is

$$J_{k,n} = \langle x_1 \cdots x_k, \ldots, x_{n-k+1} \cdots x_n \rangle \subseteq \mathbb{k}[x_1, \ldots, x_n].$$

The ideal $J_{k,n}$ has $n - k + 1$ generators, all of degree k in n variables, which correspond to the set of all k-paths of the line graph [61]. The depolarization poset of $J_{k,n}$ has a maximal element $J'_{k,n}$ whose ambient ring has $n + 2 - 2k$ variables, and we can use it to compute the reliability of consecutive k-out-of-n:G systems when n is large.

A *storehouse system* is a system with components of n types (m components of each type) and such that within each type, components are sorted so that the i'th component can be working only if the previous ones are working. The system works whenever k components are working (with $k > m$). These systems can be used to model storehouses with shared capacities, industrial straps and pipelines, for instance. The depolarization poset of the squarefree ideal $I_{n,m,k}$ corresponding to such a system (which has nm variables) has a maximal element $I'_{n,m,k}$ whose ambient ring has n variables and is generated by all monomials of degree k such that each variable appears with a degree less than or equal m. These ideals are treated in [114], Chapter 3.

Tables 3.3 and 3.4 show the timings of an algorithm implemented by the authors using the Hilbert series implementation in `Macaulay2` [57] to compute the reliability polynomial of several large consecutive k-out-of-n systems and storehouse systems respectively. The last two columns in these tables show the times used to compute the reliability polynomial using the original squarfree ideal $J_{k,n}$, resp. $I_{n,m,k}$, and the times used to compute the reliability polynomial using the maximal depolarization $J'_{k,n}$, resp. $I'_{n,m,k}$. The times are in seconds. OOT means the computation was manually stopped after 24 hours. Observe that the times are reduced due to the reduction of the number of variables in the ambient ring. Working

with the maximal depolarization makes it possible to handle bigger cases that are not possible to deal with using the squarefree reliability ideals.

n	k	Num. gens.	Time $J_{k,n}$	Time $J'_{k,n}$
100	30	71	0.54	0.18
100	15	86	34.62	17.81
200	60	141	8.12	1.63
200	30	171	1936.16	883.81
300	90	211	56.12	8.63
300	45	256	OOT	11941.60

Table 3.3: Computing times for the reliability polynomials of several consecutive k-out-of-n:G system ideals and their maximal depolarizations.

n	m	k	Num. gens.	Time $I_{n,m,k}$	Time $I'_{n,m,k}$
3	40	50	1161	0.82	0.17
3	90	100	4986	33.92	2.13
3	140	150	11311	282.33	13.91
4	30	40	11461	111.49	14.40
4	50	60	38831	1670.39	404.09
4	70	80	91001	OOT	2697.49

Table 3.4: Computing times for the reliability polynomials of several storehouse system ideals and their maximal depolarizations.

3.2.3 Multi-state k-out-of-n systems

We recall that a system is a k-out-of-n:G system (G for good) if it works whenever k of its n components work, and that it is a k-out-of-n:F (F for fail) if it fails whenever k of its n components fail. k-out-of-n systems and its variants, *e.g.* linear consecutive, circular consecutive, weighted, etc., are one of the most relevant types of systems studied in reliability theory due to their theoretical interest and wide range of applications,

cf. [69, 75, 43]. The multi-state version, which can model more general situations, has been object of intense research in the last decades and is also applied in a variety of situations [68, 69, 95, 44, 113]. Since the first definition of multi-state k-out-of-n systems [40] several authors have proposed different definitions and generalizations, together with particular methods to evaluate the reliability of these systems, see for instance [28, 68, 59, 35, 36, 107, 95, 96, 95, 82, 52] and references therein.

In this chapter we review the different definitions of multi-state k-out-of-n systems, study them in an algebraic way, and apply the algebraic method as a unified way to compute their reliability.

A problem for the reliability computation of these systems is the computational burden when complexity increases. Several algorithms have been proposed to compute the exact reliability of these systems, see [8, 30, 147, 132, 95]; also, Ding et al. propose in [35] a framework for reliability approximation. Our approach, while enumerative, shows good performance and can provide both exact reliability and bounds in the case of i.i.d components and in the case of independent non-identical components.

3.2.3.1 Simple multi-state k-out-of-n systems

The first definition of multi-state k-out-of-n systems was given by El-Neweihi et al. in the seminal work [40]. They define multi-state systems as follows:

Definition 3.2.1 (El-Neweihi *et al.*, 1978)**.** A system of n components is said to be a *multi-state coherent system (MCS)* if its structure function ϕ satisfies:

1. ϕ is increasing.

2. For level j of component i, there exists a vector $(\cdot_i, \mathbf{x})$ such that $\phi(j_i, \mathbf{x}) = j$ while $\phi(l_i, \mathbf{x}) \neq j$ for $l \neq j$ for $i = 1, \ldots, n$ and $j = 0, \ldots, M$, where $(j_i, \mathbf{x})$ means that the state of the i'th component in $\mathbf{x}$ is j.

3. $\phi(\mathbf{j}) = j$ for $j = 0, \ldots, M$, where $\mathbf{j} = (j, \ldots, j)$.

Observe that this definition is more restrictive than ours in the sense that they assume every component has the same number of states, which is in turn the number of states of the system, i.e. M.

The definiton of multi-state k-out-of-n systems in [40] is:

Definition 3.2.2 (El-Neweihi *et al.*, 1978)**.** A system is a *multi-state k-out-of-n system* if its structure function satisfies

$$\phi(\mathbf{x}) = x_{(n-k+1)} \tag{3.2.1}$$

where $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$ is a non decreasing arrangement of $x_1, \ldots, x_n$.

Observe that this definition satisfies the conditions given in Definition 3.2.1. It is easy to check that ϕ is an increasing function and $\phi(\mathbf{j}) = j$ for all $j = 0, \ldots, M$. To see condition (2) just observe that there always exists a non decreasing arrangement of $x_1, \ldots, x_n$ in which $\phi(j_i, \mathbf{x}) = j$ while $\phi(l_i, \mathbf{x}) \neq j$ for $l \neq j$ for $i = 1, \ldots, n$ and $j = 0, \ldots, M$. Taking the vector in which the first $n - k + 1$ components are lower than j and the rest of them are greater than j, we have that condition (2) is satisfied.

Remark 3.2.3. This kind of systems are called *simple multi-state k-out-of-n systems* in [75].

We describe now the j-reliability ideal of these multi-state k-out-of-n systems:

Proposition 3.2.4. *The ideal*

$$I_{(k,n),j} = \langle \prod_{\substack{\sigma \subseteq \{1,\ldots,n\} \\ |\sigma| = k}} x_i^j \mid i \in \sigma \rangle$$

is the j-reliability ideal of a multi-state k-out-of-n system as defined in Definition 3.2.2.

Proof. First of all we need to check that all $\mu \in G(I_{(k,n),j})$ satisfy $\phi(\mu) = j$. Let $x^\mu = x_{i_1}^j x_{i_2}^j \ldots x_{i_k}^j$ be a generator of $I_{(k,n),j}$, with $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$. If we make a non decreasing arrangement of $x_{i_1}, \ldots, x_{i_k}$ we obtain the

vector $(0,...,0,j,...,j)$ in which the first $n-k$ components are in state 0 and the other components are in state j. Applying the structure function ϕ to this vector we have that $\phi(0,...,0,j,...,j) = j$.

Now, if $x^\nu \in I_{(k,n),j}$, there exists $x^\mu \in G(I_{(k,n),j})$ such that $\mu \leq \nu$. This implies $\phi(\mu) \leq \phi(\nu)$ and since $\phi(\mu) = j$ and ϕ is an increasing function, we obtain $\phi(\nu) \geq j$.

Finally if $l < j$ and $\phi(\nu) = l$ we must have $x^\nu \notin I_{(k,n),j}$. Since $\phi(\nu) = l < j$ we have that there are at most, $k-1$ variables with exponent greater or equal j. This implies that there does not exist any $\sigma \in \{1,\ldots,n\}$ with $|\sigma| = k$ such that $\prod_{x_i \in \sigma} x_i^j$ *s.t.* x^ν, hence $x^\nu \notin I_{(k,n),j}$. □

In [28] Boedigheimer and Kapur define customer-driven reliability models for multi-state systems. They consider systems with M states in which component i can be in M_i states. They describe such systems using *upper and lower boundary points*, which are enough to describe the system completely and are defined as follows

Definition 3.2.5. We say $\mathbf{x}$ is a *lower boundary point* (l.b.p.) *to level* j iff $\phi(\mathbf{x}) \geq j$ and $\mathbf{y} < \mathbf{x}$ implies that $\phi(\mathbf{y}) < j$, for $j = 1,\ldots,M$. An *upper boundary point* (u.b.p) *to level* j is an n-tuple $\mathbf{x}$ such that $\phi(\mathbf{x}) \leq j$ and $\mathbf{y} > \mathbf{x}$ implies that $\phi(\mathbf{y}) > j$, for $j = 0,\ldots,M-1$.

Observe that the lower boundary points to level j are the minimal monomial generators of the j-reliability ideal of the system. To describe upper boundary points algebraically we need maximal standard pairs [130] (see Definition 1.1.7 from Chapter 1) which are in one-to-one correspondence with upper boundary points.

Theorem 3.2.6. *Let $I_{S,j}$ be the j-reliability ideal of a coherent system S for which component i can be in states $(0,\ldots,M_i)$. Then $\mu + \sum_{i\in\sigma} 1_{M_i}$ is an upper boundary point of S for level $j-1$ if and only if (x^μ,σ) is a maximal standard pair of $I_{S,j}$.*

Proof. $\Rightarrow$) Let α be an upper boundary point of S for level $j-1$. Let $\sigma \subseteq \{1,\ldots,n\}$ be the set of components of S such that $\alpha_i = M_i$. We have that $\sigma \neq \{1,\ldots,n\}$ i.e. there exists at least one component i such

that $\alpha_i \neq M_i$ hence α is of the form $\alpha = \mu + \sum_{i\in\sigma} 1_{M_i}$. $\phi(\alpha) < j$ implies $x^\alpha \notin I_{S,j}$, and we claim that (μ, σ) is a standard pair for $I_{S,j}$. To see this, let $x^\mu x^\nu$ such that $\text{supp}(x^\nu) \subseteq \sigma$. If $\nu_i \leq M_i$ then clearly $x^\mu x^\nu \notin I_{S,j}$ because $\mu + \nu \leq \alpha$ and $\phi(\alpha) < j$. Now, since $x^\alpha \notin I_{S,j}$ we know there is no minimal generator of $I_{S,j}$ that divides x^α and since $M_i = \alpha_i$ is the maximal power to which variable i can possibly be raised to in any generator of $I_{S,j}$ then no generator will divide $x^\alpha x^\nu$ for any ν such that $\text{supp}(x^\nu) \subseteq \sigma$ hence (μ, σ) is a standard pair. Assume now that (μ, σ) is not maximal. Then there is some $i' \notin \sigma$ such that $(\mu + 1_{i'}, \sigma)$ is a standard pair for $I_{S,j}$. Then $x^\mu x_{i'} \prod_{i\in\sigma} x_i^{M_i} \notin I_{S,j}$ i.e. $\phi(\alpha + 1_i) < j$ which contradicts the assumption that α is an upper boundary point of S for level $j-1$.
$\Leftarrow$) Let (x^μ, σ) be a maximal standard monomial of $I_{S,j}$, i.e. $x^\mu \notin I_{S,j}$ and $x^\mu x^\nu \notin I_{S,j}$ for all x^ν such that $\text{supp}(x^\nu) \subseteq \sigma$. Let $x^\alpha = x^\mu \prod_{i\in\sigma} x_i^{M_i}$. Since $x^\alpha \notin I_{S,j}$ we know that $\phi(\alpha) < j$. Let now $\beta > \alpha$, we can assume without loss of generality that $\beta = \alpha + 1_i$ for some $i \notin \sigma$. Suppose $x^\beta \notin I_{S,j}$. Then there is no minimal generator of $I_{S,j}$ that divides x^β but since M_i is the maximal state of component i, then there is no minimal generator of $I_{S,j}$ that divides $x^\beta x^\nu$ for any ν such that its support is a subset of σ. Finally since the difference between $x^\mu x_i$ and x^β is a monomial whose support is in σ, we have that $(x^\mu x_i, \sigma)$ is a standard pair for $I_{S,j}$, which is in contradiction with the fact that (x^μ, σ) is maximal, hence $x^\beta \in I_{S,j}$ and α is an upper boundary point of S for level $j-1$. □

Remark 3.2.7. Given a coherent system $\mathcal{S}$ its j-reliability ideal $I_j(\mathcal{S})$ is generated by the monomials corresponding to its minimal j-paths. The ideal of its dual system $I_j(\mathcal{S}^D)$ is generated by the monomials corresponding to minimal j-cuts of $\mathcal{S}$ and may be seen as the ideal generated by the maximal standard pairs of $I_j(\mathcal{S})$ [105]. These can be computed using the Alexander dual of the artinian ideal $I_j(\mathcal{S}) + \langle x_i^{m_1+1}, \ldots, x_n^{m_n+1}\rangle$ [24].

Using upper and lower boundary points, Boedigheimer and Kapur define multi-state k-out-of-n systems as follows.

Definition 3.2.8 (Boedigheimer and Kapur, 1994)**.** ϕ is a multi-state k-out-of-n : G structure function if, and only if, ϕ has $\binom{n}{k}$ lower boundary

points to level j ($j = 1, \ldots, M$) and $\binom{n}{k-1}$ upper boundary points to level j ($j = 0, \ldots, M-1$).

The minimal generating set of the ideal $I_{(k,n),j}$ in Proposition 3.2.4 has $\binom{n}{k}$ elements, i.e. this system has $\binom{n}{k}$ lower boundary points. The maximal standard pairs of $I_{(n,k),j}$ are $(\prod_{i\in\sigma} x_i^{j-1}, \{1, \ldots, n\} - \sigma)$ for all $\sigma \subseteq \{1, \ldots, n\}$ such that $|\sigma| = n - k + 1$, i.e. the number of upper boundary points of S for $j-1$ is $\binom{n}{n-k+1} = \binom{n}{k-1}$. Hence, Proposition 3.2.4 is a proof of the equivalence of definitions 3.2.2 and 3.2.8 in the case that $M_i = M$ for all i.

If we allow that the number of states of each of the components can be different, then the situation is more complicated. Let n_j be the number of components such that their maximum performance level M_i is bigger than or equal to j. If $n_j \geq k$ then the system behaves as a multi-state k-out-of-n system by setting ϕ as in Definition 3.2.2. The number of lower and upper boundary points does however vary. The lower boundary points are given by the tuples that have k components at level j and $n-k$ components at level 0, and there are $\binom{n_j}{k}$ such tuples. And if $n_j \geq k$ then the upper boundary points for level j are given by the tuples in which $k-1$ components are at their maximum level (strictly bigger than j), the other component such that its maximum level is bigger than j is exactly at level j and the rest of the components are at level $\min\{M_i, j\}$. The number of such tuples is $\binom{n_{j+1}}{k}$. Hence the system behaves at level j as a k-out-of-n_j system according to Definition 3.2.8. In fact, if we only consider those components whose maximum performance level is bigger than j then the system behaves at level j as a k-out-of-n_j system according to both definitions.

We can then generalize the ideal in Proposition 3.2.4 allowing different number of levels for each component:

Definition 3.2.9. Let S be a multi-state system with levels $\{0, \ldots, M\}$ and such that each component i has M_{i+1} levels of performance $\{0, \ldots, M_i\}$. Let $n_j \leq n$ be the number of components such that $M_i \geq j$ for each $j \in \{0, \ldots, M\}$ (for ease of notation we consider that these are components $1, \ldots, n_j$). S is a multi-state k-out-of-n system if for every $j \in \{1, \ldots, M\}$

Level	Lower boundary points	Upper boundary points
0		$(0,0,0,0,1), (0,0,0,2,0), (0,0,2,0,0),$ $(0,3,0,0,0), (4,0,0,0,0)$
1	$(0,0,0,1,1), (0,0,1,0,1), (0,1,0,0,1), (1,0,0,0,1),$ $(0,0,1,1,0), (0,1,0,1,0), (1,0,0,1,0), (0,1,1,0,0)$ $(1,0,1,0,0), (1,1,0,0,0)$	$(1,1,1,2,1), (1,1,2,1,1), (1,3,1,1,1),$ $(4,1,1,1,1)$
2	$(0,0,2,2,0), (0,2,0,2,0), (2,0,0,2,0), (0,2,2,0,0),$ $(2,0,2,0,0), (2,2,0,0,0)$	$(2,3,2,2,1), (4,2,2,2,1)$
3	$(3,3,0,0,0)$	

Table 3.5: Upper and lower boundary points for the system in Example 3.2.10

the j-reliability ideal of S, $I_{S,j}$, is of the form

$$I_{S,j} = \langle \prod_{\substack{\sigma \subseteq \{1,\ldots,n_j\} \\ |\sigma|=k}} x_i^j \mid i \in \sigma \rangle.$$

Example 3.2.10. Let S be the multi-state system such that $\mathcal{S} = \{0,1,2,3\}$, $\mathcal{S}_1 = \{0,1,2,3,4\}$, $\mathcal{S}_2 = \{0,1,2,3\}$, $\mathcal{S}_3 = \mathcal{S}_4 = \{0,1,2\}$ and $\mathcal{S}_5 = \{0,1\}$ and let $\phi(\mathbf{x}) = x_{(4)}$. Observe that $n_1 = 5$, $n_2 = 4$, $n_3 = 2$, $n_4 = 1$. The system behaves as a 2-out-of-5 for levels $j = 1,2,3$ according to Definition 3.2.2 and as a 2-out-of-n_j system for levels $j = 1,2,3$ according to Definition 3.2.8. The lower and upper boundary points are given in Table 3.5.

The reliability ideals for this system are

$$\begin{aligned} I_{S,1} &= \langle x_1x_2, x_1x_3, x_1x_4, x_1x_5, x_2x_3, x_2x_4, x_2x_5, x_3x_4, x_3x_5, x_4x_5 \rangle \\ I_{S,2} &= \langle x_1^2x_2^2, x_1^2x_3^2, x_1^2x_4^2, x_2^2x_3^2, x_2^2x_4^2, x_3^2x_4^2 \rangle \\ I_{S,3} &= \langle x_1^3x_2^3 \rangle. \end{aligned}$$

□

3.2.3.2 Generalized multi-state k-out-of-n systems

In [69] Huang, Zuo and Wu introduced *generalized multi-state k-out-of-n systems* allowing different number of components for a system to perform

at each level j naturally extending the capabilities of the systems studied in the previous section and providing more flexibility to describe practical situations. The definition in [69] is the following

Definition 3.2.11 (Huang, Zuo and Wu, 2000)**.** An n-component system is called a *generalized multi-state k-out-of-n:G system* if $\phi(\mathbf{x}) > j,\ 1 \leq j \leq M$ whenever there exists an integer value $l\ (j \leq l \leq M)$ such that at least k_l components are in state l or above.

If we denote by ϕ the structure function of the system S and by N_j the number of components in state j or above, then this definition can be rephrased by saying that $\phi(S) \geq j$ if

$$\begin{aligned} N_j &\geq k_j \\ N_{j+1} &\geq k_{j+1} \\ &\vdots \\ N_M &\geq k_M \end{aligned}$$

Hence we can denote a generalized multi-state k-out-of-n system by $S_{n,(k_1,\ldots,k_M)}$. When $k_1 \leq \cdots \leq k_m$ the system is called an *increasing* generalized multi-state k-out-of-n:G system, and if $k_1 \geq \cdots \geq k_m$ the system is said to be *decreasing*. Huang *et al.* provide formulas for both cases and an enumerative algorithm for the evaluation of the reliability of generalized multi-state k-out-of-n systems when the sequence $(k_1, \ldots, k_M)$ is monotone.

Continuing this line M. J. Zuo and Z. Tian defined in [151] generalized multi-state k-out-of-n:F systems.

Definition 3.2.12 (Zuo and Tian, 2006)**.** An n-component system is called *generalized multi-state k-out-of-n : F system* if $\phi(\mathbf{x}) < j,\ 1 \leq j \leq M$ whenever the states of at least k_l components are below l for all l such that $j \leq l \leq M$.

Using this definition they provide a correspondence between generalized multi-state k-out-of-n:G systems and generalized multi-state k-out-of-n:F systems. They study these systems when the sequence $(k_1, \ldots, k_M)$ is

not necessarily monotone and provide an efficient algorithm that is recursive on M, the number of performance levels. This algorithm outperforms the one in [69] which is recursive in n.

Using the ideals in Proposition 3.2.4 we can immediately describe the reliability ideal of a generalized multi-state k-out-of-n:G system given by $(k_1, \ldots, k_M)$.

Proposition 3.2.13. *The j-reliability ideal of a generalized multi-state k-out-of-n system $S = S_{n,(k_1,\ldots,k_M)}$ is given by*

$$I_{S,j} = I_{n,(k_j,\ldots,k_M)} = \sum_{i=j}^{M} I_{(k_i,n),i}.$$

Example 3.2.14. We study here Example 8 in [69] with the algebraic method and recover the exact same results given there. The system in this example is a generalized multi-state k-out-of-3:G system with four states $(0,1,2,3)$ such that $k_1 = 3$, $k_2 = 2$ and $k_3 = 2$, hence it is a *decreasing* generalized multi-state k-out-of-n:G system. The probabilities of the different components are given by $p_{1,0} = 0.1$, $p_{1,1} = 0.2$, $p_{1,2} = 0.3$, $p_{1,3} = 0.4$, $p_{2,0} = 0.1$, $p_{2,1} = 0.1$, $p_{2,2} = 0.2$, $p_{2,3} = 0.6$, $p_{3,0} = 0.1$, $p_{3,1} = 0.2$, $p_{3,2} = 0.4$, $p_{3,3} = 0.3$, where $p_{i,j}$ is the probability that component i is performing at level j.

- For the system to be in state 3 there must be at least 2 components in state 3 or above ($k_3 = 2$). Hence the corresponding ideal is $I_{S,3} = \langle x^3y^3, x^3z^3, y^3z^3 \rangle$. The numerator of the Hilbert series is $H_{I_{S,3}} = x^3y^3 + x^3z^3 + y^3z^3 - 2(x^3y^3z^3)$ and when plugging the probabilities in, we have that the probability that the system is in state 3 or above, denoted $R_{S,3}$, is 0.396, which equals the probability that the system is exactly in state 3, denoted $r_{S,3}$.

- The system is in state 2 or above if at least 2 components are in state 2 or above, hence $I_{S,2} = I_{(2,3),2} + I_{(2,3),3} = I_{(2,3),2} = \langle x^2y^2, x^2z^2, y^2z^2 \rangle$. The numerator of the Hilbert series is $H_{I_{S,2}} = x^2y^2 + x^2z^2 + y^2z^2 - 2(x^2y^2z^2)$ and we obtain $R_{S,2} = 0.826$ and $r_{S,2} = R_{S,2} - R_{S,3} = 0.826 - 0.396 = 0.430$.

- Since $k_1 = 3$ the system is in state 1 or above if all 3 components are in state 1 or above or if at least 2 components are in state 2 or above or if at least 2 components are in state 3 or above. The corresponding ideal is then $I_{S,1} = I_{(3,3),1} + I_{(2,3),2} + I_{(2,3),3} = I_{(3,3),1} + I_{(2,3),2} = \langle xyz, x^2y^2, x^2z^2, y^2z^2 \rangle$, $H_{I_{S,1}} = xyz + x^2y^2 + x^2z^2 + y^2z^2 - (xy^2z^2 + x^2yz^2 + x^2y^2z)$ and we obtain $R_{S,1} = 0.89$ and $r_{S,1} = R_{S,1} - R_{S,2} = 0.89 - 0.826 = 0.064$.

- Finally $r_{S,0} = R_{S,0} - R_{S,1} = 1 - 0.89 = 0.11$.

□

Using the reliability ideals of generalized multi-state k-out-of-n:G systems given in Proposition 3.2.13 we can develop a recursive method to compute their reliability. The method is recursive on M, the number of performance levels and can be used for any sequence $(k_1, \ldots, k_M)$ describing the system, not necessarily monotone. This method is an enumerative one that can be used even when the component's probabilities are not i.i.d. For the i.i.d. case our method is equivalent to the one in [151] in terms of computational complexity. We will use the technique of Mayer-Vietoris trees, which were introduced in [114, 123], see Chapter 1 for an explanation of the method. For ease of the notation we assume that the sequence $(k_1, \ldots, k_M)$ is strictly decreasing. In any other case, the only difference is that some of the summands that compose the ideal $I_{n,(k_j,\ldots,k_M)}$ will be missing, as we saw in Example 3.2.14 but this fact does not affect the algorithm description or its performance.

Let $1 \leq j \leq M$ and $I_{n,(k_j,\ldots,k_M)} = \sum_{i=j}^{M} I_{(k_i,n),i}$ the j-reliability ideal of the system. We sort the generators of $I_{n,(k_j,\ldots,k_M)}$ in ascending degree and lexicographically within each degree. For constructing the Mayer-Vietoris tree we will use as pivot always the last generator. First, we use as pivots the generators of $I_{(k_M,n),M}$. We denote each of them by $x_\sigma^M = \prod_{x_i \in \sigma} x_i^M$ for $\sigma \subseteq \{1, \ldots, n\}$ and $|\sigma| = k_M$. For each of these generators we obtain as left child in the Mayer-Vietoris tree the ideal denoted by $I_{\sigma,M}$ given by

$$I_{\sigma,M} = I_{n-k_M,(k_j-k_M,\ldots,k_{M-1}-k_M)} + \sum_{x_i \notin \sigma, x_i < \max(\sigma)} \langle x_i^M \rangle,$$

where $I_{n-k_M,(k_j-k_M,\ldots,k_{M-1}-k_M)} \subseteq \mathbb{k}[[n]-\sigma]$. On each of the nodes of the tree we use as pivots the monomials in $\sum_{x_i \notin \sigma, x_i < \max(\sigma)} \langle x_i^M \rangle$ and proceed in the same way when the node is $I_{\sigma,M} = I_{n-k_M,(k_j-k_M,\ldots,k_{M-1}-k_M)}$. Finally, after using all the generators of $I_{n,(k_j,\ldots,k_M)}$ as pivots, we are left with the ideal $I_{n,(k_j,\ldots,k_{M-1})}$. This procedure leads to the following recursive formula for the Betti number of $I_{n,(k_j,\ldots,k_M)}$ (we give here the version for i.i.d. components)

$$\begin{aligned}
&\beta_\alpha(I_{n,(k_j,\ldots,k_M)}) = \beta_\alpha(I_{n,(k_j,\ldots,k_{M-1})}) + \\
&+ \sum_{i=0}^{n-k_M-2} \binom{n}{k_M+i} \binom{i+k_M-1}{k_M-1} p_{\geq M}^{k_M+i} \beta_{\alpha-i+1}(I_{n-k_M-i,(k_j-k_M-i,\ldots,k_{M-1}-k_M-i)}) + \\
&+ \binom{n}{k_M+\alpha-1} \binom{\alpha+k_M-2}{k_M-1} p_{\geq M}^{k_M+\alpha-1} \left(\sum_{i=j}^{M-1} \binom{n-k_M-(\alpha-1)}{k_i-k_M-(\alpha-1)} p_{\geq i}^{k_i-k_M-(\alpha-1)} \right) + \\
&+ p_{\geq M}^{k_M+\alpha} \sum_{i=1}^{n-k_M} (i+1) \binom{1}{\alpha}.
\end{aligned} \tag{3.2.2}$$

The complete derivation of this formula is straightforward but somewhat tedious. It is based on the analysis of the branches of the Mayer-Vietoris tree, as described in Chapter 1. Observe that the computation for $(k_1,\ldots,k_M)$ is done in terms of cases with strictly less than M levels, and hence the recursion is on the number of performance levels, and not on the number of variables. The efficiency of this method is equivalent to the one in [151].

Remark 3.2.15. There are several algorithms to compute the reliability of generalized multi-state *k*-out-of-*n* systems. Some of them are restricted to identical independent components. Among these, the algorithm in [69] is enumerative (hence of low efficiency) and applicable to monotonic patterns, the one in [151] is also enumerative but more efficient and is applicable to monotonic and non-monotonic patterns. For the case of independent but not necessarily identical components the algorithm by [147] uses a finite Markov chain imbedding (FMCI) approach and is adequate for small size systems, as is the algorithm in [132]. Other more efficient algorithms include [30], based on conditional probabilities, or [95] using multi-valued decision diagrams. Our algebraic approach is enumerative and applicable

to both kind of systems (with independent and identical components and with independent non identical components) and produces not only the full reliability formulas but also bounds.

Quality of the algebraic bounds. For a polynomial ring $R = \mathbb{k}[x_1, \ldots, x_n]$ Hilbert's syzygy theorem (cf. [39] for instance) states that the length of any resolution of an ideal in R is bounded above by n. In our context this means that the algebraic method using the Betti numbers of reliability ideals produces a compact version of the inclusion-exclusion identity and thus a series of Bonferroni-like bounds for the system's reliability such that if the system S has n components then the reliability formula, given by the Hilbert series numerator of I_S, has at most $n+1$ summands. Every truncation of this formula provides a bound for the reliability. We compare these bounds with the lower bounds l'^j_ϕ and l^{**j}_ϕ introduced in Section 3.1.1.1.

Example 3.2.16. Let $k_1 = 4, k_2 = 2, k_3 = 1$ and let $n = 8, 11, 14$. Let us consider the multi-state generalized k-out-of-n:G systems $I_{n,(4,2,1)}$ for the following probabilities, independent but not identical:

level	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
1	0.5	0.6	0.4	0.5	0.6	0.4	0.5	0.6	0.4	0.5	0.6	0.4	0.5	0.6
2	0.2	0.15	0.1	0.2	0.15	0.1	0.2	0.15	0.1	0.2	0.15	0.1	0.2	0.15
3	0.1	0.05	0.05	0.1	0.05	0.05	0.1	0.05	0.05	0.1	0.05	0.05	0.1	0.05

Table 3.6: Probabilities $p_{i,j}$, i.e. $P(c_i \geq j)$ for the components of several generalized multistate k-out-of-n systems

The number of generators (i.e. number of minimal paths) of each of the systems considered are given in Table 3.7 we also give the number of minimal cuts.

The results are summarized in tables 3.8 and 3.9 in which we consider the probability of the system performing at levels 1 to 3. In the tables, column l_i indicates a lower bound given by the first i summands of the Hilbert series numerator of the corresponding j-reliability ideal, while column u_i denotes an upper bound given by the first i summands. An asterisk indicates that the bound is sharp. Cells with a minus sign $-$

Sytem	level	# minimal paths	# minimal cuts
$S_{8,(4,2,1)}$	1	106	168
$S_{8,(4,2,1)}$	2	36	8
$S_{8,(4,2,1)}$	3	8	1
$S_{11,(4,2,1)}$	1	396	495
$S_{11,(4,2,1)}$	2	66	11
$S_{11,(4,2,1)}$	3	11	1
$S_{14,(4,2,1)}$	1	1106	1092
$S_{14,(4,2,1)}$	2	105	14
$S_{14,(4,2,1)}$	3	14	1

Table 3.7: Number of minimal paths and cuts for several generalized multistate k-out-of-n systems

System	Level	l_2	l_4	l_6	l_8	l_{10}	l_{12}	l_{14}
$S_{8,(4,2,1)}$	1	-	-	0.419984	0.779916			
$S_{8,(4,2,1)}$	2	-	0.480262	0.530988	0.531611			
$S_{8,(4,2,1)}$	3	0.42	0.435844	0.435914*				
$S_{11,(4,2,1)}$	1	-	-	-	-	0.0.914949	0.937376*	
$S_{11,(4,2,1)}$	2	-	0,357057	0.654349	0.666748	0.666865	0.666866*	
$S_{11,(4,2,1)}$	3	0.4975	0.541256	0.541819	0.541821*			
$S_{14,(4,2,1)}$	1			-	-	-	0.870386	0.984878
$S_{14,(4,2,1)}$	2			0.670885	0.765189	0.767655	0.767675*	
$S_{14,(4,2,1)}$	3			0.627826	0.627844*			

Table 3.8: Lower bounds for several generalized multi-state k-out-of-n systems.

indicate that the bound is meaningless (i.e. upper bounds above 1 or lower bounds below 0).

The results in tables 3.8 and 3.9 allow us to discuss the strengths and weaknesses of our method. First of all, for systems with big number of generators, the first bounds are useless due to the fact that each of the first summands of the compact inclusion-exclusion formula consists of a large number of inner summands. As the number of variables increases, we obtain a collection of useful bounds, that compare well with the bounds $l'^{j}_{\phi}(\mathbf{p})$ and $l^{**j}_{\phi}(\mathbf{p})$ from Section 3.1.1.1 (see Table 3.10). Observe that $l^{**j}_{\phi}(\mathbf{p})$

System	Lvl.	u_1	u_3	u_5	u_7	u_9	u_{11}	u_{13}	u_{15}
$S_{8,(4,2,1)}$	1	-	-	-	0.825892	0.782246*			
$S_{8,(4,2,1)}$	2	-	0.750481	0.538913	0.531642	0.531612*			
$S_{8,(4,2,1)}$	3	0.55	0.43725	0.435916	0.435914*				
$S_{11,(4,2,1)}$	1	-	-	-	-	-	0.938269	0.937376*	
$S_{11,(4,2,1)}$	2	-	-	0.741715	0.668326	0.666872	0.666866*		
$S_{11,(4,2,1)}$	3	0.75	0.547875	0.541858	0.541821*				
$S_{14,(4,2,1)}$	1	-	-	-	-	-	-	0.992941	0.985126*
$S_{14,(4,2,1)}$	1	-	-	-	0.785541	0.767936	0.767677	0.767675*	
$S_{14,(4,2,1)}$	1	0.95	0.6455	0.628081	0.627845	0.627844*			

Table 3.9: Upper bounds for several generalized multi-state k-out-of-n systems.

behaves very well in case we have a multi-state parallel system, as is the case in level 3 of our systems. This is because the minimal cuts are unique in these cases. We have considered low working probabilities in our system, since our bounds are sharper in this case. In case our probabilities are high we can consider the unreliability of the dual systems and thus obtain close bounds. All our bounds were computed in less than one second on a laptop[1]. It is worth noting that the performance of our method does not depend on having identical or non-identical probability distributions in the components of the system.

□

3.2.3.3 Binary k-out-of-n system with multi-state components

The following multi-state generalization of k-out-of-n systems was introduced in [114]. Let $S_{m,n,k}$ be a system with k components, each of which can be in a set of states $\{0, 1, \ldots, m\}$. $S_{m,n,k}$ is called an m-multi-state k-out-of-n:G system if the system works whenever the sum of the states of the n components is bigger than or equal to k. Note that this kind of systems allows k to be bigger than n. This is an example of a binary system with multi-state components. This kind of systems are useful to model different situations like the following examples:

[1]CPU: intel i7-4810MQ, 2.80 GHz. RAM: 16Gb

System	Lvl.	$l_\phi^{\prime j}(\mathbf{p})$	$l_\phi^{**j}(\mathbf{p})$
$S_{8,(4,2,1)}$	1	0.108	0.0510583
$S_{8,(4,2,1)}$	2	0.1	0.0710738
$S_{8,(4,2,1)}$	3	0.1	0.435914*
$S_{11,(4,2,1)}$	1	0.1296	0.35674
$S_{11,(4,2,1)}$	2	0.1	0.125414
$S_{11,(4,2,1)}$	3	0.1	0.541821*
$S_{11,(4,2,1)}$	1	0.1296	0.762837
$S_{11,(4,2,1)}$	2	0.1	0.211015
$S_{11,(4,2,1)}$	3	0.1	0.627844*

Table 3.10: Lower bounds $l_\phi^{\prime j}(\mathbf{p})$ and $l_\phi^{**j}(\mathbf{p})$ from Section 3.1.1.1 for some generalized multi-state k-out-of-n systems

- A storehouse has n storage facilities each of which has a capacity of m units. At any given time each of the facilities is partially full, leaving a real capacity smaller than or equal to m units. The system is said to work if it is capable to store a new arriving lot that consists of k storage units.

- A set of n pumps and pipes contributes to a global pipe that covers the needs of a power plant. Each individual pipe may supply water at different levels $\{0, \ldots, m\}$ and we consider that the system is working if the combined supply (sum of all the individual supplies) is above level k.

The reliability ideal of $S_{m,n,k}$, denoted by $J_{[n,k]}^m$ is generated by all monomials x^μ in n variables such that the degree of x^μ is k and $\mu_j \leq m$ for all $1 \leq j \leq n$. To obtain the number of generators of the system (i.e. the minimal working states) and the Betti numbers, needed to compute the reliability function and bounds for it in the algebraic approach, we can proceed as follows.

First, we list all the generators in a precise ordering, following Proposition 3.2.14 in [114]: for each i from m descending to 0 and for each variable

x_j for j from 1 to n (we call x_j the *distinguished variable* in each step) we form all monomials x^μ such that

- the first $j-1$ variables have an exponent strictly smaller than i
- the variable x_j has an exponent equal to i
- the remaining last $n-j$ variables have an exponent smaller than or equal to i
- the degree of x^μ equals k

Using this ordering and Corollary 3.2.25 in [114] we can obtain the Betti numbers of $J^m_{[n,k]}$ using only one more piece of information, namely, for each generator x^μ of $J^m_{[n,k]}$ we need to know the number of variables before x_j that have a nonzero exponent in x^μ. So when we list the generators of $J^m_{[n,k]}$ we keep track of how many of the first $j-1$ variables have a nonzero exponent with the notation we just described. The method for this computation of the Betti numbers of a monomial ideal is described in detail in [114, 115].

For this, let j be the distinguished variable and $i \leq m$ fixed, the exponent of x_j in x^μ. Now, for each p between 0 and $k-i$, which represents the sum of the exponents of the first $j-1$ variables of x^μ, and for each l between 0 and $j-1$, which represents the number of variables among the first $j-1$ ones whose exponent is different from zero, we count all the possible ways to obtain the sum p using l summands each of which is between 1 and $i-1$. This number is called the *number of restricted compositions of p in l summands between* 1 *and* $i-1$ and is denoted $C(p,l,1,i-1)$ in [72]. Since we have l nonzero summands among the first $j-1$ variables, we can choose them in $\binom{j-1}{l}$ ways. For each of these choices we have that the exponents of the last $n-j$ variables sum up to $k-i-p$ and each of these exponents is between 0 and i. The number of such compositions is $C(k-i-p, n-j, 0, i)$. Hence, putting all these considerations together we have the following result.

Lemma 3.2.17. *The number of generators of $J^m_{[n,k]}$ is*

$$N^m_{[n,k]} = \sum_{i=0}^{k}\sum_{j=1}^{n}\sum_{p=0}^{k-i}\sum_{l=0}^{j-1} C(p,l,1,i-1)\binom{j-1}{l} C(k-i-p,n-j,0,i). \quad (3.2.3)$$

All these generators have degree k, hence $\beta_{0,k}(J^m_{[n,k]}) = N^m_{[n,k]}$ and $\beta_{0,j}(J^m_{[n,k]}) = 0$ for all $j \neq k$. Each generator contributes to $\beta_{i,k+i}(J^m_{[n,k]})$ with $\binom{n-l-1}{i}$ elements, hence the formula for the Betti numbers of $J^m_{[n,k]}$ is

$$\begin{aligned}&\beta_{i,k+i}(J^m_{[n,k]}) = \\ &= \sum_{i=0}^{k}\sum_{j=1}^{n}\sum_{p=0}^{k-i} C(p,l,1,i-1)\binom{j-1}{l} C(k-i-p,n-j,0,i)\binom{n-l-1}{i}\end{aligned} \quad (3.2.4)$$

and $\beta_{i,j}(J^m_{[n,k]}) = 0$ if $j \neq k+i$.

Remark 3.2.18. The number of restricted compositions of an integer with a given number of bounded summands can be obtained using a certain generating function, as shown in [7, 38, 46]. The following closed formula for some types of restricted compositions can be found in Theorem 2.1 in [72] which can be used to explicitly compute the numbers in Lemma 3.2.17 using that $C(k-i-p,n-j,0,i) = C(k-i-p+n-j,n-j,1,i+n+j)$:

$$C(n,k,1,b) = \sum_{\substack{i_2=\alpha_2,i_3,\ldots,i_b \\ \max\{0,\alpha_j\}\leq i_j\leq \min\{\beta_j,\gamma_j\}}} \prod_{l=2}^{b}\binom{k-\sum_{j=2}^{l-1} i_j}{i_l},$$

where

$$\alpha_j = n - k(j-1) - \sum_{l=j+1}^{b}(l-j+1)i_l$$

$$\beta_j = k - \sum_{l=j+1}^{b} i_l$$

$$\gamma_j = \lfloor \frac{n-k-\sum_{l=j+1}^{b}(l-1)i_l}{j-1} \rfloor.$$

In order to obtain the necessary information to construct the reliability polynomial and bounds from the Betti numbers of $J^m_{[n,k]}$ we need their multigraded version. For this, let x^μ be a minimal generator of $J^m_{[n,k]}$ and x_j its distinguished variable. Let $(x_{i_1}, \ldots, x_{i_l})$ be the l variables among the first $j-1$ that appear with a nonzero exponent in x^μ. Let $P_{x^\mu} = \{x_1, \ldots, \hat{x}_j, \ldots, x_n\} \setminus \{x_{i_1}, \ldots, x_{i_l}\}$. Then the multidegrees of the contribution of x^μ to $\beta_{i,k+i}(J^m_{[n,k]})$ are $x^\mu \prod_{x_i \in \sigma} x_i$ for each subset σ of P_{x^μ} of cardinality i. Observe that the resolution of $J^m_{[n,k]}$ is k-linear, i.e. $\beta_{i,j} J^m_{[n,k]} = 0$ for all $j \neq k+i$.

Example 3.2.19. Let S be a system with 4 components, each of which has possible states $\{0,1,2,3\}$ such that the system is working whenever the sum of the states of the components is bigger than or equal 5. The ideal of this system is $J^3_{[4,5]} \subseteq R = \mathbf{k}[x,y,z,t]$ and is minimally generated by the following 40 monomials, sorted as described before.

	$i=3$	$i=2$
x	$x^3yt, x^3zt, x^3yz, x^3y^2, x^3z^2, x^3t^2$	$x^2y^2z, x^2y^2t, x^2yz^2, x^2yt^2, x^2z^2t, x^2zt^2, x^2yzt$
y	$y^3zt, y^3z^2, y^3t^2, xy^3z, xy^3t, x^2y^3$	$y^2z^2t, y^2zt^2, xy^2zt, xy^2z^2, xy^2t^2$
z	$z^3t^2, xz^3t, yz^3t, xyz^3, x^2z^3, y^2z^3$	xz^2t^2, yz^2t^2, xyz^2t
t	$xyt^3, xzt^3, yzt^3, x^2t^3, y^2t^3, z^2t^3$	$xyzt^2$

And from this we have that $\beta_{0,5}(J^3_{[4,5]}) = 40$, $\beta_{1,6}(J^3_{[4,5]}) = 92$, $\beta_{2,7}(J^3_{[4,5]}) = 72$, $\beta_{3,8}(J^3_{[4,5]}) = 19$ and $\beta_{i,j}(J^3_{[4,5]}) = 0$ otherwise. Observe that, for instance, the multidegrees of the two contributions of xz^3t to $\beta_{1,6}(J^3_{[4,5]})$ are xyz^3t and xz^3t^2, and the multidegree of its contribution to $\beta_{2,7}(J^3_{[4,5]})$ is xyz^3t^2 since $P_{xz^3t} = \{y,t\}$.

□

We finish with an example of application of these systems.

Storage problem using binary k-out-of-n systems with multi-state components. Binary k-out-of-n systems with multi-state components can be used to model storage problems in which the storage capacity is distributed among several containers. To illustrate this, let S be the set of n tanks in a wine cellar where grape is received in the harvesting season.

Each of the tanks T_i, $i = 1, \ldots, n$ has a total capacity of C_i tons and when a tractor arrives at the cellar, the staff distributes the the new coming grapes among different tanks so that the wine produced in the tanks is sufficiently homogeneous in terms of the origin of the grapes.

The filling procedure is the following: let G be the number of loads of grapes in the incoming tractor (a load consists of 100Kg). We use a discrete measure of time, namely time t means that we have already stored in the tanks the grapes of t tractors. We denote by l_t a measure of the level of the set of tanks after time t. We can consider l_t as the average of the levels of each of the tanks, the minimum or the maximum among them. We choose a level $l \leq \min\{C_1, \ldots, C_n\}$ that we do not want to pass after storing the new coming grapes. Let $m = l - l_t$ and observe that in principle l is chosen so that $m < G$. Among all the possibilities to perform the required load, we choose one randomly. Let us denote by $p^t_{i,j}$ the probability that at time t the empty space in tank T_i is at least j. We have that $p^t_{i,0} = 1$ for all i and $p^t_{i,j} \geq 0$ for all $0 \leq j \leq m$. If one or more of the tanks is full at time t we continue with the same procedure on the remaining tanks. Our goal is to study the probability $p(l)$, $l > l_t$ that we can store the G new coming grape loads in the n tanks so that no tank is filled beyond l and assuming all tanks are already filled to level l_t. This situation can be modeled by a binary G-out-of-n system with multi-state components, in which each component can be in states $\{0, \ldots, m\}$.

Example 3.2.20. Consider a cellar with $n = 5$ tanks with a capacity of 15 tons each. After a certain time t the maximum level on any of the tanks is 12.5 tons i.e. 125 loads. A tractor arrives with 15 loads of grapes and we want to describe how $p(l)$ behaves for $l > 125$. We have modeled the probabilities $p_{i,j}$ as $p_{i,j} = 1 - (\frac{10}{150}j)^{3/2}$ for all i, and $0 \leq j \leq 15$, and $p_{i,j} = 0$ if $j > 15$, i.e. in our case all tanks have the same probability distribution. Under these conditions we have a binary 15-out-of-5 system with multi-state components such that each component can be in states $\{0, \ldots, m = l - 125\}$ for each l. Using the results in Section 3.2.3.3 we have that the ideal of this system is $J^m_{[5,15]}$. The number of generators of this ideal, according to the formula given in Lemma 3.2.17, gives the number of different ways to allocate the grapes meeting the requirements of the

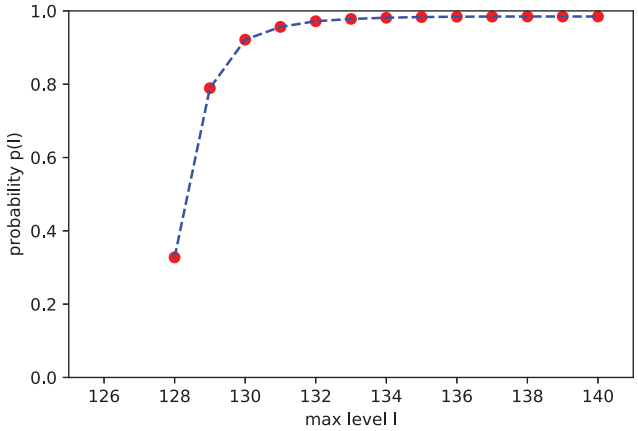


Figure 3.3: Probability that we can fill the 5 tanks in Example 3.2.20 up to level at most l for l from 125 to 140.

described procedure. Taking into account the probabilities of each of the tanks, we can compute the probability that we can meet the requirements using the multigraded Betti numbers as computed in Lemma 3.2.17. We used an implementation of the formulas (3.2.3) and (3.2.4) and algorithms to obtain the set of generators and Hilbert series of the corresponding ideals within the computer algebra system `Macaulay2` [57]. The results are shown in Figure 3.3 and Table 3.11, in which we also show the time (in seconds) taken for the computation of the full list of multigraded Betti numbers, from which we compute the probability in each case.

□

3.2.3.4 Multi-state consecutive k-out-of-n systems

A definition of consecutive multi-state k-out-of-n:F systems, in which k could take different values for different system levels was proposed in [68]. Under that definition, a possibly different number of consecutive components need to be below level j for the system to be below level j for different levels. The required number of consecutive component failures

Level l	$p(l)$	# gens	time
125	0	–	–
126	0	–	–
127	0	–	–
128	0.32768	1	0
129	0.78926	121	0.016
130	0.92148	651	0.126
131	0.95644	1451	0.532
132	0.97187	2226	1.140
133	0.97805	2826	1.594
134	0.98136	3246	2.057
135	0.98321	3526	2.274
136	0.98413	3701	2.470
137	0.98453	3801	2.799
138	0.98466	3851	2.821
139	0.98469	3871	2.834
140	0.98469	3876	2.821

Table 3.11: Probabilities, number of generators and times to compute multigraded Betti numbers for the data in Example 3.2.20

is thus dependent on the system level under consideration. The definition is formalized as follows

Definition 3.2.21. An n-component multi-state system such that its structure function ϕ satisfies that $\phi(x) < j$ for $j = 1, 2, \cdots, M$ if at least k_l consecutive components are in states below l for all $j \leq l \leq M$ is called a *multi-state consecutive k-out-of-n:F system*.

If $k_1 \geq k_2 \geq \cdots \geq k_M$ the system is called a *decreasing* multi-state consecutive k-out-of-n:F system. In this case, as j increases, the requirement on the number of consecutive components that must be below state j for the system to be below state level j also decreases.

If $k_1 \leq k_2 \leq \cdots \leq k_M$, the system is an *increasing* multi-state consecutive k-out-of-n:F system. In this case, for the system to be below a higher

state level j, a larger number of consecutive components must be below state j.

If all the k_j are the same we say the system is a *constant* consecutive k-out-of-n:F system.

Example 3.2.22 ([68] Example 2, [75] Example 12.16)**.** Consider a three-component system where both the system and the components may be in one of three possible states: $0, 1$ and 2. The system is below state 2 if and only if at least one component is below state 2 *i.e.* $k_2 = 1$. The system is below state 1 if and only if at least two consecutive components are below state 1 *i.e.* $k_1 = 2$. This is a strictly decreasing multi-state consecutive k-out-of-n system. This system has a consecutive 1-out-of-3:F structure at system state level 2 and a consecutive 2-out-of-3:F structure at system state level 1.

□

Example 3.2.23 ([68], Example 3, [75], Example 12.17)**.** Consider a three-component system where both the system and the components may be in one of four possible states: $0, 1, 2$ and 3. The system is below state 3 if and only if the three components are below state 3. The system is below state 2 if and only if at least two consecutive components are below state 2 and the three components are below state 3. The system is below state 1 if and only if at least one component is below state 1, at least two consecutive components are below state 2, and at least three components are below state 3. The system in this example has a 3-out-of-3:F structure at system state 3, a consecutive 2-out-of-3:F structure at system state level 2, and a 1-out-of-3:F structure at system state level 1.

□

As we know from Section 3.1.2, given a structure function ϕ its dual ϕ^D with respect to $\mathbf{t} \in \mathbb{N}^n$ is given by (cf. [40], and see [13] for the case $\mathbf{t} = (1, \ldots, 1)$, $M = 1$)

$$\phi^D(s_1, \ldots, s_n) = M - \phi((t_1 - s_1, \ldots, t_n - s_n)). \tag{3.2.5}$$

In our case, we use $\mathbf{t} = (M, \ldots, M)$. This means that the dual system is in state $M - j$ or above if and only if the original system is in state j or below, and that for probability evaluation, the probability of the dual component $\bar{c}_i$ to be in state greater or equal to $M - j$ is the probability that the original component c_i is at state lower than or equal to j. Observe that MS consecutive k-out-of-n:F systems are dual to consecutive k-out-of-n:G systems; this duality transforms increasing systems in decreasing ones and vice-versa. For the algebraic treatment of these systems we shall make use of their dual structure. We can treat the consecutive k-out-of-n:F structures at each level of our systems as consecutive k-out-of-n:G structures and take advantage of the ideal structure. In this setting, the system-to-ideal correspondence is clearer and more convenient.

Let $I_{k,n}$ be the reliability ideal of a binary consecutive k-out-of-n system, this ideal is given by

$$I_{k,n} = \langle x_1 \cdots x_k, x_2 \cdots x_{k+1}, \ldots, x_{n-k+1} \cdots x_n \rangle.$$

The graded Betti numbers of $I_{k,n}$ can be recursively computed by the formulas given in [118], where $\beta_{i,j,k,n}$ indicates the i-th Betti number in degree j of the ideal $I_{k,n}$:

$$\begin{aligned}
&\beta_{0,k,k,n} = n - k + 1,\\
&\beta_{1,k+1,k,n} = n - k, \text{ for } k \geq \frac{n}{2},\\
&\beta_{1,k+1,k,n} = 1 + \beta_{1,k+1,k,n-1}, \text{ for } k < \frac{n}{2},\\
&\beta_{1,2k,k,n} = n - 2k + \beta_{1,2k,k,n-1}, \text{ for } k < \frac{n}{2},\\
&\beta_{i,j,k,n} = \beta_{i-2,j-k-1,k,n-k-1} + \beta_{i-1,j-k,k,n-k-1} + \beta_{i,j,k,n-1} \ \forall i \geq 2, \text{ for } k < \frac{n}{2}.
\end{aligned}$$

Let us denote by $I_{k,n,j}$ the monomial ideal given by the generators of $I_{k,n}$ in which each variable is raised to the j-th power

$$I_{k,n,j} = \langle x_1^j \cdots x_k^j, x_2^j \cdots x_{k+1}^j, \ldots, x_{n-k+1}^j \cdots x_n^j \rangle$$

Observe that for this ideal, the graded Betti numbers are given by

$$\beta_{i,d,}(I_{k,n,j}) = \beta_{i,\frac{d}{j}}(I_{k,n}).$$

An increasing or constant multi-state consecutive k-out-of-n:G system (*i.e.* a decreasing k-out-of-n:F system) works at level j if at least k_j components work at level j or more, and these requirements do not overlap among the levels. Therefore each level has a binary consecutive k_j-out-of-n structure and the j-th reliability ideal is given by

$$\overline{I}_{k,n,j} = I_{k_j,n,j}$$

Example 3.2.24. The system in Example 3.2.22 corresponds to a multi-state consecutive k-out-of-n:G system with $k_1 = 1$, $k_2 = 2$, hence the j-reliability ideals are given by

$$\begin{aligned}\overline{I}_{k,n,1} =& \langle x, y, z\rangle,\\ \overline{I}_{k,n,2} =& \langle x^2y^2, y^2z^2\rangle\end{aligned}$$

□

The case of decreasing MS consecutive k-out-of-n:G system (*i.e.* increasing MS consecutive k-out-of-n:F systems) is more complex. The main difficulty comes from the fact that multi-state decreasing consecutive k-out-of-n:G systems consist on a set of binary consecutive k-out-of-n structures connected by `and` operators to describe the conditions under which the system is in state j or above, and these individual structures are not embedded in one another. The system is in state j or above if there are k_j consecutive components in state j or above and if there are $k_{j'}$ consecutive components in state j' or above for each $j' < j$. Since the system is decreasing, these conditions do not completely overlap.

Example 3.2.25. The dual to the system in Example 3.2.23 is a decreasing multi-state consecutive k-out-of-n system such that $k_1 = 3$, $k_2 = 2$ and $k_3 = 1$. The system is in state 1 or above if at least three components are in state 1 or above; the system is in state 2 or above if at least 2 consecutive components are in state 2 or above `and` at least 3 components are in state

1 or above. Finally, the system is in state 3 if at least 1 component is in state 3 and at least 2 components are in state 2 or above and at least 3 components are in state 1 or above. This system consists on three binary consecutive k-out-of-n structures combined using the and operator.

□

Proposition 3.2.26. *The ideal of a decreasing multi-state consecutive k-out-of-n system is of the form*

$$\overline{I}_{k,n,j} = \bigcap_{j' \leq j} I_{k_{j'},n,j'}$$

Proof. The ideal corresponding to a consecutive k-out-of-n structure in which each component is at level j is given by $I_{k,n,j}$. This models the condition of having k consecutive components out of n, operating at level j or more. The and operator between two levels with such structure implies that the monomials verifying both conditions are in the intersection of both ideals, hence the result. □

Example 3.2.27. The ideals of the system in Example 3.2.25 are given by

$$\overline{I}_{k,n,1} = \langle xyz \rangle,$$
$$\overline{I}_{k,n,2} = \langle x^2y^2, y^2z^2 \rangle \cap \langle xyz \rangle = \langle x^2y^2z, xy^2z^2 \rangle,$$
$$\overline{I}_{k,n,3} = \langle x^3, y^3, z^3 \rangle \cap \langle x^2y^2, y^2z^2 \rangle \cap \langle xyz \rangle = \langle xy^2z^3, xy^3z^2, x^2y^3z, x^3y^2z \rangle$$

□

Using these ideals to compute the reliability of the system we can improve over the enumerative method. The algebraic approach provides an algorithm that can be used for increasing, constant and decreasing as well as for non monotonic multi-state consecutive k-out-of-n systems. In the latter case the intersection in Proposition 3.2.26 runs only on the non-decreasing stretches, since $I_{k_l,n,l} \cap I_{k_{l'},n,l'} = I_{k_l,n,l}$ if $k_l \geq k_{l'}$ when $l > l'$. Huang et al. gave in [68] an algorithm for decreasing multi-state consecutive k-out-of-n:F systems and claimed that "there are no efficient algorithms for system performance evaluation of an increasing multi-state consecutive k-out-of-n:F systems". They proposed the use of duality

to obtain bounds for the reliability of such systems and the use of the enumerative method to obtain the exact reliability. Later, Belaouli and Ksir proposed in [16] a non-recursive algorithm for monotonic systems. Yamamoto et al. [141] proposed an algorithm for general multi-state consecutive k-out-of-n: G systems which do not need to be monotonic. Finally, Zhao et al. [148] used the finite Markov chain imbedding approach (see [50, 73]) for the multi-state consecutive k-out-of-n model, and more recently Yi et al. [143] used the same method for some of its variants. The algorithms in [141] and [148] are very efficient and provide the exact reliability for systems with independent components both identical and non-identical. Our algorithms based on the algebraic methods are slower than these but since they are enumerative have the advantage that can be used to obtain bounds and exact reliabilities, and that can be used in the case of non-independent components. Their efficiency is bigger than other enumerative methods since we avoid much of their redundancy (*cf.*[37]) by using the Hilbert series of the corresponding ideals in a compact form [21].

Example 3.2.28 ([141, 148])**.** Consider a non-monotone system with independent non identical components. We will consider 10 components and six levels. The number of consecutive components required at each of these levels do not follow a monotonic sequence. We have that $k_1 = 6$, $k_2 = 5$, $k_3 = 1$, $k_4 = 2$, $k_5 = 4$, and $k_6 = 4$. The probabilities of each component being in the different states are given as follows: $p_{i,0} = 0.1$, $p_{i,1} = 0.12$, $p_{i,2} = 0.13$, $p_{i,3} = 0.14$, $p_{i,4} = 0.15$, $p_{i,5} = 0.16$ and $p_{i,6} = 0.2$ if i is odd, and $p_{i,0} = 0.05$, $p_{i,1} = 0.1$, $p_{i,2} = 0.12$, $p_{i,3} = 0.13$, $p_{i,4} = 0.15$, $p_{i,5} = 0.2$ and $p_{i,6} = 0.25$ if i is even.

The j-reliability ideals of this system are

$$\overline{I}_{k,n,1} = \langle x_1x_2x_3x_4x_5x_6, x_2x_3x_4x_5x_6x_7, \ldots, x_5x_6x_7x_8x_9x_{10}\rangle,$$
$$\overline{I}_{k,n,2} = \overline{I}_{k,n,1} \cap \langle x_1^2x_2^2x_3^2x_4^2x_5^2, x_2^2x_3^2x_4^2x_5^2x_6^2, \ldots, x_6^2x_7^2x_8^2x_9^2x_{10}^2\rangle,$$
$$\overline{I}_{k,n,3} = \overline{I}_{k,n,2} \cap \langle x_1^3, x_2^3, \ldots, x_{10}^3\rangle$$
$$\overline{I}_{k,n,4} = \overline{I}_{k,n,3} \cap \langle x_1^4x_2^4, x_2^4x_3^4, \ldots, x_9^4x_{10}^4\rangle$$

$\overline{I}_{k,n,5} = \overline{I}_{k,n,4} \cap \langle x_1^5 x_2^5 x_3^5 x_4^5, x_2^5 x_3^5 x_4^5 x_5^5, \ldots, x_7^5 x_8^5 x_9^5 x_{10}^5 \rangle$
$\overline{I}_{k,n,6} = \overline{I}_{k,n,5} \cap \langle x_1^6 x_2^6 x_3^6 x_4^6, x_2^6 x_3^6 x_4^6 x_5^6, \ldots, x_7^6 x_8^6 x_9^6 x_{10}^6 \rangle$

Tables 3.12 and 3.13 show the bounds and exact reliabilities obtained by the algebraic algorihtms for $\overline{I}_{k,n,j}$, $j = 1, \ldots, 6$. In these tables, column l_i indicates a lower bound given by the first i summands of the Hilbert series numerator of the corresponding j-reliability ideal, while column u_i denotes an upper bound given by the first i summands. The bounds $l'^j(\mathbf{p})$ and $l^{**j}(\mathbf{p})$ are the ones presented in Section 3.1.1.1.

An asterisk in Table 3.12 indicates that the bound is sharp. Cells with a minus sign – indicate that the bound is meaningless (i.e., upper bounds above 1 or lower bounds below 0). Note that for systems with big number of generators, the first bounds are useless due to the fact that each of the first summands of the compact inclusion–exclusion formula consists of a large number of inner summands. Observe that our sets of bounds compare well with $l'^j(\mathbf{p})$ and $l^{**j}(\mathbf{p})$. All these bounds and reliabilities were computed in less than 0.1 seconds on a laptop [2] using the `C++` library described in [21]. It is worth noting that the performance of our method does not depend on having identical or non-identical probability distributions in the components of the system.

level	gens.	l_2	l_4	l_6	l_8	$l'^j(\mathbf{p})$	$l^{**j}(\mathbf{p})$
1	5	0.812534*				0.625026	0.807054
2	10	0.628628*				0.33627	0.496151
3	82	0.315292	0.603833	0.628417	0.628627*	0.288797	0.496148
4	58	–	–	0.560247	0.596598*	0.155202	0.411976
5	22	0.0624633	0.101436*			0.0200767	0.00178415
6	22	0.0104151	0.0123518*			0.0019125	2.73312e-07

Table 3.12: Lower bounds for the j-reliability of the consecutive k-out-of-n system in Example 3.2.28

□

[2] MacBookAir M1. 8GbRAM

level	gens.	u_1	u_3	u_5	u_7	u_9
1	5	–	0.812534*			
2	10	–	0.628628*			
3	82	0.746048	0.631709	0.628634	0.628627*	
4	58	–	–	–	0.596598*	
5	22	0.412608	0.102	0.101436*		
6	22	–	0.0123544	0.0123518*		

Table 3.13: Upper bounds for the j-reliability of the consecutive k-out-of-n system in Example 3.2.28

3.2.3.5 Sparsely connected homogeneous multi-state consecutive k-out-of-n:G systems

Sparsely connected homogeneous multi-state consecutive k-out-of-n:G systems were proposed in [52] as a generalization of the binary sparse k-out-of-n systems proposed by Zhao et al. in [142], which were themselves conceived as an extension of the consecutive k-out-of-n model. In such systems two working (resp. failing) components are said consecutive with sparse d if the number of non-working (non-failed) components between any two adjacent working (failed) components is at most d. Hence when $d = 0$ this is the usual consecutive k-out-of-n model. In the multi-state setting, the model proposed in [52] generalizes the MS consecutive k-out-of-n model, cf. [68], in the same fashion as in the binary case. One considers any two components whose states are l or above; if all the components between them are below state l and the number of such components is at most d, then these components can be called consecutive components in state l with sparse d.

Example 3.2.29 ([52], Example 1)**.** Consider a lighting system in a manufacturing workshop with ten homogeneous lamp bulbs, see Fig. 3.4. All the bulbs are arranged linearly and each of them might be in one of three different states. State 0 is a failed state, state 1 represents a partial functioning state, and state 2 is a perfect functioning state. We want to

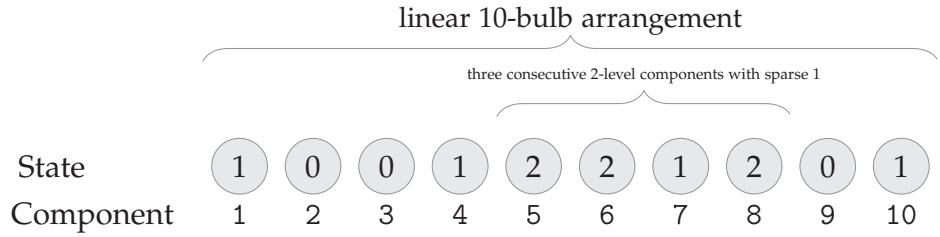


Figure 3.4: Lighting system: bulb linear arrangement in Example 3.2.29

evaluate the probability that the system can satisfy a certain requirement of brightness. According to this requirement the system may be in one of the following states: System state 0 indicates that the lighting system does not provide enough brightness for the manufacturing system to work; system state 1 indicates that the manufacturing system can work partially by a certain amount of brightness; and system state 2 means that the lighting system provides enough brightness for the manufacturing system to work perfectly. In the lighting system, the concept of sparse d can be illustrated in terms of the coverage of light. Let $d = 1$. As shown in Fig. 3.4, components 5, 6, and 8 are in state 2 while component 7 is below state 2, then they can be regarded as 3 consecutive components in state 2 because the number of components being below state 2 between components 6 and 8 does not exceed 1. However, components 1 and 4 cannot be regarded as consecutive components in state 1 because the number of components being below state 1 between them exceeds 1.

□

Let us denote by $J_{k,n,d}$ the reliability ideal of a binary consecutive k-out-of-n:G system with sparse d. It is generated by all the monomials x^μ such that x^μ is the product of k consecutive variables with sparse d, i.e. $x^\mu = x_{i_1} \cdots x_{i_k}$ such that $i_j - i_{j-1} \leq d + 1$ for all $j \in \{i_2, \ldots, i_k\}$. In order to collect all such monomials, observe that:

- i_1 can be any index in $\{1, \ldots, n - k + 1\}$

- For each of indices i_j, $j \in \{i_2, \dots, i_k\}$ the gap $i_j - i_{j-1}$ must be in $\{1, \dots, d+1\}$. Note that the sum of those $k-1$ gaps is always in $\{k-1, \dots, n-i_1\}$.

Hence, the number of generators of $J_{k,n,d}$, i.e. the number of minimal working states of the system, is given by the following result.

Proposition 3.2.30. *Let $G(J_{k,n,d})$ be the set of minimal generators of $J_{k,n,d}$, we have that*

$$\#(G(J_{k,n,d})) = \sum_{j=k-1}^{n-1} (n-j) \sum_{l=0}^{k-1} (-1)^l \binom{k-1}{l} \binom{j-l(d+1)-1}{k-2}$$

Proof. Let $C_{j,k-1,d+1}$ be the number of compositions of j in $k-1$ summands each of them in $\{1, \dots, d+1\}$. For each index i in $\{1, \dots, n-k+1\}$ we select k variables, starting in x_i, i.e. we select $k-1$ gaps, each of which is smaller than $d+1$ or equal, and the total sum cannot exceed $n-i$, being the minimal sum equal to $k-1$, since each gap is at least 1. Hence we have that the number of generators of the ideal is

$$\sum_{i=1}^{n-k+1} \sum_{j=k-1}^{n-i} C_{j,k-1,d+1},$$

and a simple reorganization of the summands leads to

$$\sum_{i=1}^{n-k+1} \sum_{j=k-1}^{n-i} C_{j,k-1,d+1} = \sum_{j=k-1}^{n-1} (n-j) C_{j,k-1,d+1}.$$

Now, by formula E in [7] we have that

$$C_{j,k-1,d+1} = \sum_{l=0}^{k-1} (-1)^l \binom{k-1}{l} \binom{j-l(d+1)-1}{k-2},$$

and hence the result. □

Let $G(J_{k,n,d}) = \{g_1, g_2, \dots\}$ be the set of minimal generators of $J_{k,n,d}$ and let it be sorted by the lex order. In order to compute the Betti numbers and Betti multidegrees of $J_{k,n,d}$ we will use Mayer-Vietoris trees and cone resolutions, cf. [123]. These are based on the iterative computation of the intersection ideals $\langle g_1, \dots, g_{i-1} \rangle \cap \langle g_i \rangle$ where i ranges in $|G(J_{k,n,d})|$.

Proposition 3.2.31. *Let* $\operatorname{set}(g_i)$ *be the set of variables in*

$$\{x_{\min(g_i)-d-1}, \ldots, x_{\max(g_i)}\}$$

such that x_i *does not divide* g_i*. Then*

$$\langle g_1, \ldots, g_{i-1}\rangle \cap \langle g_i\rangle \\ = \langle x_i \cdot g_i \mid x_i \in \operatorname{set}(g_i)\rangle + \langle g_i \cdot g_l' \mid g_l' \in G(J_{k,\min(g_i)-d-2,d})\rangle$$

Proof. Let $\operatorname{supp}(g_i)$ be the set of variables that divide g_i, $\max(g_i)$ be the biggest variable that divides g_i and $\min(g_i)$ the smallest variable that divides g_i.

For all $x_v \in \operatorname{set}(g_i)$ we have that $g_i \frac{x_v}{\max(g_i)} = g_{i'} \in G(J_{k,n,d})$ with $i' < i$ in the lex order, therefore $x_v g_i \in G(\langle g_1, \ldots, g_{i-1}\rangle \cap \langle g_i\rangle)$.

If $x_v \notin \operatorname{set}(g_i)$ and $x_w \in \operatorname{supp}(g_i)$ we have that $g_i \frac{x_v}{x_w}$ either is not in $G(J_{k,n,d})$ or it is equal to some $g_{i'} \in G(J_{k,n,d})$ with $i' > i$ in the lex order. Therefore, in any case $x_v g_i$ it is not a minimal generator of $\langle g_1, \ldots, g_{i-1}\rangle \cap \langle g_i\rangle$.

Finally, for any $g_{i'} \in G(J_{k,n,d})$ such that $\operatorname{supp}(g_{i'}) \subseteq \{1, \ldots, \min(g_i) - d - 2\}$ we have that $\operatorname{supp}(g_i) \cap \operatorname{supp}(g_{i'}) = \emptyset$ and there is no $x_v \in \operatorname{set}(g_i)$ such that x_v divides $g_{i'}$ hence $\langle g_i\rangle \cap \langle g_{i'}\rangle = \langle g_i g_{i'}\rangle$, $g_i g_{i'} \in G(\langle g_1, \ldots, g_{i-1}\rangle \cap \langle g_i\rangle$ and $g_i g_{i''}$ does not divide $g_i g_{i'}$ for any other such $g_{i''} \neq g_{i'}$. Observe that the set given by all $g_{i'} \in G(J_{k,n,d})$ such that $\operatorname{supp}(g_{i'}) \subseteq \{1, \ldots, \min(g_i) - d - 2\}$ is precisely $G(J_{k,\min(g_i)-d-2,d})$. □

Theorem 3.2.32. $\beta_i(J_{k,n,d}) = \sum_{x^\mu \in G(J_{k,n,d})} \binom{|\operatorname{set}(x^\mu)|}{i} + \beta_{i-1}(J_{k,\min(x^\mu-d-2),d})$

Proof. It is a direct consequence of Proposition 3.2.31 and the fact that

$$\operatorname{set}(x^\mu) \cap \operatorname{supp}(G(J_{k,\min(x^\mu-d-2),d})) = \emptyset.$$

□

Observe that $\beta_0(J_{k,n,d}) = \#(G(J_{k,n,d}))$ and hence the recursion in Theorem 3.2.32 is closed and it yields a procedure for the computation of the Betti numbers of $J_{k,n,d}$. From this result and applying the components' probabilities we obtain the reliability of the corresponding binary consecutive k-out-of-n:G system with sparse d.

We say that a binary consecutive k-out-of-n:G system with sparse d is *compact* if $2k \geq n - d$. For any such compact system, we have that $C_{i-d-2,k,d} = 0$ for all $i = 1, \ldots, n-k+1$, *i.e.* $J_{k,\min(x^\mu - d-2),d} = \emptyset$ for all $x^\mu \in G(J_{k,n,d})$. We therefore obtain the following consequences of Proposition 3.2.31:

Corollary 3.2.33. *Let $J_{k,n,d}$ the reliability ideal of a compact binary consecutive k-out-of-n:G system with sparse d, then $J_{k,n,d}$ has linear quotients with respect to the* lex *order.*

Proof. Having linear quotients with respect an ordering of the generators means that for such ordering we have that $\langle g_1, \ldots, g_{i-1} \rangle : \langle g_i \rangle$ is generated by a set of variables $\text{set}(g_i)$. This is equivalent to the fact that $\langle g_1, \ldots, g_{i-1} \rangle \cap \langle g_i \rangle = \langle x_i \cdot g_i \mid x_i \in \text{set}(g_i) \rangle$. Proposition 3.2.31 and the fact that for compact systems $J_{k,\min(x^\mu - d-2),d} = \emptyset$ for all $x^\mu \in G(J_{k,n,d})$ yield the result. □

Corollary 3.2.34. *Let $J_{k,n,d}$ be a compact binary consecutive k-out-of-n:G system with sparse d, then*

$$\beta_i(J_{k,n,d})) = \sum_{x^\mu \in G(J_{k,n,d})} \binom{|set(x^\mu)|}{i}.$$

Observe that a binary consecutive k-out-n ordinary system i.e. $d = 0$, is compact if and only if $k \geq n/2$.

Example 3.2.35. Let $n = 9$, $k = 3$, $d = 2$. The corresponding consecutive k-out-of-n with sparse d system is not compact. The list of minimal monomial generators of $J_{k,n,d}$ has 45 generators and is given by the monomials $x^\mu = x_a x_b x_c$ for all triples abc in the following set (given in lex order)

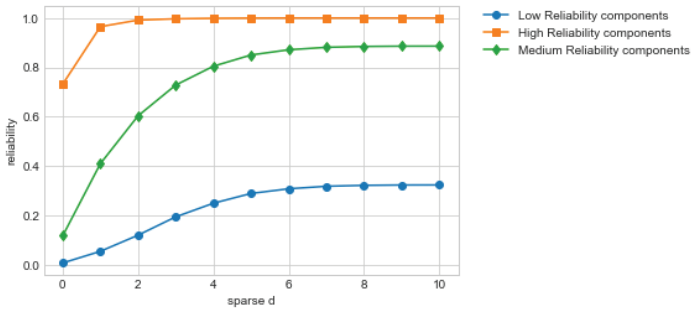


Figure 3.5: System reliability for consecutive 5-out-of-15 systems with sparse $d = \{0, \ldots, 10\}$. Low, medium and high reliability components.

$$
\begin{aligned}
&123, 124, 125, 134, 135, 136, 145, 146, 147,\\
&234, 235, 236, 245, 246, 247, 256, 257, 258,\\
&345, 346, 347, 356, 357, 358, 367, 368, 369,\\
&456, 457, 458, 467, 468, 469, 478, 479,\\
&567, 568, 569, 578, 579, 589,\\
&678, 679, 689,\\
&789.
\end{aligned}
$$

□

Example 3.2.36. Let $n = 15$, $k = 5$ and let vary d from 0 to 10. When $d = 0$ we have the usual consecutive 5-out-of-15 system, and as d increases we tend towards the traditional k-out-of-n system, which occurs when $d > n - k$. Let us assign working probabilities to the components in three ways: First consider highly reliable components, i.e. $p_{i,1} = 0.8$ if i is odd and $p_{i,1} = 0.7$ if i is even, second consider medium reliable components, i.e. $p_{i,1} = 0.5$ if i is odd and $p_{i,1} = 0.4$ if i is even and finally consider components with low reliability, $p_{i,1} = 0.3$ if i is odd and $p_{i,1} = 0.2$ if i is even. Figure 3.5 shows the behaviour of the reliability of these three kinds of systems as the sparse d varies.

□

For the multi-state case the situation is similar to multi-state consecutive k-out-of-n systems. Gao, Cui and Chen define multi-state consecutive k-out-of-n:G system with sparse d in [52] in the following way.

Definition 3.2.37. A system with n components is called a generalized multi-state consecutive k-out-of-n:G system with sparse d if $\phi(X) \geq j$, $(1 \leq j \leq M)$ whenever there exists an integer value l, $(j \leq l \leq M)$ such that at least consecutive k_l components are in state l or above with sparse d.

As in MS consecutive k-out-of-n systems, we consider increasing, decreasing and constant generalized MS consecutive k-out-of-n:G system with sparse d systems depending on the sequence of k_j for the different levels j.

Applying the same reasoning as in Section 3.2.3.4 we define $J_{k,n,d,j}$ as the ideal generated by the generators of $J_{k,n,d}$ each raised to the j-th power. We then define the ideal of a multi-state consecutive k-out-of-n:G system with sparse d as

$$\overline{J}_{k,n,d,j} = \bigcap_{j' \leq j} J_{k_{j'},n,j',d}$$

Example 3.2.38. In the system in Example 3.2.29, see [52], to reach system state 1, it is required that at least consecutive 3-out-of-10 light bulbs should be in state 1 or above with sparse 1. To reach state 2, i.e., to meet the demand of enough brightness, at least consecutive 5-out-of-10 light bulbs should be in state 2 or above with sparse 1. Therefore, we can model the mentioned example as an increasing MS consecutive (k_1, k_2)-out-of-10:G system model with sparse 1, where $k_1 = 3$ and $k_2 = 5$. Using Proposition 3.2.30 we have that the number of generators of $\overline{J}_{3,10,1,1}$ is 28 and the number of generators of $\overline{J}_{5,10,1,2}$ is 64. In both cases the computation of the Betti numbers and j-reliabilities (and bounds) of this system is computed in less than one second by our algorithms.

□

Example 3.2.39. Let $n = 15$ and $k_1 = 2, k_2 = 5, k_3 = 7, k_4 = 9$. We consider multi-state consecutive k-out-of-n:G systems with sparse d such that the systems have 4 different working levels. Take $d = 3, 5, 7$. The components

n	k	d	j	num.gens.	reliability	time(s.)
15	2	3	1	50	0.9999996	0.004018
15	5	3	2	1281	0.99801	0.152161
15	7	3	3	4470	0.785976	2.04139
15	9	3	4	4565	0.0233618	4.49723
15	2	5	1	69	0.9999996	0.003935
15	5	5	2	2499	0.999723	0.343454
15	7	5	3	6219	0.857467	4.90422
15	9	5	4	4997	0.0250292	5.43077
15	2	7	1	84	0.9999998	0.00347
15	5	7	2	2919	0.999942	0.480944
15	7	7	3	6429	0.863406	5.61369
15	9	7	4	5005	0.0250567	5.40056

Table 3.14: Number of generators of the ideals of several multi-state consecutive k-out-of-n:G systems with sparse d and times to compute their reliability

are independent but non identical. The probabilities of each component being in the different sates are given as follows: $p_{i,0} = 0.1$, $p_{i,1} = 0.15$, $p_{i,2} = 0.2$, $p_{i,3} = 0.25$ and $p_{i,4} = 0.3$ if i is odd, and $p_{i,0} = 0.05$, $p_{i,1} = 0.15$, $p_{i,2} = 0.2$, $p_{i,3} = 0.25$ and $p_{i,4} = 0.35$ if i is even. The number of generators for the corresponding ideals and the reliabilities and computation times for each of these systems are given in Table 3.14. The column `reliability` indicates the probability that the system is at level j or higher.

□

Multi-state consecutive k-out-of-n:G systems with sparse d and maximum total gap m. In multi-state consecutive systems, Huan et al. [145] consider the situation in which a system may fail when a number of consecutive failures takes place or when a total number of failures (may be not consecutive) occur, see also [80]. This situation can be applied to multi-state consecutive k-out-of-n:G systems with sparse d and therefore

extend this model to a wider range of situations. We consider as before that two components whose states are l or above are consecutive if all the components between them are below state l and the number of such components is at most d, in addition, we say that k components $c_1, \ldots, c_k$ are consecutive only if they are pairwise consecutive in this sense and the total number of components in state l or below between c_1 and c_k is at most m. With this restriction we give the following definition:

Definition 3.2.40. A system with n components is called a generalized MS consecutive k-out-of-n:G system with sparse d and maximum total gap m if $\phi(X) \geq j$ $(1 \geq j \geq M)$ whenever there exists an integer value l $(j \geq l \geq M)$ such that at least consecutive k_l components are in state l or above with sparse d and the number of components below state l within them is at most m.

Let us denote by $J'_{k,n,d,m}$ the ideal of a binary consecutive k-out-of-n:G system with sparse d and maximum total gap m. Following the proof of Proposition 3.2.30 we have that the number of generators of $J'_{k,n,d,m}$ is given by a truncation of the sum in the number of generators of $J_{k,n,d}$.

Proposition 3.2.41. *Let $G(J'_{k,n,d,m})$ be the set of minimal generators of $J'_{k,n,d,m}$, we have that*

$$\#(G(J'_{k,n,d,m})) = \sum_{j=k-1}^{m} (n-j) \sum_{l=0}^{k-1} (-1)^l \binom{k-1}{l} \binom{j-l(d+1)-1}{k-2}$$

Observe that if $m \geq n-1$ then the system is a consecutive k-out-of-n:G system with sparse d.

Example 3.2.42. Consider the system in Example 3.2.35. We have $n = 9$, $k = 3$, $d = 2$, and set $m = 3$. Now, components 1, 4 and 7 are consecutive with sparse 2 but the total number of failed components between 1 and 7 is four, hence this would be a failure state for a consecutive 3-out-of-9:G system with sparse 2 and maximum total gap 3. For such a system the list of minimal monomial generators of its reliability ideal $J'_{k,n,d,m}$ has 42 elements and is given by the monomials $x^\mu = x_a x_b x_c$ for all triples abc in the following set (given in lex order)

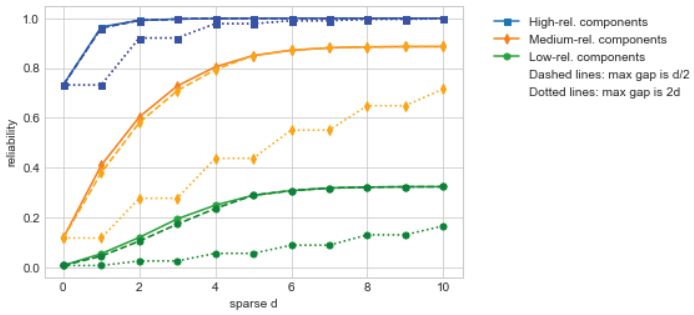


Figure 3.6: System reliability for consecutive 5-out-of-15 systems with sparse $d = \{0, \ldots, 10\}$ and maximum gap set as $d/2$ or $2d$. Systems with low, medium and high reliability components.

$$
\begin{aligned}
&123, 124, 125, 134, 135, 136, 145, 146,\\
&234, 235, 236, 245, 246, 247, 256, 257,\\
&345, 346, 347, 356, 357, 358, 367, 368,\\
&456, 457, 458, 467, 468, 469, 478, 479,\\
&567, 568, 569, 578, 579, 589,\\
&678, 679, 689,\\
&789.
\end{aligned}
$$

□

Example 3.2.43. Figure 3.6 shows the effect of setting the maximum gap at half the sparse and twice the sparse in the systems of Example 3.2.36. In both cases the reliability of the system is reduced as expected, by a small amount in case the gap is half the sparse, and by a bigger amount in case the maximum allowed gap doubles the sparse of the system.

□

3.2.3.6 Weighted multi-state k-out-of-n systems

The traditional binary k-out-of-n system model was extended by Wu and Chen [140] to *weighted* k-out-of-n systems. In a binary weighted k-out-of-

n:G system, component i has a weight w_i. The weight of each component represents the utility of the component, i.e. its contribution to the actual performance of the system. The system works if and only if the total weight of the working components is at least k, a pre-specified value. Observe that k may be larger than n. The multi-state version of weighted k-out-of-n systems was introduced by Li and Zuo in [82] where the authors define two variants of these systems.

Definition 3.2.44 (Weighted multi-state k-out-of-n system, model I)**.** In a system with n components, each component and system may be in $M+1$ possible states, $\{0,\ldots,M\}$. Component i $(1 \leq i \leq n)$, when in state j $(0 \leq j \leq M)$, has a utility value of $w_{i,j}$. The system is in state j or above if the total utility of all components is greater than or equal to k_j, a pre-specified value. Let ϕ be the structure function of the system representing the state of the system and W the total utility of all components. Then, this definition means $\Pr\{\phi \geq j\} = \Pr\{W \geq k_j\}$. Since state 0 is the worst state of the system, we have $\Pr\{\phi \geq 0\} = 1$.

For the second definition we consider only the contribution of those components in state j or above.

Definition 3.2.45 (Weighted multi-state k-out-of-n system, model II)**.** The system is in state j or above if the sum of the weights of the components whose states are in state j or above is greater than or equal to k_j. Let $\mathfrak{N}$ be the structure function of the system and W_j be the sum of the utilities of the components whose states are j or above. We then have $\Pr\{\phi \geq j\} = \Pr\{W_j \geq k_j\}$.

Since the structure functions of these kinds of systems depend strongly on the individual contributions or weights of each of the variables, the methods for computing its reliability are of an enumerative nature. Li and Zuo evaluate in [82] two such methods: a recursive one [140, 67] and the Universal Generating Function Method (UGF) [136, 85, 84]. The analysis in [82] shows that in general, the recursive approach is more effective than the UGF method for both models I and II. A key issue in the algorithmic evaluation of these systems' reliability is the efficiency in enumerating the working states for each level.

For any monomial ideal $I \subseteq \Bbbk[x_1, \ldots, x_n]$, a *quotient basis* is a basis as a $\Bbbk$-vector space of the quotient ring R/I. It amounts to a way of enumerating all the monomials that are not in I. In our case, considering $I = \langle x_1^{M+1}, \ldots, x_n^{M+1} \rangle$ we have that the $(M+1)^n$ monomials not in I correspond to all possible states of the system's components. In order to obtain the reliability of the system, we proceed state by state and add the probabilities of the states whose weight is above k_j for each j. This methodology might theoretically be less efficient than the recursive or the UGF approach, but a good implementation of the enumeration step can compensate this. This is indeed the case with the `CoCoALib` function `QuotientBasis` which is an efficient implementation of the enumeration of the elements in the $\Bbbk$-basis of R/I for any ideal I. Tables 3.15 and 3.16 show the results of some computer experiments in systems of the same kind as those in the experiments in [82]. In table 3.15 we set $n = 5$ and take M from 3 to 12. Each component's weight is a random integer number in the range $[1, 14]$ and probabilities are randomly assigned. In Table 3.16 the weights and probabilities are assigned in the same way but we keep M constant and equal to 5 while n varies from 3 to 11. In both cases we set $k_1 = 200$. In the tables, column TS indicates the total number of possible states of the system, column WS indicates the number of j-working states and column RS indicates the size of the set of minimal working states. The number of working states was obtained by exhaustive search on the possible states of the systems, and the number of minimal states was obtained by the minimization algorithm implemented in `CoCoALib`. For each of these numbers there is a column indicating the time used for its computation by the `C++` class described in [21]. Observe that [82] shows the results of another set of examples in which the weights assigned to the variables are floating point numbers in the range $[20, 50]$ this affects the performance of the UGF method, since there are less like terms to cancel, but it does not affect the recursive method. It does not affect the performance of our approach either, since our method is based on the efficiency of the algebraic approach to enumeration of all working states. Observe that in these tables the time for the computation of the reliability of the system is that in the `WS-time` column. The tables show computations

M	TS	WS	RS	TS-time(s.)	WS-time(s)	RS-time(s)
3	243	0	0	0.006094	0.006162	0
4	1024	0	0	0.007245	0.007462	0
5	3125	3	1	0.008029	0.008848	0.008869
6	7776	44	5	0.009567	0.011285	0.011382
7	16807	410	26	0.01666	0.018772	0.019281
8	32768	5398	293	0.028787	0.36573	0.044584
9	59049	16204	179	0.091856	0.101756	0.113455
10	100000	32768	390	0.150313	0.166822	0.193492
11	161051	76570	373	0.3017117	0.351811	0.413162
12	248832	121508	326	0.427333	0.482606	0.583472

Table 3.15: $n = 5$, $k_1 = 200$, average of ten runs

Model I systems, the algorithms and results are essentially equivalent for Model II.

The described procedure computes the exact reliability of weighted k-out-of-n systems. In case one is interested in the algebraic bounds as obtained in the previous sections, we need to consider the ideal generated by the monomials corresponding to working states of the system. In this case, the first step is to obtain the set of minimal working states, or equivalently the minimal set of generators of the corresponding ideal. The complexity of this procedure, starting with the complete set of working states is $O(r^2)$ were r is the total number of working states of the system. It is, therefore, an impractical procedure for large systems. In Tables 3.15 and 3.16 the time for computing the minimal generating set of the j-ideal, i.e. the minimal set of working states is given under column RS-time. The size of these sets are under column RS and one can see that these sizes are significantly smaller than that of the set of working states, hence it would be worth investigating efficient ways of obtaining these sets directly. This would imply a drastic reduction of the computing time of the reliability and bounds for weighted multi-state k-out-of-n systems.

n	TS	WS	RS	TS-time(s.)	WS-time(s)	RS-time(s)
3	125	0	0	0.00592	0.005949	0
4	625	0	0	0.006535	0.006662	0
5	3125	0	0	0.006134	0.006803	0
6	15625	20	7	0.15506	0.017679	0.017708
7	78125	3873	302	0.01859	0.036595	0.04394
8	390625	46945	2106	0.11527	0.120179	0.214819
9	1953125	224695	3561	0.155288	0.422814	0.889363
10	9765625	1662628	24131	0.738989	2.4139	18.5923
11	48828125	10022724	31389	3.812	12.1875	64.2796

Table 3.16: $M = 5$, $k_1 = 200$, average of ten runs

Chapter 4

Algorithms for Multi-state Systems

The development and implementation of efficient algorithms for system reliability computations is an important task in reliability engineering. Many algorithms exist, and are available to the community in a variety of forms. Some are included in large versatile commercial systems [3, 1, 2], others are offered as packages, functions or libraries in mathematical software systems of general purpose languages, for example `Matlab` [81, 109], `Python` [110] or `R` [125]. Others still are directly distributed by the authors as stand-alone software, like `SHARPE` [121].

This chapter is devoted to describe a `C++` class which implements the algebraic approach to system reliability described in this thesis and can be integrated in other software systems. In this way it may become available in different forms and toolboxes to researchers, software developers and reliability engineers. The language `C++` (now in its version 20, standard ISO/IEC 14882 : 2020) is a widely used [4] object-oriented general purpose computer language, both for very large systems and for small ad-hoc applications. Among the virtues of `C++` is its integrability with other languages and also its high performance, meeting the need for fast and reliable computations. Our class is implemented in the `CoCoALib` library [5], which is a `C++` library for Computations in Commutative Algebra, currently at its version `0.99800` (March 2022). It is open source and free.

The main feature of the C++ class introduced in this chapter is that it is applicable to a large variety of systems, with or without a known identifiable structure, and can be used to compute the reliability (and bounds) of systems having independent identical or non-identical components. The performance of this class is good in terms of time requirements, being able to compute the reliability of systems with hundreds of components and tens of thousands of minimal paths or cuts. Even though there exist optimized algorithms for several kinds of systems which are faster than the ones presented here, ours are useful to analyze systems for which no specialized algorithms are known, and to benchmark new algorithms for particular types of systems.

All the code of the class and the examples, which have been developed as a joint work with Anna M. Bigatti and Eduardo Sáenz-de-Cabezón, are available at

`http://www.dima.unige.it/~bigatti/data/AlgebraicReliability/`

4.1 Algebraic reliability class in CoCoALib

The good performance of an algorithm depends also on the efficiency of its implementation. In this section we give the interested reader some technical details on the implementation of our algorithms and some of the decisions we made, like the choice of data types and the structure of the algorithms. These decisions contribute to the actual performance of the algorithms in terms of memory usage and CPU time. Also, we describe the CoCoALib library, which provides convenient implementations of the main algebraic structures we need. We hope these descriptions, although not fully detailed, make it easier for engineers and reliability practitioners to practically use these algorithms or incorporate them into their own software, and also make it easy to reproduce our results, experiments and benchmarks.

4.1.1 CoCoALib

`CoCoALib`, for Computations in Commutative Algebra, is an open source `C++` software library principally based on multi-variate monomials and polynomials and devoted to algebraic geometry. It is the computational core of the CoCoA software system [6]. A crucial aspect of CoCoALib is that it was designed from the outset to be an open-source software library. This initial decision, together with the desire to help the software prosper, has many implications: *e.g.* designing a particularly clean interface for all functions with comprehensive documentation. This cleanliness makes it easy to integrate CoCoALib into other software in a trouble-free manner. The library is fully documented, and also comes with about 100 illustrative example programs. CoCoALib reports errors using `C++` exceptions, while the library itself is exception-safe and thread-safe. The current source code follows the `C++14` standard. The main features of the design of CoCoALib are:

- it is well-documented, free and open source `C++` code (under the GPL v.3 licence);
- the design is inspired by, and respects, the underlying mathematical structures;
- the source code is clean and portable;
- the user function interface is natural for mathematicians, and easy to memorize;
- execution speed is good with robust error detection.

The design of the library (and its openness) was chosen to facilitate and encourage "outsiders" to contribute. There are two categories of contribution: code written specifically to become part of CoCoALib, and stand-alone code written without considering its integration into CoCoALib. The library has combined some of the features of various external libraries into CoCoALib. Such as *Frobby* (see [111]) which is specialized for operations on monomial ideals. Other integrations are with *Normaliz*, a library for

computing with affine monoids or rational cones and *GFanLib* which is a C++ software library for computing Gröbner fans and tropical varieties.

4.1.2 The class description

We have implemented within CoCoALib a set of C++ classes for making computations in algebraic reliability. The UML class diagram is depicted in Figure 4.1.

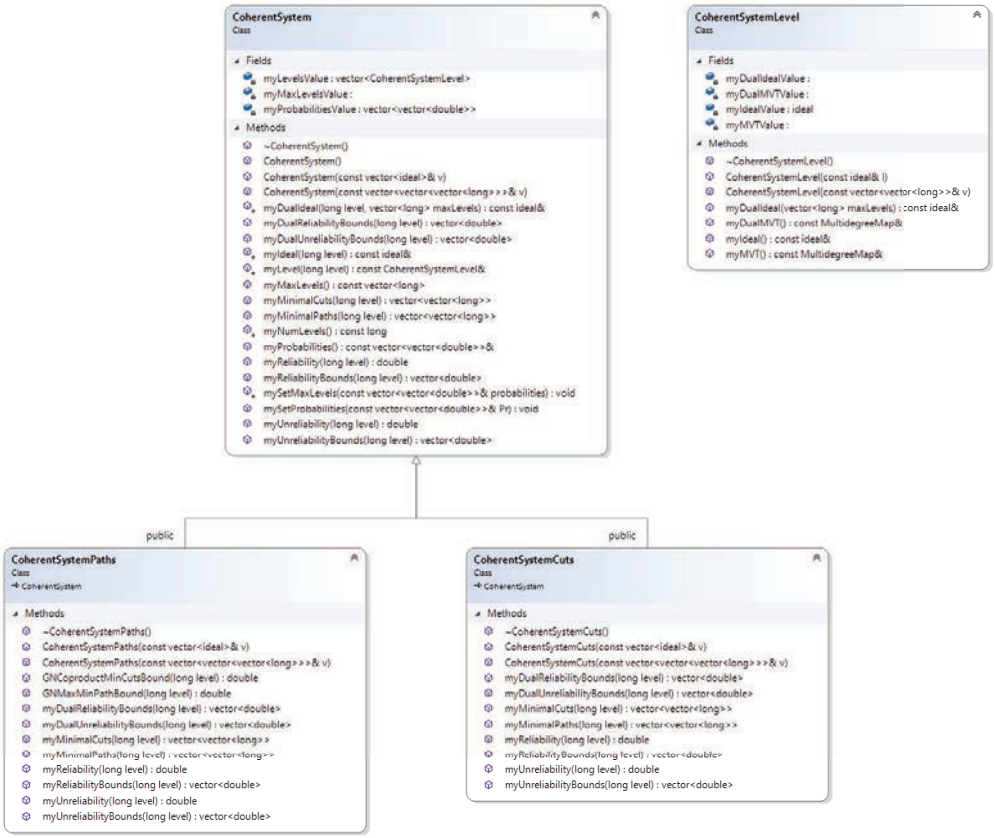


Figure 4.1: UML diagram of the CoherentSystem class

Our main class is the *abstract class* CoherentSystem which consists of a series of levels and a matrix of probabilities. The levels are stored in a std::vector (an efficient structure of the C++ language) in which

each component is an instance of the class `CoherentSystemLevel`, and the probabilities are given by a `vector` of `vectors` of type `double` where the j'th entry of the i'th vector corresponds to $p_{i,j} = p(s_i \geq j)$, the probability that the level of the i'th component of the system is bigger than or equal to j. Each instance of the class `CoherentSystemLevel` consists basically of an ideal and its dual, which are objects of the CoCoALib class `ideal`. Also, we store as member fields their Mayer-Vietoris trees, which play the role of multigraded free resolutions optimized for monomial ideals.

The *concrete classes* inheriting from the class `CoherentSystem` are the ones called `CoherentSystemPath` and `CoherentSystemCuts` which respectively represent :G systems in which the levels and probabilities denote working states, and :F systems in which the levels and probabilities represent failures.

For any instance of these two concrete classes, and hence of the abstract class `CoherentSystem` we can call the following member functions:

myMinimalPaths Receives a level and gives a vector of vectors of type `long`. Each of these vectors is a minimal path of the system at the given level.

myMinimalCuts Receives a level and gives a vector of vectors of type `long`. Each of these vectors is a minimal cut of the system at the given level.

myReliability Receives a level j and computes $R_j(\mathcal{S})$.

myUnreliability Receives a level j and computes $U_j(\mathcal{S})$.

myReliabilityBounds Receives a level j and computes bounds for $R_j(\mathcal{S})$ given by the resolution obtained by the Mayer-Vietoris tree of $I_j(\mathcal{S})$.

myUnreliabilityBounds Receives a level j and computes bounds for $U_j(\mathcal{S})$ given by the resolution obtained by the Mayer-Vietoris tree of $I_j(\mathcal{S})$.

In addition, for :G systems given by their sets of paths, we have implemented two more bounds, the ones presented in Section 3.1.1.1:

GNMaxMinPathBound Which corresponds to $l_p^{\prime j} hi(\mathbf{p})$.

GNCoproductMinCutsBound Which corresponds to $l_\phi^{**j}(\mathbf{p})$.

When computing the functions `myReliability`, `myUnreliability`, `myReliabilityBounds` or `myUnreliabilityBounds` the object checks its ideal and its dual ideal, and chooses whichever of them has a smallest number of minimal generators to perform the actual computation. To compute duals of ideals we use the `Frobby` library, in particular the function `FrbAlexanderDual` which is in general a fast computation. Once the ideal is chosen, we check whether the system has already computed its Mayer-Vietoris tree. If it is not yet computed, it is computed and stored in the corresponding class member field. Then the Mayer-Vietoris tree is used to retrieve the required value or bounds for reliability or unreliability.

4.2 Examples of use

In this section we apply our `C++` class to some examples of reliability computations. We use binary networks and multi-state systems. We consider systems in which their components have independent identically distributed probabilities as well as systems in which the components' probabilities are independent but not identically distributed. All the computations in this section have been implemented and executed in an HP Z-book laptop[1].

4.2.1 Test examples

First, we validate our algorithms with a set of diverse examples of multi-state systems found in the literature. We selected systems of different nature so that we can test our algorithms with examples featuring different characteristics. Table 4.1 shows the results of these tests. The first column of the table indicates the name of each example (see description below), n indicates the number of variables and M the number of levels of the

[1]CPU: intel i7-4810MQ, 2.80 GHz. RAM: 16Gb

system (not counting the complete failure level or level 0). Column M_i indicates the number of levels of each component and column $gens(I_j)$ indicates the number of minimal generators of the j-reliability ideal for each level $j = 1, \ldots, M$. The set of test examples consists of the following:

- `Army Battle Plan` is taken from the classical paper [28]. It is a customer-driven multi-state system with 5 different states and 4 components (two binary components and two three-level components), the probabilities of the different components are independent but not identical.

- `Bin.S-P` is a binary Series-Parallel system taken from [85] (Example 4.5) which has seven independent not identical components and two levels.

- `MAX+MIN,TIMES` is a multi-state system with 5 components and 7 levels whose structure function is given by

$$\phi(x_1, \ldots, x_5) = (\max\{x_1, x_2\} + \min\{x_3, x_4\}) \times x_5,$$

  and the details on components' and system's levels and probabilities (not i.i.d) are given in [84], Example 4.7.

- `Bridge Flow Network` is a multi-state network with 5 edges with different weights considered as flows. The states of the system are given by the possible flows through the network. The example considers the probability of a total flow of at least three units (*i.e.* the system is in level $j = 3$). The details on states and probabilities are given in [127] Example 4, see also [133] Example 5.14.

- `Dominant MS binary-imaged` system is a multi-state system with three i.i.d. components. Both the system and components have four different states. It is presented as Example 12.21 in [75] to illustrate the concept of multi-state dominant binary-imaged system.

- `MS Cons.k-out-of-n` is a multi-state consecutive k-out-of-n system with 3 components and three levels. It is example 12.18 in [75].

Example	n	M	M_i	$gens(I_j)$
`Army Battle Plan`	4	4	2,2,3,3	2,4,5,5
`Bin.S-P`	7	1	1 $\forall i$	3
`MAX+MIN,TIMES`	5	6	3,2,2,3,2	4,3,4,3,2,1
`Bridge Flow Network`	5	3	3,1,2,1,2	3 for $j = 3$
`Dominant MS binary-imaged system`	3	3	3,3,3	3,2,1
`MS Cons. k-out-of-n`	3	3	3,3,3	1,2,1
`Gen k-out-of-n:(3,2,1)`	10	3	3 $\forall i$	175,55,10
	30	3	3 $\forall i$	4525,465,30
	50	3	3 $\forall i$	20875,1275,50
`Gen.k-out-of-n:(3,4,2)`	10	3	3 $\forall i$	165,245,45
	30	3	3 $\forall i$	4495,27840,435
	50	3	3 $\forall i$	20825,231525,1225
`Gen.k-out-of-n:(4,3,2)`	10	3	3 $\forall i$	375,165,45
	30	3	3 $\forall i$	31900,4495,435
	50	3	3 $\forall i$	251125,20825,1225

Table 4.1: Test examples of multi-state systems.

4.2.2 Source to terminal networks

One of the main problems in reliability engineering is Network Reliability, see for instance [10, 133] for a comprehensive account and [56] for a recent algorithm. In this problem we consider a network in which one vertex is selected as *source vertex* and one or more vertices are selected as *target vertices*. Each of the connections in the network has a certain probability to be working, and the problem is to compute the probability that there exists at least one source-to-target path composed by operational connections. Usually the networks are binary *i.e.* the system and all of its components have only two possible levels, although the multi-state version has also been considered [9, 146].

4.2.2.1 GARR: Italian Research and Education Network

Our first example is the GARR Italian network. The motivation to use this example is to show the performance of our algorithms in a real-life

S-node	T-node	# Minpaths	i.i.d probabilities			non i.i.d.
			0.9	0.95	0.99	
TO	CT	212	0.977344	0.994704	0.999798	0.994352
TS1	NA	223	0.985203	0.996890	0.999895	0.993917
TO*	CT*	196	0.977428	0.994713	0.999797	–
TS1*	NA*	168	0.975771	0.994486	0.999795	–

Table 4.2: Reliability computations for the GARR 2008 network.

system that has already been studied in the literature. Figure 4.2 shows the official 2008 map of the backbone of the GARR network in Italy, which interconnects universities, research centers, libraries, museums, schools and other education, science, culture and innovation facilities, see `http//:www.garr.it`. The network was at the time formed by 41 nodes and 52 connections. Table 4.2 shows the results of some reliability computations in this network. First, we use `TO` as source node and `CT` as terminal node, and then we use `TS1` as source node and `NA` as terminal node. In both cases we consider an identical independent probability p for all the connections and compute the source to terminal reliability for $p = 0.9, 0.95$ and 0.99. The last two rows in Table 4.2 correspond to the same computations in [133] (Example 5.7), observe that the differences are due to the fact that the authors in [133] use a slightly different network which has 42 nodes and hence some different connections and a different number of minimal paths in each example. Since our algorithms can also treat the case of non-identical probabilities, we assigned probability 0.99 to all 10Gbps connections (4 connections), 0.95 to all 2.5 Gbps connections (14 connections) and 0.9 to the rest of the connections. The results are shown in the last column of the table. All our computations in this table took less than one second.

4.2.2.2 Random networks

Our second example is a randomly generated set of networks. This is a convenient way to generate a big number of examples not having a regular structure (like for instance series-parallel systems or k-out-of-n systems

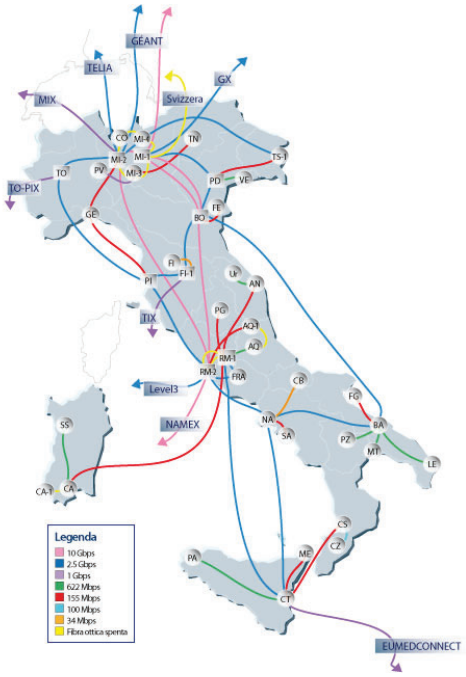


Figure 4.2: Map of the GARR network in 2008

and variants), and therefore represents a good set of benchmarks for the application to general systems. We demonstrate our algorithms' performance in several random networks generated following the Erdős-Rényi model $ER(n, p)$ [42] and Barabasi-Albert model $BA(n, m)$ [12]. These models generate networks with different characteristics such as degree distribution, modularity, etc. We compute the reliability of 100 random Erdős-Rényi networks with $n = 40$, $p = 0.05$ and 100 Barabasi-Albert networks with $n = 10$ and $m = 4$; we chose randomly one source and one terminal node in each case. The number of minimal paths varies between 100 and 1000 in both cases. However, the relation between the number of minimal paths and minimal cuts is significantly different in the two types of networks. Erdős-Rényi networks tend to have many more minimal cuts with respect to the number of minimal paths, while the situation is the opposite for Barabasi-Albert networks, see Figure 4.3. In the case of Barabasi-Albert networks our algorithms compute the reliability of

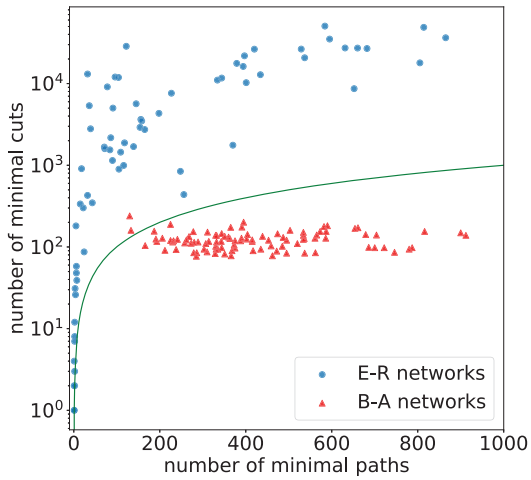


Figure 4.3: Number of minimal paths and minimal cuts in Erdős-Rényi and Barabasi-Albert networks. The line indicates number of minimal paths equals number of minimal cuts.

the network using the dual ideal, since it is smaller in most cases. The reliability of the Erdős-Rényi examples was always computed using the minimal path ideal. Times for the computation of the reliability of these networks are shown in Figure 4.4. The figures show that the times depend greatly on the number of minimal paths, but the topology of the network influences the algebraic characteristics of the ideals. Observe that there are two cases of Barabasi-Albert graphs in which the number of minimal paths is smaller than the number of minimal cuts and hence the path ideal was used for the computation, which results in higher computation times compared with the cases in which the dual was used. The resolutions of these networks ideals are much shorter in the dual case and hence the results.

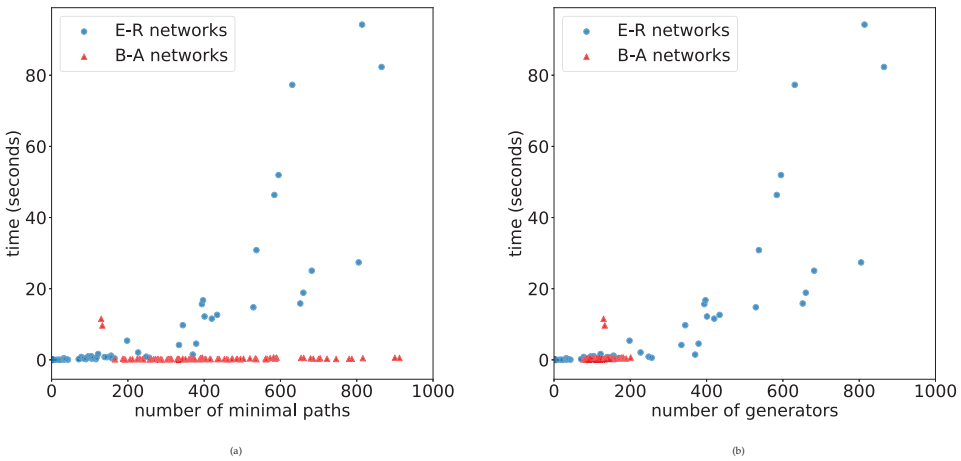


Figure 4.4: Times for reliability computation on Erdős-Rényi and Barabasi-Albert graphs.

4.2.3 Multi-state generalized k-out-of-n systems

Our final example is multi-state generalized k-out-of-n systems. We include this example since they are one of the most important types of systems studied in the reliability engineering literature, both in their binary and multi-state versions.

We have used our `C++` class to compute the reliability of several generalized k-out-of-n systems. Since each of these systems is given by a vector $(k_1, \ldots, k_M)$ we generated randomly 100 vectors for systems with four levels, and 10 variables. Figures 4.5 (a) and (b) show the number of minimal paths and minimal cuts of these systems and the computing time of these examples vs. the number of generators used for its computation in each case. All systems considered have components with independent, non-identical working probabilities. The figures show that most of these systems have a smaller number of minimal paths compared to the number of its minimal cuts, and that the computation time depends greatly on the structure of the system. Let us denote by k the maximum of the integers k_l for $l \in \{1, \ldots, M\}$. Figures 4.6 (a) and (b) show the number of minimal

paths and cuts for systems with 12 components, 4 levels and $k = 4$, $k = 6$. The number of minimal cuts and paths of multi-state generalized k-out-of-n grows as $\binom{n}{k}$. The performance of our algorithms depend greatly on the number of minimal paths or minimal cuts, as can be seen in Figure 4.6 (c). There exist specialized algorithms for this kind of systems that are recursive on M, see [151, 105] or based on Decision Diagrams [95].

4.2.4 Computational complexity

The algebraic method is (in its general form) an enumerative method, similar to the inclusion-exclusion approach but less redundant. The compact form of the Hilbert series provided by our algorithms gives some computational advantages, but there exist certain intrinsic limitations due to the complexity of the problem. The computation of network reliability (either k-terminal, 2-terminal or all-terminal) is a $\#P$-hard problem [11] and hence there is no hope of finding an efficient algorithm for computing the reliability of general systems unless $P = NP$, even in the binary

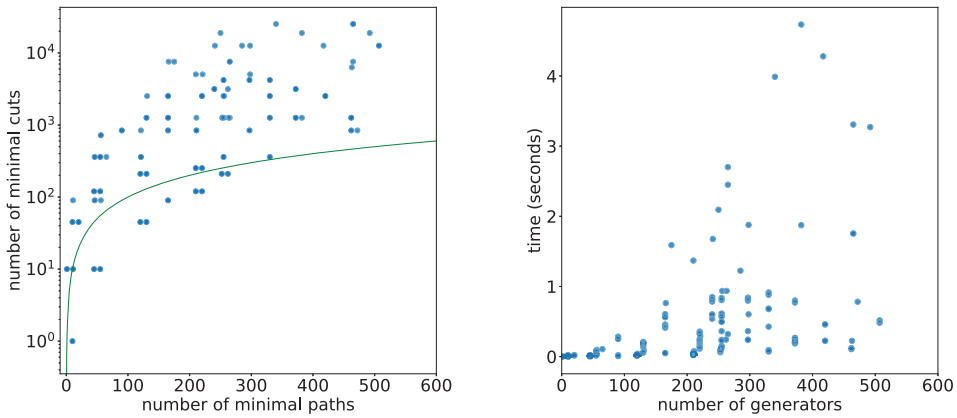


Figure 4.5: Number of minimal paths vs. number of minimal cuts and computing time for generalized multi-state k-out-of-n systems with 10 components, 4 levels, and non-identical probabilities.

case. The algebraic method in which our algorithms are based shows that the problem of computing the reliability of a multi-state system can be polynomially reduced to the computation of the multigraded Hilbert series of a monomial ideal. This problem belongs to the class of #P-hard

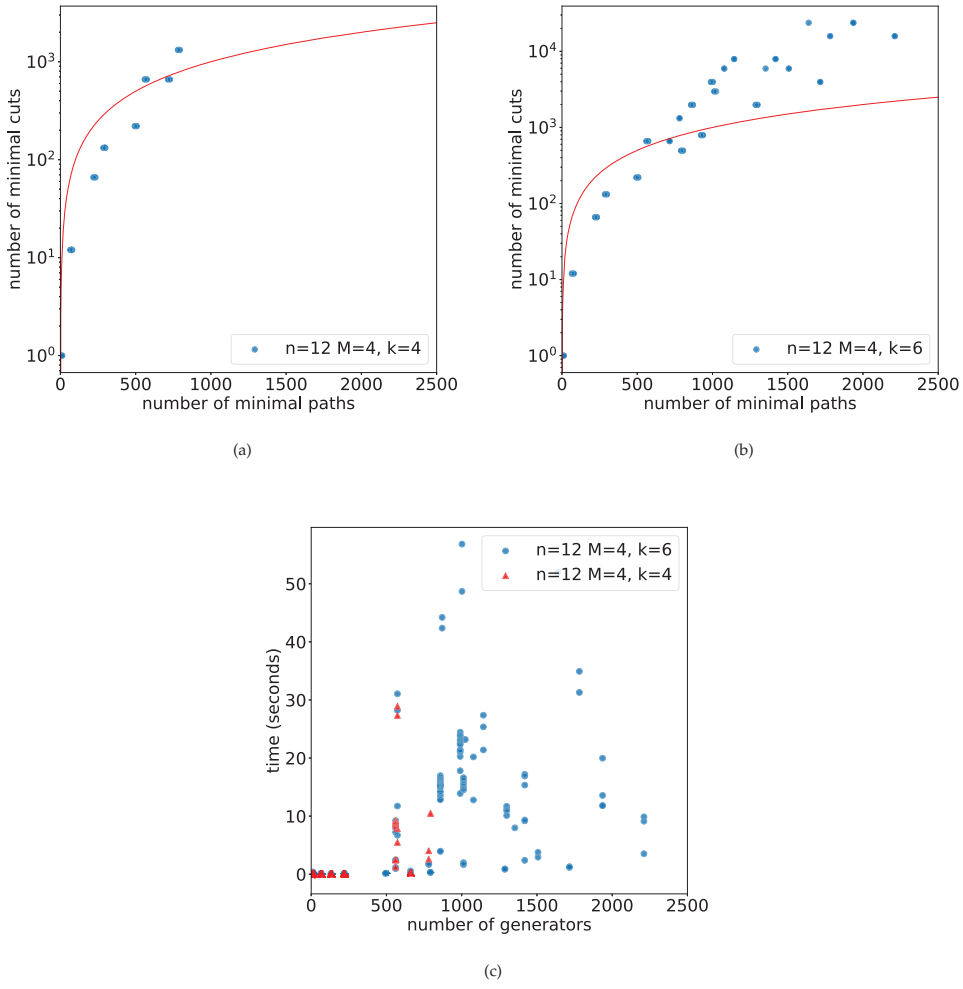


Figure 4.6: Number of minimal paths vs. number of minimal cuts and computing time for generalized multi-state k-out-of-n systems with 12 components, 4 levels, $k = 4, 6$, and non-identical probabilities.

problems and there exist several sub-problems of it that are known to be #P-complete or NP-complete. In particular, the problem of computing the Euler characteristic of an abstract simplicial complex is equivalent to the computation of the coefficient of the monomial $x_1 \cdots x_n$ in the multigraded Hilbert series of a (square-free) monomial ideal, and this problem belongs to the #P-complete complexity class [112].

There are two complementary directions to follow for finding satisfactory solutions for these problems. One is to develop specialized polynomial algorithms for particular families of systems. The other is to find algorithms showing good heuristic behavior when applied to general problems. In these two directions it is of paramount importance to develop good implementations in terms of data types, memory management, etc. that make the algorithms applicable to practical problems.

The algebraic method for system reliability contributes to both of the directions described above. On the one hand, the study of the structure of the ideals of particular classes of systems can provide efficient algorithms or even formulas (explicit or recursive) for their Hilbert series, see [117, 118] for k-out-of-n and consecutive k-out-of-n binary systems, and [105] for the multi-state version. As an example, the formulas for k-out-of-n systems have complexity $O(n^2)$ which is quadratic, but not optimal when restricted to systems with statistically independent components, for which the algorithm in [17], based on the Fast Fourier Transform, runs in complexity $O(n(\log n)^2)$. On the other hand, for the general case we used efficient algorithms for computing the multi-graded Hilbert series of monomial ideals and Alexander duals. These algorithms avoid much of the redundancy that shows up in reliability computation of general systems, when we have no evident structure to take advantage of. Besides, they make use of the recursive nature of the problem, which has also been used in other approaches like the Universal Generating Function method. However, there is still room for improvement. As the UGF and other methods show, it is important, for the sake of efficiency, to identify good base cases for the recursion, and for simplification techniques. The algorithms provided use only *algebraic* base cases and simplifications, and hence it is expected to gain efficiency by exploring other base cases that arise from

the knowledge of system reliability. Finally, as it is common in computer algebra, implementations of general algorithms which are good enough for NP-hard or $\#P$-hard problems offer good performance in practice. A paradigmatic example of this are the good algorithms for Gröbner bases, a problem whose complexity is known to be doubly exponential. This is the case of the class presented, in which we took advantage of the data types and optimized routines provided by `CoCoALib` together with good implementations for the Hilbert series and Alexander dual algorithms. This allows us to efficiently compute the reliability of general systems of big size with an affordable use of time and memory resources.

Chapter 5

Future work

The directions explored in this thesis have opened some ways of continuity, both in a theoretical, applied and computational framework.

5.1 Theoretical aspects

Regarding the ways we can follow on the theoretical research, from Section 2.3 we know that not every poset is the support poset of a monomial ideal. In this thesis we partially answered the question

Question 5.1.1. *How can we characterize those posets that are realizable as the support poset of a monomial ideal?*

We have seen that forests and collections of disjoint diamonds are in the category of realizable posets. However, we did not characterize all of them. From this point we will look for more support posets that are realizable as a support poset of some monomial ideal and, then if there are characteristics shared between the families found. However, this problem is very complex and will take long time to properly answer this question. In fact, maybe is easier to try to answer the same question about posets that cannot be realized as support posets:

Question 5.1.2. *How can we characterize those posets that are not realizable as the support poset of a monomial ideal?*

Another open question arises from the fact that a given poset can be seen a the support poset of several ideals which have different properties, cf. Remark 2.3.33. Even within the class of forests and series-parallel ideals we have seen that a given tree can be the support poset of different series-parallel ideals, cf. Example 2.3.38, these may have different number of generators, hence different Betti numbers, projective dimension, regularity, etc. A wide open question is then to find relations between ideals having the same support poset and properties of the ideal that can be read off the support poset:

Question 5.1.3. *What properties are shared by ideals that have the same support poset?*

Attempts to answer these questions will prove the usefulness of the support poset as a tool to study monomial ideals and their structure.

Furthermore, in a talk given by Professor Federico Ardila at University of Genova about matroids (see [104]), we found that support posets are possibly related with these objects. A finite matroid is equivalent to a geometric lattice ([129]) *i.e.* a finite, atomistic (every element is a join) and semimodular (for every two elements a and b, if $a \wedge b <: a$, then $b <: a \vee b$, where) lattice. We believe there exists a way for turning a support poset into a matroid and we can obtain valuable information from both of them. The topic was not closely related with the goals of the research so we leave it as a future direction to explore.

Another interesting question, although out of the scope of this thesis, is to compare the bound obtained for projdim in Theorem 2.3.13 with other bounds for the projective dimension of monomial ideals, like the ones in [32, 33].

5.2 Applications to system reliability

When thinking on the applications of this research to Reliability Theory, we also have some open paths for going further.

- During a visit of Professor Fatemeh Mohammadi to University of La Rioja, we started to look into measures of importance in a system

(see, for instance [74]). We saw that it might be possible to use support posets for explore this field.

- Another way to explore is modules of systems. A module of a system is, summing up, a subsystem. It is interesting to explore when one can gen some profit of using them or not, *i.e.* when is convenient to decompose a big system into smaller ones and compute the reliability of them. Support posets are believed to be a useful tool for answering this question. We believe that each component of the support poset corresponds, at least, with a module of a system and could ease computations.

- Together with Rodrigo Iglesias and Professor Eduardo Sáenz-de-Cabezón, we are exploring an algebraic version of the sum-of-disjoint-products method (see [47, 88]) for multi-state system reliability analysis. For going further on this topic, we employ involutive basis ([122, 124]).

- Future research on multi-state systems will continue to combine Markov and other models of movement between states with the ideal theory describing the detailed structure of failure. In addition, we would like to study the relations and possible combination with other related methods like the Universal Generating Function method ([76]).

5.3 Computational aspects

Regarding computations and computer implementations, our future work includes the design of specialized algebraic algorithms for particular kinds of systems. The structure of particular systems induces a particular structure in the associated ideals which can be studied using algebraic and combinatorial tools allowing the design of more efficient algorithms.

Another direction of improvement is to adapt the algorithm for systems with non-independent components or try to use continue variables rather than discrete (then, instead of working with probabilities we will use

probability distributions). The algebraic theory is exactly the same and only the probability assignment to the computed monomials need to be changed. This would give a wider flexibility to the C++ class presented on Chapter 4.

The last point of improvement regarding our C++ class, is tuning and optimizing the existing algorithms in order to improve their efficiency and reduce the computing times.

Finally, we are developing another C++ class for compute polarization and depolarization for monomial ideals.

Summary

In this thesis we study the reliability of multi-state systems using an algebraic approach based on monomial ideals.

Chapter one is devoted to introduce the tools needed along this research. Despite the fact that it is a chapter with no original results, it serves to settle the theoretical foundations both in the algebraic and in the reliability field.

The first section of the chapter is dedicated to monomial ideals, an indispensable object for the development of this doctoral thesis.

Further on the chapter, resolutions are presented. Resolutions are exact sequences defined over a module and the relations existing between its generators. When working with graded modules, another relevant notion for this research can be defined, which is the minimal resolution. The minimal resolution will allow us to compute, in addition to the reliability os a system as other resolutions, the tightest bounds within the enumerative ones.

Then, we make a review on Hilbert series, whose numerator enumerates all the monomials belonging to a monomial ideal. The Hilbert series of a monomial ideal can be calculated in different ways. One of them is computing a free resolution of the ideals and writing its numerator as the alternate sum of its ranks. When the resolution is minimal, the ranks have an especial name: Betti numbers. In fact, Betti numbers are going to proportionate the tightest bounds.

For a deeper study, references [62, 31, 114] are recommended.

Chapter one ends exploring reliability of binary systems. It starts making a little review of the development of the field. Then the proper

definition of a binary system is given, together with some important objects related to them such as paths, cuts or state space. Then, the algebraic method with an algebraic approach is presented by exploring the relationship between the squarefree monomial ideals and binary systems and, then, by computing its reliability or bounds for it. For going further on this topic we recommend [75, 116, 117, 118, 119, 120].

We start chapter two dealing with polarization, an operation that transforms an arbitrary monomial ideal $I = \langle m_1, \ldots, m_r \rangle \subseteq R = \mathbb{k}[x_1, \ldots, x_n]$ into a squarefree one $I = \langle \overline{m_1}, \ldots, \overline{m_r} \rangle \subseteq S = \mathbb{k}[y_1, \ldots, y_m]$, where $m \geq n$. When one polarizes a monomial ideal, the property of being squarefree is obtain but, in return, the polynomial ring in which the new ideal lies has a bigger number of variables.

Depolarization is the inverse operation if polarization,*i.e.* given a squarefree monomial ideal, depolarization turns it into a monomial ideal with exponents. Although polarization is unique, depolarization is not: when a squarefree monomial ideal is depolarized it could be more than one monomial ideals as a result. Figure 5.1 sums up this idea.

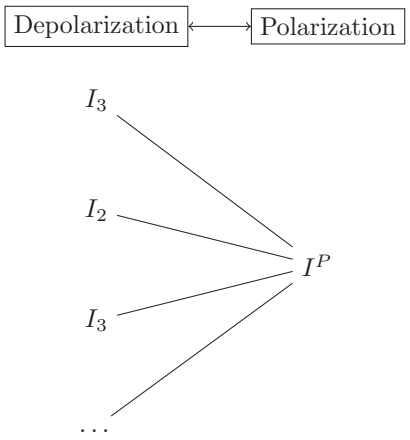


Figure 5.1: Graphic representation of que uniqueness and not uniqueness of the operations polarization and depolarization

The aim of using polarization and depolarization operations is that sometimes exist explicit formulas for computing certain properties of a monomial ideal which are better in the case of being squarefree (or vicev-

ersa). Polarization and depolarization preserve some algebraic invariants such as Betti numbers or the Hilbert series (not are always exactly the same but ease the calculations). Section 2.3.2.3 is devoted to investigate properties transferred by the ideals who shares the same polarization.

Then, support posets are introduced. The support poset is a combinatoric tool which allows to compute all the depolarizations of a squarefree monomial ideals (see Theorem 2.3.8).

The question that may naturally arise is if all the posets are realizables as a support poset of a monomial ideal, *i.e.* if given a poset C there exist a monomial ideal I such that its support poset is C. Following this idea, Proposition 2.3.1 gives a sufficient condition to construct monomials ideals with a given support poset.

Going further in the chapter, some families of posets that are realizable as support posets are shown. Some of this families are set of lines or diamonds and forest (see Proposition 2.3.18, Proposition 2.3.23 and Theorem 2.3.30). Furthermore, certain properties of the ideals which has the support poset as just mentioned are explicitly calculate such as its Betti numbers.

The chapter ends describing the support poset of k-out-of-n and series-parallel ideals.

In the following chapter we study of algebraic reliability of multi-state systems . A multi-state system is a system which can reach more than to levels of performance. For example, given the lighting system of a city, the failure of the lights of a street does not mean the failure of the entire system, but it does not mean that it is perfectly working. From Chapter 1 it is knows that the reliability of a binary system is the probability of being in a working state. Yet, for multi-state systems the notion of j-reliability is needed, which is the probability that the system is working, at least, at level j.

In Section 3.1.1 the algebraic method based on monomial ideals for computing multi-state system reliability is showed, together with some examples of its functioning.

Going further in the chapter we deal with multi-state k-out-of-n systems, a kind of system which has high relevance in the field due to its wide

range of applications. Firstly, a review of the different definitions they have received along the literature is done. In Proposition 3.2.4, Definition 3.2.13 and Proposition 3.2.13 the definitions aforementioned are proposed in terms of monomial ideals. Besides, a way to compute these ideals and its Betti numbers on a recursive way. Section 3.2.3.2 shows a comparative between the bounds presented on Section 3.1.1.1 and the ones obtained with the algebraic method.

Finally, some variants of k-out-of-n are treated, such as the binary k-out-of-n systems with multi-state components, the multi-state consecutive k-out-of-n, the sparsely connected homogeneous multi-state k-out-of-n and the weighted multi-state k-out-of-n systems. For all of them, we define its associated algebraic structures and explicit formulas for its Betti numbers.

This thesis ends showing a `C++` class implemented with the free software `CoCoALib` and devoted to compute the reliability (and bounds) of multi-state systems. The algorithm for computing the reliability is based on the algebraic approach studied in this thesis. The code and some examples are available at

`http://www.dima.unige.it/~bigatti/data/AlgebraicReliability/`

We first introduce `CoCoALib`, a `C++` library created to make Computations in Commutative Algebra. It was developed by John Abbot and Anna Maria Bigatti. It is based on multivariate monomials and polynomials and, currently, is at its `0.998000` version.

Going further in the chapter, an explicit description of the implemented class can be found. Its `UML` is the one depicted on Figure 4.1 on Chapter 4.

Next section shows some examples and the results using the class. Firstly, the class is validated with diverse multi-state systems appearing in the literature, such us the *Army Battle Plan* from [26]. Then, we compute network reliability. In particular, the examples of networks used are the *GAAR: Italian Research and Education Network* and randomly generated networks based on Erdős-Rényi $ER(n, p)$ and Barabasi-Albert $BA(n, m)$ models. This section ends showing how the implemented class behaves

with two multi-state k-out-of-12 systems with 4 levels of performance and taking $k = 4$ and $k = 6$.

Finally, computational complexity of the algebraic method implemented in the class are treated.

The results of this thesis have been published on

- Mohammadi, F., Pascual-Ortigosa, P., Sáenz-de-Cabezón, E., and Wynn, H. Polarization and depolarization of monomial ideals with application to multi-state system reliability. *Journal of Algebraic Combinatorics 51* (2020), 617-639.
  JCR: Q3 (JIF), Q2 (JCI), 2020; SJR: Q1, 2020.
  https://doi.org/10.1007/s10801-019-00887-6

- Pascual-Ortigosa, P., and Sáenz-de-Cabezón, E.. Support posets of some monomial ideals. *Applicable Algebra in Engineering, Communication and Computing* (2020), 1-19.
  JCR: Q3 (JIF), Q4 (JCI), 2020; SJR: Q2 (2020).
  https://doi.org/10.1007/s00200-020-00461-9

- Pascual-Ortigosa, P., Sáenz-de-Cabezón, E., and Wynn, H. Algebraic reliability of multi-state k-out-of-n systems. *Probability in the Engineering and Informational Sciences 35* (2021), 1005.
  JCR: Q3 (JIF), Q3 (JCI), 2020; SJR: Q3 (2021).
  https://doi:10.1017/S0269964820000455

- Pascual-Ortigosa, P., and Sáenz-de-Cabezón, E.. Algebraic Analysis of variants of multi-state k-out-of-n systems. *Mathematics* (2021), 9(17), 2042.
  JCR: Q1 (JIF), Q1 (JCI), 2020; SJR: Q2 (2021).
  https://doi.org/10.3390/math9172042

- Bigatti, A., Pascual-Ortigosa, P., and Sáenz-de-Cabezón, E.. A C++ class for multi-state algebraic reliability computations. *Reliability Engineering & System Safety 213* (2021), 107751.
  JCR: Q1 (JIF), Q1 (JCI), 2020; SJR: Q1 (2021).
  https://doi.org/10.1016/j.ress.2021.107751

Resumen

Esta tesis se centra en el estudio de la fiabilidad de sistemas multi-estado con un acercamiento algebraico basado en el uso de ideales monomiales.

En el capítulo uno se presentan las herramientas necesarias en esta tesis doctoral. A pesar de que es un capítulo en el que no se aportan resultados novedosas, sirve para asentar las bases teóricas, tanto en el ámbito algebraico como en el relacionado con la fiabilidad de sistemas.

La primera parte del capítulo está dedicada a ideales monomiales, objeto indispensable en el desarrollo de esta tesis.

Más adelante nos encontramos con el concepto de resolución, que es una secuencia exacta definida sobre un módulo y las relaciones existentes entre sus generadores. Si trabajamos con módulos graduados podemos definir otro concepto de especial relevancia tanto en el campo del Álgebra como para este trabajo de investigación: la resolución mínima. La resolución mínima nos va a permitir calcular, además de la fiabilidad de un sistema como con cualquier otra resolución, las cotas más ajustadas dentro de las de naturaleza enumerativa.

Después se introduce la serie de Hilbert, cuyo numerador nos va a permitir enumerar los monomios de un ideal monomial. La serie de Hilbert de un ideal se puede calcular de diferentes formas. Una de ellas es calculando una resolución libre del ideal y escribir su numerador como la suma alternada de sus rangos. En caso de que la resolución sea mínima, los rangos tienen un nombre especial: números de Betti. Estos rangos nos van a ofrecer las cotas más ajustadas de entre todas las resoluciones.

Para profundizar en los resultados algebraicos presentados se recomienda consultar [62, 31, 114], entre otros.

Este capítulo preeliminar finaliza con una pequeña revisión de fiabilidad de sistemas binarios. En ella se hace una pequeña revisión de la evolución del campo de estudio y definimos qué es un sistema binario, así como objetos relevantes asociados a él (caminos, cortes, espacio de estados...). Finalmente se muestra la relación existente entre los ideales monomiales libres de cuadrados y los sistemas binarios y cómo se puede calcular su fiabilidad - o cotas para ella - utilizando dicha relación. Se recomienda la lectura de [75, 116, 117, 118, 119, 120].

Comenzamos el capítulo dos tratando la operación polarización la cual, dado un ideal monomial $I = \langle m_1, \ldots, m_r \rangle \subseteq R = \mathbb{k}[x_1, \ldots, x_n]$, lo transforma en un ideal monomial libre de cuadrados $I = \langle \overline{m_1}, \ldots, \overline{m_r} \rangle \subseteq S = \mathbb{k}[y_1, \ldots, y_m]$, donde $m \geq n$. Cuando polarizamos un ideal monomial ganamos la propiedad de que el ideal resultante sea libre de cuadrados pero, a cambio, trabajamos en un anillo con un mayor número de variables.

La operación depolarización es la inversa de la polarización, *i.e.* dado un ideal monomial libre de cuadrados lo transformamos en un ideal monomial con exponentes. Hay que tener en cuenta que, aunque la operación polarización es única, la depolarización no lo es: cuando depolarizamos un ideal monomial libre de cuadrados puede haber varios ideales monomiales como resultado. La Figura 5.2 resume esta idea.

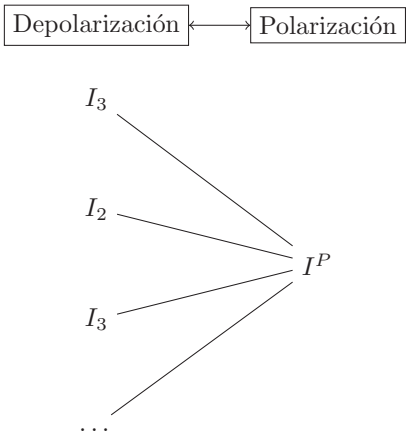


Figure 5.2: Idea gráfica de la unicidad y no unicidad de las operaciones polarización y depolarización

El objetivo del uso de la operación polarización y depolarización es que, en algunos casos, existen fórmulas explícitas para calcular ciertas propiedades de un ideal en el caso libre de cuadrados (o al contrario). Las operaciones polarización y depolarización conserva, aunque no siempre exactamente igual si que es fácil calcularlos, ciertos invariantes algebraicos (como pueden ser los números de Betti o la serie de Hilbert). En la Sección 2.3.2.3 se realiza un estudio de diferentes propiedades compartidas entre los ideales que tienen la misma polarización.

Más adelante se presentan los *support posets*, una herramienta de naturaleza combinatoria que va a permitir calcular todas las depolarizaciones de un ideal monomial (ver Teorema 2.3.8).

De manera natural surge la pregunta de si todos los posets son realizables como *support poset* de un ideal monomial,*i.e.* si dado un poset C va a existir un ideal monomial I tal que su C. En esta línea, la Proposición 2.3.1 proporciona una condición suficiente para construir ideales que tengan un *support poset* dado.

Después, se muestran familias de posets que son realizables como *support poset* de un ideal monomial, como pueden ser los conjuntos de líneas o diamantes y los bosques (ver Proposición 2.3.18, Proposición 2.3.23 y Teorema 2.3.30) y se dan, de manera explícita, propiedades fundamentales de dichos ideales, como los números de Betti.

Finalmente se describe el *suppor poset* de los ideales k-entre-n y de los series-paralelo.

El siguiente capítulo está dedicado al estudio de fiabilidad algebraica en sistemas multi-estado. Un sistema multi-estado es aquel que puede tomar más de dos niveles de funcionamiento. Por ejemplo, dado un sistema de alumbrado de una ciudad, que fallen las luces de una calle no quiere decir que el sistema no funcione, aunque tampoco que lo haga a pleno rendimiento. Sabemos del Capítulo 1 que la fiabilidad de un sistema es la probabilidad de que esté funcione. Sin embargo, para los sistemas multi-estado necesitamos el concepto de j-fiabilidad, que es la probabilidad de que un sistema esté funcionando, al menos, a nivel j.

El método algebraico basado en ideales monomiales para calcular la fiabilidad de este tipo de sistemas se explica en la Sección 3.1.1 junto con ejemplos que muestran su funcionamiento.

Avanzando en el capítulo nos encontramos con sistemas multi-estado k-entre-n, un tipo específico de sistemas de gran interés debido a su amplio abanico de aplicaciones. Para ellos, revisamos las diferentes definiciones que se han dado en la literatura y cómo interpretarlas en términos de ideales monomiales (ver Proposición 3.2.4, Definición 3.2.13 y Proposición 3.2.13). También se muestra un modo de calcular estos ideales de manera recursiva, así como sus números de Betti. Además, en la Sección 3.2.3.2 se hace una comparativa entre las cotas presentadas en la Sección 3.1.1.1 y las que se obtienen con empleando el método algebraico.

El capítulo finaliza tratando variantes de los sistemas k-entre-n multi-estado, como los sistemas binarios k-entre-n con componentes multi-estado, los sistemas multi-estado k-entre-n consecutivos, los sistemas homogéneos *sparsely* conectados k-entre-n o sistemas multi-estado k-entre-n con pesos. Para todos ellos definimos sus estructuras algebraicas asociadas y fórmulas para calcular sus números de Betti.

En el último capítulo se presenta una clase en `C++` desarrollada con el software libre `CoCoALib` para el cálculo de fiabilidad (y cotas) de sistemas multi-estado basado en el acercamiento algebraico propuesto en esta tesis. El código de la clase así como ejemplos están disponibles en

`http://www.dima.unige.it/~bigatti/data/AlgebraicReliability/`

En primer lugar se hace una pequeña introducción de `CoCoALib`, una librería de `C++` pensada para realizar Cálculos en Álgebra Conmutativa. Ha sido implementada por John Abott y Anna Maria Bigatti. Está basada en monomios y polinomios multi variables y, actualmente, se encuentra en la versión `0.998000`.

Avanzando en el capítulo, nos encontramos con una descripción detalla de la clase, cuyo diagrama `UML` aparece en la Figura 4.1 del Capítulo 4.

Más adelante se realizan unos ejemplos utilizando la clase desarrollada. Primero se valida la clase con sistemas de diferente naturaleza que apare-

cen en la literatura, como puede ser el *Army Battle Plan* de [26]. Después, se calcula la fiabilidad de redes empleando la clase implementada. En particular, los ejemplos de redes utilizados son la *GAAR: Italian Research and Education Network* y redes generadas de manera aleatoria utilizando los modelos de Erdős-Rényi $ER(n,p)$ y Barabasi-Albert $BA(n,m)$. Los ejemplos finalizan mostrando el comportamiento de la clase en sistemas en dos sistemas k-entre-12 multi-estado con 4 niveles de funcionamiento y diferenciando $k=4$ y $k=6$.

Para finalizar, se trata la complejidad computacional del método algebraico implementado con la clase.

Los resultados presentados en esta tesis doctoral han sido publicados en

- Mohammadi, F., Pascual-Ortigosa, P., Sáenz-de-Cabezón, E., and Wynn, H. Polarization and depolarization of monomial ideals with application to multi-state system reliability. *Journal of Algebraic Combinatorics 51* (2020), 617-639.
  JCR: Q3 (JIF), Q2 (JCI), 2020; SJR: Q1, 2020.
  `https://doi.org/10.1007/s10801-019-00887-6`

- Pascual-Ortigosa, P., and Sáenz-de-Cabezón, E.. Support posets of some monomial ideals. *Applicable Algebra in Engineering, Communication and Computing* (2020), 1-19.
  JCR: Q3 (JIF), Q4 (JCI), 2020; SJR: Q2 (2020).
  `https://doi.org/10.1007/s00200-020-00461-9`

- Pascual-Ortigosa, P., Sáenz-de-Cabezón, E., and Wynn, H. Algebraic reliability of multi-state k-out-of-n systems. *Probability in the Engineering and Informational Sciences 35* (2021), 1005.
  JCR: Q3 (JIF), Q3 (JCI), 2020; SJR: Q3 (2021).
  `https://doi:10.1017/S0269964820000455`

- Pascual-Ortigosa, P., and Sáenz-de-Cabezón, E.. Algebraic Analysis of variants of multi-state k-out-of-n systems. *Mathematics* (2021), $9(17)$, 2042.
  JCR: Q1 (JIF), Q1 (JCI), 2020; SJR: Q2 (2021).
  `https://doi.org/10.3390/math9172042`

- Bigatti, A., Pascual-Ortigosa, P., and Sáenz-de-Cabezón, E.. A C++ class for multi-state algebraic reliability computations. *Reliability Engineering &*

System Safety 213 (2021), 107751.
JCR: Q1 (JIF), Q1 (JCI), 2021; SJR: Q1 (2021).
https://doi.org/10.1016/j.ress.2021.107751

Conclusions and results

In this thesis we have focused on the reliability of multi-state systems with and algebraic approach based on monomial ideals. The main goals have been

- Research on polarization and depolarization of monomial ideals.
- Investigate the application of polarization and depolarization operations for the analysis of multi-state system reliability.
- Study different kinds of systems and its associated algebraic structures.
- Algorithms' development.

This works started with a background chapter in which the tools needed for the later chapters have been presented: monomial ideals, which is the most basic tool in this research, but the most needed due to they are in charge of relating Algebra and Reliability Theory; resolutions and Hilbert series, which have been used for computing reliability and bounds for it; Mayer-Vietoris trees, a fast and efficient algorithm (implemented in `CoCoALib` by Eduardo Sáenz-de-Cabezón en [114]) which allows us to compute the numerator of the Hilbert series; and a brief introduction to Reliability Theory, where binary systems ([75]) and how the algebraic method based on monomial ideals ([116, 117, 118, 120]) are presented.

The second chapter is devoted to explore the relation between square-free monomial ideals and arbitrary ones. For doing that, polarization and depolarization operations were investigated.

Polarization transforms a monomial ideal $I \subseteq R = \mathbb{k}[x_1, \ldots, x_n]$ into a squarefree one $I^P \subseteq S = \mathbb{k}[y_1, \ldots, y_m]$. When polarazing, we change from a polynomial ring with n variables to another one with m, where $m \geq n$. The inverse operation, depolarization, transforms a squarefree monomial ideals into monomial one. However, the result of depolarization is not unique: for a squarefree monomial ideal there exist some depolarizations. In Section 2.2 one can find a deep study of this operations.

The main interest of monomial ideals and its polarization is that they share some algebraic invariants (see Corollary 1.6.3, [62]). For some families of monomial ideals there exist explicit formulas for computing these invariants, such us the minimal Eliahou-Kervaire resolution [41] for stable ideals. Taking this into account, some natural questions arises: Are we able to compute all the depolarizations of a monomial ideal? Is it possible to characterize the squarefree monomial ideals which has a special kind of monomial ideal as depolarization? What properties are shared between the ideals sharing the same polarization?

For answering these questions, we develop a tool called support poset of a monomial ideal:

Definition. Let I be a squarefree monomial ideal with $G(I) = \{m_1, \ldots, m_r\}$. For each i in $\operatorname{supp}(I)$ we define the set $C_i \subseteq \operatorname{supp}(I)$ as,

$$C_i = \bigcap_{\substack{m \in G(I) \\ x_i \text{ divides } m}} \operatorname{supp}(m).$$

Let $C_I = \{C_1, \ldots, C_n\}$. The poset on the elements of C_I ordered by inclusion is called the *support poset* of I and is denoted $\operatorname{suppPos}(I)$. We define the *support poset* of a general monomial ideal as the support poset of its polarization.

With the support posets we are able to obtain all the posible depolarizations of a squarefree monomial ideal:

Theorem (Theorem 2.3.8)**.** Let $I = \langle m_1, \ldots, m_r \rangle \subseteq R = \mathbb{k}[x_1, \ldots, x_n]$ be a squarefree monomial ideal. Every depolarization of I can be obtained from a depolarization order of I.

We call copolar ideals to all the monomial ideals obtained from the same support poset. We explored some properties shared by copolar ideals in section 2.3.2.3, such as the results below:

Lemma (Lemma 2.3.12)**.** Let I and J be two copolar ideals. Then $\text{lcm}(I) \cong \text{lcm}(J)$.

Theorem (Theorem 2.3.13)**.** The width of $\text{suppPos}(I^P)$ is an upper bound for $\text{projdim}(I)$.

We say that a poset C is relizable as s support poset of a monomial ideal I when C is its support poset. It is known that not all posets are realizable as a support poset of a monomial ideals. Even we did not characterize which are and which are not, we obtain relevant results.

The first of them is a proposition which gives give a sufficient condition for building a monomial ideal given a support poset under certain conditions:

Proposition (Proposition 2.3.1)**.** Let $(\mathcal{C} = \{C_1, \ldots, C_n\}, \subseteq)$ be a poset such that $\{i\} \subseteq C_i \subseteq [n]$ for each i, and if $k \in C_i$ and $i \in C_j$ then $k \in C_j$ for all i, j, k. Let $R = \Bbbk[x_1, \ldots, x_n]$ and let $m_i = \prod_{j \in C_i} x_j$ for each i. For any $\sigma \subseteq [n]$ let $m_\sigma = \text{lcm}(m_i | i \in \sigma)$, and for any collection Σ of subsets of $[n]$, consider the monomial ideal $I_\Sigma = \langle m_\sigma | \sigma \in \Sigma \rangle$. Then $(\mathcal{C}, \subseteq)$ is the support poset of I_Σ if the following properties hold:

1. $\forall i \in [n]$ there is some $\sigma \in \Sigma$ such that $x_i | m_\sigma$.
2. If $\{\sigma : \ x_i | m_\sigma\} \subseteq \{\sigma : \ x_j | m_\sigma\}$, then $C_j \subseteq C_i$.

Proposition 2.3.14 gives a sufficient condition for a poset to be the support poset of a zero-dimensional monomial ideal

Proposition (Proposition 2.3.14)**.** Let $n, m_1, \ldots, m_n$ be some positive integers with $1 \leq m_i \leq n$ for all i and let $m = \sum_i m_i$. Consider a poset $(\mathcal{P}, \subseteq)$ on subsets of $\{1, \ldots, m\}$ formed by n disjoint paths each of length m_i. Then there is a squarefree monomial ideal I whose support poset is $\mathcal{P}$ except if $n = 2$ and $m_1 \neq m_2$. Moreover, if $m_i > 1$ for all i, then there is a zero-dimensional monomial ideal copolar to I.

Proposition 2.3.18 and Proposition 2.3.23 show that given a poset formed by lines or diamonds, it will always exist a monomial ideal for which the poset is its support poset.

Proposition (Proposition 2.3.18)**.** Let n and m be two positive integers and let $(\mathcal{P}, \subseteq)$ be a poset of subsets of the set $[nm] = \{1, \ldots, nm\}$ formed by n disjoint lines each of length m. Then there is at least one squarefree monomial ideal $I_{n,m}$ such that $\mathcal{P}$ is its support poset and there is a zero-dimensional monomial ideal $J_{n,m}$ copolar to $I_{n,m}$.

In particular, the ideal $J_{n,m} \subseteq \Bbbk[y_1, \ldots, y_n]$ given by

$$J_{n,m} = \langle y_1^m, \ldots, y_n^m, y_1^{m-1}y_2, \ldots, y_1y_2^{m-1}, \ldots, y_1^{m-1}y_n, \ldots, y_1y_n^{m-1} \rangle.$$

is a zero dimensional ideal having $\mathcal{P}$ as its support poset.

Proposition (Proposition 2.3.23)**.** Let m be a positive integer, let $(\mathcal{P}, \subseteq)$ be a poset of subsets of the set $[4m]$ formed by $m > 1$ disjoint diamonds $D_1, \ldots, D_m$, $D_i = \{a_{i1}, \ldots, a_{i4}\}$ with $a_{i1} < a_{i2}$, $a_{i1} < a_{i3}$, $a_{i2} < a_{i4}$, $a_{i3} < a_{i4}$. Then there is at least one squarefree monomial ideal I_m such that $\mathcal{P}$ is its support poset.

Besides, Proposition 2.3.21 and Proposition 2.3.24 gives an explicit formula for the Betti numbers of those ideals.

Proposition 2.3.27 and Theorem 2.3.30 proof that trees and forest are realizable as a support poset of a monomial ideal and the Betti numbers of its monomial ideals are given.

Proposition (Proposition 2.3.27)**.** Let $\mathcal{P}$ be a tree with nodes $\{1, \ldots, n\}$ and let $\{l_1, \ldots, l_k\} \subseteq \{1, \ldots, n\}$ be the set of leaves of the tree. There exists a squarefree monomial ideal $I_L(\mathcal{P}) \subseteq \Bbbk[x_1, \ldots, x_n]$ with k generators such that $\mathcal{P}$ is it support poset. The Taylor resolution of $I_L(\mathcal{P})$ minimally resolves it and therefore $\beta_i(I_L(\mathcal{P})) = \binom{k}{i+1}$ for all $i \geq 0$.

Theorem (Theorem 2.3.30)**.** Let $\mathcal{P}$ be a forest whose trees $\mathcal{P}_1, \ldots, \mathcal{P}_m$ have n_i nodes and l_i leaves each, for $i = 1, \ldots, m$. Then there is a squarefree monomial ideal $I_L(\mathcal{P}) \subseteq \Bbbk[x_1, \ldots, x_n]$, $n = \sum_{i=1}^m n_i$ whose support poset is $\mathcal{P}$. The ideal $I_L(\mathcal{P})$ has $g = \sum_{i=1}^m l_i$ minimal monomial generators, its Taylor complex minimally resolves it, and $\beta_i(I_L(\mathcal{P})) = \sum_{j=1}^m \binom{l_j}{i+1}$ for all i.

This chapter ends showing that monomial ideals associated to binary consecutive k-out-of-n systems has a set of lines as a support poset when $k < n - k + 1$ and a tree otherwise (see Proposition 2.3.31) meanwhile series-parallel ideals has a forest as a support poset (Theorem 2.3.35).

Chapter three is focused on algebraic reliability of multi-state systems. After introducing basic notions related to multi-state systems, the functioning of the algebraic method based on monomial ideals is showed. It can be summarized as

1. Associate to the system S its j-reliability ideal $I_{S,j}$.
2. Obtain the minimal generating set of $I_{S,j}$ to get the set $\overline{\mathcal{F}}_{S,j}$.
3. Compute the Hilbert series of $I_{S,j}$ to have the j-reliability of S.

3' Compute any free resolution of $I_{S,j}$. The alternating sum of the ranks of this resolution gives a formula for the Hilbert series of $I_{S,j}$ i.e., the unreliability of S, which provides bounds by truncation at each summand.

Then, some examples of how the algebraic method works when working with multi-state systems are showed: the reliability of a multi-state system given by its set of minimal paths and of a network is computed. Besides, it is showed how to compute the reliability of a multi-state system via binary systems using the operations polarization and depolarization.

From this point, the chapter is devoted to the particular case of multi-state k-out-of-n systems.

Firstly, a review of the different definitions given in the literature, for both the simple ([40, 28]) and the general ([69, 151]) case, is done. The j-reliability ideals associated to each definition are described as

Proposition (Proposition 3.2.4)**.** The ideal

$$I_{(k,n),j} = \langle \prod_{\substack{\sigma \subseteq \{1,\dots,n\} \\ |\sigma| = k}} x_i^j \mid i \in \sigma \rangle$$

is the j-reliability ideal of a multi-state k-out-of-n system as defined in Definition 3.2.2.

Definition (Definition 3.2.12)**.** An n-component system is called *generalized multi-state k-out-of-n : F system* if $\phi(\mathbf{x}) < j$, $1 \leq j \leq M$ whenever the states of at least k_l components are below l for all l such that $j \leq l \leq M$.

Proposition (Proposition 3.2.13)**.** The j-reliability ideal of a generalized multi-state k-out-of-n system $S = S_{n,(k_1,\ldots,k_M)}$ is given by

$$I_{S,j} = I_{n,(k_j,\ldots,k_M)} = \sum_{i=j}^{M} I_{(k_i,n),i}.$$

Furthermore, it is showed how to compute in a recursive way these ideals and its Betti numbers. Section 3.2.3.2 compares the bounds for reliability obtained with the algebraic method with other given in the literatura (see Section 3.1.1.1). Weaknesses and strengths of the algebraic method are shown. For example, if the number of generator is quite big, the very first bounds are not good, while the greater the number of variables, the better the bounds are.

Chapter three ends treating variants of k-out-of-n systems such as binary k-out-of-n system with multi-state components, multi-state consecutive k-out-of-n systems, sparsely connected homogeneous k-out-of-n systems and weighted multi-state k-out-of-n systems. For all of them, a description of it j-reliability ideals and formulas for computing its Betti numbers are given.

In chapter four a `C++` class implemented with the library `CoCoALib`, currently in its `0.998000` version, for computing the reliability of a multi-state system via the algebraic method. The class is available at

`http://www.dima.unige.it/~bigatti/data/AlgebraicReliability/`.

Then, a detailed description of the class is given (see Figure 4.1 on Chapter 4).

The main class is `CoherentSystem` which consist of a set of levels and a matrix of probabilities. Each level, stored in a `std::vector`, is a instance of the class `CoherentSystemLevel`. Each instance of this class has an ideal and it dual and its Mayer-Vietoris trees. `CoherentSystemPath` and `CoherentSystemCuts` inherits from the main class. An instance of this classes can call to the following functions:

myMinimalPaths Receives a level and gives a vector of vectors where each vector is a minimal path of the system.

myMinimalCuts Receives a level and gives a vector of vectors where each vector is a minimal cut of the system.

myReliability Receives a level j and computes $\mathbf{R}_{S,j}$.

myUnreliability Receives a level j and computes $\mathbf{U}_{S,j}$.

myReliabilityBounds Receives a level j and computes bounds for $\mathbf{R}_{S,j}$ using the Mayer-Vietoris tree of $I_{S,j}$.

myUnreliabilityBounds Receives a level j and computes bounds for $\mathbf{U}_{S,j}$ using the Mayer-Vietoris tree of $I_{S,j}$.

Going further on this chapter, some examples are developed with the class aforementioned. Firstly, we validate the class with different systems appearing in the literature, as *Army Battle Plan* from [26]. Then, we compute the reliability of some networks. The networks chosen are *GAAR: Italian Research and Education Network* and randomly generated networks with Erdős-Rényi $ER(n,p)$ and Barabasi-Albert $BA(n,m)$ models. We taste our class in a k-out-of-12 system with four levels of performance and takin $k=4$ and $k=6$. The chapter ends dealing with the computational complexity of the algebraic method.

In this thesis we have accomplished the aims fixed and we have opened new directions of research on this area.

Conclusiones y resultados

En esta tesis nos hemos centrado en el estudio de fiabilidad de sistemas multi-estado con un acercamiento algebraico basado en ideales monomiales. Los objetivos principales han sido:

- Estudio de la polarización y depolarización de ideales monomiales.
- Definir la aplicabilidad de las operaciones polarización y depolarización al análisis de la fiabilidad de un sistema.
- Estudio de los distintos tipos de sistemas multi-estado y sus estructuras algebraicas asociadas.
- Implementación de algoritmos.

La tesis ha comenzado con un capítulo en el que se presentan las herramientas necesarias en los capítulos posteriores: los ideales monomiales, que es la herramienta más básica, aunque la más necesaria puesto que son los encargados de relacionar Álgebra y Teoría de Fiabilidad; resoluciones y serie de Hilbert, los cuales han sido empleados para el cómputo de la fiabilidad y cotas para ella; árboles de Mayer-Vietories, un algoritmo rápido y eficiente (implementado en el software libre `CoCoALib` por Eduardo Sáenz-de-Cabezón en [114]) que nos permite calcular el numerador de la serie de Hilbert; y una breve introducción de Teoría de la Fiabilidad, en la que se presentan los sistemas binarios y cómo funciona el método algebraico basado en ideales monomiales para ellos [75, 116, 117, 118, 120].

El capítulo segundo se centra en la relación existe entre los ideales monomiales libres de cuadrados y los ideales monomiales arbitrarios. Para

estudiar dicha relación, hemos investigado las operaciones polarización y depolarización.

La operación polarización transforma un ideal monomial $I \subseteq R = \mathbb{k}[x_1, \ldots, x_n]$ en un ideal monomial libre de cuadrados $I^P \subseteq S = \mathbb{k}[y_1, \ldots, y_m]$. Al polarizar, pasamos de un anillo de polinomios en n variables a otro en m, donde $m \geq n$. La operación inversa, la depolarización, transforma un ideal monomial libre de cuadrados en un ideal monomial. Sin embargo, el resultado de esta operación no es único: para un ideal libre de cuadrados existen varias depolarizaciones. En la Sección 2.2 se tratan ambas operaciones en profundidad.

El interés principal de los ideales monomiales y su polarización es que comparten ciertos invariantes algebraicos (Corolario 1.6.3, [62]). Para ciertas familias de ideales monomiales existen fórmulas explícitas para el cómputo de estos invariantes, como por ejemplo la resolución de Eliahou-Kervaire [41] para ideales estables, que es mínima. Teniendo esto en cuenta surgen, de manera natural, varias preguntas: ¿podemos calcular todas las depolarizaciones de un ideal monomial? ¿se pueden caracterizar los ideales libres de cuadrados que cuentan ideales determinados como depolarización? ¿qué propiedades comparten los ideales que comparten la misma polarización?

Para responder estas preguntas, introducimos el concepto de *support poset* de un ideal monomial:

Definición. Sea I un ideal monomial libre de cuadrados cuyo conjunto generador es $G(I) = \{m_1, \ldots, m_r\}$. Para cada i en $\operatorname{supp}(I)$ definimos el conjunto $C_i \subseteq \operatorname{supp}(I)$ como

$$C_i = \bigcap_{\substack{m \in G(I) \\ x_i \text{ divide } m}} \operatorname{supp}(m).$$

Sea $C_I = \{C_1, \ldots, C_n\}$. El poset de elementos de C_I ordenador por inclusión se llama *support poset* de I y lo denotamos por $\operatorname{suppPos}(I)$.

Definimos el *support poset* de un ideal monomial general como el *support poset* de su polarización.

Con la herramienta *support poset* definida ya somos capaces de calcular todas las posibles depolarizaciones de un ideal monomial libre de cuadrados:

Teorema (Teorema 2.3.8)**.** Sea $I = \langle m_1, \ldots, m_r \rangle \subseteq R = \mathbb{k}[x_1, \ldots, x_n]$ un ideal monomial libre de cuadrados. Todas las depolarizaciones de I se pueden obtener con un irden de depolarización de I.

Llamamos ideales copolares a aquellos ideales monomiales obtenidos del mismo *support poset*. Algunas propiedades compartidas por ideales copolares se han explorado en la Sección 2.3.2.3, como son:

Lema (Lema 2.3.12)**.** Sean I y J dos ideales copolares. Entonces $\text{lcm}(I) \cong \text{lcm}(J)$.

Teorema (Teorema 2.3.13)**.** La anchura de $\text{suppPos}(I^P)$ es una cota superior de $\text{projdim}(I)$.

Diremos que un poset C es realizable como *support poset* de un ideal monomial I cuando C es su *support poset*. Se sabe que no todos los poset son realizables como *support poset* de un ideal monomial y, aunque no se ha caracterizado cuáles lo son y cuáles no lo son, se han obtenido resultados relevantes.

El primero de ellos es una proposición que da una condición suficiente para poder construir un ideal monomial dado un poset que cumple ciertas condiciones:

Proposición (Proposición 2.3.1)**.** Sea $(\mathcal{C} = \{C_1, \ldots, C_n\}, \subseteq)$ un *support poset* tal que $\{i\} \subseteq C_i \subseteq [n]$ para cada i, y si $k \in C_i$ e $i \in C_j$ entonces $k \in C_j$ para todos i, j, k. Sea $R = \mathbb{k}[x_1, \ldots, x_n]$ y sea $m_i = \prod_{j \in C_i} x_j$ para cada i. Para cualquier $\sigma \subseteq [n]$ sea $m_\sigma = \text{lcm}(m_i | i \in \sigma)$, y para cada colección Σ de subconjuntos de $[n]$, consideremos el ideal monomial $I_\Sigma = \langle m_\sigma | \sigma \in \Sigma \rangle$. Entonces $(\mathcal{C}, \subseteq)$ es el *support poset* de I_Σ si se cumplen las siguientes condiciones:

1. $\forall i \in [n]$ existe algún $\sigma \in \Sigma$ tal que $x_i | m_\sigma$.
2. Si $\{\sigma : \ x_i | m_\sigma\} \subseteq \{\sigma : \ x_j | m_\sigma\}$, entonces $C_j \subseteq C_i$.

La Proposición 2.3.14 da una condición suficiente para que un poset sea el *support poset* de un ideal monomial 0-dimensional:

Proposición (Proposición 2.3.14)**.** Sean $n, m_1, \ldots, m_n$ enteros positivos con $1 \leq m_i \leq n$ para todo i y sea $m = \sum_i m_i$. Consideremos el *poset* $(\mathcal{P}, \subseteq)$ en subconjuntos de $\{1, \ldots, m\}$ formado por n caminos disjuntos cada uno de ellos de longitud m_i. Entonces existe un ideal monomial libre de cuadrados I cuyo support poset es $\mathcal{P}$ excepto si $n = 2$ y $m_1 \neq m_2$. Ademas, si $m_i > 1$ para todo i, entonces existe un ideal monomial 0-dimensional copolar a I.

La Proposición 2.3.18 y Proposición 2.3.23 muestran que, si tenemos un poset formado por líneas y diamantes, siempre va a realizable como *support poset* de un ideal monomial.

Proposición (Proposición 2.3.18)**.** Sean n y m dos enteros positivos y sea $(\mathcal{P}, \subseteq)$ un *poset* de subconjuntos de $[nm] = \{1, \ldots, nm\}$ formado por n líneas disjuntas de longitud m. Entonces, existe al menos un ideal monomial libre de cuadrados $I_{n,m}$ tal que $\mathcal{P}$ es su *support poset* y existe un ideal monomial 0-dimensional $J_{n,m}$ copolar a $I_{n,m}$.

En particular, el ideal $J_{n,m} \subseteq \mathbb{k}[y_1, \ldots, y_n]$ dado por

$$J_{n,m} = \langle y_1^m, \ldots, y_n^m, y_1^{m-1}y_2, \ldots, y_1y_2^{m-1}, \ldots, y_1^{m-1}y_n, \ldots, y_1y_n^{m-1} \rangle.$$

es un ideal monomial 0-dimensional cuyo *support poset* es $\mathcal{P}$.

Proposición (Proposición 2.3.23)**.** Sea m un entero positivo, sea $(\mathcal{P}, \subseteq)$ un *poset* de subconjuntos del conjunto $[4m]$ formado por $m > 1$ diamantes disjuntos $D_1, \ldots, D_m$, $D_i = \{a_{i1}, \ldots, a_{i4}\}$ con $a_{i1} < a_{i2}$, $a_{i1} < a_{i3}$, $a_{i2} < a_{i4}$, $a_{i3} < a_{i4}$. Entonces existe al menos un ideal monomial libre de cuadrados I_m tal que $\mathcal{P}$ es su *support poset*.

Además, en la Proposición 2.3.21 y Proposición 2.3.24 se da una fórmula explícita para calcular los números de Betti de dichos ideales.

La Proposición 2.3.27 y el Teorema 2.3.30 prueban que los árboles y los bosques también son *support poset* de un ideal monomial y se da una fórmula explícita para sus números de Betti.

Proposición (Proposición 2.3.27)**.** Sea $\mathcal{P}$ un árbol con nodos $\{1,\dots,n\}$ y sea $\{l_1,\dots,l_k\} \subseteq \{1,\dots,n\}$ el conjunto de hojas del árbol. Existe un ideal monomial libre de cuadrados $I_L(\mathcal{P}) \subseteq \Bbbk[x_1,\dots,x_n]$ con k generadores tal que $\mathcal{P}$ es su *support poset*. La resolución de Taylor de $I_L(\mathcal{P})$ es mínima y, por lo tanto, $\beta_i(I_L(\mathcal{P})) = \binom{k}{i+1}$ para todo $i \geq 0$.

Teorema (Teorema 2.3.30)**.** Sea $\mathcal{P}$ un bosque cuyos árboles $\mathcal{P}_1,\dots,\mathcal{P}_m$ tienen n_i nodos y l_i hojas cada uno, para $i = 1,\dots,m$. Entonces existe un ideal monomial libre de cuadrados $I_L(\mathcal{P}) \subseteq \Bbbk[x_1,\dots,x_n]$, $n = \sum_{i=1}^m n_i$ cuyo *support poset* e $\mathcal{P}$. El ideal $I_L(\mathcal{P})$ tiene $g = \sum_{i=1}^m l_i$ generadores monomiales mínimos, su complejo de Taylor lo resuelve de forma mínima, y $\beta_i(I_L(\mathcal{P})) = \sum_{j=1}^m \binom{l_j}{i+1}$ para todo i.

Finalizamos este capítulo mostrando que los ideales monomiales asociados a sistemas binario k-entre-n consecutivos tienen como *support poset* un conjunto de líneas cuando $k < n-k+1$ y un árbol en el caso contrario (Proposición 2.3.31) mientras que los ideales series-paralelo tienen un bosque (ver Teorema 2.3.35).

El capítulo tres está dedicado al estudio de fiabilidad algebraica en sistemas multi-estado. Tras presentar las definiciones básicas relacionadas con sistemas multi-estado, se explica cómo funciona el método algebraico aplicado a sistemas multi-estado que, en resumidas cuentas, funciona de la siguiente forma:

1. Se asocia al sistema S sus ideales de j-fiabilidad $I_{S,j}$.

2. Obtenemos el sistema mínimo generador de $I_{S,j}$, para cada j, para conseguir el conjunto de los caminos mínimos $\overline{\mathcal{F}}_{S,j}$.

3. Se calcula la serie de Hilbert de $I_{S,j}$, con el que ya podemos calcular la j-fiabilidad del sistema.

3′ Calculamos una resolución libre de $I_{S,j}$. La suma alternada de los rangos de dicha disolución dan una fórmula para obtener el numerador de la seria de Hilbert. El numerador de la serie de Hilbert calculado así proporciona cotas truncando en cada sumando.

Después, se muestran varios ejemplos de cómo funciona el método algebraico cuando se trabaja con sistemas multi-estado: se calcula la fiabilidad de un sistema multi-estado dado por su conjunto de caminos mínimos, de una red y se muestra cómo se puede calcular la fiabilidad de un sistema multi-estado mediante un sistema binario y viceversa (estos últimos casos gracias al trabajo realizado en el capítulo dos).

Desde este punto, el capítulo se centra en el caso particular de sistemas multi-estado k-entre-n.

En primer lugar, se hace un repaso de las diferentes definiciones que se han dado en la literatura, tanto para el caso simple ([40, 28]) como para el caso generalizado ([69, 151]). Los ideales de j-fiabilidad asociados a cada una de las definiciones son:

Proposición (Proposición 3.2.4)**.** El ideal

$$I_{(k,n),j} = \langle \prod_{\substack{\sigma \subseteq \{1,\dots,n\} \\ |\sigma|=k}} x_i^j \mid i \in \sigma \rangle$$

es el ideal de j-fiabilidad de un sistema multi-estado k-entre-n definido como en la Definición 3.2.2.

Definición (Definición 3.2.12)**.** Un sistema de n componentes se llama *sistema multi-estado k-entre-n : F generalizado* si $\phi(\mathbf{x}) < j$, $1 \leq j \leq M$ cuando el estado de al menos k_l componentes están por encima del nivel l para todo l tal que $j \leq l \leq M$.

Proposición (Definición 3.2.13)**.** El ideal de j-fiabilidad ideal de un sistema multi-estado k-entre-n generalizado $S = S_{n,(k_1,\dots,k_M)}$ viene dado por

$$I_{S,j} = I_{n,(k_j,\dots,k_M)} = \sum_{i=j}^{M} I_{(k_i,n),i}.$$

Además, vemos cómo calcular estos ideales de forma recursiva y calculamos sus números de Betti.

En la Sección 3.2.3.2 se hace una comparativa de la calidad de las cotas que se obtienen cuando se calcula la fiabilidad usando el método

algebraico presentado con otras conocidas en la literatura (ver Sección 3.1.1.1). En esta sección se muestran fortalezas y debilidades de nuestro método: por ejemplo, si el número de generadores es muy grande, las primeras cotas no son buenas, mientras que cuando el número de variables va aumentando, las cotas van mejorando.

El resto del capítulo tres está dedicado al estudio de casos particulares de sistemas k-entre-n, como son el caso de sistemas binarios k-entre-n con componentes multi-estado, sistemas multi-estado k-entre-n consecutivos o sistemas multi-estado k-entre-n con pesos. Para todos ellos mostramos sus ideales de j-fiabilidad y damos fórmulas para calcular sus números de Betti.

En el capítulo cuatro presentamos una clase de `C++` implementada con la librería `CoCoALib`, actualmente en su versión `0.998000`, para el cálculo de la fiabilidad de sistemas por medio del método algebraico presentado en esta tesis. La clase está disponible en

`http://www.dima.unige.it/~bigatti/data/AlgebraicReliability/`.

Avanzando en el capítulo, nos encontramos con una descripción detalla de la clase (ver Figura 4.1 del Capítulo 4).

La clase principal es la clase `CoherentSystem` que consiste en una serie de niveles y una matriz de probabilidades. Cada uno de los niveles, almacenados en un `std::vector`, es una instancia de la clase `CoherentSystemLevel`. Cada instancia de esta clase consta de un ideal y su dual y sus árboles de Mayer-Vietoris. Las clases `CoherentSystemPath` y `CoherentSystemCuts` heredan de la clase principal. Cualquier instancia de estas clases puede llamar a las funciones

myMinimalPaths Recibe un nivel y devuelve un vector de vectores, en el que cada uno de estos vectores es un camino mínimo.

myMinimalCuts Recibe un nivel y devuelve un vector de vectores, en el que cada uno de estos vectores es un corte mínimo.

myReliability Recibe un nivel j y calcula $\mathbf{R}_{S,j}$.

myUnreliability Recibe un nivel j y calcula $\mathbf{U}_{S,j}$.

myReliabilityBounds Recibe un nivel j y calcula cotas para $\mathbf{R}_{S,j}$ mediante el árbol de Mayer-Vietoris de $I_j(\mathcal{S})$.

myUnreliabilityBounds Recibe un nivel j y calcula cotas para $\mathbf{U}_{S,j}$ mediante el árbol de Mayer-Vietoris de $I_j(\mathcal{S})$.

Más adelante se realizan unos ejemplos utilizando la clase desarrollada. Primero se valida la clase con sistemas de diferente naturaleza que aparecen en la literatura, como puede ser el *Army Battle Plan* de [26] y, después, se calcula la fiabilidad de redes empleando la clase implementada. En particular, se utiliza la red *GAAR: Italian Research and Education Network* y redes generadas de manera aleatoria mediante los modelos de Erdős-Rényi $ER(n,p)$ y Barabasi-Albert $BA(n,m)$. Finalmente, se muestra el comportamiento de la clase en sistemas en dos sistemas k-entre-12 multi-estado con 4 niveles de funcionamiento y diferenciando $k=4$ y $k=6$.

El capítulo finaliza tratando la complejidad computacional del método algebraico implementado con la clase.

Así, esta tesis cumple los objetivos planteados. Además, durante su desarrollo han surgido nuevas vías de investigación que serán tratadas en un futuro.

Bibliography

[1] ALD. Available at http://reliability-analysis-software.com/.

[2] ITEM software. Available at http://www.itemsoft.com.

[3] Reliasoft. Available at http://www.reliasoft.com.

[4] TIOBE. Available at http://www.tiobe.com/tiobe-index, april 2020.

[5] Abbott, J., and Bigatti, A. M. CoCoALib: a C++ library for doing Computations in Commutative Algebra. Available at http://cocoa.dima.unige.it/cocoalib.

[6] Abbott, J., Bigatti, A. M., and Robbiano, L. CoCoA: a system for doing Computations in Commutative Algebra. Available at http://cocoa.dima.unige.it.

[7] Abramson, M. Restricted combinations and compositions. *Fibonacci Quart 14*, 5 (1976), 439.

[8] Amari, S. V., Zuo, M. J., and Dill, G. A fast and robust reliability evaluation algorithm for generalized multi-state *k*-out-of-*n* systems. *IEEE Transactions on reliability 58*, 1 (2009), 88–97.

[9] Bai, G., Tian, Z., and Zuo, M. J. Reliability evaluation of multistate networks: An improved algorithm using state-space decomposition and experimental comparison. *IISE Transactions 50*, 5 (2018), 407–418.

[10] Ball, M. Computing network reliability. *Oper. Res. 27* (1979), 823–838.

[11] Ball, M. Computational complexity of network reliability analysis: An overview. *IEEE Trans. Rel. R-35* (1986), 230–239.

[12] Barabási, A.-L., and Albert, R. Emergence of scaling in random networks. *Science 286* (1999), 509–512.

[13] Barlow, R. E., and Proschan, F. *Statistical theory of reliability and life testing: Probability models*. Holt, Rinehart and Winston, 1975.

[14] Barlow, R. E., and Proschan, F. *Mathematical theory of reliability*. SIAM, 1996.

[15] Batzies, E., and Welker, V. Discrete morse theory for cellular resolutions. *Journal für die reine undangewandte Mathematik 543* (2002), 147–168.

[16] BELALOUI, S., and KSIR, B. Reliability of a multi-state consecutive k-out-of-n: G system. *International Journal of Reliability, Quality and Safety Engineering 14*, 04 (2007), 361–377.

[17] Belfore, L. An $o(n(\log n)^2)$ algorithm for computing the reliability of k-out-of-n : g and & k-to$l$-out-of-n : g systems. *IEEE Trans. Rel. R-44* (1995), 132–136.

[18] Benito, M. personal communication.

[19] Berkesch, C., and Matusevich, L. F. A-graded methods for monomial ideals. *Journal of Algebra 322*, 8 (2009), 2886–2904.

[20] Bermejo, I., and Gimenez, P. Saturation and Castelnuovo–Mumford regularity. *Journal of Algebra 303*, 2 (2006), 592–617.

[21] Bigatti, A., Pascual-Ortigosa, P., and Sáenz-de Cabezón, E. A c++ class for multi-state algebraic reliability computations. *Reliability Engineering & System Safety* (2021), 107751.

[22] Bigatti, A. M., Caboara, M., and Robbiano, L. On the computation of Hilbert-Poincaré series. *Applicable algebra in engineering, communication and computing 2*, 1 (1991), 21–33.

[23] Bigatti, A. M., Gimenez, P., and Sáenz-de Cabezón, E. *Monomial ideals, computations and applications*, vol. 2083. Springer, 2013.

[24] Bigatti, A. M., and Sáenz-de-Cabezón. Computation of the (n-1)-st Koszul Homology of monomial ideals and related algorithms. In *International Symposium on Symbolic and Algebraic Computation, ISSAC'09, Seoul, South Korea, 2009* (2009), ACM, pp. 31–37.

[25] Billinton, R., and Wenyuan, L. Hybrid approach for reliability evaluation of composite generation and transmission systems using monte-carlo simulation and enumeration technique. In *IEE Proceedings C (Generation, Transmission and Distribution)* (1991), vol. 138, IET, pp. 233–241.

[26] Birnbaum, Z. W., Esary, J. D., and Saunders, S. C. Multi-component systems and structures and their reliability. *Technometrics 3*, 1 (1961), 55–77.

[27] Bobbio, A., and Trivedi, K. S. *Reliability and availability engineering: modeling, analysis, and applications*. Cambridge University Press, 2017.

[28] Boedigheimer, R. A., and Kapur, K. C. Customer-driven reliability models for multistate coherent systems. *IEEE Transactions on Reliability 43*, 1 (1994), 46–50.

[29] Caldarola, L. Coherent systems with multistate components. *Nuclear Engineering and Design 58*, 1 (1980), 127–139.

[30] Chaturvedi, S., Basha, S., Amari, S., and Zuo, M. Reliability analysis of generalized multi-state k-out-of-n systems. *Proceedings of the Institution of Mechanical Engineers, Part O: Journal of Risk and Reliability 226*, 3 (2012), 327–336.

[31] Cox, D. A., Little, J., and O'shea, D. *Using algebraic geometry*, vol. 185. Springer Science & Business Media, 2006.

[32] Dao, H., Huneke, C., and Schweig, J. Bounds on the regularity and projective dimension of ideals associated to graphs. *Journal of Algebraic Combinatorics 38*, 1 (2013), 37–55.

[33] Dao, H., and Schweig, J. Bounding the projective dimension of a squarefree monomial ideal via domination in clutters. *Proceedings of the American Mathematical Society 143*, 2 (2015), 555–565.

[34] Dilworth, R. P. A decomposition theorem for partially ordered sets. *Annals of Mathematics* (1950), 161–166.

[35] Ding, Y., Zuo, M. J., Lisnianski, A., and Li, W. A framework for reliability approximation of multi-state weighted *k*-out-of-*n* systems. *IEEE Transactions on reliability 59*, 2 (2010), 297–308.

[36] Ding, Y., Zuo, M. J., Tian, Z., and Li, W. The hierarchical weighted multi-state *k*-out-of-*n* system model and its application for infrastructure management. *IEEE Transactions on Reliability 59*, 3 (2010), 593–603.

[37] Dohmen, K. *Improved Bonferroni inequalities via abstract tubes: inequalities and identities of inclusion-exclusion type*. Springer Science & Business Media, 2003.

[38] Eger, S. Restricted weighted integer compositions and extended binomial coefficients. *J. Integer Seq 16*, 13.1 (2013), 3.

[39] Eisenbud, D. *Commutative algebra with a view towards algebraic geometry*. Springer, 1995.

[40] El-Neweihi, E., Proschan, F., and Sethuraman, J. Multistate coherent systems. *Journal of Applied Probability 15*, 4 (1978), 675–688.

[41] Eliahou, S., and Kervaire, M. Minimal resolutions of some monomial ideals. *Journal of Algebra 129*, 1 (1990), 1–25.

[42] Erdős, P., and Rényi, A. On random graphs i. *Publicationes Mathematicae Debrecen 6* (1959), 290–297.

[43] Eryilmaz, S. Review of recent advances in reliability of consecutive k-out-of-n and related systems. *Proceedings of the Institution of Mechanical Engineers, Part O: Journal of Risk and Reliability 224*, 3 (2010), 225–237.

[44] Eryilmaz, S. Reliability analysis of multi-state system with three-state components and its application to wind energy. *Reliability Engineering & System Safety 172* (2018), 58–63.

[45] Esary, J. D., and Proschan, F. Coherent structures of non-identical components. *Technometrics 5*, 2 (1963), 191–209.

[46] Flajolet, P., and Sedgewick, R. *Analytic combinatorics*. Cambridge University press, 2009.

[47] Fratta, L., and Montanari, U. A boolean algebra method for computing the terminal reliability in a communication network. *IEEE Transactions on Circuit Theory 20*, 3 (1973), 203–211.

[48] Fröberg, R. Some complex constructions with applications to Poincaré series. In *Séminaire d'Algèbre Paul Dubreil*. Springer, 1979, pp. 272–284.

[49] Fröberg, R. A study of graded extremal rings and of monomial rings. *Mathematica Scandinavica* (1982), 22–34.

[50] Fu, J. C. Reliability of consecutive-k-out-of-n: F systems with (k-1)-step Markov dependence. *IEEE Transactions on Reliability 35*, 5 (1986), 602–606.

[51] Funnemark, E., and Natvig, B. Bounds for the availabilities in a fixed time interval for multistate monotone systems. *Adv. Appl. Prob 17* (1985), 638–665.

[52] Gao, H., Cui, L., and Chen, J. Reliability modeling for sparsely connected homogeneous multistate consecutive-k-out-of-n: G systems. *IEEE Transactions on Systems, Man, and Cybernetics: Systems 51*, 3 (2021), 1844–1854.

[53] Gasemyr, J., and Natvig, B. Improved availability bounds for binary and multi-state monotone systems with independent component processes. *Journal of Applied Probability 54*, 3 (2017), 750–762.

[54] Gertsbakh, I. B., and Shpungin, Y. *Models of network reliability: analysis, combinatorics, and Monte Carlo*. CRC press, 2016.

[55] Giglio, B., Wynn, H. P., et al. Monomial ideals and the scarf complex for coherent systems in reliability theory. *The Annals of Statistics 32*, 3 (2004), 1289–1311.

[56] Goharshady, A. K., and Mohammadi, F. An efficient algorithm for computing network reliability in small treewidth. *Reliability Engineering & System Safety 193* (2020), 106665.

[57] Grayson, D. R., and Stillman, M. E. Macaulay2, a software system for research in algebraic geometry. Available at `http://www.math.uiuc.edu/Macaulay2/`, 2002.

[58] Hà, H. T., and Morey, S. Embedded associated primes of powers of square-free monomial ideals. *Journal of Pure and Applied Algebra 214*, 4 (2010), 301–308.

[59] Habib, A., Al-Seedy, R., and Radwan, T. Reliability evaluation of multi-state consecutive k-out-of-r-from-n: G system. *Applied Mathematical Modelling 31*, 11 (2007), 2412–2423.

[60] Hartshorne, R. Connectedness of the Hilbert scheme. *Publications Mathématiques de l'Institut des Hautes Études Scientifiques 29*, 1 (1966), 7–48.

[61] He, J., and Tuyl, A. V. Algebraic properties of the path ideal of a tree. *Communications in Algebra® 38*, 5 (2010), 1725–1742.

[62] Herzog, J., and Hibi, T. Monomial ideals. In *Monomial Ideals*. Springer, 2011.

[63] Herzog, J., Hibi, T., Murai, S., and Takayama, Y. Componentwise linear ideals with minimal or maximal Betti numbers. *Arkiv för Matematik 46*, 1 (2008), 69–75.

[64] Herzog, J., Hibi, T., and Qureshi, A. A. Polarization of koszul cycles with applications to powers of edge ideals of whisker graphs. *arXiv preprint arXiv:1310.7984* (2013).

[65] Herzog, J., Popescu, D., and Vladoiu, M. On the ext-modules of ideals of Borel type. *Contemporary Mathematics 331* (2003), 171–186.

[66] Herzog, J., and Takayama, Y. Resolutions by mapping cones. *Homology, Homotopy and Applications 4*, 2 (2002), 277–294.

[67] Higashiyama, Y. A factored reliability formula for weighted-k-out-of-n system. *Asia-Pacific Journal of Operational Research 18*, 1 (2001), 61.

[68] Huang, J., Zuo, M. J., and Fang, Z. Multi-state consecutive-k-out-of-n systems. *IIE Transactions 35*, 6 (2003), 527–534.

[69] Huang, J., Zuo, M. J., and Wu, Y. Generalized multi-state k-out-of-n: G systems. *IEEE Transactions on Reliability 49*, 1 (2000), 105–111.

[70] Hudson, J. C. The structure and reliability of multistate systems with multistate components. *PhD. Thesis, Wayne State University* (1982).

[71] Ichim, B., Katthän, L., and Moyano-Fernández, J. J. The behavior of Stanley depth under polarization. *Journal of Combinatorial Theory, Series A 135* (2015), 332–347.

[72] Jaklič, G., Vitrih, V., and ŽAGAR, E. Closed form formula for the number of restricted compositions. *Bulletin of the Australian Mathematical Society 81*, 2 (2010), 289–297.

[73] Koutras, M. V. On a Markov chain approach for the study of reliability structures. *Journal of Applied Probability 33*, 2 (1996), 357–367.

[74] Kuo, W., and Zhu, X. *Importance measures in reliability, risk, and optimization: principles and applications*. John Wiley & Sons, 2012.

[75] Kuo, W., and Zuo, M. J. *Optimal reliability modeling: principles and applications*. John Wiley & Sons, 2003.

[76] Levitin, G. *The Universal Generating Function in Reliability Analysis and Optimization*. Springer, 2005.

[77] Levitin, G., and Lisnianski, A. Optimal multistage modernization of power system subject to reliability and capacity requirements. *Electric Power Systems Research 50*, 3 (1999), 183–190.

[78] Levitin, G., and Xing, L. Reliability and performance of multi-state systems with propagated failures having selective effect. *Reliability Engineering & System Safety 95*, 6 (2010), 655–661.

[79] Levitin, G., Xing, L., Ben-Haim, H., and Dai, Y. Multi-state systems with selective propagated failures and imperfect individual and group protections. *Reliability Engineering & System Safety 96*, 12 (2011), 1657–1666.

[80] Levitin, G., Xing, L., and Dai, Y. Linear multistate consecutively-connected systems subject to a constrained number of gaps. *Reliability Engineering & System Safety 133* (2016), 246–252.

[81] Li, H.-S., and Cao, Z.-J. Matlab codes of subset simulation for reliability analysis and structural optimization. *Struct. Multidisc. Optim. 54* (2016), 391–410.

[82] Li, W., and Zuo, M. J. Reliability evaluation of multi-state weighted k-out-of-n systems. *Reliability Engineering & System Safety 93*, 1 (2008), 160–167.

[83] Lisnianski, A. Extended block diagram method for a multi-state system reliability assessment. *Reliability Engineering & System Safety 92*, 12 (2007), 1601–1607.

[84] Lisnianski, A., Frenkel, I., and Ding, Y. *Multi-state system reliability analysis and optimization for engineers and industrial managers*. Springer, 2010.

[85] Lisnianski, A., and Levitin, G. *Multi-state system reliability: assessment, optimization and applications*, vol. 6. World Scientific Publishing Company, 2003.

[86] Lisnianski, A., Levitin, G., Ben-Haim, H., and Elmakis, D. Power system structure optimization subject to reliability constraints. *Electric Power Systems Research 39*, 2 (1996), 145–152.

[87] Liu, Y., Pedrielli, G., Li, H., Lee, L. H., Chen, C.-H., and Shortle, J. F. Optimal computing budget allocation for stochastic n–*k* problem in the power grid system. *IEEE Transactions on Reliability 68*, 3 (2019), 778–789.

[88] Locks, M. O. A minimizing algorithm for sum of disjoint products. *IEEE Transactions on Reliability 36*, 4 (1987), 445–453.

[89] Lyubeznik, G. A new explicit finite free resolution of ideals generated by monomials in an r-sequence. *Journal of Pure and Applied Algebra 51*, 1-2 (1988), 193–195.

[90] Martínez-Bernal, J., Morey, S., and Villarreal, R. H. Associated primes of powers of edge ideals. *Collectanea Mathematica 63*, 3 (2012), 361–374.

[91] Martínez-Bernal, J., Morey, S., Villarreal, R. H., and Vivares, C. E. Depth and regularity of monomial ideals via polarization and combinatorial optimization. *Acta Mathematica Vietnamica 44*, 1 (2019), 243–268.

[92] McCulloch, W. S., and Pitts, W. A logical calculus of the ideas immanent in nervous activity. *The bulletin of mathematical biophysics 5*, 4 (1943), 115–133.

[93] Miller, E. Topological cohen-macaulay criteria for monomial ideals. *Contemporary Mathematics 502* (2009), 137.

[94] Miller, E., and Sturmfels, B. *Combinatorial commutative algebra*, vol. 227. Springer Science & Business Media, 2004.

[95] Mo, Y., Xing, L., Amari, S. V., and Dugan, J. B. Efficient analysis of multi-state k-out-of-n systems. *Reliability Engineering & System Safety 133* (2015), 95–105.

[96] Mo, Y., Xing, L., Cui, L., and Si, S. Mdd-based performability analysis of multi-state linear consecutive-k-out-of-n: F systems. *Reliability Engineering & System Safety 166* (2017), 124–131.

[97] Mohammadi, F. Combinatorial and geometric view of the system reliability theory. In *International Congress on Mathematical Software* (2016), Springer, pp. 148–153.

[98] Mohammadi, F. Divisors on graphs, orientations, syzygies, and system reliability. *Journal of Algebraic Combinatorics 43*, 2 (2016), 465–483.

[99] Mohammadi, F., Sáenz-de Cabezón, E., and Wynn, H. P. Types of signature analysis in reliability based on Hilbert series. *Journal of Symbolic Computation 79* (2017), 140–155.

[100] Mohammadi, F., and Shokrieh, F. Divisors on graphs, binomial and monomial ideals, and cellular resolutions. *Mathematische Zeitschrift 283*, 1-2 (2016), 59–102.

[101] Moore, E. F., and Shannon, C. E. Reliable circuits using less reliable relays. *Journal of the Franklin Institute 262*, 3 (1956), 191–208.

[102] Natvig, B. *Multi-state systems reliability theory with applications*. John Wiley & sons, 2011.

[103] Natvig, B., and Streller, A. The steady-state behaviour of multistate monotone systems. *Journal of applied probability 21*, 4 (1984), 826–835.

[104] Oxley, J. G. *Matroid theory*, vol. 3. Oxford University Press, USA, 2006.

[105] Pascual-Ortigosa, P., Sáenz-de-Cabezón, E., and Wynn, H. Algebraic reliability of multi-state *k*-out-of-*n* systems. *Probability in the Engineering and Informational Sciences* (2020).

[106] Phan, J. P. Order properties of monomial ideals and their free resolutions.

[107] Radwan, T., Habib, A., Alseedy, R., and Elsherbeny, A. Bounds for increasing multi-state consecutive k-out-of-r-from-n: F system with equal components probabilities. *Applied Mathematical Modelling 35*, 5 (2011), 2366–2373.

[108] Ram, M., and Dohi, T. *Systems engineering: Reliability analysis using k-out-of-n structures*. CRC Press, 2019.

[109] Rao, S. *Reliability engineering*. Pearson, 2015.

[110] Reid, M. Reliability. a Python library for reliability engineering. Available at `http://github.com/MatthewReid854/reliability`, 2020.

[111] Roune, B. H. Frobby. Available at `http://www.broune.com/frobby`.

[112] Roune, B. H., and Sáenz-de-Cabezón, E. Complexity and algorithms for the Euler characteristic of simplicial complexes. *Journal of Symbolic Computation 50* (2013), 170–196.

[113] Rushdi, A. M. A. Utilization of symmetric switching functions in the symbolic reliability analysis of multi-state k-out-of-n systems. *International Journal of Mathematical, Engineering and Management Sciences (IJMEMS) 4*, 2 (2019), 306–326.

[114] Saenz-de Cabezon, E. Combinatorial koszul homology: Computations and applications. *arXiv preprint arXiv:0803.0421* (2008).

[115] Sáenz-de-Cabezón, E. Multigraded betti numbers without computing minimal free resolutions. *Appl. Alg. Eng. Commun. Comput. 20* (2009), 481–495.

[116] Sáenz-de Cabezón, E., and Wynn, H. P. Betti numbers and minimal free resolutions for multi-state system reliability bounds. *Journal of symbolic computation 44*, 9 (2009), 1311–1325.

[117] Sáenz-de Cabezón, E., and Wynn, H. P. Mincut ideals of two-terminal networks. *Applicable Algebra in Engineering, Communication and Computing 21*, 6 (2010), 443–457.

[118] Sáenz-de Cabezón, E., and Wynn, H. P. Computational algebraic algorithms for the reliability of generalized k-out-of-n and related systems. *Mathematics and Computers in Simulation 82*, 1 (2011), 68–78.

[119] Sáenz-de Cabezón, E., and Wynn, H. P. Algebraic reliability based on monomial ideals: A review. In *Harmony of Gröbner Bases and the Modern Industrial Society*. 2012, pp. 314–335.

[120] Sáenz-de Cabezón, E., and Wynn, H. P. Hilbert functions in design for reliability. *IEEE transactions on reliability 64*, 1 (2015), 83–93.

[121] Sahner, R., Trivedi, K. S., and Pullafito, A. SHARPE, (Symbolic Hierarchical Automated Reliability and Performance Evaluator). Available at `http://sharpe.pratt.duke.edu`.

[122] Seiler, W. M. A combinatorial approach to involution and δ-regularity i: Involutive bases in polynomial algebras of solvable type. *Applicable Algebra in Engineering, Communication and Computing 20*, 3 (2009), 207–259.

[123] Seiler, W. M. A combinatorial approach to involution and δ-regularity ii: Structure analysis of polynomial modules with Pommaret bases. *Applicable Algebra in Engineering, Communication and Computing 20*, 3 (2009), 261–338.

[124] Seiler, W. M. A combinatorial approach to involution and δ-regularity ii: Structure analysis of polynomial modules with pommaret bases. *Applicable Algebra in Engineering, Communication and Computing 20*, 3 (2009), 261–338.

[125] Shaffer, L. B., Young, T. M., Guess, F. M., Bensmail, H., and León, R. V. Using R software for reliability data analysis. *International Journal of Reliability and Applications 9* (2008), 53–70.

[126] Sharma, V. K., Agarwal, M., and Sen, K. Reliability evaluation and optimal design in heterogeneous multi-state series-parallel systems. *Information Sciences 181*, 2 (2011), 362–378.

[127] Shrestha, A., Xing, L., and Dai, Y. Decision diagram based methods and complexity analysis for multi-state systems. *IEEE Trans. Rel. 59* (2010), 145–160.

[128] Singh, C., Jirutitijaroen, P., and Mietra, J. *Introduction to power system reliability*. Wiley-IEEE Press, 2019.

[129] Stern, M. *Semimodular lattices: theory and applications*, vol. 73. Cambridge University Press, 1999.

[130] Sturmfels, B., Trung, N. V., and Vogel, W. Bounds on degrees of projective schemes. *Mathematische Annalen 302* (1995), 417–432.

[131] Taylor, D. *Ideals Generated by monomials in an R-sequence*. PhD thesis, University of Chicago, 1960.

[132] Tian, Z., Zuo, M. J., and Yam, R. C. Multi-state k-out-of-n systems and their performance evaluation. *IIE Transactions 41*, 1 (2008), 32–44.

[133] Trivedi, K. S., and Bobbio, A. *Reliability and availability engineering: modeling, analysis, and applications*. Cambridge University Press, 2017.

[134] Turing, A. M., et al. On computable numbers, with an application to the entscheidungsproblem. *J. of Math 58*, 345-363 (1936), 5.

[135] Ushakov, I. A universal generating function. *Soviet Journal of Computer and Systems Sciences 24*, 5 (1986), 118–129.

[136] Ushakov, I. Optimal standby problem and a universal generating function. *Sov. J. Comput. Sys. Sci. 25* (1987), 61–73.

[137] Villarreal, R. H. *Monomial algebras*, vol. 238. Marcel Dekker New York, 2001.

[138] Von Neumann, J. Probabilistic logics and the synthesis of reliable organisms from unreliable components. *Automata studies 34* (1956), 43–98.

[139] Welker, V. G.-I. P.-V. The lcm-lattice in monomial resolutions. *Mathematical Research Letters 6* (1999), 521–532.

[140] Wu, J. S., and Chen, R. J. An algorithm for computing the reliability of a weighted-k-out-of-n system. *IEEE Trans, Rel. 43*, 2 (1994), 327–328.

[141] Yamamoto, H., Zuo, M. J., Akiba, T., and Tian, Z. G. Recursive formulas for the reliability of multistate consecutive k-out-of-n: G systems. *IEEE Trans, Rel. 55*, 1 (2006), 98–104.

[142] Yamamoto, H., Zuo, M. J., Akiba, T., and Tian, Z. G. Reliability for sparsely connected consecutive k systems. *IEEE Trans, Rel. 56*, 3 (2007), 516–524.

[143] Yamamoto, H., Zuo, M. J., Akiba, T., and Tian, Z. G. Reliabilities of some multistate consecutive k systems. *IEEE Trans, Rel. 69*, 2 (2020), 414–429.

[144] Yingkui, G., and Jing, L. Multi-state system reliability: A new and systematic review. *Procedia Engineering 29* (2012), 531–536.

[145] Yu, H., Yang, J., Peng, R., and Zhao, Y. Reliability evaluation of linear multi-state consecutively-connected systems constrained by m consecutive and n total gaps. *Reliability Engineering & System Safety 150* (2016), 35–43.

[146] Zhang, C., Liu, T., and Bai, G. An improved algorithm for reliability bounds of multistate networks. *Communications in Statistics-Theory and Methods* (2020), 1–20.

[147] Zhao, X., and Cui, L. Reliability evaluation of generalised multi-state k-out-of-n systems based on fmci approach. *International Journal of Systems Science 41*, 12 (2010), 1437–1443.

[148] Zhao, X., Xu, Y., and Liu, F. Y. State distributions of multistate consecutive k-out-of-n systems. *IEEE Trans, Rel. 61*, 2 (2012), 274–281.

[149] Zio, E., Marella, M., and Podofillini, L. A Monte Carlo simulation approach to the availability assessment of multi-state systems with operational dependencies. *Reliability Engineering & System Safety 92*, 7 (2007), 871–882.

[150] Zio, E., Podofillini, L., and Levitin, G. Estimation of the importance measures of multi-state elements by Monte Carlo simulation. *Reliability Engineering & System Safety 86*, 3 (2004), 191–204.

[151] Zuo, M. J., and Tian, Z. Performance evaluation of generalized multi-state k-out-of-n systems. *IEEE Transactions on Reliability 55*, 2 (2006), 319–327.

Index