TESIS DOCTORAL

Homología efectiva y sucesiones espectrales

Ana Romero Ibáñez

UNIVERSIDAD DE LA RIOJA

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Universidad de La Rioja
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Effective Homology and Spectral Sequences

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of Doctor of Philosophy

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To my grandmother Concordia. The first lines of this memoir were written in her home, during her last days. She would have needed a few more months to enjoy the end of this work with me.

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Contents

Introduction

Algebraic Topology consists in trying to use as much as possible "algebraic" methods to attack topological problems; in the simplest cases, it consists in associating to topological spaces some *algebraic invariants* which describe their essential properties. For instance, one can define some special groups associated with a topological space, in a way that respects the relation of homeomorphism of spaces. This allows one to study some interesting properties about topological spaces by means of statements about groups, which are often easier to prove. More generally, there exist several *functors* which assign to some topological spaces some *algebraic objects*. Very frequently, if this functor works on a topological space of "finite type", then the result is also an algebraic object of finite type. But in general there do not exist *algorithms* capable of computing these algebraic objects of finite type according to the different functors of Algebraic Topology.

Two important algebraic invariants are homotopy groups and homology (and cohomology) groups. The simplest one is the computation of the usual homology groups (with coefficients in $\mathbb{Z}$) of finite simplicial complexes. It is not hard to write such an algorithm: a simplicial complex determines a chain complex of finite type and its homology groups are deduced from elementary operations with the differential operators, as explained for example in [KMM04]. A more difficult problem is to compute the homotopy groups of a finite simplicial complex X, denoted $\pi_n(X)$.

The definition of homotopy group was given by Hurewicz in [Hur35] and [Hur36] as a generalization of the fundamental group, originally due to Poincaré in [Poi95], a paper which can be considered the origin of Algebraic Topology. At this time only some groups of the first non-trivial space, namely the 2-sphere S^2, were known; to be precise, Heinz Hopf [Hop35] computed $\pi_2(S^2) = \mathbb{Z}$ and $\pi_3(S^2) = \mathbb{Z}$. The group $\pi_4(S^2) = \mathbb{Z}_2$ was determined by Hans Freudenthal in 1937 [Fre37], but then little more was known about homotopy groups of spaces until 1950. The following groups $\pi_n(S^2)$ were obtained by Jean-Pierre Serre for $5 \leq n \leq 9$ [Ser51]. In fact, for $n = 6$, Serre proved the group $\pi_6(S^2)$ has twelve elements, but he was not able to choose between both possible solutions $\mathbb{Z}_{12}$ and $\mathbb{Z}_2 \oplus \mathbb{Z}_6$, the first historical example where a topologist faced a serious *extension problem*. Two years later, Barratt and Paechter [BP52] proved there exists an element of order 4 in $\pi_6(S^2)$, so that finally $\pi_6(S^2) = \mathbb{Z}_{12}$. Other references about computation of homotopy groups of spheres are, for instance, [Tod62], [Mah67], and [Rav86].

Serre also obtained a general *finiteness result* [Ser53] which asserts that, if X is a simply connected space such that the homology groups $H_n(X;\mathbb{Z})$ are of *finite type*, then the homotopy groups $\pi_n(X)$ are also Abelian groups of finite type. In 1957, Edgar Brown published in [Bro57] a *theoretical* algorithm for the computation of these groups, based on the Postnikov tower and making use of finite approximations of infinite simplicial sets, transforming in this way the finiteness results of Serre into a computability result. Nevertheless, Edgar Brown himself quoted in his paper that his algorithm has no practical use, even with the most powerful computer you can imagine: it is a consequence of the hyper-exponential complexity of the algorithm designed by Brown.

The effective homology method appeared in the eighties trying to make available *real* algorithms for the computation of homology and homotopy groups. Introduced by Francis Sergeraert in [Ser87] and [Ser94], the present state of this technique is described in [RS97] and [RS06]. It is based on the notion of *object with effective homology*, which connects a space with its *homology* by means of chain equivalences, and it is closely related with the homology perturbation theory, whose fundamental references are the classical works of Shi Weishu [Shi62] and Ronnie Brown [Bro67], and those of Victor Gugenheim, Larry Lambe, and Jim Stasheff [GL89] [GLS91].

The effective homology method has been concretely implemented in the system Kenzo [DRSS99] (whose previous version was called EAT [RSS97]), a Common Lisp program which has made it possible to compute some complicated homology groups so far unreachable. In particular, Kenzo can compute, for instance, the homology groups of total spaces of fibrations, of iterated loop spaces, of classifying spaces, etc. Other useful papers about the effective homology method and the Kenzo system are [RS88], [Rub91], [RS02], and [RS05a].

Spectral sequences are a different technique traditionally considered to calculate homology and homotopy groups of spaces (see, for instance, [McC85] or [Hat04]). For example, the Serre spectral sequence [Ser51] gives information about the homology groups of the total space of a fibration when the homology groups of the base and fiber spaces are known. On the other hand, the Eilenberg-Moore spectral sequence [EM65b] gives information about the homology groups of the base space (resp. the fiber space) from the homologies of the total space and of the fiber (resp. base space). For the computation of homotopy groups, the spectral sequences of Adams [Ada60] or Bousfield-Kan [BK72a] can be considered.

But the various classical spectral sequences pose a very important problem: they are not *algorithms*. A spectral sequence is a family of "pages" $(E^r_{p,q}, d^r)_{r\geq 1}$ of differential bigraded modules, each page being made of the homology groups of the preceding one. Then, as expressed by John McCleary in [McC85]:

> *It is worth repeating the caveat about differentials mentioned in Chapter 1: knowledge of $E^r_{*,*}$ and d^r determines $E^{r+1}_{*,*}$ but not d^{r+1}. If we think of a spectral sequence as a black box, then the input is a differential bigraded module, usually $E^1_{*,*}$, and, with each turn of the handle, the machine computes a successive homology according to a sequence of differentials. If some differential is unknown, then some other (any*

other) principle is needed to proceed. From Chapter 1, the reader is acquainted with several algebraic tricks that allow further calculation. In the nontrivial cases, it is often a deep geometric idea that is caught up in the knowledge of a differential.

In most cases, it is in fact a matter of computability: the higher differentials of the spectral sequence are mathematically defined, but their definition is not constructive. In other words, the differentials are not computable with the usually provided information.

Another different problem of spectral sequences is the extension problem at abutment. A spectral sequence gives one a filtration of the looked-for (homology or homotopy) groups, but then in some cases several solutions are possible. This was the problem Jean-Pierre Serre found when trying to compute $\pi_6(S^2)$, a problem which can be solved making use of the effective homology technique.

The goal of this work has been to relate spectral sequences and effective homology, showing that the effective homology method can be used to produce *algorithms* computing the various components of some spectral sequences, *higher differentials* included.

The organization of the memoir is as follows. The first chapter includes some preliminary notions and results that will be used in the rest of this work. To be precise, in the first section we introduce chain complexes and spectral sequences, two fundamental notions in Homological Algebra. The second section is devoted to simplicial topology, focusing on simplicial sets, homotopy groups, and Eilenberg-MacLane spaces. Finally, the effective homology method and the Kenzo system are explained in the third section.

After this first chapter, the memoir is divided into two different parts. Chapters 2 and 3 are devoted to spectral sequences associated with filtered complexes, which under favorable *natural* conditions converge to their homology groups. On the other hand, Chapters 4 and 5 deal with the Bousfield-Kan spectral sequence, related with the computation of homotopy groups.

Chapter 2 contains several algorithms for the computation of the different components of spectral sequences associated with filtered complexes with effective homology: the groups $E^r_{p,q}$, the differential maps $d^r_{p,q}$ for every level r, as well as the stage r at which the convergence has been reached for each degree n, and the filtration of the homology groups induced by the filtration of the chain complex. Our results can be applied, for instance, for the computation of spectral sequences associated with bicomplexes. These algorithms have been implemented as a new module for the Kenzo system, which is also explained in this chapter by means of some elementary examples.

The results presented in Chapter 2 make it also possible to *compute* two classical examples of spectral sequences, those of Serre and Eilenberg-Moore, which are studied in Chapter 3. If the spaces involved in the corresponding fibrations are objects with effective homology, the different components of the associated spectral sequences can be determined making use of our algorithms. In this way, we make *constructive* these spectral sequences that, up to now, were not algorithms. Both situations have been illustrated by means of several examples implemented in Common Lisp.

Other spectral sequences of great interest are not associated with filtered complexes. This is the case of the Bousfield-Kan spectral sequence, which first appeared in [BK72a] trying to generalize the Adams spectral sequence [Ada60]. Although there exists a formal definition of the Adams spectral sequence, problems are found in order to *compute* it, as explained in the introduction of [Tan85]. First of all, we must determine the cohomology of the Steenrod algebra [Ste62], then we must find the higher differentials, and finally extension problems at abutment can appear. In this case, our previous algorithms for spectral sequences associated with filtered complexes cannot be used, but the effective homology method can also be useful to develop a constructive version of the Bousfield-Kan spectral sequence.

As a first step toward this *effective* version of the Bousfield-Kan spectral sequence, the main result of Chapter 4 is an algorithm computing the *effective* homology of the $\mathbb{Z}$-free simplicial Abelian group RX generated by a 1-reduced simplicial set X. The "ordinary" homology of RX is easily deduced from Cartan's work [Car55] about Eilenberg-MacLane spaces, but this information is unsufficient for our purpose: *effective* homology is definitely required here. Our algorithm makes use of several constructions of Algebraic Topology such as the Dold-Kan correspondence between the categories of chain complexes and simplicial Abelian groups, fibrations or Eilenberg-MacLane spaces. Some parts of this algorithm have been implemented by means of a set of programs in Common Lisp which are also explained in this chapter.

Chapter 5 is devoted to the Bousfield-Kan spectral sequence associated with a simplicial set X, trying to construct an algorithm (based on the effective homology technique) computing the whole set of its components. The first part of this chapter contains some algorithms to deal with cosimplicial structures, which are one of the main ingredients in this spectral sequence; the second one is focused on the *construction* of the Bousfield-Kan spectral sequence. We begin this part with a proof of the convergence, based on elementary computations of Homological Algebra. Then the results of Chapter 4 are used to *compute* its first two stages. For the computation of the higher levels, we include the sketch of a new algorithm which is not yet finished.

The memoir ends with a chapter which includes conclusions and further work, and the bibliography.

Chapter 1

Preliminaries

In the first chapter of this memoir we include the definitions, notations, and basic results that we will use in the rest of this work. In particular, the first section is devoted to two fundamental notions in Homological Algebra: chain complexes and spectral sequences. The second section contains some definitions and results of simplicial topology. Finally, we present the fundamental ideas of the effective homology method.

1.1 Basics on Homological Algebra

1.1.1 Chain complexes

The following basic definitions can be found, for instance, in [Mac63].

Definition 1.1. Let R be a ring with a unit element $1 \neq 0$. A *left R-module M* is an additive Abelian group together with a map $p : R \times M \to M$, denoted by $p(r, m) \equiv rm$, such that for every $r, r' \in R$ and $m, m' \in M$

$$\begin{aligned} (r + r')m &= rm + r'm \\ r(m + m') &= rm + rm' \\ (rr')m &= r(r'm) \\ 1m &= m \end{aligned}$$

A similar definition is given for a *right R-module.* Unless the distinction being necessary, we will talk of an *R-module M* without specifying if it is a right or a left R-module.

For $R = \mathbb{Z}$ (the integer ring), a $\mathbb{Z}$-module M is simply an Abelian group. The map $p : \mathbb{Z} \times M \to M$ is given by

$$p(n, m) = \begin{cases} m + \overset{n}{\cdots} + m & \text{if } n > 0 \\ 0 & \text{if } n = 0 \\ (-m) + \overset{-n}{\cdots} + (-m) & \text{if } n < 0 \end{cases}$$

Definition 1.2. A subset S of an R-module M is a *submodule* if S is closed under addition and for all elements $r \in R$ and $s \in S$, one has $rs \in S$.

One can easily observe that a submodule S of and R-module M is itself an R-module.

Definition 1.3. Let R be a ring and M and N be R-modules. An *R-module morphism* $\alpha : M \to N$ is a function from M to N such that for every $m, m' \in M$ and $r \in R$

$$\alpha(m + m') = \alpha(m) + \alpha(m')$$
$$\alpha(rm) = r\alpha(m)$$

Definition 1.4. Given a ring R, a *chain complex* C_* of R-modules is a pair of sequences $C_* = (C_n, d_n)_{n\in\mathbb{Z}}$ where, for each degree $n \in \mathbb{Z}$, C_n is an R-module, the homogeneous component of degree n of $C = \bigoplus_{n\in\mathbb{Z}} C_n$, and $d_n : C_n \to C_{n-1}$ (*the differential map*) is an R-module morphism (of degree -1) such that $d_{n-1} \circ d_n = 0$ for all n.

The module C_n is called the module of *n-chains.* The image $B_n = \operatorname{Im} d_{n+1} \subseteq C_n$ is the (sub)module of *n-boundaries.* The kernel $Z_n = \operatorname{Ker} d_n \subseteq C_n$ is the (sub)module of *n-cycles.*

Given a chain complex $C_* = (C_n, d_n)_{n\in\mathbb{Z}}$, the identities $d_{n-1} \circ d_n = 0$ are equivalent to the inclusion relations $B_n \subseteq Z_n$: every boundary is a cycle. But the converse in general is not true. Thus the next definition makes sense.

Definition 1.5. Let $C_* = (C_n, d_n)_{n\in\mathbb{Z}}$ be a chain complex of R-modules. For each degree $n \in \mathbb{Z}$, the *n-homology module* of C_* is defined as the quotient module

$$H_n(C_*) = \frac{Z_n}{B_n}$$

Definition 1.6. A chain complex $C_* = (C_n, d_n)_{n\in\mathbb{Z}}$ is *acyclic* if $H_n(C_*) = 0$ for all n, that is to say, if $Z_n = B_n$ for every $n \in \mathbb{Z}$.

Definition 1.7. A *morphism of chain complexes of R-modules* (or a *chain complex morphism*) $f : C_* \to D_*$ between two chain complexes of R-modules $C_* = (C_n, d_{C_n})_{n\in\mathbb{Z}}$ and $D_* = (D_n, d_{D_n})_{n\in\mathbb{Z}}$ is a graded R-module morphism (degree 0) which commutes with the differential map. In other words, f consists of R-module morphisms $f_n : C_n \to D_n$ satisfying $d_{D_n} \circ f_n = f_{n-1} \circ d_{C_n}$ for each n.

It is not difficult to prove that a chain complex morphism $f : C_* \to D_*$ induces an R-module morphism on the corresponding homology modules

$$H_*(f) : H_*(C_*) \longrightarrow H_*(D_*)$$

Definition 1.8. Let $f, g : C_* \to D_*$ be morphisms of chain complexes of R-modules. A *(chain) homotopy* h from f to g, written $h : f \simeq g$, is a set of R-module morphisms $h_n : C_n \to D_{n+1}$ such that $h_{n-1} \circ d_{C_n} + d_{D_{n+1}} \circ h_n = f_n - g_n$ for all n.

Theorem 1.9. [Mac63] Given $f, g : C_* \to D_*$ chain complex morphisms and $h : f \simeq g$ a chain homotopy, then the morphisms induced by f and g on homology are the same:

$$H_n(f) = H_n(g) : H_n(C_*) \longrightarrow H_n(D_*) \quad \text{for all } n \in \mathbb{Z}$$

Definition 1.10. A chain complex morphism $f : C_* \to D_*$ is said to be a *chain equivalence* if there exist a morphism $g : D_* \to C_*$ and homotopies $h_1 : \mathrm{Id}_{C_*} \simeq g \circ f$ and $h_2 : \mathrm{Id}_{D_*} \simeq f \circ g$.

Corollary 1.11. [Mac63] If $f : C_* \to D_*$ is a chain equivalence, the induced map

$$H_n(f) : H_n(C_*) \longrightarrow H_n(D_*)$$

is an isomorphism for each dimension n.

Definition 1.12. Let M be a right R-module, and N a left R-module. The *tensor product* $M \otimes_R N$ is the Abelian group generated by the symbols $m \otimes n$ for every $m \in M$ and $n \in N$, subject to the relations

$$\begin{aligned} (m + m') \otimes n &= m \otimes n + m' \otimes n \\ m \otimes (n + n') &= m \otimes n + m \otimes n' \\ mr \otimes n &= m \otimes rn \end{aligned}$$

for all $r \in R,\ m, m' \in M$, and $n, n' \in N$.

If $R = \mathbb{Z}$ (the integer ring), then M and N are Abelian groups and their tensor product will be denoted simply by $M \otimes N$.

Definition 1.13. Let $C_* = (C_n, d_{C_n})_{n\in\mathbb{Z}}$ and $D_* = (D_n, d_{D_n})_{n\in\mathbb{Z}}$ be chain complexes of right and left R-modules respectively. The *tensor product* $C_* \otimes_R D_*$ is the chain complex of $\mathbb{Z}$-modules $C_* \otimes_R D_* = ((C_* \otimes_R D_*)_n, d_n)_{n\in\mathbb{Z}}$ with

$$(C_* \otimes_R D_*)_n = \bigoplus_{p+q=n} (C_p \otimes_R D_q)$$

where the differential map is defined on the generators $x \otimes y$ with $x \in C_p$ and $y \in D_q$, according to the Koszul rule for the signs, by

$$d_n(x \otimes y) = d_{C_p}(x) \otimes y + (-1)^p x \otimes d_{D_q}(y)$$

Definition 1.14. Let C_* and C'_* be chain complexes of right R-modules, D_* and D'_* chain complexes of left R-modules, and $f : C_* \to C'_*$ and $g : D_* \to D'_*$ chain complex morphisms. The *tensor product* $f \otimes_R g : C_* \otimes_R D_* \to C'_* \otimes_R D'_*$ is the morphism of chain complexes given by

$$(f \otimes_R g)(x \otimes y) = f_p(x) \otimes g_q(y)$$

for a generator $x \otimes y$ of $C_p \otimes_R D_q$.

Proposition 1.15. [Mac63] Let C_* and C'_* be chain complexes of right R-modules, D_* and D'_* chain complexes of left R-modules, $f^1, f^2 : C_* \to C'_*$ and $g^1, g^2 : D_* \to D'_*$ morphisms of chain complexes. Given s and t chain homotopies such that $s : f^1 \simeq f^2$ and $t : g^1 \simeq g^2$, then there exists a chain homotopy $u : f^1 \otimes g^1 \simeq f^2 \otimes g^2$. Concretely, $u : C_* \otimes_R D_* \to C'_* \otimes_R D'_*$ is defined for any generator $x \otimes y \in C_p \otimes_R D_q$, respecting the Koszul rule for the signs, by

$$u(x \otimes y) = s_p(x) \otimes g^1_q(y) + (-1)^p f^2_p(x) \otimes t_q(y)$$

Chain complexes together with chain complex morphisms form a category that we will denote by $\mathcal{C}$. In a similar way, it is possible to define the category of cochain complexes.

Definition 1.16. Given a ring R, a *cochain complex* of R-modules is a pair of sequences $C^* = (C^n, \delta^n)_{n\in\mathbb{Z}}$ where, for each $n \in \mathbb{Z}$, C^n is an R-module and $\delta^n : C^{n-1} \to C^n$ (*the coboundary map*) is an R-module morphism (in this case of degree $+1$) such that $\delta^{n+1} \circ \delta^n = 0$ for all n.

The kernel $Z^n = \operatorname{Ker} \delta^{n+1} \subseteq C^n$ is called the module of n-*cocycles*, $B^n = \operatorname{Im} \delta^n \subseteq C^n$ is the module of n-*coboundaries*, and the quotient $H^n(C) = Z^n/B^n$ is the n-*cohomology module* of C^*.

The corresponding definitions can also be given for *cochain complex morphism*, *cochain homotopy* or *tensor product* of two cochain complexes. See [Wei94] for details.

In many situations the ring R is the integer ring, $R = \mathbb{Z}$. In this case, a chain complex C_* is given by a graded Abelian group $\{C_n\}_{n\in\mathbb{Z}}$ and a graded group morphism of degree -1, $\{d_n : C_n \to C_{n-1}\}_{n\in\mathbb{Z}}$, satisfying $d_{n-1} \circ d_n = 0$ for all n. In most part of this work, we will choose $R = \mathbb{Z}$; unless otherwise stated, the integer ring must be considered.

From now on in this memoir, we will only work with *non-negative* chain complexes, that is to say, $C_* = (C_n, d_n)_{n\in\mathbb{Z}}$ such that $C_n = 0$ if $n < 0$. A non-negative chain complex C_* will be therefore denoted by $C_* = (C_n, d_n)_{n\in\mathbb{N}}$. Moreover, the chain (and cochain) complexes we work with are supposed to be free.

Definition 1.17. A chain complex $C_* = (C_n, d_n)_{n\in\mathbb{N}}$ of $\mathbb{Z}$-modules is said to be *free* if C_n is a free $\mathbb{Z}$-module for each $n \in \mathbb{N}$.

The same definition can be given for a cochain complex $C^* = (C^n, \delta^n)_{n\in\mathbb{N}}$.

Most often, our free $\mathbb{Z}$-modules will be provided with a natural *distinguished basis* of generators g_i. Every n-chain (or n-cochain) can be expressed then as a linear combination

$$\sum \lambda_i g_i$$

where each $\lambda_i \in \mathbb{Z}$, and g_i is a *generator* of C_n. A product $\lambda_i g_i$ is called a *term*, and a sum of terms is a *combination*.

The next definition is necessary for Theorem 1.19, which expresses a very important property of free complexes that will be used later.

Definition 1.18. A chain complex $C_* = (C_n, d_n)_{n \in \mathbb{N}}$ is called *short* if there exists $m \in \mathbb{N}$ such that $C_n = 0$ for $n \neq m, m+1$, and $d_{m+1} : C_{m+1} \to C_m$ is monomorphic. A chain complex C_* is called *elementary* if C_* is short and, moreover, $C_m \cong \mathbb{Z}$ (which implies that $C_{m+1} \cong \mathbb{Z}$ or $C_{m+1} \cong 0$).

Theorem 1.19. [Dol72] Every free chain complex $C_* = (C_n, d_n)_{n \in \mathbb{N}}$ is a direct sum of short (free) complexes. Furthermore, if every C_n is finitely generated, then C_* is a direct sum of elementary complexes.

1.1.2 Spectral sequences

We include in this section some basic definitions and properties about spectral sequences, which have been mostly extracted from [Mac63]. A more complete reference is of course [McC85].

Definition 1.20. Let R be a ring, a *bigraded R-module* is a family of R-modules $E = \{E_{p,q}\}_{p,q \in \mathbb{Z}}$. A *differential* $d : E \to E$ of bidegree $(-r, r-1)$ is a family of morphisms of R-modules $d_{p,q} : E_{p,q} \to E_{p-r,q+r-1}$ for each $p, q \in \mathbb{Z}$, such that $d_{p-r,q+r-1} \circ d_{p,q} = 0$. The pair (E, d) is called a *differential bigraded module.*

The relations $d_{p-r,q+r-1} \circ d_{p,q} = 0$ allow us to define the *homology* of E as the bigraded R-module $H(E, d) = H(E) = \{H_{p,q}(E)\}_{p,q \in \mathbb{Z}}$ with

$$H_{p,q}(E) = \frac{\operatorname{Ker} d_{p,q}}{\operatorname{Im} d_{p+r,q-r+1}}$$

Definition 1.21. A *spectral sequence* $E = (E^r, d^r)_{r \geq 1}$ is a sequence of bigraded R-modules $E^r = \{E^r_{p,q}\}_{p,q \in \mathbb{Z}}$, each provided with a differential $d^r = \{d^r_{p,q}\}_{p,q \in \mathbb{Z}}$ of bidegree $(-r, r-1)$ and with isomorphisms $H(E^r, d^r) \cong E^{r+1}$ for every $r \geq 1$.

In some situations, only the first levels of a spectral sequence are given; in other words, only $(E^r, d^r)_{1 \leq r \leq k}$ are known (very frequently $k = 1$ or 2). Since each E^{r+1} in the spectral sequence is (up to isomorphism) the bigraded homology module of the preceding differential bigraded module (E^r, d^r), the stage k in the spectral sequence, given by $E^k = \{E^k_{p,q}\}$ and $d^k = \{d^k_{p,q}\}$, allows us to build the bigraded module at the level $k+1$, $E^{k+1} = \{E^{k+1}_{p,q}\}$. But then our information is not sufficient to define the next differential d^{k+1}, which therefore must be independently defined too. In this way, a finite number of stages of the spectral sequence does not allow us to compute the *whole* spectral sequence, some extra information is necessary to determine the successive differential maps.

Definition 1.22. Let $E = (E^r, d^r)_{r \geq 1}$ and $E' = (E'^r, d'^r)_{r \geq 1}$ be two spectral sequences. A *morphism of spectral sequences* $f : E \to E'$ is a family of morphisms of bigraded modules $\{f^r : E^r \to E'^r\}_{r \geq 1}$ of bidegree $(0, 0)$, with $d^r \circ f^r = f^r \circ d^r$, and such that each f^{r+1} is the map induced by f^r on the homology module $H(E^r, d^r) \cong E^{r+1}$.

A spectral sequence E can be presented as a tower

$$0 = B^0 \subseteq B^1 \subseteq B^2 \subseteq \cdots \subseteq Z^2 \subseteq Z^1 \subseteq Z^0 = E^1$$

of bigraded submodules of E^1, where $E^{r+1} = Z^r/B^r$ and the differential d^{r+1} can be taken as a mapping $Z^r/B^r \to Z^r/B^r$, with kernel Z^{r+1}/B^r and image B^{r+1}/B^r.

We say that the module Z^{r-1} is the set of elements that *live till stage* r, while B^{r-1} is the module of elements that *bound by stage* r. Let $Z^\infty = \bigcap_r Z^r$ be the submodule of E^1 of elements that *survive forever* and $B^\infty = \bigcup_r B^r$ the submodule of those elements which *eventually bound.* It is clear that $B^\infty \subseteq Z^\infty$ and therefore the spectral sequence determines a bigraded module $E^\infty = \{E^\infty_{p,q}\}_{p,q\in\mathbb{Z}}$ given by

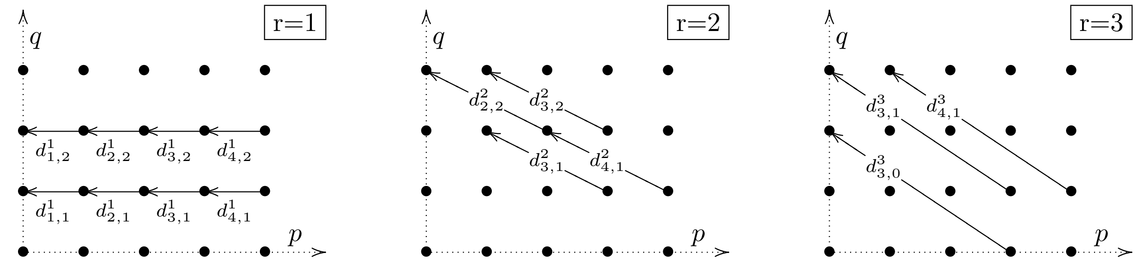
$$E^\infty_{p,q} = \frac{Z^\infty_{p,q}}{B^\infty_{p,q}}$$

which is the bigraded module that remains after the computation of the infinite sequence of successive homologies. The modules $E^\infty_{p,q}$ are called the *final modules* of the spectral sequence E.

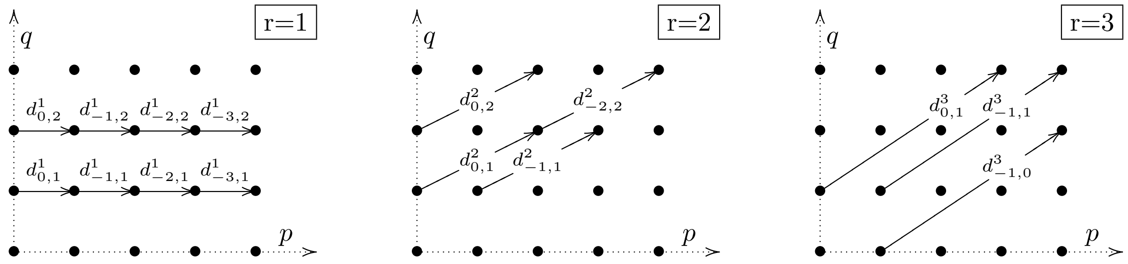
Definition 1.23. A spectral sequence $E = (E^r, d^r)_{r\geq 1}$ is a *first quadrant spectral sequence* if for all $r \geq 1$ $E^r_{p,q} = 0$ when $p < 0$ or $q < 0$. A *second quadrant spectral sequence* E is one with $E^r_{p,q} = 0$ if $p > 0$ or $q < 0$.

Let us remark that these conditions for $r = 1$ imply the same conditions for higher r, in other words, $E = (E^r, d^r)_{r\geq 1}$ is a first quadrant spectral sequence if $E^1_{p,q} = 0$ when $p < 0$ or $q < 0$. And in the same way for a second quadrant spectral sequence.

If E is a first quadrant spectral sequence, it is useful to represent the bigraded modules $E^r = \{E^r_{p,q}\}_{p,q\in\mathbb{Z}}$ at the lattice points of the first quadrant of the plane. In the figures that follow we consider the levels $r = 1, 2$, and 3, but only some differential maps $d^r_{p,q}$ are included.



Similarly, in the case of a second quadrant spectral sequence, the bigraded modules $E^r = \{E^r_{p,q}\}_{p,q\in\mathbb{Z}}$ can be displayed at the lattice points of the second quadrant of the (p, q)-plane. However, we consider more convenient to represent them also in the first quadrant. To this aim, we simply change the sign of the first index p, that is to say, we represent the module $E^r_{p,q}$ at the point $(-p, q)$ (which is in the first quadrant). In this way the differential maps have shift $(r, r - 1)$.



Definition 1.24. A spectral sequence $E = (E^r, d^r)_{r\geq 1}$ is said to be *convergent* if for every $p, q \in \mathbb{Z}$ there exists $r_{p,q} \geq 1$ such that $d^r_{p,q} = 0 = d^r_{p+r,q-r+1}$ for all $r \geq r_{p,q}$.

If $E = (E^r, d^r)_{r\geq 1}$ is convergent, one has $E^r_{p,q} = E^{r_{p,q}}_{p,q}$ for all $r \geq r_{p,q}$, and therefore $E^\infty_{p,q} = E^{r_{p,q}}_{p,q}$.

A first quadrant spectral sequence $E = (E^r, d^r)_{r\geq 1}$ is always convergent: $r > p$ implies $d^r_{p,q} = 0$, and for $r > q + 1$ it is clear $d^r_{p+r,q-r+1} = 0$. However, a second quadrant spectral sequence is not necessarily convergent.

Definition 1.25. A spectral sequence $(E^r, d^r)_{r\geq 1}$ is said to be *bounded below* if for each degree $n \in \mathbb{Z}$ there exists an integer $s = s(n)$ such that $E^1_{p,q} = 0$ when $p < s$ and $p + q = n$.

For instance, a first quadrant spectral sequence is bounded below, it suffices to consider $s(n) = 0$ for all $n \in \mathbb{Z}$.

Theorem 1.26. [Mac63] Let $E = (E^r, d^r)_{r\geq 1}$ and $E' = (E'^r, d'^r)_{r\geq 1}$ be two spectral sequences and $f : E \to E'$ a morphism between them such that the bigraded module morphism $f^k : E^k = \{E^k_{p,q}\}_{p,q\in\mathbb{Z}} \to E'^k = \{E'^k_{p,q}\}_{p,q\in\mathbb{Z}}$ is an isomorphism for some $k \geq 1$. Then $f^r : E^r \to E'^r$ is an isomorphism for every $r \geq k$. Furthermore, if E and E' are bounded below, then $f^\infty : E^\infty \to E'^\infty$ is also an isomorphism.

1.2 Simplicial Topology

1.2.1 Simplicial sets

Simplicial sets were first introduced by Eilenberg and Zilber [EZ50], who called them *semi-simplicial complexes*. They can be used to express some topological properties of spaces by means of combinatorial notions. A good reference for the definitions and results of this section is [May67].

Definition 1.27. Let $\mathcal{D}$ be a category, the category $s\mathcal{D}$ of *simplicial objects over* $\mathcal{D}$ is defined as follows.

- An object $K \in s\mathcal{D}$ consists of
  - for each integer $n \geq 0$, an object $K_n \in \mathcal{D}$;
  - for every pair of integers (i, n) such that $0 \leq i \leq n$, *face* and *degeneracy* maps $\partial_i : K_n \to K_{n-1}$ and $\eta_i : K_n \to K_{n+1}$ (which are morphisms in the category $\mathcal{D}$) satisfying the *simplicial identities*:

$$\begin{aligned}
\partial_i \partial_j &= \partial_{j-1} \partial_i && \text{if } i < j \\
\eta_i \eta_j &= \eta_{j+1} \eta_i && \text{if } i \leq j \\
\partial_i \eta_j &= \eta_{j-1} \partial_i && \text{if } i < j \\
\partial_i \eta_j &= \mathrm{Id} && \text{if } i = j, j+1 \\
\partial_i \eta_j &= \eta_j \partial_{i-1} && \text{if } i > j+1
\end{aligned}$$

- Let K and L be simplicial objects, a *simplicial map* (or *simplicial morphism*) $f : K \to L$ consists of maps $f_n : K_n \to L_n$ (which are morphisms in $\mathcal{D}$) which commute with the face and degeneracy operators, that is to say, $f_{n-1} \circ \partial_i = \partial_i \circ f_n$ and $f_{n+1} \circ \eta_i = \eta_i \circ f_n$ for all $0 \leq i \leq n$.

If a $\mathcal{D}$-object has *elements*, the elements of K_n are called the *n-simplices* of K. An n-simplex x is *degenerate* if $x = \eta_j y$ with $y \in K_{n-1}$, $0 \leq j < n$; otherwise x is called *non-degenerate.*

Definition 1.28. A *simplicial set* is a simplicial object over the category of sets. The category of simplicial sets will be denoted by $\mathcal{S}$.

The following property is satisfied by every simplicial set. It is a very important feature that will be used several times in this memoir.

Property 1.29. [May67] Let K be a simplicial set. Any degenerate n-simplex $x \in K_n$ can be expressed in a unique way as a (possibly) iterated degeneracy of a non-degenerate simplex y in the following way:

$$x = \eta_{j_k} \dots \eta_{j_1} y$$

with $y \in K_r$, $k = n - r > 0$, and $0 \leq j_1 < \cdots < j_k < n$.

This canonical form can easily be obtained by means of the simplicial identities.

An important example of simplicial set is the one associated with every topological space X. Let Δ_n be the *standard geometric n-simplex*, that is, the subset of $\mathbb{R}^{n+1}$

$$\Delta_n = \{(t_0, \dots, t_n) |\ 0 \leq t_i \leq 1,\ \sum_{i=0}^{n} t_i = 1\}$$

Definition 1.30. Given a topological space X, the *singular simplicial set* of X, written $S(X)$, is defined as the simplicial set with *singular n-simplices*

$$S_n(X) = \{f : \Delta_n \to X|\ f \text{ is continuous}\}$$

where the faces and degeneracies $\partial_i : S_n(X) \to S_{n-1}(X)$ and $\eta_i : S_n(X) \to S_{n+1}(X)$ are defined as

$$\begin{aligned}(\partial_i f)(t_0, \dots, t_{n-1}) &= f(t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{n-1})\\ (\eta_i f)(t_0, \dots, t_{n+1}) &= f(t_0, \dots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \dots, t_{n+1})\end{aligned}$$

Another useful example of simplicial set is the *standard m-simplex* $\Delta[m]$.

Definition 1.31. For $m \geq 0$, the *standard m-simplex* $\Delta[m]$ is a simplicial set built as follows. An n-simplex of $\Delta[m]$ is any $(n+1)$-tuple $(a_0, \dots, a_n)$ of integers such that $0 \leq a_0 \leq \cdots \leq a_n \leq m$, and the face and degeneracy operators are defined as

$$\begin{aligned}\partial_i(a_0, \dots, a_n) &= (a_0, \dots, a_{i-1}, a_{i+1}, \dots, a_n)\\ \eta_i(a_0, \dots, a_n) &= (a_0, \dots, a_i, a_i, a_{i+1} \dots, a_n)\end{aligned}$$

It is not hard to show that $\Delta[m]$ has exactly one non-degenerate m-simplex, the element $(0, 1, \dots, m) \equiv i_m$. The usefulness of this simplicial set is due to the following property.

Property 1.32. [May67] Let K be a simplicial set and $x \in K_n$. There exists a unique simplicial morphism

$$\Delta(x) : \Delta[n] \longrightarrow K$$

which maps $i_n = (0, \dots, n)$ into x.

This universal property allows us to define the *standard maps*

$$\begin{aligned}\partial^j = \Delta(\partial_j i_n) :&\quad \Delta[n-1] \longrightarrow \Delta[n] \qquad 0 \leq j \leq n\\ \eta^j = \Delta(\eta_j i_n) :&\quad \Delta[n+1] \longrightarrow \Delta[n] \qquad 0 \leq j \leq n\end{aligned}$$

Definition 1.33. For a simplicial set K, the *n-skeleton* $K^{[n]} \subseteq K$ is the (sub)simplicial set generated by all the simplices of K of dimension $\leq n$.

Let K be a simplicial set and $\star \in K_0$ a chosen 0-simplex (called the *base point*). We will also denote by $\star$ the degenerate simplices $\eta_{n-1} \dots \eta_0 \star \in K_n$ for every n.

Definition 1.34. A simplicial set K is said to be *reduced* (or 0-*reduced*) if $K_0 = \{\star\}$, in other words, if K has only one 0-simplex. Given $m \geq 1$, K is *m-reduced* if $K_n = \{\star\}$ for all $n \leq m$.

Definition 1.35. Let f and g be simplicial maps from a simplicial set K to a simplicial set L. A *(simplicial) homotopy* from f to g, written $h : f \simeq g$, is a set of functions $h_i : K_n \to L_{n+1}$ for each pair of integers (i, n) such that $0 \leq i \leq n$, satisfying:

$$\begin{aligned}
\partial_0 h_0 &= f \\
\partial_{n+1} h_n &= g \\
\partial_i h_j &= h_{j-1}\partial_i & \text{if } i < j \\
\partial_{j+1} h_{j+1} &= \partial_{j+1} h_j \\
\partial_i h_j &= h_j \partial_{i-1} & \text{if } i > j+1 \\
\eta_i h_j &= h_{j+1}\eta_i & \text{if } i \leq j \\
\eta_i h_j &= h_j \eta_{i-1} & \text{if } i > j
\end{aligned}$$

The categories $\mathcal{S}$ of simplicial sets and $\mathcal{C}$ of chain complexes of $\mathbb{Z}$-modules are closely connected: given a simplicial set K, it is possible to construct, in a very easy way, an associated chain complex.

Definition 1.36. Let K be a simplicial set, we define the *chain complex associated with* K, $C_*(K) = (C_n(K), d_n)_{n\in\mathbb{N}}$, in the following way:

- $C_n(K) = \mathbb{Z}[K_n]$ is the free $\mathbb{Z}$-module generated by K_n. Therefore an n-chain $c \in C_n(K)$ is a combination $c = \sum_{i=1}^{m} \lambda_i x_i$ with $\lambda_i \in \mathbb{Z}$ and $x_i \in K_n$ for $1 \leq i \leq m$;
- the differential map $d_n : C_n(K) \to C_{n-1}(K)$ is given by

$$d_n(x) = \sum_{i=0}^{n} (-1)^i \partial_i(x) \text{ for } x \in K_n$$

  and it is extended by linearity to the combinations $c = \sum_{i=1}^{m} \lambda_i x_i \in C_n(K)$.

Let us remark that if a simplex $x \in K_n$ is degenerate, $x = \eta_j y$ with $0 \leq j < n$ and $y \in K_{n-1}$, then $d_n(x)$ is a sum of degenerate $(n-1)$-simplices:

$$\begin{aligned}
d_n(\eta_j y) = \sum_{i=0}^{n} (-1)^i \partial_i \eta_j y &= \sum_{i=0}^{j-1} (-1)^i \eta_{j-1}\partial_i y + (-1)^j y + (-1)^{j+1} y \\
+ \sum_{i=j+2}^{n} (-1)^i \eta_j \partial_{i-1} y &= \sum_{i=0}^{j-1} (-1)^i \eta_{j-1}(\partial_i y) + \sum_{i=j+2}^{n} (-1)^i \eta_j (\partial_{i-1} y)
\end{aligned}$$

As a consequence, the next definition makes sense.

Definition 1.37. The *normalized (non-degenerate) chain complex associated with* K, $C_*^N(K) = (C_n^N(K), d_n^N)_{n\in\mathbb{N}}$, is given by

- $C_n^N(K) = C_n(K)/\mathbb{Z}[D_n(K)]$, where $D_n(K)$ is the set of degenerate elements of K_n. We can also think of $C_n^N(K)$ as the free $\mathbb{Z}$-module generated by the set of non-degenerate n-simplices of K, denoted by $ND_n(K)$. This means that an n-chain $c \in C_n^N(K)$ is a combination $c = \sum_{i=1}^m \lambda_i x_i$ where $\lambda_i \in \mathbb{Z}$ and x_i is a non-degenerate n-simplex of K for all $1 \leq i \leq m$;
- the differential map $d_n^N : C_n^N(K) \to C_{n-1}^N(K)$ is given by

$$d_n^N(x) = \sum_{i=0}^{n}(-1)^i \partial_i(x) \mod ND_{n-1}(K) \text{ for } x \in ND_n(K)$$

  We observe that $d_n^N(x)$ is obtained from $d_n(x)$ by canceling the degenerate simplices. The definition is extended by linearity to the combinations of $C_n^N(K)$.

Definition 1.38. Given a simplicial set K, the *n-homology group* of K, $H_n(K)$, is the n-homology group of the chain complex $C_*(K)$:

$$H_n(K) = H_n(C_*(K))$$

Analogously, given a ring R one can consider the chain complex of R-modules $(R[K])_*$, where $(R[K])_n$ is the free R-module generated by the n-simplices of K, and the differential map d_n is given again by the alternate sum of the faces. This allows us to define the *n-homology group of K with coefficients in R* as the n-homology module of the chain complex $(R[K])_*$, or equivalently, that of $C_*(K) \otimes R_*$ where $R_* = (R_n, d_n)_{n \in \mathbb{N}}$ is the chain complex of R-modules given by $R_0 = R$ and $R_n = 0$ for all $n \neq 0$ (with d_n the null map for all n):

$$H_n(K; R) = H_n((R[K])_*) = H_n(C_*(K) \otimes R_*)$$

If $R = \mathbb{Z}$, then $(\mathbb{Z}[K])_* = C_*(K) \otimes \mathbb{Z}_*$ is simply the chain complex associated with K, that is, $C_*(K)$.

Theorem 1.39 (Normalization Theorem). [Mac63] Given a simplicial set K, the canonical projection $C_*(X) \to C_*^N(X) \cong C_*(K)/D_*(K)$ is a chain equivalence.

This theorem implies the homology groups of $C_*(K)$ are isomorphic to those of $C_*^N(K)$, and therefore

$$H_n(K) = H_n(K; \mathbb{Z}) = H_n(C_*(K)) \cong H_n(C_*^N(K))$$

If K is m-reduced for some $m \in \mathbb{N}$, then we observe that $H_0(K) = \mathbb{Z}$ and $H_n(K) = 0$ for all $0 < n \leq m$.

We can also consider the *reduced n-homology group* of K, $\widetilde{H}_n(K)$, defined as

$$\widetilde{H}_n(K) = H_n(\widetilde{C}_*(K))$$

with $\widetilde{C}_*(K) = C_*(K)/C_*(\star)$. Clearly one has $\widetilde{H}_n(K) = H_n(K)$ for $n \neq 0$, and $\widetilde{H}_0(K)$ has one less free generator than $H_0(K)$.

A simplicial map $f : K \to L$ defines a chain complex morphism on the associated chain complexes, $\bar{f} : C_*(K) \to C_*(L)$. The group morphism $\bar{f}_n : C_n(K) \to C_n(L)$ is defined by linear extension on the generators of $C_n(K)$, which are the n-simplices of K. Since f is compatible with the face operators, then $\bar{f}$ commutes with the differential maps, that is to say, $\bar{f}$ is a chain complex morphism. On the other hand, as far as f also commutes with the degeneracy operators, then the chain complex morphism $\bar{f}$ is also well-defined on the normalized chain complexes, $\bar{f} : C^N_*(K) \to C^N_*(L)$.

Remark 1.40. Let $f, g : K \to L$ be simplicial maps, and $\bar{f}, \bar{g} : C_*(K) \to C_*(L)$ the corresponding chain complex morphisms. Let $h : K \to L$ be a simplicial homotopy, $h : f \simeq g$. Then we can construct $\bar{h} : C_*(K) \to C_{*+1}(L)$ which is a chain homotopy, $\bar{h} : \bar{f} \simeq \bar{g}$. Given $x \in K_n$ a generator of $C_n(K)$, we define

$$\bar{h}(x) = \sum_{i=0}^{n} (-1)^i h_i(x) \in C_{n+1}(L)$$

and we extend the definition to the elements of $C_n(X)$ by linearization. In this way, the map $\bar{h}$ satisfies $d \circ \bar{h} + \bar{h} \circ d = f - g$.

We can also consider the maps $\bar{f}$ and $\bar{g}$ defined on the normalized chain complexes $C^N_*(K)$ and $C^N_*(L)$. Thanks to the identities of Definition 1.35, it is clear that if x is a degenerate n-simplex of K then $\bar{h}(x) = \sum_{i=0}^{n} (-1)^i h_i(x)$ is a combination of degenerate $(n+1)$-simplices of L. This allows us to define the map $\bar{h}$ on the normalized chain complexes, $\bar{h} : C^N_*(K) \to C^N_{*+1}(L)$ given by

$$\bar{h}(x) = \sum_{i=0}^{n} (-1)^i h_i(x) \mod ND_{n+1}(L) \text{ for } x \in ND_n(K)$$

and by linear extension we define $\bar{h}(c)$ for every combination $c \in C^N_n(K) = \mathbb{Z}[ND_n(K)]$. It is clear that the equation $d \circ \bar{h} + \bar{h} \circ d = f - g$ holds and in this way $\bar{h} : C^N_*(K) \to C^N_{*+1}(L)$ is also a chain homotopy, $\bar{h} : \bar{f} \simeq \bar{g}$.

Definition 1.41. A *simplicial (Abelian) group* G is a simplicial object over the category of (Abelian) groups, in other words, it is a simplicial set where each G_n is an (Abelian) group and the face and degeneracy operators are group morphisms. The category of simplicial Abelian groups will be denoted by $\mathcal{A}$.

Every simplicial Abelian group can be considered as a chain complex, written G_*. The group of n-chains is the set of n-simplices G_n (which in this case is an Abelian group), and the differential map $d_n : G_n \to G_{n-1}$ is given by $d_n = \sum_{i=0}^{n} (-1)^i \partial_i$.

Definition 1.42. Given two simplicial sets K and L, the Cartesian product $K \times L$ is a simplicial set with n-simplices

$$(K \times L)_n = K_n \times L_n$$

and if $(x, y) \in K_n \times L_n$, the face and degeneracy operators are defined as

$$\begin{aligned} \partial_i(x,y) &= (\partial_i x, \partial_i y) \quad \text{for } 0 \leq i \leq n \\ \eta_i(x,y) &= (\eta_i x, \eta_i y) \quad \text{for } 0 \leq i \leq n \end{aligned}$$

If K and L are simplicial Abelian groups, then $K \times L$ is also a simplicial Abelian group, and it can also be seen as the direct sum of K and L: $K \times L \cong K \oplus L$.

Definition 1.43. Let G be a simplicial group (called the *fiber* space), B a simplicial set (the *base* space), and $\tau = \{\tau_n : B_n \to G_{n-1}\}_{n\geq 1}$ (called the *twisting operator*) satisfying

$$\begin{aligned} \partial_0\tau(b) &= \tau(\partial_1 b) \cdot \tau(\partial_0 b)^{-1} \\ \partial_i\tau(b) &= \tau(\partial_{i+1} b), \quad \text{if } 0 < i \leq n-1 \\ \eta_i\tau(b) &= \tau(\eta_{i+1} b), \quad \text{if } 0 \leq i \leq n-1 \\ \tau(\eta_0 b) &= e_n \end{aligned}$$

where e_n is the null element of the group G_n, and $b \in B_n$.

Then the *(principal) twisted (Cartesian) product* $E(\tau) \equiv G \times_\tau B$ is the simplicial set defined by

$$\begin{aligned} E(\tau)_n &= G_n \times B_n, \text{ and for each } (g,b) \in G_n \times B_n \\ \partial_i(g,b) &= (\partial_i g, \partial_i b), \quad \text{if } 0 < i \leq n \\ \partial_0(g,b) &= (\partial_0 g \cdot \tau(b), \partial_0 b) \\ \eta_i(g,b) &= (\eta_i g, \eta_i b), \quad \text{for } 0 \leq i \leq n \end{aligned}$$

It is not difficult to prove, thanks to the identities that the twisting operator τ satisfies, that $E(\tau)$ is a simplicial set. In fact, the condition that $E(\tau)$ defined in this way is a simplicial set is equivalent to the requirement that τ satisfies the necessary equations.

The twisting operator $\tau : B_* \to G_{*-1}$ defines a *(principal) fibration* $G \hookrightarrow E \to B$ of base space B, fiber space G and total space $E = E(\tau) = G \times_\tau B$.

A fibration $G \hookrightarrow E \to B$ can also be defined by the *projection* $p : E \to B$ (which must satisfy some specific properties). For details, see [May67].

1.2.2 Homotopy groups

The n-homotopy group of a topological space X with a base point x_0 is defined as the set of homotopy classes of continuous maps $f : S^n \to X$ that map a chosen base point $a \in S^n$ to the base point $x_0 \in X$. For a Kan simplicial set K with a base point $\star \in K_0$, a more algebraic definition of homotopy groups can be given. In fact, it can be seen that both definitions are closely connected by means of the *realization functor*. All the definitions and results of this section (and details about the connection with the usual definition of homotopy group of a topological space) can be found in [May67].

Definition 1.44. A simplicial set K is said to satisfy the *extension condition* if for every collection of $n+1$ n-simplices $x_0, x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{n+1}$ which satisfy the compatibility condition $\partial_i x_j = \partial_{j-1} x_i$ for all $i < j, i \neq k$, and $j \neq k$, there exists an $(n+1)$-simplex $x \in K_{n+1}$ such that $\partial_i x = x_i$ for every $i \neq k$. A simplicial set which satisfies the extension condition is called a *Kan simplicial set.*

Definition 1.45. Let K be a simplicial set. Two n-simplices x and y of K are said to be *homotopic*, written $x \sim y$, if $\partial_i x = \partial_i y$ for $0 \leq i \leq n$, and there exists an $(n+1)$-simplex z such that $\partial_n z = x, \partial_{n+1} z = y$, and $\partial_i z = \eta_{n-1}\partial_i x = \eta_{n-1}\partial_i y$ for $0 \leq i < n$.

If K is a Kan simplicial set, then $\sim$ is an equivalence relation on the set of n-simplices of K for every $n \geq 0$.

Let $\star \in K_0$ be a base point, we recall that we also denote by $\star$ the degeneracies $\eta_{n-1} \ldots \eta_0 \star \in K_n$ for every n. We define $\widetilde{K}_n$ as the set of all $x \in K_n$ such that $\partial_i x = \star$ for every $0 \leq i \leq n$.

Definition 1.46. Given a Kan simplicial set K and a base point $\star \in K_0$, we define

$$\pi_n(K, \star) = \widetilde{K}_n / (\sim)$$

The set $\pi_n(K, \star)$ admits a group structure for $n \geq 1$ and it is Abelian for $n \geq 2$. It is called the *n-homotopy group* of K.

Very frequently, if no confusion is possible, we will omit the base point and denote the n-homotopy group of K by $\pi_n(K)$.

It is not hard to observe that every Kan simplicial set morphism $f : K \to L$ induces a group morphism on the corresponding homotopy groups

$$\pi_*(f) : \pi_*(K) \longrightarrow \pi_*(L)$$

Definition 1.47. A Kan simplicial set K with a base point $\star \in K_0$ is said to be *contractible* if $\pi_n(K, \star) = 0$ for all n.

Definition 1.48. A Kan simplicial set K is said to be *minimal* if, for every n-simplices $x, y \in K_n$ and $0 \leq k \leq n$ such that $\partial_i x = \partial_i y$ for all $0 \leq i \leq n$ with $i \neq k$, then $\partial_k x = \partial_k y$.

It can be proved that a Kan simplicial set K is minimal if and only if $x \sim y$ implies $x = y$, so that each homotopy class has only one element.

In most cases the computation of the homotopy groups of a simplicial set K, $\pi_n(K)$, is more difficult than that of the homology groups, $H_n(K)$. The Hurewicz theorem, which was proved in the series of papers [Hur35] and [Hur36], expresses an interesting relation between both invariants that will be useful to determine some homotopy groups of K (in the first degrees) when the homology groups $H_n(K)$ are known. Before introducing this important theorem, we need the following definition.

Definition 1.49. Let K be a Kan simplicial set with a base point $\star \in K_0$. Let $[x] \in \pi_n(K, \star)$, then $\partial_i x = \star$ for all $0 \leq i \leq n$ so that x can be considered as a cycle of $\widetilde{C}_n(K) = C_n(K)/C_n(\star)$. We define

$$h : \pi_n(K, \star) \longrightarrow \widetilde{H}_n(K) = H_n(\widetilde{C}_*(K))$$

by $h([x]) = \{x\}$, where $\{x\}$ denotes the homology class of x in $\widetilde{H}_n(K)$. It can be seen that the maps h are well-defined and are morphisms of groups. They are called the *Hurewicz homomorphisms.*

Theorem 1.50 (Hurewicz Theorem). [Whi78] Let K be an $(m-1)$-reduced Kan simplicial set (with a base point $\star \in K_0$), for $m \geq 2$. Then $H_n(K) = 0$ for every $0 < n \leq m-1$, the map $h : \pi_m(K, \star) \to \widetilde{H}_m(K)$ is an isomorphism, and $h : \pi_{m+1}(K, \star) \to \widetilde{H}_{m+1}(K)$ is an epimorphism.

The case where K is a simplicial Abelian group is more favorable. First of all, it can be seen that every simplicial group G is a Kan simplicial set (with base point equal to the null element e_0 of the group G_0), and therefore it makes sense to consider its homotopy groups $\pi_n(G, e_0) \equiv \pi_n(G)$. Furthermore, John Moore defined the so-called *Moore complex* allowing one to express the homotopy groups of a simplicial group as homology groups of a chain complex.

Definition 1.51. Let G be a simplicial Abelian group, the *normalization* $N_*(G) = (N_n(G), d_n)_{n \in \mathbb{N}}$ is a chain complex, the *Moore complex* of G, defined by

$$N_n(G) = G_n \cap \operatorname{Ker} \partial_0 \cap \ldots \cap \operatorname{Ker} \partial_{n-1}$$

with differential map $d_n : N_n(G) \to N_{n-1}(G)$ given by $d_n = (-1)^n \partial_n$.

Proposition 1.52. [May67] The n-homotopy group of a simplicial Abelian group G coincides with the n-homology group of the chain complex $N_*(G)$ for every degree n

$$\pi_n(G) = H_n(N_*(G))$$

1.2.3 Eilenberg-MacLane spaces

Eilenberg-MacLane spaces were introduced in [EM53] and play an important role in many contexts in Algebraic Topology, as cohomology operations or computations of homotopy groups. Furthermore, it is known that every simplicial Abelian group has the homotopy type of a product of Eilenberg-MacLane spaces. All the definitions and results that follow can be found in [May67].

Definition 1.53. An *Eilenberg-MacLane space* of type (π, n) is a simplicial group K (with base point $e_0 \in K_0$) such that $\pi_n(K) = \pi$ and $\pi_i(K) = 0$ if $i \neq n$. The simplicial group K is called a $K(\pi, n)$ if it is an Eilenberg-MacLane space of type (π, n) and in addition it is minimal.

Various methods can be used to construct the spaces $K(\pi, n)$'s, although the results are necessarily isomorphic [May67, Theorem 23.6]. Let us consider the following one.

Let π be an Abelian group. First of all, we construct a simplicial Abelian group $K = K(\pi, 0)$ given by $K_n = \pi$ for all $n \geq 0$, and with face and degeneracy operators $\partial_i : K_n = \pi \to K_{n-1} = \pi$ and $\eta_i : K_n = \pi \to K_{n+1} = \pi$, $0 \leq i \leq n$, equal to the identity map of the group π.

For $n \geq 0$, we build recursively $K(\pi, n)$ by means of the classifying space constructor.

Definition 1.54. Let G be a simplicial Abelian group. The *classifying space* of G, written $\overline{\mathcal{W}}(G)$, is the simplicial Abelian group built as follows. The n-simplices of $\overline{\mathcal{W}}(G)$ are the elements of the Cartesian product

$$\overline{\mathcal{W}}(G)_n = G_{n-1} \times G_{n-2} \times \cdots \times G_0$$

In this way, $\overline{\mathcal{W}}(G)_0$ is the null group and has only one element that we denote by [ ]. For $n \geq 1$, an element of $\overline{\mathcal{W}}(G)_n$ has the form $[g_{n-1}, \ldots, g_0]$, with $g_i \in G_i$. The face and degeneracy operators are given by

$$\begin{aligned}
\eta_0[\ ] &= [e_0] \\
\partial_i[g_0] &= [\ ], \quad i = 0, 1 \\
\partial_0[g_{n-1}, \ldots, g_0] &= [g_{n-2}, \ldots, g_0] \\
\partial_i[g_{n-1}, \ldots, g_0] &= [\partial_{i-1} g_{n-1}, \ldots, \partial_1 g_{n-i+1}, \partial_0 g_{n-i} + g_{n-i-1}, g_{n-i-2}, \ldots, g_0], \quad 0 < i \leq n \\
\eta_0[g_{n-1}, \ldots, g_0] &= [e_n, g_{n-1}, \ldots, g_0] \\
\eta_i[g_{n-1}, \ldots, g_0] &= [\eta_{i-1} g_{n-1}, \ldots, \eta_0 g_{n-i}, e_{n-i}, g_{n-i-1}, \ldots, g_0], \quad 0 < i \leq n
\end{aligned}$$

where e_n denotes the null element of the Abelian group G_n.

We define inductively $\overline{\mathcal{W}}^n(G) = \overline{\mathcal{W}}(\overline{\mathcal{W}}^{n-1}(G))$ for all $n \geq 1$, $\overline{\mathcal{W}}^0(G) = G$.

Theorem 1.55. [May67] Let π be an Abelian group and $K = K(\pi, 0)$ as explained before. Then $\overline{\mathcal{W}}^n(K)$ is a $K(\pi, n)$.

This particular type of spaces have some fundamental properties which lead to a proof that every simplicial Abelian group has the homotopy type of a product of $K(\pi, n)$'s. For details, see [May67] or [GJ99].

Theorem 1.56. [May67] Let G be a simplicial Abelian group, and $\pi_n \equiv \pi_n(G)$. Then G is homotopy equivalent to the infinite Cartesian product of Eilenberg-MacLane spaces

$$\prod_{n \geq 0} K(\pi_n, n)$$

Eilenberg-MacLane spaces will appear several times in this work. In this section we have only included the definition and some basic information about them, in Section 4.1.2 some other useful remarks will be considered.

1.3 Effective Homology

The computation of homology groups of topological spaces is one of the first problems in Algebraic Topology, and these groups can be difficult to reach, for example when loop spaces or classifying spaces are involved. The methods of *effective homology* (introduced in [Ser87] and [Ser94]) give in particular to their user *algorithms* computing for example the homology groups of the total space of a fibration, of an arbitrarily iterated loop space (Adams' problem), of a classifying space, etc. The main idea consists in keeping systematically a deep and subtle connection between the *homology* of any object and the object itself.

1.3.1 Definitions and fundamental results

In this section, we present some definitions (including the notion of object with effective homology) and fundamental results about the effective homology method. More details can be found in [RS02] and [RS06].

Definition 1.57. An *effective chain complex* is a *free* chain complex of $\mathbb{Z}$-modules $C_* = (C_n, d_n)_{n\in\mathbb{N}}$ where each group C_n is finitely generated, a provided algorithm returns a (distinguished) $\mathbb{Z}$-basis in each degree n, and each differential map d_n is also given by an algorithm.

If a chain complex $C_* = (C_n, d_n)_{n\in\mathbb{N}}$ is effective, the differential maps $d_n : C_n \to C_{n-1}$ can be expressed as finite integer matrices, and then it is possible to know *everything* about C_*: we can compute the subgroups $\operatorname{Ker} d_n$ and $\operatorname{Im} d_{n+1}$, we can determine whether an n-chain $c \in C_n$ is a cycle or a boundary, and in the last case, we can obtain $z \in C_{n+1}$ such that $c = d_{n+1}(z)$. In particular an elementary algorithm computes its homology groups using, for example, the Smith Normal Form technique (for details, see [KMM04]).

On the other hand, in many situations we must deal with *locally effective* chain complexes. In this case we can have an infinite number of generators for each group C_n, so that no *global* information is available. For example, it is not possible in general to compute the subgroups $\operatorname{Ker} d_n$ and $\operatorname{Im} d_{n+1}$, which can have infinite nature. However, "local"[1] information can be obtained: we can compute, for instance, the boundary of a given element.

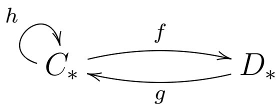
More generally, we talk of *locally effective objects* when only "local" computations are possible. For instance, we can consider a locally effective simplicial set; the set of n-simplices is not necessarily of finite type, but we can compute the faces of any specific n-simplex.

The effective homology technique consists in combining locally effective objects with effective chain complexes by means of chain equivalences. In this way, we will be able

[1] The qualifier "componentwise" would be more appropriate, but a little heavy.

to compute homology groups of locally effective objects even if we cannot obtain global information about them.

Definition 1.58. A *reduction* ρ (also called *contraction* by other authors) between two chain complexes C_* and D_*, denoted in this memoir by $\rho : C_* \Rightarrow\!\!\!\!\Rightarrow D_*$, is a triple $\rho = (f, g, h)$



where f and g are chain complex morphisms, h is a graded group morphism of degree +1, and the following relations are satisfied:

1) $f \circ g = \mathrm{Id}_{D_*}$;

2) $d_C \circ h + h \circ d_C = \mathrm{Id}_{C_*} - g \circ f$;

3) $f \circ h = 0; \quad h \circ g = 0; \quad h \circ h = 0.$

We observe that this is a particular case of chain equivalence (Definition 1.10), where $h_1 = h : \mathrm{Id}_{C_*} \simeq g \circ f$ and the second homotopy $h_2 : \mathrm{Id}_{D_*} \simeq f \circ g$ is the null map.

These relations express that C_* is the direct sum of D_* and an acyclic chain complex. This decomposition is simply $C_* = \mathrm{Ker}\, f \oplus \mathrm{Im}\, g$, with $\mathrm{Im}\, g \cong D_*$ and $H_*(\mathrm{Ker}\, f) = 0$. In particular, this implies that the graded homology groups $H_*(C_*)$ and $H_*(D_*)$ are canonically isomorphic.

Very frequently, the *small* chain complex D_* is effective, so that we can compute its homology groups by means of elementary operations with integer matrices. On the other hand, in many situations the *big* chain complex C_* is locally effective and therefore its homology groups cannot directly be determined. However, if we know a reduction from C_* over D_* and D_* is effective, then we are also able to compute the homology groups of C_* by means of those of D_*.

Given a chain complex C_*, a *trivial reduction* $\rho = (f, g, h) : C_* \Rightarrow\!\!\!\!\Rightarrow C_*$ can be constructed, where f and g are the identity map and $h = 0$.

It can be seen that in the definition of reduction the *important* equations are 1) and 2). As we show in the next Remark, given a triple (f, g, h) satisfying 1) and 2) it is possible to modify them weakly so that they also satisfy 3).

Remark 1.59. Let C_* and D_* be chain complexes, $f : C_* \to D_*$ and $g : D_* \to C_*$ chain complex morphisms, and $h : C_* \to C_{*+1}$ a graded group morphism of degree +1 such that

1) $f \circ g = \mathrm{Id}_{D_*}$;

2) $d_C \circ h + h \circ d_C = \mathrm{Id}_{C_*} - g \circ f$.

Then it is possible to define $h' : C_* \to C_{*+1}$ (a graded group morphism of degree $+1$) such that $\rho' = (f, g, h')$ is a reduction $\rho' : C_* \Rrightarrow D_*$.

Proof. First, let us consider h^1 given by $h^1 = h - g \circ f \circ h$ which satisfies $d_C \circ h^1 + h^1 \circ d_C = d_C \circ h + h \circ d_C = \mathrm{Id}_{C_*} - g \circ f$ and also $f \circ h^1 = 0$. Then, we define $h^2 = h^1 - h^1 \circ g \circ f$, for which the equations $d_C \circ h^2 + h^2 \circ d_C = d_C \circ h + h \circ d_C = \mathrm{Id}_{C_*} - g \circ f$, $f \circ h^2 = 0$, and $h^2 \circ g = 0$ hold. Finally, let us define $h' = h^2 \circ d_C \circ h^2$. In this way, h' satisfies $d_C \circ h' + h' \circ d_C = d_C \circ h + h \circ d_C = \mathrm{Id}_{C_*} - g \circ f$, $f \circ h' = 0$, $h' \circ g = 0$, and $h' \circ h' = 0$. □

As we see in the next two propositions, we can easily construct the composition and the tensor product of two reductions.

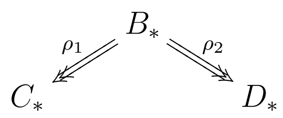
Proposition 1.60. Let $\rho = (f, g, h) : C_* \Rrightarrow D_*$ and $\rho' = (f', g', h') : D_* \Rrightarrow E_*$ be two reductions. Another reduction $\rho'' = (f'', g'', h'') : C_* \Rrightarrow E_*$ is defined by:

$$\begin{aligned} f'' &= f' \circ f \\ g'' &= g \circ g' \\ h'' &= h + g \circ h' \circ f \end{aligned}$$

Proposition 1.61. Let $\rho = (f, g, h) : C_* \Rrightarrow D_*$ and $\rho' = (f', g', h') : C'_* \Rrightarrow D'_*$ be two reductions. Another reduction $\rho'' = (f'', g'', h'') : C_* \otimes C'_* \Rrightarrow D_* \otimes D'_*$ is defined by:

$$\begin{aligned} f'' &= f \otimes f' \\ g'' &= g \otimes g' \\ h'' &= h \otimes \mathrm{Id}_{C'_*} + (g \circ f) \otimes h' \end{aligned}$$

Definition 1.62. A *strong chain equivalence* ε between two chain complexes C_* and D_*, denoted by $\varepsilon : C_* \Lleftarrow\!\!\Rrightarrow D_*$, is a triple (B_*, ρ_1, ρ_2) where B_* is a chain complex, and ρ_1 and ρ_2 are reductions from B_* over C_* and D_* respectively:



A strong chain equivalence $\varepsilon : C_* \Lleftarrow\!\!\Rrightarrow D_*$ can also be seen as a particular case of the *classical* notion of chain equivalence (Definition 1.10). If $\varepsilon = (B_*, \rho_1, \rho_2)$ with $\rho_1 = (f_1, g_1, h_1)$ and $\rho_2 = (f_2, g_2, h_2)$, then one can see that the compositions $f_2 \circ g_1$ and $f_1 \circ g_2$ define a chain equivalence with chain homotopies $f_1 \circ h_2 \circ g_1$ and $f_2 \circ h_1 \circ g_2$.

From now on in this memoir, we will use the word *equivalence* for a strong chain equivalence $\varepsilon : C_* \Lleftarrow\!\!\Rrightarrow D_*$.

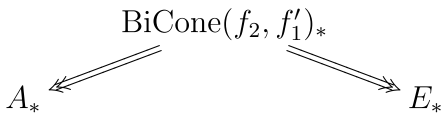
To define the composition of two equivalences, the Bicone constructor is necessary.

Definition 1.63. Let $C_* = (C_n, d_{C_n})_{n \in \mathbb{N}}$ and $D_* = (D_n, d_{D_n})_{n \in \mathbb{N}}$ be chain complexes, and $f : C_* \to D_*$ a chain complex morphism. The *Cone* of f, written $\mathrm{Cone}(f)_*$, is the chain complex $\text{Cone}(f)_* = A_* = (A_n, d_{A_n})_{n\in\mathbb{N}}$ given by $A_n = C_n \oplus D_{n+1}$[2], with differential map $d_{A_n}(c, d) = (d_{C_n}(c), f_n(c) - d_{D_{n+1}}(d))$.

Definition 1.64. Let $B_* = (B_n, d_{B_n})_{n\in\mathbb{N}}$, $C_* = (C_n, d_{C_n})_{n\in\mathbb{N}}$, and $D_* = (D_n, d_{D_n})_{n\in\mathbb{N}}$ be chain complexes, and $f : B_* \to C_*$, $g : D_* \to C_*$ chain complex morphisms. The *Bicone* of f and g, written $\text{BiCone}(f, g)_*$, is constructed from $\text{Cone}(f)_*$ and $\text{Cone}(g)_*$ by identifying both target chain complexes C_*. That is to say, $\text{BiCone}(f, g)_*$ is a chain complex $\text{BiCone}(f, g)_* = A_* = (A_n, d_{A_n})_{n\in\mathbb{N}}$ with n-chain group $A_n = B_n \oplus C_{n+1} \oplus D_n$ and differential map $d_{A_n}(b, c, d) = (d_{B_n}(b), f_n(b) - d_{C_{n+1}}(c) + g_n(d), d_{D_n}(d))$.

Proposition 1.65. [RS97] Let $\varepsilon : A_* \Longleftrightarrow C_*$ and $\varepsilon' : C_* \Longleftrightarrow E_*$ be two equivalences as in the following diagram:

with $\rho_1 = (f_1, g_1, h_1) : B_* \Rightarrow A_*$, $\rho_2 = (f_2, g_2, h_2) : B_* \Rightarrow C_*$, $\rho_1' = (f_1', g_1', h_1') : D_* \Rightarrow C_*$, and $\rho_2' = (f_2', g_2', h_2') : D_* \Rightarrow E_*$. We consider the Bicone $\text{BiCone}(f_2, f_1')_*$. Then, it is possible to build a new equivalence ε'' as follows:



Details about the different components in the new reductions can be found in [RS06].

In a similar way we can define *reductions* and *(strong) equivalences of cochain complexes*. All the results included in this section are also true in this case.

Once we have introduced the notion of equivalence, it is possible to give the definition of *object with effective homology*, which is the fundamental idea of the effective homology technique.

Definition 1.66. An *object with effective homology* X is a quadruple $(X, C_*(X), HC_*, \varepsilon)$ where

- X is a locally effective object;
- $C_*(X)$ is a (locally effective) chain complex canonically associated with X, that allows us to study the homological nature of X;
- HC_* is an effective chain complex;
- ε is an equivalence $\varepsilon : C_*(X) \Longleftrightarrow HC_*$.

[2]Usually, $A_n = C_{n-1} \oplus D_n$ is preferred, but this choice is necessary when the Cone is used to construct the composition of two equivalences.

For instance, if K is a simplicial set, then the chain complex canonically associated with K, $C_*(K)$, is described in Definition 1.36. Equivalently, we can also consider the normalized chain complex $C_*^N(K)$ introduced in Definition 1.37. It has been already said that the homology groups of both chain complexes are isomorphic, and in fact there exists a reduction $C_*(K) \Rightarrow\!\!\!\!\!\Rightarrow C_*^N(K)$.

Theorem 1.67. Let K be a simplicial set, $C_*(K)$ the chain complex associated with K, and $C_*^N(K)$ the normalized chain complex. Then it is possible to build a reduction $\rho : C_*(K) \Rightarrow\!\!\!\!\!\Rightarrow C_*^N(K)$.

In this way, K is a simplicial set with effective homology if an equivalence between $C_*(K)$ or $C_*^N(K)$ and an effective chain complex is known. Clearly, using Theorem 1.67 and the composition of reductions and equivalences, from and equivalence $\varepsilon : C_*(K) \Lleftarrow\!\!\!\Rrightarrow HC_*$ we can determine $\varepsilon' : C_*^N(K) \Lleftarrow\!\!\!\Rrightarrow HC_*$ and reciprocally an equivalence $\varepsilon' : C_*^N(K) \Lleftarrow\!\!\!\Rrightarrow HC_*$ allows us to construct $\varepsilon : C_*(K) \Lleftarrow\!\!\!\Rrightarrow HC_*$.

It is clear that if X is an object with effective homology, then the homology groups of X (which are those of the associated chain complex $C_*(X)$) are isomorphic to the homology groups of the effective chain complex HC_*, that can easily be computed using some elementary operations. But it is important to understand that in general the HC_* component of an object with effective homology is not made of the homology groups of X; this component HC_* is a *free* $\mathbb{Z}$-chain complex of finite type, in general with a non-null differential.

The main problem now is the following: given a chain complex $C_* = (C_n, d_n)_{n\in\mathbb{N}}$, is it possible to determine its effective homology? We must distinguish three cases.

- First of all, if a chain complex C_* is by chance effective, then we can choose the trivial effective homology: ε is the equivalence $C_* \Leftarrow\!\!\!\!\!\Leftarrow C_* \Rightarrow\!\!\!\!\!\Rightarrow C_*$, where the two components ρ_1 and ρ_2 are both the trivial reduction on C_*.
- In some cases, some theoretical results are available providing an equivalence between some chain complex C_* and an *effective* chain complex. Typically, the Eilenberg-MacLane space $K(\mathbb{Z}, 1)$ has the homotopy type of the circle S^1 and a reduction $C_*(K(\mathbb{Z}, 1)) \Rightarrow\!\!\!\!\!\Rightarrow C_*(S^1)$ can be built.
- The most important case: let $X_1, \ldots, X_n$ be objects with effective homology and Φ a constructor that produces a new space $X = \Phi(X_1, \ldots, X_n)$ (for example, the Cartesian product of two simplicial sets, the classifying space of a simplicial group, etc). In *natural* "reasonable" situations, there exists an effective homology version of Φ that allows us to deduce a version with effective homology of X, the result of the construction, from versions with effective homology of the arguments $X_1, \ldots, X_n$.

For instance, given two simplicial sets K and L with effective homology, then the Cartesian product $K \times L$ is an object with effective homology too, and this is also valid for twisted Cartesian products. We will see in Section 3.1.2 how this effective homology is obtained.

1.3.2 Perturbation theorems

The next two theorems will be useful when obtaining the effective homology version of some topological constructors. The main idea is that given a reduction, if we *perturb* one of the complexes then it is possible to perturb the other one so that we obtain a new reduction between the *perturbed* complexes. The first theorem (the Trivial Perturbation Lemma) is very easy, but it can be useful. The Basic Perturbation Lemma is not trivial at all. It was discovered by Shih Weishu [Shi62], although the abstract modern form was given by Ronnie Brown [Bro67].

Definition 1.68. Let $C_* = (C_n, d_n)_{n\in\mathbb{N}}$ be a chain complex. A *perturbation* δ of the differential d is a collection of group morphisms $\delta = \{\delta_n : C_n \to C_{n-1}\}_{n\in\mathbb{N}}$ such that the sum $d + \delta$ is also a differential, that is to say, $(d + \delta) \circ (d + \delta) = 0$.

The perturbation δ produces a new chain complex $C'_* = (C_n, d_n + \delta_n)_{n\in\mathbb{N}}$; it is the *perturbed* chain complex.

Theorem 1.69 (Trivial Perturbation Lemma, TPL). Let $C_* = (C_n, d_{C_n})_{n\in\mathbb{N}}$ and $D_* = (D_n, d_{D_n})_{n\in\mathbb{N}}$ be two chain complexes, $\rho = (f, g, h) : C_* \Rightarrow\!\!\!\!\!\Rightarrow D_*$ a reduction, and δ_D a perturbation of d_D. Then a new reduction $\rho' = (f', g', h') : C'_* \Rightarrow\!\!\!\!\!\Rightarrow D'_*$ can be constructed where:

1) C'_* is the chain complex obtained from C_* by replacing the old differential d_C by the perturbed differential $(d_C + g \circ \delta_D \circ f)$;

2) the new chain complex D'_* is obtained from the chain complex D_* only by replacing the old differential d_D by $(d_D + \delta_D)$;

3) $f' = f$;

4) $g' = g$;

5) $h' = h$.

The perturbation δ_D of the *small* chain complex D_* is naturally transferred (using the reduction ρ) to the *big* chain complex C_*, obtaining in this way a new reduction ρ' (which in fact has the same components as ρ) between the perturbed chain complexes. On the other hand, if we consider a perturbation d_C of the top chain complex C_*, in general it is not possible to perturb the small chain complex D_* so that there exists a reduction between the perturbed chain complexes. As we will see, we need an additional hypothesis.

Theorem 1.70 (Basic Perturbation Lemma, BPL). [Bro67] Let us consider a reduction $\rho = (f, g, h) : C_* \Rrightarrow D_*$ between two chain complexes $C_* = (C_n, d_{C_n})_{n\in\mathbb{N}}$ and $D_* = (D_n, d_{D_n})_{n\in\mathbb{N}}$, and δ_C a perturbation of d_C. Furthermore, the composite function $h \circ \delta_C$ is assumed *locally nilpotent*, in other words, given $x \in C_*$ there exists $m \in \mathbb{N}$ such that $(h \circ \delta_C)^m(x) = 0$. Then a new reduction $\rho' = (f', g', h') : C'_* \Rrightarrow D'_*$ can be constructed where:

1) C'_* is the chain complex obtained from the chain complex C_* by replacing the old differential d_C by $(d_C + \delta_C)$;

2) the new chain complex D'_* is obtained from D_* by replacing the old differential d_D by $(d_D + \delta_D)$, with $\delta_D = f \circ \delta_C \circ \phi \circ g = f \circ \psi \circ \delta_C \circ g$;

3) $f' = f \circ \psi = f \circ (\mathrm{Id}_{C_*} - \delta_C \circ \phi \circ h)$;

4) $g' = \phi \circ g$;

5) $h' = \phi \circ h = h \circ \psi$;

with the operators ϕ and ψ defined by

$$\phi = \sum_{i=0}^{\infty} (-1)^i (h \circ \delta_C)^i$$
$$\psi = \sum_{i=0}^{\infty} (-1)^i (\delta_C \circ h)^i = \mathrm{Id}_{C_*} - \delta_C \circ \phi \circ h,$$

the convergence of these series being ensured by the locally nilpotency of the compositions $h \circ \delta_C$ and $\delta_C \circ h$.

These two theorems can be applied in many important cases for the computation of the effective homology of several complicated spaces (loop spaces, classifying spaces, total spaces of fibrations, etc). Furthermore, they have been intensively used in this memoir, as we will see in the following chapters.

1.3.3 The Kenzo program

Kenzo is a 16,000 lines program written in Common Lisp [Gra96], which is devoted to Symbolic Computation in Algebraic Topology. It was developed by Francis Sergeraert and some coworkers, and is www-available (see [DRSS99] for documentation and details). It works with rich and complex algebraic structures (chain complexes, differential graded algebras, simplicial sets, simplicial groups, morphisms between these objects, reductions, etc.) and has obtained some results (for example homology groups of iterated loop spaces of a loop space modified by a cell attachment, components of complex Postnikov towers, etc.) which had never been determined before.

The fundamental idea of the Kenzo system for the computation of homology groups is the notion of object with effective homology. Specifically, to obtain the homology groups of a space X, the program proceeds in the following way: if the complex is effective, then its homology groups can be determined by means of elementary operations with differential matrices. Otherwise, the program uses the effective homology of the space, which is located in one of its slots.

Chain complexes are the simplest algebraic structure implemented in Kenzo. From them, by inheritance, the rest of structures (such as simplicial sets, simplicial groups, algebras, coalgebras, etc.) are built. The definition of this class is included in the following lines.

```
(DEFCLASS CHAIN-COMPLEX ()
    ((cmpr  :type cmprf :initarg :cmpr  :reader cmpr1)
     (basis :type basis :initarg :basis :reader basis1)
     ;; BaSe GeNerator
     (bsgn :type gnrt :initarg :bsgn :reader bsgn)
     ;; DiFFeRential
     (dffr :type morphism :initarg :dffr :reader dffr1)
     ;; GRound MoDule
     (grmd :type chain-complex :initarg :grmd :reader grmd)
     ;; EFfective HoMology
     (efhm :type homotopy-equivalence :initarg :efhm :reader efhm)
     ;; IDentification NuMber
     (idnm :type fixnum :initform (incf *idnm-counter*) :reader idnm)
     ;; ORiGiN
     (orgn :type list :initarg :orgn :reader orgn)))
```

The relevant slots are `cmpr`, a function coding the equality between the generators; `basis`, the function defining the distinguished ordered basis of each group of n-chains, or the keyword `:locally-effective` if the chain complex is not effective; `dffr`, the differential morphism, which is an instance of the class `MORPHISM`; `efhm`, which stores information about the effective homology of the chain complex; and `orgn`, used to keep record of information about the object.

The class `CHAIN-COMPLEX` is extended by inheritance with new slots, obtaining more elaborate structures. For instance, extending it with an `aprd` (algebra product) slot, we obtain the `ALGEBRA` class. Multiple inheritance is also available; for example, the class `SIMPLICIAL-GROUP` is obtained by inheritance from the classes `KAN` and `HOPF-ALGEBRA`.

It is worth emphasizing here that simplicial sets have also been implemented as a subclass of `CHAIN-COMPLEX`. To be precise, the class `SIMPLICIAL-SET` inherits from the class `COALGEBRA`, which is a direct subclass of `CHAIN-COMPLEX`, with a slot `cprd` (the coproduct). The class `SIMPLICIAL-SET` has then one slot of its own: `face`, a Lisp function computing any face of a simplex of the simplicial set. The basis is in this case (when working with effective objects) the list of non-degenerate simplices, and the differential map of the associated chain complex is given by the alternate sum of the faces, where the degenerate simplices are canceled.

To explain roughly the general style of Kenzo computations, let us consider a didactic example. According to Section 1.2.3, the "minimal" simplicial model of the Eilenberg-MacLane space $K(\mathbb{Z}, 1)$ is defined by $K(\mathbb{Z}, 1)_n = \mathbb{Z}^n$; an infinite number of simplices is required in every dimension $n \geq 1$. This does not prevent such an object from being installed and handled by the Kenzo program.

```
> (setf kz1 (k-z 1))
[K1 Abelian-Simplicial-Group]
```

The `k-z` Kenzo function constructs the standard Eilenberg-MacLane space and this object is assigned to the symbol `kz1`. In ordinary mathematical notation (as seen in Section 1.2.3), a 3-simplex of `kz1` could be for example $[3|5|-5]$, denoted by (3 5 -5) in Kenzo. The faces of this simplex can be determined:

```
> (dotimes (i 4)
    (print (face kz1 i 3 '(3 5 -5))))
<AbSm - (5 -5)>
<AbSm - (8 -5)>
<AbSm 1 (3)>
<AbSm - (3 5)>
nil
```

The faces are computed as explained in Section 1.2.3; in particular the face of index 2 is degenerate: $\partial_2[3|5|-5] = \eta_1[3]$. *Local* (in fact simplex-wise) computations are so possible, the object `kz1` is *locally effective*. But no global information is available. For example if we try to obtain the list of non-degenerate simplices in dimension 3, we obtain an error.

```
> (basis kz1 3)
Error: The object [K1 Abelian-Simplicial-Group] is locally-effective.
```

This basis in fact is $\mathbb{Z}^3$, an infinite set whose element *list* cannot be explicitly stored nor displayed. So that the homology groups of `kz1` cannot elementarily be computed. But $K(\mathbb{Z}, 1)$ has the homotopy type of the circle S^1; the Kenzo program knows this fact, reachable as follows. We can ask for the effective homology of $K(\mathbb{Z}, 1)$:

```
> (efhm kz1)
[K22 Homotopy-Equivalence K1 <= K1 => K16]
```

A reduction `K1` $= K(\mathbb{Z}, 1) \Rrightarrow$ `K16` is constructed by Kenzo. What is `K16`?

```
> (orgn (k 16))
(circle)
```

What about the basis of this circle in dimensions 0, 1 and 2?

```
>(dotimes (i 3)
   (print (basis (k 16) i)))
(*)
(s1)
nil
nil
```

The first `nil` means $\emptyset$ and the second `nil` is "technical" (independently produced by the iterative `dotimes`). The basis are available, the boundary operators too:

```
> (? (k 16) 1 'sl)
----------------------------------------------------------------------{CMBN 0}
------------------------------------------------------------------------------
```

The boundary of the unique non-degenerate 1-simplex is the null combination of degree 0. So that the homology groups of $K(\mathbb{Z}, 1)$ are computable through the *effective* equivalent object `K16`:

```
> (homology kz1 0 3)
Homology in dimension 0 :
Component Z
---done---

Homology in dimension 1 :
Component Z
---done---

Homology in dimension 2 :
---done---
```

In this way, Kenzo computes the homology groups of *complicated* spaces by means of the effective homology method. In particular, as we will see in Chapter 3, Kenzo is able to compute the homology groups of twisted products or loop spaces.

Chapter 2

Effective homology and spectral sequences of filtered complexes: algorithms and programs

In the first chapter of this memoir (concretely, in Section 1.1.2) we have introduced the definition and some general results about spectral sequences, but we have not yet said when (or where) a spectral sequence can arise. As we will see in this work, many classical examples of spectral sequences are defined by means of filtrations.

Given a filtered chain complex (and under certain *good* conditions), there exists a spectral sequence which converges to its homology groups. However, the formal expression that defines the groups $E^r_{p,q}$ includes some subgroups which are not necessarily of finite type and in many cases one cannot compute them. Thus, this formal expression is not always sufficient to *compute* the spectral sequence.

On the other hand, if the filtered chain complex is an object with effective homology, we have seen in Section 1.3 that it is possible to compute its homology groups. In this chapter we show that the effective homology method can also be useful to determine spectral sequences. Combining the notions of object with effective homology and spectral sequence, we have developed an algorithm that allows us to determine every component of the spectral sequence associated with a filtered chain complex: the groups $E^r_{p,q}$, the differential maps $d^r_{p,q}$ in every stage r, the convergence level, and the induced filtration of the homology groups.

The work explained in this chapter has been presented in [RRS06] and [Rom06b].

2.1 Filtrations and spectral sequences

In this section we present the definition and some useful results about the spectral sequence associated with a filtered chain complex, most of them extracted from [Mac63].

Definition 2.1. An *increasing filtration* F of a chain complex $C_* = (C_n, d_n)_{n\in\mathbb{N}}$ is a family of sub-chain complexes $F_pC_* = (F_pC_n, d_n)_{n\in\mathbb{N}}$ (that is, $F_pC_n \subseteq C_n$ satisfying $d_n(F_pC_n) \subseteq F_pC_{n-1}$) for each $p \in \mathbb{Z}$ such that

$$\cdots \subseteq F_{p-1}C_n \subseteq F_pC_n \subseteq F_{p+1}C_n \subseteq \cdots \quad \text{for all } n \in \mathbb{N}$$

Similarly, a *decreasing filtration* F of $C_* = (C_n, d_n)_{n\in\mathbb{N}}$ is a family of sub-chain complexes $F^pC_* = (F^pC_n, d_n)_{n\in\mathbb{N}}$, for each $p \in \mathbb{Z}$, such that

$$\cdots \subseteq F^{p+1}C_n \subseteq F^pC_n \subseteq F^{p-1}C_n \subseteq \cdots \quad \text{for all } n \in \mathbb{N}$$

We will mostly work with increasing filtrations and we will call them simply *filtrations*. The pair (C_*, F) is said to be a *filtered (chain) complex*.

Note 2.2. A filtration F of a chain complex C_* induces a filtration F_H of the graded homology group $H_*(C_*)$. Let $i_p : F_pC_* \hookrightarrow C_*$ be the p-injection; then one defines $F_{H_p}H_n(C_*) = H_n(i_p)(H_n(F_pC_*))$. In other words, the filtration F_H is given by

$$F_{H_p}H_n(C_*) = \frac{F_p(\operatorname{Ker} d_n) \cup \operatorname{Im} d_{n+1}}{\operatorname{Im} d_{n+1}}$$

where $F_p(\operatorname{Ker} d_n) = \operatorname{Ker} d_n \cap F_pC_n$.

Definition 2.3. A filtration F of $C_* = (C_n, d_n)_{n\in\mathbb{N}}$ is said to be *bounded below* if for each degree n there exists $s = s(n)$ such that $F_sC_n = 0$. F is called *bounded* if for each n there are integers $s = s(n) < t = t(n)$ such that $F_sC_n = 0$ and $F_tC_n = C_n$. F is *canonically bounded* if $F_{-1}C_n = 0$ and $F_nC_n = C_n$ for all n.

Definition 2.4. A filtration F of a chain complex C_* is called *convergent above* if each C_n is the union of all F_pC_n. F is said to be *convergent below* if for all $n \in \mathbb{N}$ the intersection of all F_pC_n is equal to zero. Obviously F bounded implies F is convergent above and convergent below.

Since $F_{p-1}C_*$ is a chain subcomplex of F_pC_* for each $p \in \mathbb{Z}$, it makes sense to consider the quotient $F_pC_*/F_{p-1}C_*$. We say that the elements of $F_pC_*/F_{p-1}C_*$ have *filtration index* (or *filtration degree*) equal to p.

It is convenient to write the indices of the grading as (p, q), where $q = n - p$ is the *complementary degree*. For every pair of integers p, q, we can consider

$$C_{p,q} = \frac{F_pC_{p+q}}{F_{p-1}C_{p+q}}$$

If F is convergent above and convergent below then it is clear that $C_n = \bigoplus_{p+q=n} C_{p,q}$. The filtration F produces in this way a bigraded $\mathbb{Z}$-module, $\{C_{p,q}\}_{p,q\in\mathbb{Z}}$, with differential maps $d^i_{p,q} : C_{p,q} \to C_{p-i,q+i-1}$ for each $i \geq 0$; this is called a *multicomplex*.

Definition 2.5. A *multicomplex* $C_{*,*} = \{C_{p,q}, d^i_{p,q}\}_{p,q\in\mathbb{Z},i\geq 0}$ is a bigraded $\mathbb{Z}$-module with maps $d^i_{p,q} : C_{p,q} \to C_{p-i,q+i-1}$ for all $p, q \in \mathbb{Z}$ and $i \geq 0$, such that for each $p, q \in \mathbb{Z}$ and $k \geq 0$ the following equation holds

$$\sum_{i+j=k} d^j_{p-i,q+i-1} \circ d^i_{p,q} = 0$$

We note that, for each fixed p, the operator $d^0 : C_{p,*} \to C_{p,*-1}$ is a differential map, that is to say, for every $q \in \mathbb{Z}$ one has $d^0_{p,q-1} \circ d^0_{p,q} = 0$. Nevertheless, for $i \geq 1$ the operator d^i does not necessarily satisfy $d^i_{p-i,q+i-1} \circ d^i_{p,q} = 0$.

Definition 2.6. Given a multicomplex $C_{*,*} = \{C_{p,q}, d^i_{p,q}\}_{p,q\in\mathbb{Z},i\geq 0}$, the *total graded module* $T_* = T_*(C_{*,*})$ is defined by

$$T_n = \bigoplus_{p+q=n} C_{p,q}$$

The formal infinite sum $d_n = \sum_{i\geq 0} d^i : T_n \to T_{n-1}$ given by

$$d_n(x) = \sum_{i\geq 0} d^i_{p,q}(x) \quad \text{ if } x \in C_{p,q}, \text{ with } p + q = n$$

defines an operator of degree -1 on the total graded module whenever the sum is finite for each element. In this case, the equation $\sum_{i+j=k} d^j_{p-i,q+i-1} \circ d^i_{p,q} = 0$ implies that $d_{n-1} \circ d_n = 0$, and therefore $T_* = T_*(C_{*,*}) = (T_n, d_n)_{n\in\mathbb{Z}}$ is a chain complex, which is called the *total (chain) complex* of $C_{*,*}$.

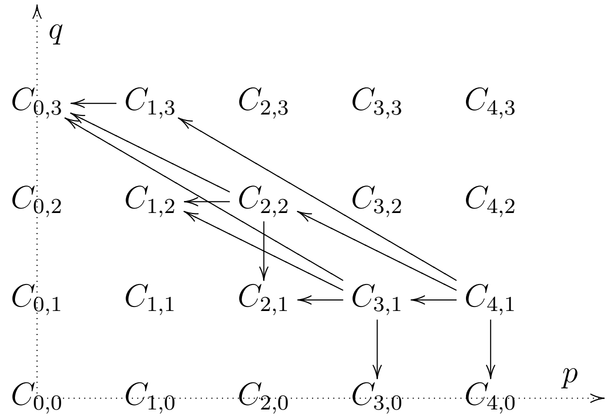
If $C_{*,*}$ is the multicomplex associated with a filtration F of a chain complex C_* (which is convergent above and convergent below), then $d_n(x) = \sum_{i\geq 0} d^i_{p,q}(x)$ has only a finite number of terms for each $x \in C_{p,q}$. Therefore $T_*(C_{*,*})$ is a chain complex.

In this way, filtrations can be seen as multicomplexes. Reciprocally, a multicomplex $C_{*,*} = \{C_{p,q}, d^i_{p,q}\}_{p,q\in\mathbb{Z},i\geq 0}$ can be considered as defining a filtration F of the total chain complex $T_*(C_{*,*})$. The filtration F is given by the first degree p:

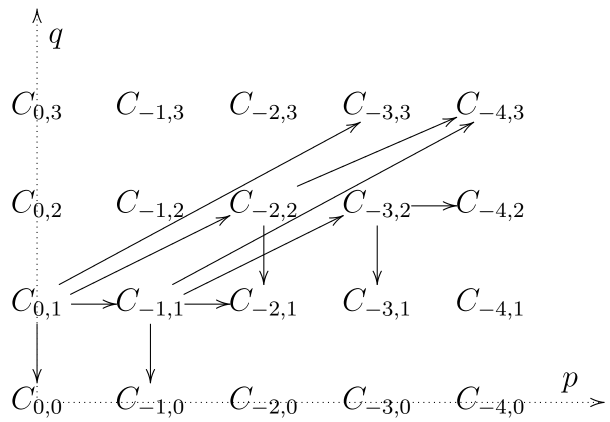
$$F_pT_n = \bigoplus_{h\leq p} C_{h,n-h}$$

A similar filtration can also be defined by means of the second degree q.

Canonically bounded filtrations correspond to *first quadrant multicomplexes*, that is, $C_{*,*} = \{C_{p,q}, d^i_{p,q}\}_{p,q\in\mathbb{Z},i\geq 0}$ such that $C_{p,q} = 0$ when $p < 0$ or $q < 0$. A first quadrant multicomplex $C_{*,*}$ can be represented with a diagram as the following one. Only some maps $d^i_{p,q}$ are included.



Similarly, a *second quadrant multicomplex* $C_{*,*} = \{C_{p,q}, d^i_{p,q}\}_{p,q\in\mathbb{Z},i\geq 0}$ is one with $C_{p,q} = 0$ when $p > 0$ or $q < 0$. As in the case of second quadrant spectral sequences, second quadrant multicomplexes will be represented in the first quadrant of the (p, q)-plane, drawing $C_{p,q}$ with $p < 0$ at the point $(-p, q)$.



Definition 2.7. Let $H_* = \{H_n\}_{n\in\mathbb{N}}$ be a graded module. A spectral sequence $(E^r, d^r)_{r\geq 1}$ is said to *converge* to H_* (denoted by $E^1 \Rightarrow H_*$) if there is a filtration F_H of H_* and for each p one has isomorphisms of graded modules

$$E^\infty_{p,*} \cong \frac{F_{H_p} H_*}{F_{H_{p-1}} H_*}$$

The collection $H_* = \{H_n\}_{n\in\mathbb{N}}$ is called the *abutment* of the spectral sequence.

The next theorem gives the definition of the spectral sequence associated with a filtered complex, that will appear many times in this memoir.

Theorem 2.8. [Mac63] Let F be a filtration of a chain complex $C_* = (C_n, d_n)_{n\in\mathbb{N}}$. There exists a spectral sequence $E = E(C_*, F) = (E^r, d^r)_{r\geq 1}$, defined by

$$E^r_{p,q} = \frac{Z^r_{p,q} \cup F_{p-1}C_{p+q}}{d_{p+q+1}(Z^{r-1}_{p+r-1,q-r+2}) \cup F_{p-1}C_{p+q}}$$

where $Z^r_{p,q}$ is the submodule $Z^r_{p,q} = \{a \in F_pC_{p+q} | \ d_{p+q}(a) \in F_{p-r}C_{p+q-1}\} \subseteq F_pC_{p+q}$, and $d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}$ is the morphism induced on these subquotients by the differential map $d_{p+q} : C_{p+q} \to C_{p+q-1}$. The first level of the spectral sequence satisfies

$$E^1_{p,q} \cong H_{p+q}\left(\frac{F_pC_*}{F_{p-1}C_*}\right)$$

Furthermore, if F is bounded, then $E^1 \Rightarrow H_*(C_*)$; to be precise, there are natural isomorphisms

$$E^\infty_{p,q} \cong \frac{F_{H_p}(H_{p+q}(C_*))}{F_{H_{p-1}}(H_{p+q}(C_*))}$$

with $F_{H_p}(H_*(C_*))$ induced by the filtration F of C_*, as explained in Note 2.2.

The convergence of the spectral sequence holds under weaker conditions than bounded, for instance when the filtration F is convergent above and convergent below.

Theorem 2.9. [McC04] Let $C_* = (C_n, d_n)_{n\in\mathbb{N}}$ be a chain complex and F a filtration of C_* which is convergent below and convergent above. Then the associated spectral sequence $E = E(C_*, F) = (E^r, d^r)_{r\geq 1}$ converges to the graded homology group $H_*(C_*)$.

Theorem 2.8 gives a formal expression for the groups and the differential maps of the spectral sequence associated with a filtered complex, which converges to its homology groups if the filtration is convergent below and convergent above. Nevertheless, we must bear in mind that in many cases this expression is not sufficient to *compute* $E^r_{p,q}$ and $d^r_{p,q}$. If C_* is not of finite type, it may occur that the subgroups $Z^r_{p,q}$ are not computable and therefore we cannot always determine, using this formula, the groups $E^r_{p,q}$.

Definition 2.10. Given two chain complexes C_* and D_* with filtrations F_C and F_D respectively, a *filtered chain complex morphism* $f : (C_*, F_C) \to (D_*, F_D)$ is a chain complex morphism $f : C_* \to D_*$ which is compatible with the filtrations, that is to say,

$$f(F_{C_p}C_*) \subseteq F_{D_p}D_*$$

A filtered chain complex morphism $f : (C_*, F_C) \to (D_*, F_D)$ induces a morphism of spectral sequences $f : E(C_*, F_C) \to E(D_*, F_D)$. For each level r we have a morphism of bigraded modules

$$f^r : E(C_*, F_C)^r = \{E(C_*, F_C)^r_{p,q}\}_{p,q\in\mathbb{Z}} \longrightarrow E(D_*, F_D)^r = \{E(D_*, F_D)^r_{p,q}\}_{p,q\in\mathbb{Z}}$$

The spectral sequence construction is functorial:

- the map induced by the identity of a filtered chain complex is the identity map of the associated spectral sequence. In other words, given $\mathrm{Id}_{C_*} : C_* \to C_*$ one has

$$(\mathrm{Id}_{C_*})^r = \mathrm{Id} : E(C_*, F_C)^r_{p,q} \longrightarrow E(C_*, F_C)^r_{p,q} \quad \text{for all } r \geq 1 \text{ and } p, q \in \mathbb{Z}$$

- the map induced by a composition is the composition of the induced maps. That is to say, if $f : (B_*, F_B) \to (C_*, F_C)$ and $g : (C_*, F_C) \to (D_*, F_D)$ are filtered chain complex morphisms, then

$$(g \circ f)^r = g^r \circ f^r : E(B_*, F_B)^r_{p,q} \longrightarrow E(D_*, F_D)^r_{p,q} \quad \text{for all } r \geq 1 \text{ and } p, q \in \mathbb{Z}$$

Theorem 2.11. [Mac63] Let C_* and D_* be chain complexes with filtrations F_C and F_D, both of them bounded below and convergent above. Let $f : (C_*, F_C) \to (D_*, F_D)$ be a filtered chain complex morphism such that for some $k \geq 1$ the induced morphism of bigraded modules

$$f^k : E(C_*, F_C)^k \longrightarrow E(D_*, F_D)^k$$

is an isomorphism. Then f^r is an isomorphism for all $\infty \geq r \geq k$. Moreover, the induced map on the graded homology groups $H_*(f) : H_*(C_*) \to H_*(D_*)$ is also an isomorphism.

Definition 2.12. Given two filtered complex morphisms $f, g : (C_*, F_C) \to (D_*, F_D)$ and a chain homotopy $h : f \simeq g$, we say that h has *order* $\leq k$ if

$$h(F_{C_p} C_*) \subseteq F_{D_{p+k}} D_{*+1}$$

Proposition 2.13. [Mac63] Let $f, g : (C_*, F_C) \to (D_*, F_D)$ be filtered chain complex morphisms, and $h : f \simeq g$ a chain homotopy of order $\leq k$. Then the induced maps on the corresponding spectral sequences coincide for every level $r > k$, that is,

$$f^r = g^r : E(C_*, F_C)^r \longrightarrow E(D_*, F_D)^r \quad \text{for all } r > k$$

2.2 Main theoretical results

As stated by Theorem 2.8, the groups $E^r_{p,q}$ of the spectral sequence associated with a filtered chain complex (C_*, F) are given by the following subquotient of $F_p C_{p+q}$:

$$E^r_{p,q} = \frac{Z^r_{p,q} \cup F_{p-1} C_{p+q}}{d_{p+q+1}(Z^{r-1}_{p+r-1,q-r+2}) \cup F_{p-1} C_{p+q}}$$

However, it is worth emphasizing that the subgroups $Z^r_{p,q}$ and $d_{p+q+1}(Z^{r-1}_{p+r-1,q-r+2})$ which appear in this formula are not always computable and therefore in many cases this formal expression does not provide an algorithm for the construction of the spectral sequence.

This problem can be solved making use of the effective homology method. In this section we include several results which relate the notions of spectral sequence and effective homology, that will be used in Section 2.3 for the development of an algorithm computing spectral sequences of filtered complexes with effective homology.

First of all, the next theorem explains that reductions of filtered complexes (satisfying certain *natural* conditions) have a good behavior with respect to the associated spectral sequences.

Theorem 2.14. Let $\rho = (f, g, h) : C_* \Rrightarrow D_*$ be a reduction between two chain complexes $C_* = (C_n, d_{C_n})_{n\in\mathbb{N}}$ and $D_* = (D_n, d_{D_n})_{n\in\mathbb{N}}$, such that filtrations F_C and F_D are defined on C_* and D_* respectively. If the maps f and g are filtered complex morphisms (that is, they are compatible with the filtrations) and the homotopy operator h has order $\leq k$, then the induced morphism of spectral sequences $f : E(C_*, F_C) \to E(D_*, F_D)$ gives an isomorphism of bigraded modules at each stage $r > k$:

$$f^r : E(C_*, F_C)^r \cong E(D_*, F_D)^r \quad \text{for all } r > k$$

Proof. First of all, since $f \circ g = \mathrm{Id}_{D_*}$, the induced morphisms on the spectral sequence coincide for every level r, that is:

$$(f \circ g)^r = (\mathrm{Id}_{D_*})^r : E(D_*, F_D)^r \longrightarrow E(D_*, F_D)^r \quad \text{for all } r \geq 1$$

Then, due to the functoriality of the spectral sequence construction, it follows that $(f \circ g)^r = f^r \circ g^r$ and $(\mathrm{Id}_{D_*})^r = \mathrm{Id}_{E(D_*,F_D)^r}$, and therefore

$$f^r \circ g^r = \mathrm{Id}_{E(D_*,F_D)^r} : E(D_*, F_D)^r \longrightarrow E(D_*, F_D)^r \quad \text{for all } r \geq 1$$

On the other hand, the equation $d_C \circ h + h \circ d_C = \mathrm{Id}_{C_*} - g \circ f$ implies $h : g \circ f \simeq \mathrm{Id}_{C_*}$, and h has order $\leq k$, so that using Proposition 2.13 we obtain

$$(g \circ f)^r = (\mathrm{Id}_{C_*})^r : E(C_*, F_C)^r \longrightarrow E(C_*, F_C)^r \quad \text{if } r > k$$

and therefore

$$g^r \circ f^r = \mathrm{Id}_{E(C_*,F_C)^r} : E(C_*, F_C)^r \longrightarrow E(C_*, F_C)^r \quad \text{if } r > k$$

Thus, the maps $f^r : E(C_*, F_C)^r \longrightarrow E(D_*, F_D)^r$ and $g^r : E(D_*, F_D)^r \longrightarrow E(C_*, F_C)^r$ are inverse morphisms of bigraded modules for all $r > k$. Therefore one has

$$f^r : E(C_*, F_C)^r \cong E(D_*, F_D)^r \quad \text{for } r > k$$

□

The next corollary is a very useful result that combines both spectral sequence and effective homology concepts and is one of the main results on which our algorithm for computing spectral sequences associated with filtered complexes (which will be explained in Section 2.3) is based.

Corollary 2.15. Let $C_* = (C_n, d_{C_n})_{n\in\mathbb{N}}$ be a chain complex with a filtration F_C. Let us suppose that C_* is an object with effective homology, such that there exists an equivalence $C_* \overset{\rho_1}{\Lleftarrow} D_* \overset{\rho_2}{\Rrightarrow} HC_*$ with $\rho_1 = (f_1, g_1, h_1)$ and $\rho_2 = (f_2, g_2, h_2)$, and such that filtrations F_D and F_{HC} are also defined on the chain complexes D_* and HC_*. If the maps f_1, f_2, g_1, and g_2 are morphisms of filtered chain complexes and both homotopies h_1 and h_2 have order $\leq k$, then the spectral sequences of the complexes C_* and HC_* are isomorphic for $r > k$:

$$E(C_*, F_C)^r_{p,q} \cong E(HC_*, F_{HC})^r_{p,q} \quad \text{for all } p, q \in \mathbb{Z} \text{ and } r > k$$

Proof. The result is obtained applying Theorem 2.14 to the reductions $\rho_1 : D_* \Rrightarrow C_*$ and $\rho_2 : D_* \Rrightarrow HC_*$. The inverse morphisms are $(f_2 \circ g_1)^r : E(C_*, F_C)^r_{p,q} \to E(HC_*, F_{HC})^r_{p,q}$ and $(f_1 \circ g_2)^r : E(HC_*, F_{HC})^r_{p,q} \to E(C_*, F_C)^r_{p,q}$. □

This corollary can be applied for the computation of the spectral sequence associated with a filtered complex C_* with effective homology, whenever D_* and HC_* are also filtered chain complexes.

In some cases, (C_*, F_C) is a filtered complex but filtrations are not defined on the chain complexes D_* and HC_*. The results that follow allow us to propagate the filtration F_C to D_* and HC_* (defining filtrations F_D and F_{HC} respectively), in a way that the three associated spectral sequences are isomorphic from some stage r on.

Proposition 2.16. Let $\rho = (f, g, h) : C_* \Rrightarrow D_*$ be a reduction between two chain complexes $C_* = (C_n, d_{C_n})_{n\in\mathbb{N}}$ and $D_* = (D_n, d_{D_n})_{n\in\mathbb{N}}$. Let us suppose that D_* is a filtered chain complex, with a filtration F_D. Then it is possible to define a filtration F_C of C_* such that the spectral sequences $E(C_*, F_C)$ and $E(D_*, F_D)$ are isomorphic at every level $r \geq 1$:

$$E(C_*, F_C)^r_{p,q} \cong E(D_*, F_D)^r_{p,q} \quad \text{for all } p, q \in \mathbb{Z} \text{ and } r \geq 1$$

Proof. We define

$$F_{C_p} C_n = \{x \in C_n |\ f(x) \in F_{D_p} D_n\}$$

It is clear that $F_{C_p} C_n \subseteq F_{C_{p+1}} C_n$ and $d_{C_n}(F_{C_p} C_n) \subseteq F_{C_p} C_{n-1}$, so that F_C is a filtration of the chain complex C_*. Furthermore, it is not difficult to prove that the three components of the reduction, f, g, and h, are then compatible with the filtrations F_C and F_D. We can apply therefore Theorem 2.14 (in this case the order of the homotopy operator h is ≤ 0) and one has that $f : E(C_*, F_C) \to E(D_*, F_D)$ is an isomorphism for every stage $r > 0$. Therefore one has

$$E(C_*, F_C)^r_{p,q} \cong E(D_*, F_D)^r_{p,q} \quad \text{for } p, q \in \mathbb{Z} \text{ and } r \geq 1$$

□

Proposition 2.17. Let $\rho = (f, g, h) : C_* \Rrightarrow D_*$ be a reduction between two chain complexes $C_* = (C_n, d_{C_n})_{n\in\mathbb{N}}$ and $D_* = (D_n, d_{D_n})_{n\in\mathbb{N}}$. We suppose now that C_* is a filtered complex (with a filtration F_C) such that the composition $g \circ f$ is compatible with the filtration, that is to say, $g \circ f(F_{C_p} C_*) \subseteq F_{C_p} C_*$, and such that the homotopy h has order $\leq k$. Then there exists a filtration F_D of D_* such that the spectral sequences $E(C_*, F_C)$ and $E(D_*, F_D)$ are isomorphic at every level $r > k$:

$$E(C_*, F_C)^r_{p,q} \cong E(D_*, F_D)^r_{p,q} \quad \text{for all } p, q \in \mathbb{Z} \text{ and } r > k$$

Proof. The filtration F_D of the chain complex D_* is given by

$$F_{D_p} D_n = \{x \in D_n |\ g(x) \in F_{C_p} C_n\}$$

As in the previous proposition, we observe that $F_{D_p}D_n \subseteq F_{D_{p+1}}D_n$ and $d_{D_n}(F_{D_p}D_n) \subseteq F_{D_p}D_{n-1}$. Furthermore, the definition of F_D and the condition that $g \circ f$ is compatible with F_C imply that g and f respect the filtrations too. Since h has order $\leq k$, thanks to Theorem 2.14 one has

$$E(C_*, F_C)^r_{p,q} \cong E(D_*, F_D)^r_{p,q} \quad \text{for all } p, q \in \mathbb{Z} \text{ and } r > k$$

□

It is worth remarking the difference between both propositions. In the first one, a filtration is defined on the small chain complex, and it is then naturally transferred to the big chain complex without any extra hypotheses. Moreover, both spectral sequences are isomorphic at every level r. On the other hand, in the second proposition a filtration of the big chain complex is given. In this case, one cannot directly propagate it to the small one, we need $g \circ f$ being compatible with the filtration. The spectral sequences are then isomorphic from some stage k on, and in general k can be greater than 1.

Corollary 2.18. Let $C_* = (C_n, d_{C_n})_{n \in \mathbb{N}}$ be a chain complex with effective homology, with an equivalence $C_* \overset{\rho_1}{\Lleftarrow} D_* \overset{\rho_2}{\Rrightarrow} HC_*$ where $\rho_1 = (f_1, g_1, h_1)$ and $\rho_2 = (f_2, g_2, h_2)$. Let F_C be a filtration of C_*, such that given an element $x \in D_n$ with $f_1(x) \in F_{C_p}C_n$, then $f_1 \circ g_2 \circ f_2(x) \in F_{C_p}C_n$ and $f_1 \circ h_2(x) \in F_{C_{p+k}}C_{n+1}$ for some $k \geq 0$. Then we can define a filtration F_{HC} of the chain complex HC_* such that the spectral sequences $E(C_*, F_C)$ and $E(HC_*, F_{HC})$ are isomorphic after level k:

$$E(C_*, F_C)^r_{p,q} \cong E(HC_*, F_{HC})^r_{p,q} \quad \text{for all } p, q \in \mathbb{Z} \text{ and } r > k$$

Proof. First of all, we define a filtration F_D of D_* by

$$F_{D_p}D_n = \{x \in D_n |\ f_1(x) \in F_{C_p}C_n\}$$

As seen in Proposition 2.16, the maps f_1, g_1, and h_1 are compatible with the filtrations F_C and F_D and then the spectral sequences $E(C_*, F_C)$ and $E(D_*, F_D)$ are isomorphic at every level:

$$E(C_*, F_C)^r_{p,q} \cong E(D_*, F_D)^r_{p,q} \quad \text{for all } p, q \in \mathbb{Z} \text{ and } r \geq 1$$

Then, the filtration F_{HC} of HC_* is given by

$$F_{HC_p}HC_n = \{x \in HC_n |\ g_2(x) \in F_{D_p}D_n\} = \{x \in HC_n |\ f_1 \circ g_2(x) \in F_{C_p}C_n\}$$

If we want the reduction ρ_2 to have a good behavior with respect to the filtrations F_D and F_{HC}, we must suppose that the composition $g_2 \circ f_2$ is compatible with the filtration F_D. This means that given $x \in F_{D_p}D_n$, then $g_2 \circ f_2(x) \in F_{D_p}D_n$; equivalently, given $x \in D_n$ such that $f_1(x) \in F_{C_p}C_n$, then $f_1 \circ g_2 \circ f_2(x) \in F_{C_p}C_n$. Furthermore, the homotopy operator h_2 must have order $\leq k$ for some $k \geq 0$, in other words, given $x \in F_{D_p}D_n$, then $h_2(x) \in F_{D_{p+k}}D_{n+1}$. This means that each $x \in D_n$ with $f_1(x) \in F_{C_p}C_n$ satisfies $f_1 \circ h_2(x) \in F_{C_{p+k}}C_{n+1}$.

If these hypotheses are satisfied, Proposition 2.17 allows us to affirm that the spectral sequences $E(D_*, F_D)$ and $E(HC_*, F_{HC})$ are isomorphic at every stage $r > k$, and composing with the previous result we have isomorphisms

$$E(C_*, F_C)^r_{p,q} \cong E(HC_*, F_{HC})^r_{p,q} \quad \text{for all } p, q \in \mathbb{Z} \text{ and } r > k$$

The inverse isomorphisms are $(f_2 \circ g_1)^r : E(C_*, F_C)^r_{p,q} \to E(HC_*, F_{HC})^r_{p,q}$ and $(f_1 \circ g_2)^r : E(HC_*, F_{HC})^r_{p,q} \to E(C_*, F_C)^r_{p,q}$. □

Corollaries 2.15 and 2.18 will make it possible to compute spectral sequences of (complicated) filtered complexes with effective homology, obtaining in this way a real algorithm, as will be explained in Section 2.3. This algorithm is based on the following idea: if a filtered complex is effective, then its spectral sequence can be computed by means of elementary operations with matrices (in a similar way to the computation of homology groups); otherwise, the effective homology is needed to compute the groups $E^r_{p,q}$ by means of an analogous spectral sequence deduced of an appropriate filtration of the associated effective complex, which is isomorphic to the spectral sequence of the initial complex after some level r.

2.3 An algorithm computing spectral sequences of filtered complexes

Using the theoretical results presented in Section 2.2, it is possible to develop an algorithm computing spectral sequences associated with filtered complexes with effective homology (under suitable hypotheses). This algorithm will allow us to determine not only the groups $E^r_{p,q}$, but also the differential maps $d^r_{p,q}$, as well as the convergence level of the spectral sequence for each degree $n \in \mathbb{N}$, and the filtration of the graded homology group induced by the filtration of the chain complex.

Let $C_* = (C_n, d_n)_{n\in\mathbb{N}}$ be a chain complex. We only work with free chain complexes, so that for each degree n a (possibly infinite) set of generators $G_n = \{g^n_i\}$ is known. A filtration F of C_* will be given by means of the *filtration index* of each generator $g \in G_n$, $\text{Flin}(g) \in \mathbb{Z}$, and then the filtration index of a combination $c = \sum_{i=1}^m \lambda_i g_i$ (with $\lambda_i \in \mathbb{Z}, \lambda_i \neq 0, g_i \in G_n$) will be

$$\text{Flin}(c) = \max\{\text{Flin}(g_i)|\ 1 \leq i \leq m\}$$

Then we take $F_pC_n = \{x \in C_n|\ \text{Flin}(x) \leq p\}$, which defines a filtration F of C_* which is convergent above and convergent below. Moreover, for every $c \in C_n$, one has $\text{Flin}(c) = \min\{p \in \mathbb{Z}|\ c \in F_pC_n\}$.

In order to compute the spectral sequence $E(C_*, F)$ we must distinguish two situations: if C_* is effective, then the spectral sequence can be determined by means of elementary operations; if C_* is not effective (C_* is locally effective), then we must use the effective homology.

2.3.1 Effective chain complexes

Let $C_* = (C_n, d_n)_{n\in\mathbb{N}}$ be an effective chain complex with a filtration F. In this case all the components of the spectral sequence can directly be determined as follows.

2.3.1.1 Groups

For each degree $n \in \mathbb{N}$, the group C_n is finitely generated. We have a finite set of generators $G_n = \{g_1^n, \dots, g_{m_n}^n\}$ (for some $m_n \geq 0$), and we consider it ordered by the filtration index, that is to say, $\text{Flin}(g_i^n) \leq \text{Flin}(g_{i+1}^n)$ for all $1 \leq i < m_n$.

Let us remark that every subgroup $A \subseteq C_n$ is a free subgroup and it is generated by a set of combinations $\{c_1, \dots, c_l\}$ for some $0 \leq l \leq m_n$, with $c_j = \sum_{i=1}^{m_n} \lambda_i^j g_i^n$ for $1 \leq j \leq l$. This subgroup can then be identified with a matrix (that we also denote by A) with m_n rows and l columns that has the coefficient λ_i^j in the (i,j)-position (i-row and j-column).

On the other hand, for each $n \in \mathbb{N}$, the differential $d_n : C_n \cong \mathbb{Z}^{m_n} \to C_{n-1} \cong \mathbb{Z}^{m_{n-1}}$ can be expressed as a matrix D_n with m_{n-1} rows and m_n columns, which is given by the coefficients of $d_n(g_1^n), \dots, d_n(g_{m_n}^n)$ in the base $\{g_1^{n-1}, \dots, g_{m_{n-1}}^{n-1}\}$.

For each $r \geq 1$ and $p, q \in \mathbb{Z}$, we want to compute the group $E_{p,q}^r$ (defined in Theorem 2.8), which is given by the formula:

$$E_{p,q}^r = \frac{Z_{p,q}^r \cup F_{p-1}C_{p+q}}{d_{p+q+1}(Z_{p+r-1,q-r+2}^{r-1}) \cup F_{p-1}C_{p+q}}$$

where $Z_{p,q}^r$ is the submodule $Z_{p,q}^r = \{a \in F_pC_{p+q} | \ d_{p+q}(a) \in F_{p-r}C_{p+q-1}\}$.

First of all, the subgroup $Z_{p,q}^r$ can be determined as the kernel of a submatrix of D_n with $n = p+q$. To be precise, we take only the columns of D_n corresponding to the generators g_j^n with $\text{Flin}(g_j^n) \leq p$, and the rows of the elements g_i^{n-1} which satisfy $p-r < \text{Flin}(g_i^{n-1}) \leq p$. The kernel of this submatrix (that can be computed, for example, using the Smith Normal Form method [KMM04]) is a subgroup of C_n, generated by a list of combinations. Therefore it can be represented by a matrix, that we call also $Z_{p,q}^r$.

Similarly, the subgroup $Z_{p+r-1,q-r+2}^{r-1}$ can be expressed as a matrix $Z_{p+r-1,q-r+2}^{r-1}$, and the group $d_{p+q+1}(Z_{p+r-1,q-r+2}^{r-1})$ is given simply by the matrix multiplication $D_{n+1} \cdot Z_{p+r-1,q-r+2}^{r-1}$.

Furthermore, we can also think in the subgroup $F_{p-1}C_{p+q} = F_{p-1}C_n$ as the matrix $F_{p-1,n}$ whose only non-null elements are those at the positions (i,i) with $\text{Flin}(g_i^n) \leq p-1$, which are equal to 1.

The subgroup in the *numerator* of the formula, $Z_{p,q}^r \cup F_{p-1}C_{p+q}$, is given then by the concatenation of the columns of the two corresponding matrices (in this case the generators are not necessarily linearly independent). Similarly, one can determine the generators of the *denominator*, $d_{p+q+1}(Z_{p+r-1,q-r+2}^{r-1}) \cup F_{p-1}C_{p+q}$.

Once we have the generators of both groups, an elementary algorithm based on the Smith Normal Form technique allows us to determine a *basis-divisors* representation that describes completely the quotient $E^r_{p,q}$.

The group $E^r_{p,q}$ is a finitely generated group and therefore it is isomorphic to a group of the form

$$\mathbb{Z}^\alpha \oplus \mathbb{Z}_{\beta_1} \oplus \cdots \oplus \mathbb{Z}_{\beta_k}$$

where $\mathbb{Z}_\beta$ denotes the group of integers modulo β, α is a non-negative integer, $\beta_i > 1$ for all i, and each β_i divides β_{i+1}. The number α is called the *Betti number* and $\beta_0, \ldots, \beta_k$ are the *torsion coefficients* of the group.

The basis-divisors description of this group consists in a list of combinations $(c_1, \ldots, c_{\alpha+k})$ which generate the group, as well as the list of non-negative integers $(\beta_1, \ldots, \beta_k, 0, \overset{\alpha}{\ldots}, 0)$ that contains the torsion coefficients and α 0's corresponding to the free factor. The list of *divisors* can be seen as the list of the coefficients of the elements that appear in the denominator with regard to the list of combinations that generate the group.

In this way, the basis-divisors representation of $E^r_{p,q}$ can be obtained by means of elementary operations with matrices and determines completely the group, so that we have obtained the following algorithm.

Algorithm 1.
Input:

- an effective chain complex $C_* = (C_n, d_n)_{n\in\mathbb{N}}$, with a filtration F defined by means of the filtration index of the generators of each C_n,
- the numbers $r \geq 1$ and $p, q \in \mathbb{Z}$.

Output: a basis-divisors description of the quotient group $E^r_{p,q}$ of the spectral sequence $E = E(C_*, F)$, in other words,

- a list of combinations $(c_1, \ldots, c_{\alpha+k})$ which generate the group,
- and a list of non-negative integers $(\beta_1, \ldots, \beta_k, 0, \overset{\alpha}{\ldots}, 0)$ (such that β_i divides β_{i+1}) where α is the Betti number of the group and $\beta_1, \ldots, \beta_k$ are the torsion coefficients.

We will show several examples of the computation of the groups $E^r_{p,q}$ of some filtered chain complexes in Sections 2.4, 3.1 and 3.2.

2.3.1.2 Differential maps

Given $r \geq 1$ and $p, q \in \mathbb{Z}$, the computation of the differential map $d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}$ is not difficult once we have determined the groups $E^r_{p,q}$ and $E^r_{p-r,q+r-1}$ (with their basis-divisors representation).

Let us suppose that $E^r_{p,q}$ is generated by a set of combinations $\{c_1, \ldots, c_t\}$ with divisors $(b_1, \ldots, b_t) = (\beta_1, \ldots, \beta_k, 0, \overset{\alpha}{\ldots}, 0)$ (where $t = k + \alpha$). Similarly, $E^r_{p-r,q+r-1}$ is generated by $\{c'_1, \ldots, c'_{t'}\}$, with divisors $(b'_1, \ldots, b'_{t'}) = (\beta'_1, \ldots, \beta'_k, 0, \overset{\alpha'}{\ldots}, 0)$.

Then, let a be a class of the quotient

$$E^r_{p,q} = \frac{Z^r_{p,q} \cup F_{p-1}C_{p+q}}{d_{p+q+1}(Z^{r-1}_{p+r-1,q-r+2}) \cup F_{p-1}C_{p+q}}$$

This class must be given by means of its coefficients $\{\lambda_1, \ldots, \lambda_t\}$ (with $\lambda_i \in \mathbb{Z}$) with respect to the set of generators $\{c_1, \ldots, c_t\}$. Then, we consider the element $x = \sum_{i=1}^{t} \lambda_i c_i \in a$.

To apply the differential map $d^r_{p,q}$ to the class $a = [x]$, we need an element $z \in a = [x]$ such that $z \in Z^r_{p,q}$, in other words, we must build the *projection* of $x \in Z^r_{p,q} \cup F_{p-1}C_{p+q}$ over the factor $Z^r_{p,q}$. To this aim, we consider the set of generators of the numerator given by the concatenation of the two matrices $Z^r_{p,q}$ and $F_{p-1,p+q}$, and compute a set of coefficients of x with respect to this new generator system (in general these coefficients are not unique since the generators are not necessarily linearly independent). These coefficients can be computed using again the Smith Normal Form technique. This provides a decomposition $x = z + y$ with $z \in Z^r_{p,q}$ and $y \in F_{p-1}C_{p+q}$; we consider only the first factor $z \in Z^r_{p,q}$.

If we apply now the differential map $d_{p+q} = d_n$ to the element $z \in Z^r_{p,q}$ we obtain $d_{p+q}(z) \in Z^r_{p-r,q+r-1}$, and therefore one can consider $[d_{p+q}(z)] \in E^r_{p-r,q+r-1}$. Moreover it is clear that $a = [x] = [z]$, so that we can define $d^r_{p,q}(a) = d^r_{p,q}([x]) = [d_{p+q}(z)] \in E^r_{p-r,q+r-1}$. Finally, if we want to express the class $d^r_{p,q}(a) = [d_{p+q}(z)]$ with regard to the basis-divisors description of the quotient $E^r_{p-r,q+r-1}$, we only have to compute the coefficients of $d_{p+q}(z)$ with respect to the set of generators $\{c'_1, \ldots, c'_{t'}\}$, and "simplify" them considering the corresponding divisors $(b'_1, \ldots, b'_{t'})$.

Algorithm 2.
Input:

- an effective chain complex $C_* = (C_n, d_n)_{n \in \mathbb{N}}$, with a filtration F defined by means of the filtration index of the generators of each C_n,
- the integers p, q, and $r \geq 1$,
- a class $a \in E^r_{p,q}$, given by means of the coefficients $\{\lambda_1, \ldots, \lambda_t\}$ with respect to the basis $\{c_1, \ldots, c_t\}$ of the group $E^r_{p,q}$ determined by Algorithm 1.

Output: the coefficients of the class $d^r_{p,q}(a) \in E^r_{p-r,q+r-1}$ with respect to the basis $\{c'_1, \ldots, c'_{t'}\}$ computed by means of Algorithm 1.

2.3.1.3 Convergence level

As far as the chain complex C_* is effective, the filtration F is bounded and therefore the associated spectral sequence $E(C_*, F)$ is convergent. This implies that for each pair of integers (p, q) there exists $r = r(p, q) \geq 1$ such that $E^r_{p,q} = E^\infty_{p,q}$. Given a degree $n \in \mathbb{N}$ there exist $s = s(n) < t = t(n)$ such that $F_sC_n = 0$ and $F_tC_n = C_n$, and therefore for $r = r(n) = \max\{r(p, q) | \; s < p \leq t, q = n - p\}$, one has that every $E^r_{p,q}$ with $p + q = n$ satisfies

$$E^r_{p,q} = E^\infty_{p,q} \cong \frac{F_{H_p} H_n(C_*)}{F_{H_{p-1}} H_n(C_*)}$$

On the other hand, $H_n(C_*)$ is computable and it is a finitely generated Abelian group, of the form

$$H_n(C_*) \cong \mathbb{Z}^\alpha \oplus \mathbb{Z}_{\beta_1} \oplus \cdots \oplus \mathbb{Z}_{\beta_k}$$

for some $\alpha \geq 0$ and $\beta_i > 1$ such that β_i divides β_{i+1}.

The induced filtration of $H_n(C_*)$ is finite:

$$0 = F_{H_s} H_n(C_*) \subseteq F_{H_{s+1}} H_n(C_*) \subseteq \cdots \subseteq F_{H_t} H_n(C_*) = H_n(C_*)$$

Then it is not difficult to observe that the Betti number α must coincide with the sum of the Betti numbers of the groups $E^r_{p,q} \cong F_{H_p}(H_n(C_*))/F_{H_{p-1}}(H_n(C_*))$ (for $r = r(n)$, $s < p \leq t$, and $q = n - p$). Similarly, the product of all the torsion elements $\beta_1 \cdots \beta_k$ must be equal to the product of the torsion elements of all the groups $E^r_{p,q}$.

Reciprocally, if for some stage r the sum of the Betti numbers and the product of the torsion coefficients of the groups $E^r_{p,q}$ coincide with the Betti number and the torsion coefficients of the homology group $H_n(C_*)$, then for every $r' \geq r$ one has $E^{r'}_{p,q} = E^r_{p,q}$ for all $p+q = n$, and therefore $E^\infty_{p,q} = E^r_{p,q}$, so that the convergence of the spectral sequence has been reached.

Thus, given a degree n one can compute the Betti number and the torsion elements of $H_n(C_*)$ and compare them with those of the groups $E^1_{p,q}$ (for $s < p \leq t$ and $q = n-p$), $E^2_{p,q}$, and so on, until we reach a stage r for which they *coincide*. This will be the smallest r such that $E^r_{p,q} = E^\infty_{p,q}$ for all $p, q \in \mathbb{Z}$ with $p + q = n$, which is the *convergence level* of the spectral sequence for degree n.

Algorithm 3.
Input:

- an effective chain complex $C_* = (C_n, d_n)_{n\in\mathbb{N}}$, with a filtration F defined by means of the filtration index of the generators of each C_n,
- a degree $n \geq 0$.

Output: the smallest $r \geq 1$ such that $E^r_{p,q} = E^\infty_{p,q}$ for all $p, q \in \mathbb{Z}$ such that $p + q = n$.

2.3.1.4 Filtration of the homology groups

The filtration F of the chain complex $C_* = (C_n, d_n)_{n\in\mathbb{N}}$ induces a filtration F_H of the graded homology group $H_*(C_*)$ given by

$$F_{H_p}H_n(C_*) = \frac{F_p(\operatorname{Ker} d_n) \cup \operatorname{Im} d_{n+1}}{\operatorname{Im} d_{n+1}}$$

where $F_p(\operatorname{Ker} d_n) = \operatorname{Ker} d_n \cap F_pC_n$.

Given $n \in \mathbb{N}$, there exist $s = s(n) < t = t(n)$ such that $F_sC_n = 0$ and $F_tC_n = C_n$. Then we observe that $F_p(\operatorname{Ker} d_n) = Z^r_{p,n-p}$ for $r \geq p - s$, and therefore we can compute a basis of the subgroup $F_p(\operatorname{Ker} d_n)$ as explained in Section 2.3.1.1. A basis of $\operatorname{Im} d_{n+1}$ is given simply by the differential matrix D_{n+1}, and therefore an elementary algorithm provides us the basis-divisors representation of the quotient group defining $F_{H_p}H_n(C_*)$.

Algorithm 4.
Input:

- an effective chain complex $C_* = (C_n, d_n)_{n\in\mathbb{N}}$, with a filtration F given by the filtration index of the generators of each C_n,
- a degree $n \geq 0$ and a filtration index $p \in \mathbb{Z}$.

Output: a basis-divisors representation of the group $F_{H_p}H_n(C_*)$.

2.3.2 Locally effective chain complexes

Let us consider now a filtered chain complex (C_*, F_C), where $C_* = (C_n, d_{C_n})_{n\in\mathbb{N}}$ is not effective, so that for some $m \in \mathbb{N}$ the group of m-chains C_m can have an infinite number of generators. Therefore, some differential maps $d_{C_n} : C_n \to C_{n-1}$ cannot be represented as matrices and the process explained in Section 2.3.1 for computing the different components of the spectral sequence cannot be applied.

Let us suppose that C_* is a chain complex with effective homology

$$C_* \overset{\rho_1}{\Lleftarrow} D_* \overset{\rho_2}{\Rrightarrow} HC_*$$

where $\rho_1 = (f_1, g_1, h_1)$ and $\rho_2 = (f_2, g_2, h_2)$.

If the different components of the reductions ρ_1 and ρ_2 satisfy the hypotheses of Corollary 2.18 (that is to say, there exists $k \in \mathbb{Z}$ such that for each element $x \in D_n$ with $f_1(x) \in F_{C_p}C_n$, then $f_1 \circ g_2 \circ f_2(x) \in F_{C_p}C_n$ and $f_1 \circ h_2(x) \in F_{C_{p+k}}C_{n+1}$), then we can propagate the filtration F_C and define filtrations F_D and F_{HC} of the chain complexes D_* and HC_* respectively such that the spectral sequences $E(C_*, F_C)$ and $E(HC_*, F_{HC})$ are isomorphic after the stage k:

$$E(C_*, F_C)^r_{p,q} \cong E(HC_*, F_{HC})^r_{p,q} \quad \text{for all } r > k$$

In this way, (HC_*, F_{HC}) becomes an effective filtered chain complex an therefore the algorithms explained in Section 2.3.1 can be used for the computation of the associated spectral sequence $E(HC_*, F_{HC})$. Concretely, it is possible to determine the groups $E^r_{p,q}$ with the corresponding generators, the differential maps $d^r_{p,q}$, the convergence level for each degree $n \in \mathbb{N}$, and the filtration of the homology groups $H_*(HC_*)$ induced by the filtration F_{HC}.

Then, thanks to the isomorphisms between the spectral sequences $E(C_*, F_C)$ and $E(HC_*, F_{HC})$ (which are induced by the compositions $f_2 \circ g_1$ and $f_1 \circ g_2$), we can also compute every component of the spectral sequence associated with C_*.

Nevertheless, in many classical situations it is not necessary to transfer the filtration F_C of C_* to the chain complexes D_* and HC_*, since natural filtrations F_D and F_{HC} (different in general from those deduced from F_C) are also defined on these chain complexes. Furthermore, in most cases it is not difficult to prove that both reductions ρ_1 and ρ_2 in the effective homology of C_* have a good behavior with respect to the three filtrations F_C, F_D, and F_{HC}. Then, applying Corollary 2.15, one has that the spectral sequences associated with (C_*, F_C) and (HC_*, F_{HC}) will be isomorphic after some stage k (very frequently $k = 0$ or 1). In this way, it is possible to determine the spectral sequence $E(C_*, F_C)$ (groups and differential maps after the stage k, convergence level, filtration of the homology groups) by means of the spectral sequence $E(HC_*, F_{HC})$ which can be computed following the algorithms of Section 2.3.1.

The definition of the filtrations F_D and F_{HC} and their compatibility with the different components of the reductions must be studied in each particular situation. Several examples will be presented in Sections 2.4, 3.1, and 3.2.

2.4 Bicomplexes

A bicomplex is a particular case of filtered chain complex (or equivalently, a particular case of multicomplex). As we will see in this memoir, many useful spectral sequences arise from bicomplexes.

First of all, let us include here some basic definitions, which can be found in [Mac63].

Definition 2.19. A *bicomplex* (or *double complex*) $C_{*,*}$ is a bigraded free $\mathbb{Z}$-module $C_{*,*} = \{C_{p,q}\}_{p,q\in\mathbb{Z}}$ provided with morphisms $d'_{p,q} : C_{p,q} \to C_{p-1,q}$ and $d''_{p,q} : C_{p,q} \to C_{p,q-1}$ satisfying $d'_{p-1,q} \circ d'_{p,q} = 0$, $d''_{p,q-1} \circ d''_{p,q} = 0$, and $d'_{p,q-1} \circ d''_{p,q} + d''_{p-1,q} \circ d'_{p,q} = 0$.

We can picture this by means of a diagram of the form

$$
\begin{array}{ccccccc}
 & & \downarrow & & \downarrow & & \downarrow & & \\
\longleftarrow & & C_{p-1,q+1} & \xleftarrow{d'_{p,q+1}} & C_{p,q+1} & \xleftarrow{d'_{p+1,q+1}} & C_{p+1,q+1} & \longleftarrow \\
 & & \downarrow{\scriptstyle d''_{p-1,q+1}} & & \downarrow{\scriptstyle d''_{p,q+1}} & & \downarrow{\scriptstyle d''_{p+1,q+1}} & \\
\longleftarrow & & C_{p-1,q} & \xleftarrow{d'_{p,q}} & C_{p,q} & \xleftarrow{d'_{p+1,q}} & C_{p+1,q} & \longleftarrow \\
 & & \downarrow{\scriptstyle d''_{p-1,q}} & & \downarrow{\scriptstyle d''_{p,q}} & & \downarrow{\scriptstyle d''_{p+1,q}} & \\
\longleftarrow & & C_{p-1,q-1} & \xleftarrow{d'_{p,q-1}} & C_{p,q-1} & \xleftarrow{d'_{p+1,q-1}} & C_{p+1,q-1} & \longleftarrow \\
 & & \downarrow & & \downarrow & & \downarrow &
\end{array}
$$

We observe that it is a particular case of multicomplex with $d^0_{p,q} = d''_{p,q}$ (the *vertical differential map*), $d^1_{p,q} = d'_{p,q}$ (the *horizontal differential map*), and $d^i_{p,q} = 0$ for $i > 1$. Note that here both operators $d' : C_{*,q} \to C_{*-1,q}$ (for a fixed q) and $d'' : C_{p,*} \to C_{p,*-1}$ (for each p) are differential maps, so that both columns and rows are chain complexes.

The *total (chain) complex* $T_* = T_*(C_{*,*}) = (T_n, d_n)_{n\in\mathbb{Z}}$ is in this case the chain complex given by

$$T_n = \bigoplus_{p+q=n} C_{p,q}$$

and differential map $d_n(x) = d'_{p,q}(x) + d''_{p,q}(x)$ for $x \in C_{p,q}$.

Definition 2.20. The *n-homology group* of a bicomplex $C_{*,*} = \{C_{p,q}, d'_{p,q}, d''_{p,q}\}_{p,q\in\mathbb{Z}}$ is the n-homology group of the total complex $T_* = T_*(C_{*,*})$

$$H_n(C_{*,*}) = H_n(T_*)$$

We can also consider *approximations* of the homology of a bicomplex by means of the homology of the columns and the rows.

Definition 2.21. Given a bicomplex $C_{*,*} = \{C_{p,q}, d'_{p,q}, d''_{p,q}\}_{p,q\in\mathbb{Z}}$, the *second homology* $H''(C_{*,*})$ is defined with respect to d'' in the usual way as

$$H''_{p,q}(C_{*,*}) \equiv H''_{p,q} = \frac{\operatorname{Ker} d''_{p,q}}{\operatorname{Im} d''_{p,q+1}}$$

The equations $d' \circ d'' + d'' \circ d' = 0$ and $d' \circ d' = 0$ allows us to consider, for each $q \in \mathbb{Z}$, a differential $d' : H''_{*,q} \to H''_{*-1,q}$ induced by the original d'. We obtain in this way a (non-free) chain complex $H''_{*,q} = (H''_{p,q}, d'_{p,q})_{p\in\mathbb{Z}}$ and therefore it makes sense to construct the homology groups

$$H'_p H''_q(C_{*,*}) = H_p(H''_{*,q}) = \frac{\operatorname{Ker} d'_{p,q} : H''_{p,q} \to H''_{p-1,q}}{\operatorname{Im} d'_{p+1,q} : H''_{p+1,q} \to H''_{p,q}}$$

Analogously, we can define the *first homology* $H'(C_{*,*})$ given by

$$H'_{p,q}(C_{*,*}) \equiv H'_{p,q} = \frac{\operatorname{Ker} d'_{p,q}}{\operatorname{Im} d'_{p+1,q}}$$

and then

$$H''_q H'_p(C_{*,*}) = H_q(H'_{p,*}) = \frac{\operatorname{Ker} d''_{p,q} : H'_{p,q} \to H'_{p,q-1}}{\operatorname{Im} d''_{p,q+1} : H'_{p,q+1} \to H'_{p,q}}$$

Two different filtrations can be canonically associated with bicomplexes, producing two different spectral sequences. They are directly related with the iterated homologies $H''H'(C_{*,*})$ and $H'H''(C_{*,*})$. Details about the construction of these spectral sequences can be found in [Mac63].

Definition 2.22. Let $C_{*,*} = \{C_{p,q}, d'_{p,q}, d''_{p,q}\}_{p,q\in\mathbb{Z}}$ be a bicomplex and $T_* = T_*(C_{*,*})$ the total chain complex. The *first filtration* F' of T_* is defined by means of the column number p, that is to say,

$$F'_pT_n = \bigoplus_{h\leq p} C_{h,n-h}$$

The associated spectral sequence $E' = E(T_*, F')$ is called the *first spectral sequence* of the bicomplex $C_{*,*}$.

In a similar way, one can consider the *second filtration* F'' given by the row number,

$$F''_qT_n = \bigoplus_{h\leq q} C_{n-h,h}$$

and construct the *second spectral sequence* $E'' = E(T_*, F'')$.

Theorem 2.23. [Mac63] Let $C_{*,*} = \{C_{p,q}, d'_{p,q}, d''_{p,q}\}_{p,q\in\mathbb{Z}}$ be a bicomplex and $E' = (E'^r, d'^r)_{r\geq 1}$ the associated first spectral sequence. There are natural isomorphisms

$$E'^1_{p,q} \cong H''_{p,q}(C_{*,*}) \quad \text{and} \quad E'^2_{p,q} \cong H'_pH''_q(C_{*,*})$$

If $C_{p,q} = 0$ for $p < 0$, then $E' \Rightarrow H_*(C_{*,*}) = H_*(T_*)$.

Similarly, for the second spectral sequence $E'' = (E''^r, d''^r)_{r\geq 1}$ one has

$$E''^1_{p,q} \cong H'_{p,q}(C_{*,*}) \quad \text{and} \quad E''^2_{p,q} \cong H''_qH'_p(C_{*,*})$$

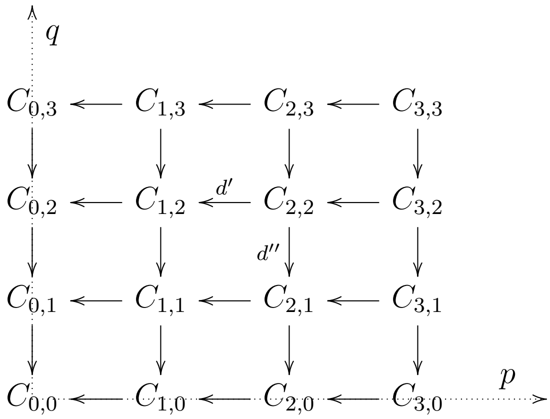
Moreover if $C_{p,q} = 0$ for $q < 0$, then $E'' \Rightarrow H_*(C_{*,*})$.

This theorem shows that the iterated homologies $H'H''(C_{*,*})$ and $H''H'(C_{*,*})$ approximate the homology of the bicomplex, $H_*(C_{*,*})$.

As in the general case, the spectral sequence of a bicomplex is directly computable when the total complex $T_*(C_{*,*})$ is effective. In this case, all the components of the spectral sequence can be determined applying the algorithms of Section 2.3.1. However, if the total complex is not effective, then its effective homology is necessary. In the next section we explain how the effective homology of a first quadrant bicomplex is computed when the effective homology of each column is known.

2.4.1 Effective homology of a bicomplex

Let us consider a first quadrant bicomplex $C_{*,*} = \{C_{p,q}, d'_{p,q}, d''_{p,q}\}_{p,q\in\mathbb{Z}}$, which satisfies $C_{p,q} = 0$ if $p < 0$ or $q < 0$.



The identity $d''_{p,q-1} \circ d''_{p,q} = 0$ implies that each column $C^p_* = (C_{p,q}, d''_{p,q})_{q\in\mathbb{N}}$ (p fixed) is a chain complex, so it makes sense to look for the relation between the homologies of the columns C^p_* and that of the bicomplex (which is the homology of the total complex $T_* = T_*(C_{*,*})$).

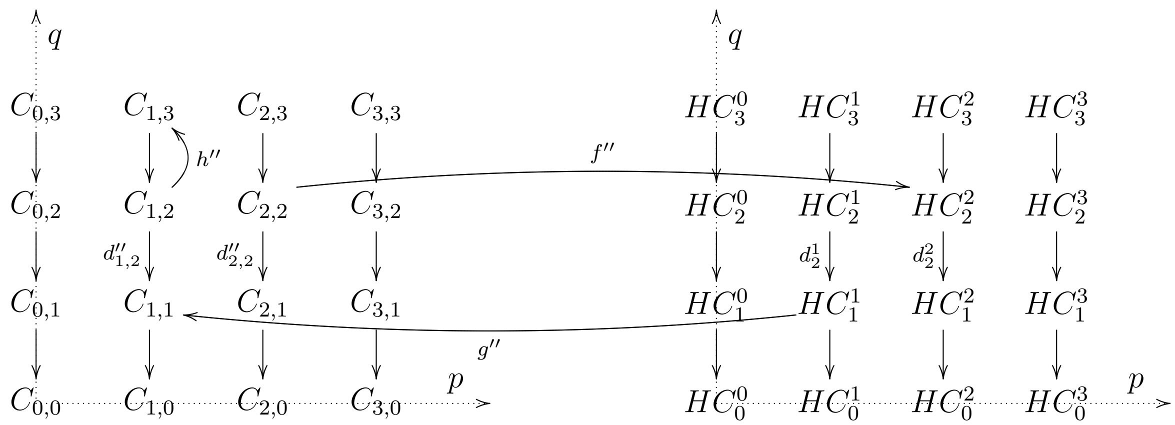
Let us suppose that the columns C^p_* are objects with effective homology, such that there exist reductions $\rho^p = (f^p, g^p, h^p) : C^p_* \Rrightarrow HC^p_*$ where $HC^p_* = (HC^p_q, d^p_q)_{q\in\mathbb{N}}$ is an effective chain complex for all $p \in \mathbb{N}$ (the left reduction in the effective homology of each column is trivial). Then we are going to construct a new effective chain complex HC_* which provides us the effective homology of the total complex T_*. This is one of the typical examples of application of the Basic Perturbation Lemma, which was introduced in Section 1.3.2.

As a first step, we build a chain complex $T''_* = (T''_n, d''_n)_{n\in\mathbb{N}}$, total complex of the bicomplex $C''_{*,*} = \{C_{p,q}, 0, d''_{p,q}\}_{p,q\in\mathbb{N}}$, where only the vertical arrows $d''_{p,q}$ are considered. This means that T''_* is given by

$$T''_n = \bigoplus_{p+q=n} C_{p,q}$$

with differential map $d''_n(x) = d''_{p,q}(x)$ for $x \in C_{p,q}$.

Using the reductions $\rho^p = (f^p, g^p, h^p) : C^p_* \Rrightarrow HC^p_*$, it is not hard to construct a reduction $\rho'' = (f'', g'', h'')$ from T''_* over the total complex $T_*(HC''_{*,*})$ of a new bicomplex $HC''_{*,*} = \{HC^p_q, 0, d^p_q\}_{p,q\in\mathbb{N}}$. This new bicomplex $HC''_{*,*}$ has in the p-column the chain complex HC^p_*, the vertical arrows are given by the differential maps of each HC^p_*, and the horizontal arrows are null. The three components f'', g'', and h'' of the reduction ρ'' coincide with the corresponding maps f^p, g^p, and h^p of each column. In other words, given x a generator of $T_n = T_n(C''_{*,*})$ such that $x \in C_{p,q}$, and y a generator of $T_n(HC''_{*,*})$ with $y \in HC''_{p,q} = HC^p_q$, then $f''(x) = f^p(x) \in HC''_{p,q} = HC^p_q$, $g''(y) = g^p(y) \in C_{p,q}$, and $h''(x) = h^p(x) \in C_{p,q+1}$.



It is clear that for each degree n the component $T_n(HC''_{*,*}) = \bigoplus_{p+q=n} HC^p_q$ is a (finite) sum of finite type groups, so that the chain complex $T_*(HC''_{*,*})$ is an effective chain complex.

The reduction $\rho'' : T''_* \Rightarrow\!\!\!\!\!\Rightarrow T_*(HC''_{*,*})$ is the first ingredient for the application of the Basic Perturbation Lemma. We also need a perturbation δ of the differential map d'', which is defined by means of the horizontal arrows, the maps $d'_{p,q}$. In this way we obtain the initial total complex $T_* = T_*(C_{*,*})$, where now all the arrows are considered.

The homotopy operator h'' in the reduction ρ'' maps an element $x \in C_{p,q}$ to an element $h''(x) \in C_{p,q+1}$, while the perturbation $\delta = d'$ decreases by one unit the first degree p. Therefore it is not difficult to see that the composition $h'' \circ \delta$ is locally nilpotent, so that the conditions of the BPL are satisfied. A reduction $\rho = (f, g, h) : T_* \Rightarrow\!\!\!\!\!\Rightarrow T_*(HC_{*,*})$ is deduced, where the bicomplex $HC_{*,*}$ is obtained from $HC''_{*,*}$ by replacing the initial differential map d^p_q (with only vertical arrows) by a perturbed differential $d^p_q + \delta_{HC}$.

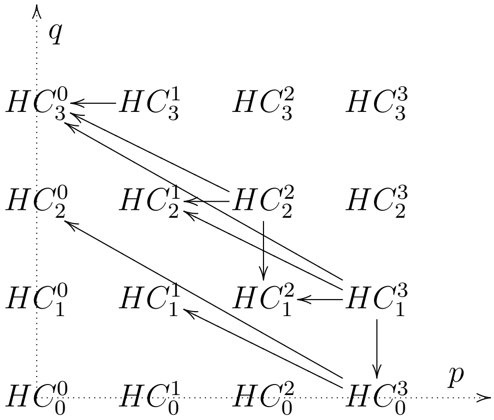
The perturbation δ_{HC} is defined as

$$\delta_{HC} = f'' \circ d' \circ \phi \circ g'' = f'' \circ \psi \circ d' \circ g''$$

where ϕ and ψ are given by the series

$$\phi = \sum_{i=0}^{\infty} (-1)^i (h'' \circ d')^i$$
$$\psi = \sum_{i=0}^{\infty} (-1)^i (d' \circ h'')^i = \mathrm{Id}_{T''_*} - d' \circ \phi \circ h''$$

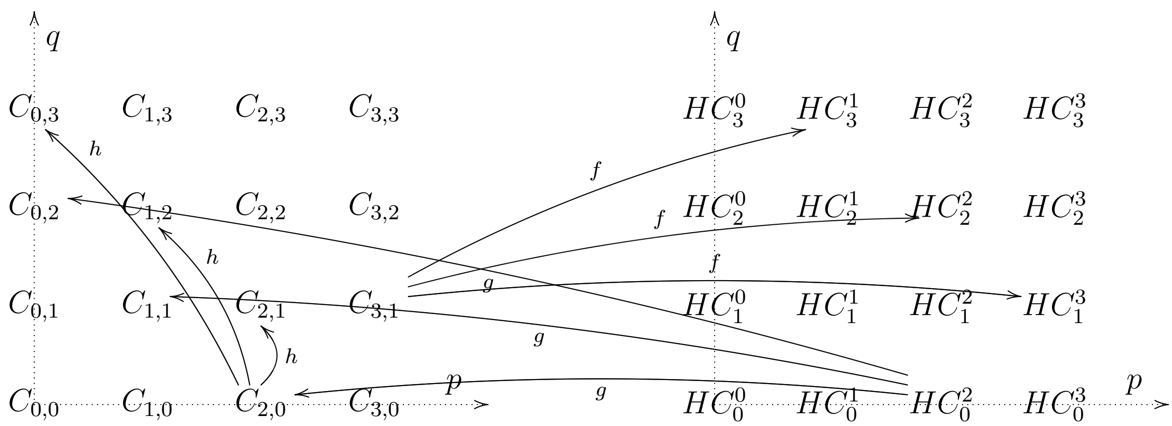
This adds new arrows of different shifts to the bicomplex $HC''_{*,*}$, which produces a multicomplex $HC_{*,*}$. In the following figure only some arrows are included:



The components of the reduction $\rho = (f, g, h)$ are

$$\begin{aligned} f &= f'' \circ \psi = f'' \circ (\mathrm{Id}_{T''_*} - d' \circ \phi \circ h''), \\ g &= \phi \circ g'', \\ h &= \phi \circ h'' = h'' \circ \psi \end{aligned}$$

Given $x \in C_{p,q}$, then $f(x) \in HC_q^p \oplus HC_{q+1}^{p-1} \oplus \cdots \oplus HC_{p+q}^0$. Similarly, if $y \in HC_q^p$, then $g(y) \in C_{p,q} \oplus C_{p-1,q+1} \oplus \cdots \oplus C_{0,p+q}$. Finally, $h(x) \in C_{p,q+1} \oplus C_{p-1,q+2} \oplus \cdots \oplus C_{0,p+q+1}$. In the following figure, we represent some arrows which take part of the maps f, g, and h. The differential maps of the bicomplex $C_{*,*}$ and the multicomplex $HC_{*,*}$ are not drawn.



It is clear that the multicomplex $HC_{*,*}$ is an effective chain complex, and therefore the reduction $\rho = (f, g, h) : T_* \Rrightarrow T_*(HC_{*,*})$ provides us the searched effective homology of $T_* = T_*(C_{*,*})$.

The method explained in this section can also be applied for the computation of the effective homology of a first quadrant multicomplex $C_{*,*} = \{C_{p,q}, d^i_{p,q}\}_{p,q \in \mathbb{N}, i \geq 0}$ when the columns $C_*^p = (C_{p,q}, d^0_{p,q})_{q \in \mathbb{N}}$ are objects with effective homology. Again, we consider first the multicomplex where the only non-null arrows are the vertical maps d^0, and then the perturbation is given by the sum $\sum_{i\geq 1} d^i$. Using the BPL as before it is possible to construct a new effective chain complex HC_* and a reduction $T_* = T_*(C_{*,*}) \Rrightarrow HC_*$, obtaining in this way the effective homology of the multicomplex $C_{*,*}$.

We can consider also the case in which for each column C^p_*, instead of a reduction $\rho^p : C^p_* \Rrightarrow HC^p_*$, we have an equivalence

$$C^p_* \overset{\rho^p_1}{\Lleftarrow} D^p_* \overset{\rho^p_2}{\Rrightarrow} HC^p_*$$

where each HC^p_* is an effective chain complex. Then the effective homology of the total complex $T_* = T_*(C_{*,*})$ is given by an equivalence

$$T_* \overset{\rho_1}{\Lleftarrow} D_* \overset{\rho_2}{\Rrightarrow} HC_*$$

where D_* and HC_* are the total complexes of a bicomplex and a multicomplex obtained by application of the Trivial Perturbation Lemma and the Basic Perturbation Lemma respectively, following the same method as explained before.

The condition of $C_{*,*}$ being a first quadrant bicomplex (or multicomplex) guarantees the local nilpotency which is necessary for the application of the BPL. In fact the result holds under weaker conditions than first quadrant, it is sufficient that $C_{*,*}$ is a bicomplex associated with a bounded filtration, that is to say, for every $n \in \mathbb{Z}$ there exist $s = s(n) < t = t(n)$ such that $C_{p,q} = 0$ if $p < s$ or $p > t$. We will see an interesting example of a bounded second quadrant bicomplex in Chapter 3, concretely in Section 3.2.

We have seen therefore that the effective homology of the total complex $T_* = T_*(C_{*,*})$ of a first quadrant bicomplex can be computed when the columns are objects with effective homology. In particular, the effective homology of T_* allows us to compute the homology groups of the bicomplex. Furthermore, it can be useful to determine the associated spectral sequence, as we explain in the next section.

2.4.2 Spectral sequence of a bicomplex

Let $C_{*,*} = \{C_{p,q}, d'_{p,q}, d''_{p,q}\}_{p,q\in\mathbb{N}}$ be a first quadrant bicomplex and $T_* = (T_n, d_n)_{n\in\mathbb{N}}$ its total complex. We consider the first spectral sequence $E' = (E'^r, d'^r)_{r\geq 1}$, which is the spectral sequence determined by the filtration F' of T_* given by the first degree (the column number), in other words,

$$F'_p T_n = \bigoplus_{h\leq p} C_{h,n-h}$$

If each group $C_{p,q}$ is of finite type for every $p, q \in \mathbb{N}$, then the total complex T_* is effective (as far as $C_{*,*}$ is a first quadrant bicomplex, T_n has a finite number of factors

$C_{p,q}$ with $p+q=n$), and therefore the different components of the spectral sequence can be computed by means of elementary operations as explained in Section 2.3.1. On the other hand, if for some pair (p,q) the group $C_{p,q}$ is not of finite type, then T_* is not an effective chain complex and therefore the associated spectral sequence E' cannot directly be determined, the effective homology is necessary.

We have seen in Section 2.4.1 that, given reductions $\rho^p = (f^p, g^p, h^p) : C^p_* \Rightarrow\!\!\!\!\Rightarrow HC^p_*$ for each column $p \geq 0$, where $HC^p_* = (HC^p_q, d^p_q)_{q\in\mathbb{N}}$ is an effective chain complex, it is possible to construct a reduction $\rho = (f,g,h) : T_* \Rightarrow\!\!\!\!\Rightarrow T_*(HC_{*,*})$, $HC_{*,*}$ being also an effective chain complex (concretely, an effective multicomplex).

In particular, the multicomplex $HC_{*,*}$ can be considered as a filtered chain complex, where the filtration F_{HC} of the total chain complex $T_*(HC_{*,*})$ is given again by the first degree (the column number). This produces a spectral sequence $E(T_*(HC_{*,*}), F_{HC})$ which converges to the homology groups of $T_*(HC_{*,*})$. Since $T_*(HC_{*,*})$ is effective, this spectral sequence can directly be computed as explained in Section 2.3.1.

Furthermore, we recall that given two elements $x \in C_{p,q}$ and $y \in HC^p_q$, then $f(x) \in HC^p_q \oplus HC^{p-1}_{q+1} \oplus \cdots \oplus HC^0_{p+q}$, $g(y) \in C_{p,q} \oplus C_{p-1,q+1} \oplus \cdots \oplus C_{0,p+q}$, and $h(x) \in C_{p,q+1} \oplus C_{p-1,q+2} \oplus \cdots \oplus C_{0,p+q+1}$. Thus, the three components f, g, and h of the reduction ρ are compatible with the filtrations of $T_* = T_*(C_{*,*})$ and $T_*(HC_{*,*})$ (both of them defined by means of the column number), and therefore from Theorem 2.14 we deduce that both spectral sequences $E' = E(T_*, F')$ and $E(T_*(HC_{*,*}), F_{HC})$ associated with $C_{*,*}$ and $HC_{*,*}$ are isomorphic for all the stages $r \geq 1$, in other words, for every level. The isomorphisms are induced by the maps f and g.

We can compute in this way all the components of the first spectral sequence E' of the bicomplex $C_{*,*}$ by means of the spectral sequence associated with the multicomplex $HC_{*,*}$ (which is easily calculable) and the morphisms f and g.

Algorithm 5.
Input:

- a first quadrant bicomplex $C_{*,*} = \{C_{p,q}, d'_{p,q}, d''_{p,q}\}_{p,q\in\mathbb{N}}$,
- reductions $\rho^p = (f^p, g^p, h^p) : C^p_* \Rightarrow\!\!\!\!\Rightarrow HC^p_*$ for each column $p \geq 0$, where $HC^p_* = (HC^p_q, d^p_q)_{q\in\mathbb{N}}$ is an effective chain complex.

Output: every component of the first spectral sequence $E' = E(T_*(C_{*,*}), F')$:

- the groups $E'^r_{p,q}$ for every $p,q \in \mathbb{Z}$ and $r \geq 1$, with a basis-divisors description,
- the differential maps $d^r_{p,q}$ for all $p,q \in \mathbb{Z}$ and $r \geq 1$,
- the convergence level for each degree $n \in \mathbb{N}$,
- the filtration of the homology groups $H_*(C_{*,*}) = H_*(T_*)$, that is, the groups $F_{H_p}H_n(C_{*,*})$ for each degree $n \in \mathbb{N}$ and filtration index $p \in \mathbb{Z}$.

The same algorithm is also valid when $C_{*,*}$ is a bounded bicomplex. If $C_{*,*}$ is not bounded it is also possible to compute the groups $E^r_{p,q}$ (which only depend on the columns

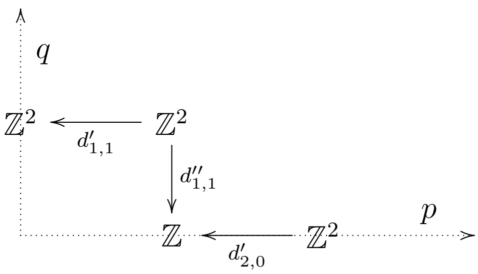
$p-r+1,\ldots,p+r-1$) and the differential maps $d^r_{p,q}$ for every $p,q,r \in \mathbb{Z}$, although in this case one cannot always determine the final groups $E^\infty_{p,q}$.

2.4.3 Examples

In this section we consider two examples of bicomplexes, with their associated spectral sequences. First of all, we introduce a didactic example with only a few non-null (finite type) groups, so that its spectral sequence can easily be computed by hand. The second example is the Bar construction of an algebra, which in some cases is a locally effective (not effective) bicomplex, and therefore in general it is not possible to determine directly its spectral sequence.

2.4.3.1 A didactic example

Let us consider the following first quadrant bicomplex $C_{*,*}$ (only the non-null groups have been drawn), where the differential maps $d'_{1,1}$, $d'_{2,0}$, and $d''_{1,1}$ are given by the matrices $D'_{1,1} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, $D'_{2,0} = \begin{pmatrix} 2 & 0 \end{pmatrix}$, and $D''_{1,1} = \begin{pmatrix} 1 & 0 \end{pmatrix}$.



We choose a set of generators of each group, for instance $C_{0,1} = \mathbb{Z}[a_1, a_2]$, $C_{1,0} = \mathbb{Z}[b]$, $C_{1,1} = \mathbb{Z}[c_1, c_2]$, and $C_{2,0} = \mathbb{Z}[d_1, d_2]$. Then, the first spectral sequence of the bicomplex can easily be computed.

The different groups $E^r_{p,q}$ are given by the formula

$$E^r_{p,q} = \frac{Z^r_{p,q} \cup F'_{p-1}T_{p+q}}{d_{p+q+1}(Z^{r-1}_{p+r-1,q-r+2}) \cup F'_{p-1}T_{p+q}}$$

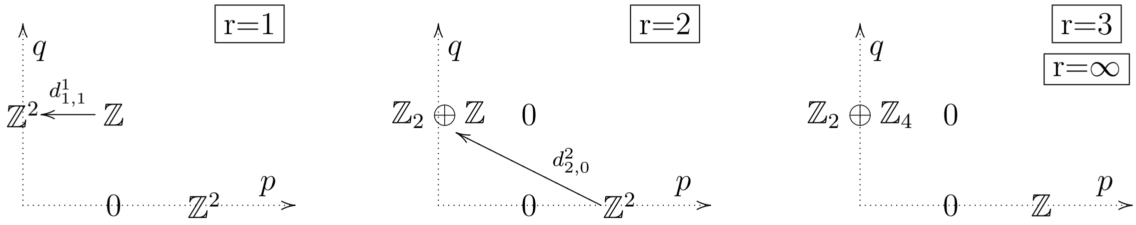
where $T_* = (T_n, d_n)_{n\in\mathbb{N}}$ is the total complex of $C_{*,*}$ and F' is the first filtration of T_* defined by the column number.

At level $r = 1$ one has $E^1_{0,1} = \mathbb{Z}[a_1, a_2] \cong \mathbb{Z}^2$, $E^1_{1,0} = \mathbb{Z}[a_1, a_2, b]/\mathbb{Z}[a_1, a_2, b] = 0$, $E^1_{1,1} = \mathbb{Z}[c_2] \cong \mathbb{Z}$, and $E^1_{2,0} = \mathbb{Z}[c_1, c_2, d_1, d_2]/\mathbb{Z}[c_1, c_2] \cong \mathbb{Z}^2$. The rest of the groups $E^1_{p,q}$ are necessarily equal to zero. We have therefore only one possibly non-null differential map, $d^1_{1,1} : E^1_{1,1} \cong \mathbb{Z} \to E^1_{0,1} \cong \mathbb{Z}^2$, and it is not difficult to see that it is given by the matrix $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$.

For $r = 2$, one has the groups $E^2_{0,1} \cong E^1_{0,1}/\operatorname{Im} d^1_{1,1} = \mathbb{Z}[a_1, a_2]/\mathbb{Z}[2a_2] \cong \mathbb{Z}_2 \oplus \mathbb{Z}$, $E^2_{1,0} = E^1_{1,0} = 0$, $E^2_{1,1} \cong \operatorname{Ker} d^1_{1,1} = 0$, and $E^2_{2,0} \cong E^1_{2,0} \cong \mathbb{Z}[d_1, d_2] \cong \mathbb{Z}^2$. The differential map $d^2_{2,0} : E^2_{2,0} \cong \mathbb{Z}^2 \to E^2_{0,1} \cong \mathbb{Z}_2 \oplus \mathbb{Z}$ can be expressed by means of the matrix $\begin{pmatrix} 0 & 0 \\ -4 & 0 \end{pmatrix}$.

The only two groups that remain in stage $r = 3$ are $E^3_{0,1} \cong E^2_{0,1}/\operatorname{Im} d^2_{2,0} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_4$ and $E^3_{2,0} \cong \operatorname{Ker} d^2_{2,0} \cong \mathbb{Z}$. Since all the maps $d^r_{p,q}$ are necessarily null for $r \geq 3$, the groups $E^3_{p,q}$ coincide with the corresponding $E^\infty_{p,q}$, and therefore the convergence level of the spectral sequence is less than or equal to 3 for every degree n.

We can represent the *whole* spectral sequence by means of the three following diagrams:



On account of the isomorphisms $E^\infty_{p,q} \cong F'_{H_p}(H_{p+q}(T_*))/F'_{H_{p-1}}(H_{p+q}(T_*))$, the filtration of the homology groups $H_* = H_*(C_{*,*})$ is necessarily given by

$$F'_{-1}H_1 = 0 \subset F'_0 H_1 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_4 \cong H_1(C_{*,*})$$
$$F'_1 H_2 = 0 \subset F'_2 H_2 \cong \mathbb{Z} \cong H_2(C_{*,*})$$
$$0 \cong H_n(C_{*,*}) \text{ for all } n \neq 1, 2$$

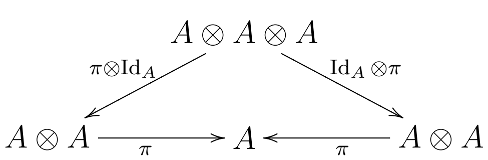
In this particular case the spectral sequence determines completely the homology groups of the filtered complex. Nevertheless this is not the general situation, in Chapter 3 we will see some examples where, for the computation of the looked-for homology groups, we must deal with an extension problem with several possible solutions.

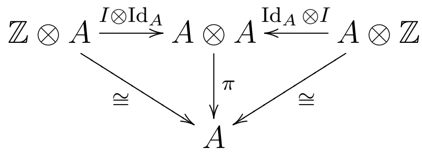
The next section includes an example of bicomplex which is not effective. The effective homology is then necessary to determine the different components of the associated spectral sequence.

2.4.3.2 The Bar construction

The Bar construction of an algebra is the algebraic version of the classifying space of a simplicial group and it provides an example of bicomplex which is used, for instance, in the computation of the effective homology of Eilenberg-MacLane spaces. The definitions that follow can be found in [Mac63].

Definition 2.24. A $\mathbb{Z}$-*algebra* A is a $\mathbb{Z}$-module with two morphisms, an associative *product* $\pi : A \otimes A \to A$ and a *unit* $I : \mathbb{Z} \to A$, such that the following diagrams are commutative:





Definition 2.25. A *differential (graded) algebra* is a chain complex $A_* = (A_n, d_n)_{n\in\mathbb{N}}$ together with a $\mathbb{Z}$-module morphism $I : \mathbb{Z} \to A_0$ (the *coaugmentation*) and a chain complex morphism $\pi : A_* \otimes A_* \to A_*$, such that they satisfy the commutativity properties of an algebra.

Definition 2.26. Let $A_* = (A_n, d_n)_{n\in\mathbb{N}}$ be a differential graded algebra with $A_0 \cong \mathbb{Z}$. Then it is possible to define a chain complex, $\text{Bar}(A_*)_*$, whose n-component $\text{Bar}(A_*)_n$ is the free $\mathbb{Z}$-module generated by the elements of the form $[g_1|g_2|\dots|g_k]$ such that $g_j \in C_{n_j}$ and $\sum_{j=1}^{k}(n_j+1) = n$, and whose differential map is given as sum of two components, the *tensorial differential* d_t and the *simplicial differential* d_s defined as:

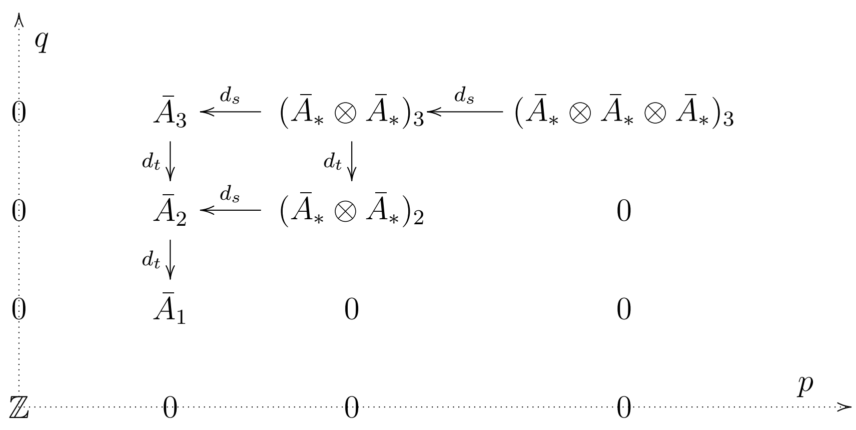
$$d_t([g_1|\dots|g_k]) = -\sum_{i=1}^{k}(-1)^{\alpha_i}[g_1|\dots|g_{i-1}|d_{n_i}(g_i)|g_{i+1}|\dots|g_n]$$

$$d_s([g_1|\dots|g_k]) = \sum_{i=2}^{k}(-1)^{\alpha_i}[g_1|\dots|g_{i-2}|\pi(g_{i-1}\otimes g_i)|g_{i+1}|\dots|g_k]$$

where $d_{n_i}(g_i)$ is the differential of a generator $g_i \in C_{n_i}$ in the original chain complex A_*, and $\alpha_i = \sum_{j=1}^{i-1} n_j$.

The object $[g_1|g_2|\dots|g_k]$ with $\sum_{j=1}^{k}(n_j+1) = n$ is called a *bar*, and it is an n-chain of the k-th iterated suspension of the tensor product $A \otimes \overset{k}{\cdots} \otimes A$. The integer n is the *total degree*, k is the *simplicial degree*, while $n-k$ is the *tensorial degree*.

The chain complex $\text{Bar}(A_*)_*$ can be represented as the following first quadrant bicomplex, where $\bar{A}_*$ is equal to A_* without its component of degree 0. The vertical differential is given by the component d_t, and d_s is the horizontal differential.



This first quadrant bicomplex produces a spectral sequence that converges to the homology groups of $\mathrm{Bar}(A_*)_*$. If A_* is not an effective chain complex, neither is $\mathrm{Bar}(A_*)_*$, and therefore the effective homology is necessary for the computation of the associated spectral sequence.

Let us suppose that A_* is an object with effective homology such that there exists an equivalence $\varepsilon : A_* \Lleftarrow D_* \Rrightarrow HA_*$, where HA_* is an effective chain complex. Then it is not difficult to determine the effective homology of each column of the bicomplex $\mathrm{Bar}(A_*)_*$, which is given by the iterated tensor product of the equivalence ε. Once we have the effective homology of the columns we can apply the process explained in Section 2.4.1 for computing the effective homology of the total complex of a bicomplex, which in our case is the chain complex $\mathrm{Bar}(A_*)_*$.

Using this effective homology, we can determine the homology groups of $\mathrm{Bar}(A_*)_*$ and also the spectral sequence associated with this bicomplex, as explained in Section 2.4.2, even if A_* is not an effective chain complex.

In Section 2.5.2.2 we will show an example of the computation of the effective homology of $\mathrm{Bar}(A_*)_*$ and the corresponding spectral sequence for the case $A_* = C_*(K(\mathbb{Z}, 2))$.

2.5 Implementation

The algorithms explained in Section 2.3 have been implemented as a new module for the Kenzo system. The set of programs we have developed (with about 2500 lines) allows computations of spectral sequences of filtered complexes, when the effective homology of this complex is available. The programs determine not only the groups $E^r_{p,q}$, but also the differential maps $d^r_{p,q}$ of the spectral sequence, as well as the level r on which the convergence has been reached for each degree n, and the filtration of the homology groups induced by the filtration of the chain complex.

In Section 2.5.1 we explain the essential part of these programs, describing the functions with the same format as in the Kenzo documentation [DRSS99]. In Section 2.5.2 we will see some examples of calculations.

2.5.1 A new module for the Kenzo system

In the development of the new module for Kenzo that allows one to compute spectral sequences associated with filtered complexes, the first step has been to enhance the class system of Kenzo with the class `FILTERED-CHAIN-COMPLEX`, whose definition is:

```
(DEFCLASS FILTERED-CHAIN-COMPLEX (chain-complex)
    ((flin :type chcm-flin :initarg :flin :reader flin1)))
```

This class inherits from the class `CHAIN-COMPLEX`, and has one slot of its own:

`flin` (FiLtration INdex function) a Lisp function that, from a degree n and a generator $g \in C_n$, returns the filtration index $\text{Flin}(g) = \min\{p \in \mathbb{Z} | \; g \in F_pC_n\}$.

We have also written several functions that allow us to construct filtered complexes and to obtain some useful information about them (when they are finitely generated in each degree). The description of some of these methods is shown here:

`build-flcc :cmpr` *cmpr* `:basis` *basis* `:bsgn` *bsgn* `:intr-dffr` *intr-dffr* `:dffr-strt` *dffr-strt* `:flin` *flin* `:orgn` *orgn*
The returned value is an instance of type `FILTERED-CHAIN-COMPLEX`. The keyword arguments are similar to those of the function `build-chcm` (that constructs a chain complex), with the new argument `flin` which is the filtration index function.

`change-chcm-to-flcc` *chcm flin flin-orgn*
This method builds a `FILTERED-CHAIN-COMPLEX` instance from an already created chain complex *chcm*. The user must introduce the filtration index function, *flin*, and *flin-orgn*, a list explaining the *origin* of the filtration, that will be used to define the origin of the filtered chain complex.

`fltrd-basis` *flcc n p*
Returns the elements of the basis of F_pC_n, where C_* is the *effective* filtered chain complex *flcc*.

`flcc-dffr-mtrx` *flcc n p*
Matrix of the differential map of degree n of the subcomplex F_pC_*, where $C_* =$ *flcc* is an effective chain complex.

The core of this new module consists in several functions that construct the different elements of the spectral sequence associated with a filtered complex (groups, differential maps, convergence level, and filtration of the homology groups), implementing the algorithms presented in Section 2.3. The main functions are:

`spsq-group` *flcc r p q*
Displays on the screen the components ($\mathbb{Z}$ or $\mathbb{Z}_m$) of the group $E^r_{p,q}$ of the spectral sequence of the filtered chain complex *flcc*.

`spsq-basis-dvs` *flcc r p q*
Returns a basis-divisors description of the group $E^r_{p,q}$, with a list of combinations which generate it and a list of non-negative integers corresponding to the Betti number and the torsion coefficients of the group.

`spsq-dffr` *flcc r p q int-list*
Computes the differential map $d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}$. The list *int-list* specifies the coefficients of the element we want to apply the differential to with respect to the generators of the group $E^r_{p,q}$ (the role of *int-list* can be better understood by means of the examples of Section 2.5.2).

`spsq-cnvg` *flcc n*
Determines the level r at which the convergence of the spectral sequence has been reached for a specific degree n.

`hmlg-fltr` *flcc n p*
Computes the group $F_{H_p}H_n(C_*)$, with $C_* =$ *flcc* an *effective* filtered chain complex.

To provide a better understanding of these new tools, some elementary examples of their use are shown in the next section. Furthermore, in Chapter 3 we will present more interesting examples where the application of the programs allows the computation of some groups and differential maps which are beyond the calculations appearing in the literature.

2.5.2 Examples

In this section we present some examples of application of the programs we have developed for computing spectral sequences associated with filtered complexes. First, we consider as a didactic example the effective bicomplex introduced in Section 2.4.3.1, computing in a detailed way all the components of the associated spectral sequence. As a second example, we will show the computation of some elements of the spectral sequence associated with the Bar construction of the differential graded algebra $A_* = C_*(K(\mathbb{Z}, 2))$.

2.5.2.1 Bicomplexes

Bicomplexes are a particular case of filtered complexes and therefore they can be implemented using the general function `build-flcc` presented in Section 2.5.1. However, we have also developed a set of programs that allow us to work with bicomplexes in an easier way. The most important functions are:

`build-bicm :bcbasis` *bcbasis* `:dffr1` *dffr1* `:dffr2` *dffr2* `:cmpr` *cmpr*
`:orgn` *orgn*
The returned value is an instance of type `CHAIN-COMPLEX`. The keyword arguments are: *cmpr*, the comparison function for generators; *bcbasis*, the bigraded basis of the bicomplex (it can be a function of two integer parameters defining the distinguished basis, or the keyword `:locally-effective`); *dffr1*, a Lisp function describing the horizontal differential morphism; *dffr2*, a Lisp function defining the vertical differential map; and *orgn*, the origin description of the bicomplex.

`change-bicm-to-flcc` *bicm*
This method builds a `FILTERED-CHAIN-COMPLEX` instance from the already created bicomplex *bicm*.

Let us construct the bicomplex introduced in Section 2.4.3.1. It can be built by means of the following instructions.

```
> (defun bcbasis (degr1 degr2)
   (if (and (= degr1 0) (= degr2 1))    '(a1 a2)
      (if (and (= degr1 1) (= degr2 0))    '(b)
         (if (and (= degr1 1) (= degr2 1))    '(c1 c2)
              (if (and (= degr1 2) (= degr2 0))    '(d1 d2)
                 '( )
                 )))))
bcbasis
> (defun dif1 (degr1 degr2 gnrt)
   (if (and (= degr1 1) (= degr2 1) (eql gnrt 'c1))    (list (cons 2 'a1))
      (if (and (= degr1 1) (= degr2 1) (eql gnrt 'c2))    (list (cons 2 'a2))
         (if (and (= degr1 2) (= degr2 0) (eql gnrt 'd1))    (list (cons 2 'b))))))
dif1
> (defun dif2 (degr1 degr2 gnrt)
   (if (and (= degr1 1) (= degr2 1) (eql gnrt 'c1))    (list (cons 1 'b))))
dif2
> (setf bc (build-bicm :cmpr 's-cmpr :bcbasis #'bcbasis :dffr1 #'dif1 :dffr2 #'dif2
         :orgn '(Bicomplex1)))
[K23 Chain-Complex]
```

We observe that the result is a chain complex. We can ask for its basis, for instance in dimension 1 we obtain a list with the three generators, a_1, a_2, and b, each of them identified with the corresponding bidegree:

```
> (basis bc 1)
(<BcGnrt [0 1] a1> <BcGnrt [0 1] a2> <BcGnrt [1 0] b>)
```

Then, we turn `bc` into a filtered complex.

```
> (change-bicm-to-flcc bc)
[K23 Filtered-Chain-Complex]
```

Once we have a filtered chain complex, the new module for spectral sequences of filtered complexes can be used to compute the different components of the spectral sequence associated with the bicomplex `bc`. First of all, let us determine some groups:

```
> (spsq-group bc 1 0 1)
Spectral sequence E^1_{0,1}
Component Z
Component Z
> (spsq-group bc 1 1 0)
Spectral sequence E^1_{1,0}
nil
> (spsq-group bc 3 0 1)
Spectral sequence E^3_{0,1}
Component Z/2Z
Component Z/4Z
```

The corresponding generators can also be determined:

```
> (spsq-basis-dvs bc 1 0 1)
((
----------------------------------------------------------------------{CMBN 1}
<1 * <BcGnrt [0 1] a1>>
------------------------------------------------------------------------------

----------------------------------------------------------------------{CMBN 1}
<1 * <BcGnrt [0 1] a2>>
------------------------------------------------------------------------------
) (0 0))
> (spsq-basis-dvs bc 1 1 1)
((
----------------------------------------------------------------------{CMBN 2}
<1 * <BcGnrt [1 1] c2>>
------------------------------------------------------------------------------
) (0))
> (spsq-basis-dvs bc 3 0 1)
((
----------------------------------------------------------------------{CMBN 1}
<1 * <BcGnrt [0 1] a2>>
------------------------------------------------------------------------------

----------------------------------------------------------------------{CMBN 1}
<-1 * <BcGnrt [0 1] a1>>
------------------------------------------------------------------------------
) (2 4))
```

In the first case, $E^1_{0,1}$, we obtain a list of two combinations, $1 * a_1$ and $1 * a_2$. The list of divisors is (0 0) because the group is free, isomorphic to $\mathbb{Z}^2$. The second group, $E^1_{1,1} \cong \mathbb{Z}$, has only one generator, the combination $1 * c_2$, and the corresponding divisor is again 0. For $E^3_{0,1}$ we have the same generators as for $E^1_{0,1}$, $1 * a_1$ and $1 * a_2$, but now both of them are torsion elements. The combination $1 * a_1$ has finite order equal to 4, while $1 * a_2$ has order 2.

The differential maps $d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}$ can be computed using the function `spsq-dffr`. The last argument must be a list including the coordinates of the element we want to apply the differential to, with regard to the generators of the group. In the case of $d^1_{1,1} : E^1_{1,1} \cong \mathbb{Z} \to E^1_{0,1} \cong \mathbb{Z}^2$ we must give a list with one element. For instance, the differential map applied to the generator $1 * c_2$ of $E^1_{1,1}$ is determined by

```
> (spsq-dffr bc 1 1 1 '(1))
(0 2)
```

The result is the list (0 2), which corresponds to the second generator of the group $E^1_{0,1}$ multiplied by 2, that is, the combination $2 * a_2$.

Similarly, we can determine the differential map $d^2_{2,0} : E^2_{2,0} \cong \mathbb{Z}^2 \to E^2_{0,1} \cong \mathbb{Z}_2 \oplus \mathbb{Z}$. In this case, to apply the function `spsq-dffr`, we must specify a list with two coefficients:

the list (1 0) corresponds to the first generator, $1 * d_1$, and the list (0 1) to the second one, $1 * d_2$.

```
> (spsq-dffr bc 2 2 0 '(1 0))
(0 -4)
> (spsq-dffr bc 2 2 0 '(0 1))
(0 0)
```

It is also possible to compute the convergence level of the spectral sequence for each degree n. For degrees 1 and 2 the convergence is reached at stage 3; for $n = 3$ all the groups $E^1_{p,q}$ are already null so that the convergence level is 1.

```
> (spsq-cnvg bc 1)
3
> (spsq-cnvg bc 2)
3
> (spsq-cnvg bc 3)
1
```

Finally, we can determine the filtration of the homology groups. For example, for degree $n = 2$, we obtain $F_0H_2 = F_1H_2 = 0$ and $F_2H_2 \cong \mathbb{Z} \cong H_2$.

```
> (hmlg-fltr bc 2 0)
Filtration F_0 H_2
nil
> (hmlg-fltr bc 2 1)
Filtration F_1 H_2
nil
> (hmlg-fltr bc 2 2)
Filtration F_2 H_2
Component Z
```

This is a very simple example of bicomplex where all the components of the spectral sequence can be determined by hand. In the following section we consider the Bar construction of the algebra $A_* = C_*(K(\mathbb{Z}, 2))$. It is not an effective chain complex and therefore the spectral sequence cannot directly be computed.

2.5.2.2 The Bar construction

The Eilenberg-MacLane space $K(\mathbb{Z}, 2)$ is a simplicial Abelian group so that the associated chain complex $C_*(K(\mathbb{Z}, 2))$ can be seen as an algebra. Hence, we can consider the chain complex $B_* = \text{Bar}(C_*(K(\mathbb{Z}, 2)))_*$, whose homology groups are in fact isomorphic to the homology groups of $\overline{\mathcal{W}}(K(\mathbb{Z}, 2)) = K(\mathbb{Z}, 3)$.

Taking into account that $C_*(K(\mathbb{Z}, 2))$ is not an effective chain complex, neither is B_*. Nevertheless, Kenzo allows us to construct this space in a simple way with the following statements:

```
> (setf kz2 (k-z 2))
[K29 Abelian-Simplicial-Group]
> (setf bkz2 (bar kz2))
[K46 Chain-Complex]
```

The simplicial Abelian group `kz2` is not effective but its effective homology is available by means of the function `efhm`.

```
> (setf kz2-efhm (efhm kz2))
[K166 Homotopy-Equivalence K29 <= K156 => K152]
```

The associated effective complex, `K152`, is the right bottom chain complex of the equivalence `kz2-efhm`.

```
> (setf kz2-efcc (rbcc kz2-efhm))
[K152 Chain-Complex]
```

The effective homology of `bkz2` can then be obtained applying the function `bar` to the effective homology of `kz2`.

```
> (setf (slot-value bkz2 'efhm) (bar kz2-efhm))
[K203 Homotopy-Equivalence K46 <= K186 => K200]
```

What is the effective chain complex `K200`?

```
> (orgn (k 200))
(add [K174 Chain-Complex] [K198 Morphism (degree -1): K174 -> K174])
```

It has been obtained by "adding" a perturbation (the morphism `K198`) to the chain complex `K174`, that is to say, applying the Basic Perturbation Lemma. Now, we can inspect what `K174` is.

```
> (orgn (k 174))
(vrtc-bar [K152 Chain-Complex])
```

It is the vertical Bar of the chain complex `K152`, in other words, a chain complex generated in each degree n by the *bars* $[g_1|g_2|\dots|g_k]$ with $\sum_{j=1}^{k}(\deg(g_j)+1) = n$, but where only the vertical (tensorial) differential map d_t has been considered. Finally, we recall that `K152` is the effective chain complex of our space `kz2`, and therefore we observe that Kenzo computes the effective homology of `bkz2` as explained in Section 2.4.3.2.

The effective homology of `kz2` allows us to determine its homology groups. For instance, in dimensions $7, 8, 9$, and 10 we obtain the following groups:

```
> (homology bkz2 7 11)
Homology in dimension 7 :
Component Z/3Z
---done---
Homology in dimension 8 :
Component Z/2Z
---done---
Homology in dimension 9 :
Component Z/2Z
---done---
Homology in dimension 10 :
Component Z/3Z
---done---
```

Furthermore, this effective homology is also necessary for the computation of the associated spectral sequence. First of all, we must define filtrations on both chain complexes `bkz2` and `bkz2-efcc`, making use of the function `abar-flin`.

```
> (change-chcm-to-flcc bkz2 abar-flin '(abar-flin))
[K46 Filtered-Chain-Complex]
> (change-chcm-to-flcc (k 200) abar-flin '(abar-flin))
[K200 Filtered-Chain-Complex]
```

And once we have defined the filtrations, we can start the computation of the spectral sequence. In this case we do not explain the details of the calculations.

Some groups:

```
> (spsq-group bkz2 1 1 6)
Spectral sequence E^1_{1,6}
Component Z
> (spsq-group bkz2 1 2 6)
Spectral sequence E^1_{2,6}
Component Z
Component Z
> (spsq-group bkz2 2 1 6)
Spectral sequence E^2_{1,6}
Component Z/3Z
```

Some differential maps:

```
> (spsq-dffr bkz2 1 2 6 '(1 0))
(3)
> (spsq-dffr bkz2 1 2 6 '(0 1))
(3)
> (spsq-dffr bkz2 1 3 6 '(1))
(-2 2)
> (spsq-dffr bkz2 1 2 8 '(1 0 0))
(4)
```

```
> (spsq-dffr bkz2 1 2 8 '(0 1 0))
(6)
> (spsq-dffr bkz2 1 2 8 '(0 0 1))
(4)
```

Convergence levels:

```
> (spsq-cnvg bkz2 7)
2
> (spsq-cnvg bkz2 8)
2
> (spsq-cnvg bkz2 9)
2
```

Filtration of the homology group $H_8 \cong \mathbb{Z}_2$:

```
> (hmlg-fltr bkz2 8 0)
Filtration F_0 H_8
nil
> (hmlg-fltr bkz2 8 1)
Filtration F_1 H_8
nil
> (hmlg-fltr bkz2 8 2)
Filtration F_2 H_8
Component Z/2Z
```

Chapter 3

Effective homology and spectral sequences of filtered complexes: applications

In this chapter we consider two classical examples of spectral sequences: the Serre and Eilenberg-Moore spectral sequences. Both of them were built by means of filtered complexes and have been used to compute homology groups of some complicated spaces. Nevertheless, in many cases these spectral sequences are not algorithms and cannot be completely determined.

On the other hand, the Serre and Eilenberg-Moore spectral sequences can be replaced by real algorithms based on the effective homology technique, allowing one to compute the homology groups of the associated complexes. In this chapter we will show that this effective homology can also be used to compute the spectral sequences themselves. If the spaces involved in the constructions are objects with effective homology, then the algorithms presented in Chapter 2 can be applied to determine all the components of the associated Serre and Eilenberg-Moore spectral sequences.

The chapter is divided into two parts. The first one is devoted to the Serre spectral sequence; the work explained there has been presented in [RRS06] and [Rom06b]. The second part deals with the Eilenberg-Moore spectral sequence; this material was announced in [RRS06] but has not been published yet.

3.1 Serre spectral sequence

3.1.1 Introduction

One of the first examples of spectral sequence is due to Jean-Pierre Serre [Ser51], using previous work by Jean Leray [Ler46] [Ler50] and Jean-Louis Koszul [Kos47]. In order to introduce this famous spectral sequence, in this section we give a brief overview of some of the initial examples considered by Serre. Some technical details are skipped.

The Serre spectral sequence involves fibrations $G \hookrightarrow E \to B$, where G is the fiber space, B is the base space, and E is the total space that can be seen as a twisted Cartesian product of B and G. The associated spectral sequence is given in the following theorem.

Theorem 3.1 (Serre spectral sequence). [Ser51] Let $G \hookrightarrow E \to B$ be a fibration where the base space B is a 1-reduced simplicial set. Then there exists a first quadrant spectral sequence $(E^r, d^r)_{r\geq 1}$ with

$$E^2_{p,q} = H_p(B; H_q(G))$$

which converges to the graded homology group $H_*(E)$, that is to say, there exists a filtration F_H of $H_*(E)$ such that

$$E^\infty_{p,q} \cong \frac{F_{H_p} H_{p+q}(E)}{F_{H_{p-1}} H_{p+q}(E)}$$

It is frequently thought this spectral sequence is a process making it possible to compute the groups $H_*(E)$ when the groups $H_*(B)$ and $H_*(G)$ are known, but this is false in general. The definition of the spectral sequence allows one to construct the groups at level $r = 2$, but the differential maps $d^r_{p,q}$ are unknown and in many cases we do not have the necessary information to compute them. And even if we know all the differentials $d^r_{p,q}$ and we can reach the final groups $E^\infty_{p,q}$, we must deal with an extension problem not always solvable to determine the homology groups $H_*(E)$.

This means that the Serre spectral sequence is not an algorithm that enables us to compute the homology groups of the total space of the fibration, but in fact it is a (rich and interesting) set of relations between the homology groups $H_*(G)$, $H_*(E)$, and $H_*(B)$. In addition, one must bear in mind that in many cases this spectral sequence cannot be determined. To illustrate this non-constructive nature, we include here one of the initial examples of Serre, considering the beginning of his calculations.

The computation of sphere homotopy groups is known as a difficult problem in Algebraic Topology. It is not hard to prove that $\pi_n(S^k) = 0$ for $n < k$ and $\pi_k(S^k) \cong \mathbb{Z}$, but the computation of the higher groups $\pi_n(S^k)$ for $n > k$ becomes more complicated. Making use of his famous spectral sequence, Serre computed many homotopy groups at the beginning of the fifties. For instance, how can one use the Serre spectral sequence to determine the homotopy groups of S^3? First of all, as explained before, $\pi_n(S^3) = 0$ for $n < 3$ and $\pi_3(S^3) \cong \mathbb{Z}$. In order to compute $\pi_4(S^3)$, we consider a fibration

$$G_2 \hookrightarrow X_4 \to S^3$$

where $G_2 = K(\mathbb{Z}, 2)$ is an Eilenberg-MacLane space, induced by the universal fibration $K(\mathbb{Z}, 2) \hookrightarrow E(\mathbb{Z}, 3) \to K(\mathbb{Z}, 3)$ (see [May67] for details). The beginning of the spectral sequence (the groups $E^2_{p,q}$) is determined by means of the well-known homology groups of S^3 and G_2; the result is shown in the next figure.

$r = 2$

q			
$\mathbb{Z}$	0	0	$\mathbb{Z}$
0	0	0	0
$\mathbb{Z}$	0	0	$\mathbb{Z}$
0	0	0	0
$\mathbb{Z}$	0	0	$\mathbb{Z}$ p

One can easily observe that all the arrows $d^2_{p,q} : E^2_{p,q} \to E^2_{p-2,q+1}$ are necessarily null and therefore the groups $E^3_{p,q}$ are equal to the corresponding $E^2_{p,q}$. But problems arise when trying to determine the differentials $d^3_{p,q}$. The arrow $d^3_{3,0} : E^3_{3,0} \cong \mathbb{Z} \to E^3_{0,2} \cong \mathbb{Z}$ must be an isomorphism, but to know the arrows $d^3_{3,2q}$ some other (extra) information than which is given by the spectral sequence itself is necessary. In this particular case, a specific tool (the multiplicative structure of the cohomology) gives the solution, the arrow $d^3_{3,2q} : E^3_{3,2q} \cong \mathbb{Z} \to E^3_{0,2q+2} \cong \mathbb{Z}$ is the multiplication by $q+1$. Thus, it can be deduced that all the groups $E^3_{3,2q}$ die and the only non-null final groups are $E^\infty_{0,0} \cong \mathbb{Z}$ and $E^\infty_{0,2q} \cong \mathbb{Z}_q$ for $q \geq 2$.

On account of the isomorphisms $E^\infty_{p,q} \cong F_{H_p} H_{p+q}(X_4)/F_{H_{p-1}} H_{p+q}(X_4)$, in this case the Serre spectral sequence entirely gives the homology groups of the total space X_4: $H_0(X_4) \cong \mathbb{Z}$, $H_{2n}(X_4) \cong \mathbb{Z}_n$ for $n \geq 2$, and the other $H_n(X_4)$ are null. Furthermore, the Hurewicz Theorem 1.50 and the long exact sequence of homotopy (see [Whi78] or [May67] for details) imply that $\pi_4(S^3) \cong \pi_4(X_4) \cong H_4(X_4) \cong \mathbb{Z}_2$.

Then, a new fibration

$$G_3 \hookrightarrow X_5 \to X_4$$

should be considered to determine $\pi_5(S^3)$, where $G_3 = K(\mathbb{Z}_2, 3)$ is chosen because $\pi_4(X_4) \cong \mathbb{Z}_2$. In this case Serre was also able to obtain all the necessary ingredients to compute the maps $d^r_{p,q}$ which play an important role in the beginning of the associated spectral sequence. The main tool (extra information) is again the multiplicative structure in cohomology and more generally the module structure with respect to the Steenrod algebra $\mathcal{A}_2$ [Ste62]. The final groups $E^\infty_{p,q}$ (with $p + q \leq 8$) of this spectral sequence are shown in the following figure.

$r = \infty$

q								
$\mathbb{Z}_2$	0							
$\mathbb{Z}_2$	0	0						
$\mathbb{Z}_2$	0	0	0					
0	0	0	0	0				
0	0	0	0	0	$\mathbb{Z}_2$			
0	0	0	0	0	0	0		
0	0	0	0	0	0	0	0	
$\mathbb{Z}$	0	0	0	0	0	$\mathbb{Z}_3$	0	0 p

For $p + q = 5$, one can observe that there is only one non-null group, $E^\infty_{0,5} \cong \mathbb{Z}_2$, and therefore $H_5(X_5) \cong \mathbb{Z}_2$. Again the Hurewicz theorem and the long homotopy exact sequence imply $\pi_5(S^3) \cong \pi_5(X_4) \cong \pi_5(X_5) \cong H_5(X_5) \cong \mathbb{Z}_2$; it was the first important result obtained by Serre.

Let us stress that the filtration of $H_6(X_5)$ has two stages, which give the short exact sequence

$$0 \to \mathbb{Z}_2 \to H_6(X_5) \to \mathbb{Z}_3 \to 0$$

The group $H_6(X_5)$ is then an extension of $\mathbb{Z}_3$ by $\mathbb{Z}_2$, and fortunately there is a unique solution: $H_6(X_5) \cong \mathbb{Z}_6$.

In order to compute $\pi_6(S^3)$, we consider a new fibration

$$G_4 \hookrightarrow X_6 \to X_5$$

with $G_4 = K(\mathbb{Z}_2, 4)$. In this case there are three non-null groups $E^\infty_{p,q}$ for $p + q \leq 6$: $E^\infty_{0,0} \cong \mathbb{Z}$, $E^\infty_{0,6} \cong \mathbb{Z}_2$, and $E^\infty_{6,0} \cong \mathbb{Z}_6$. In degree 6 we obtain a short exact sequence

$$0 \to \mathbb{Z}_2 \to H_6(X_6) \to \mathbb{Z}_6 \to 0$$

but now there are two possible extensions (the trivial one $\mathbb{Z}_2 \oplus \mathbb{Z}_6$ and the twisted one $\mathbb{Z}_{12}$), and the Serre spectral sequence does not give any information that allows one to make the correct choice. In this way, Serre proved $\pi_6(S^3)$ has 12 elements, but he was unable to choose between the two possible options $\mathbb{Z}_{12}$ and $\mathbb{Z}_2 \oplus \mathbb{Z}_6$. Two years later, Barratt and Paechter [BP52] proved that the group $\pi_6(S^3)$ has an element of order 4, and consequently $\pi_6(S^3) \cong \mathbb{Z}_{12}$ is the correct answer.

These examples illustrate the fact that the computation of the Serre spectral sequence is not an easy task and in some situations some other information is needed to overcome the ambiguities in the spectral sequence itself. In other cases, this computation is in fact *not possible*, since some differential maps $d^r_{p,q}$ cannot be determined by any other means (we do not have the necessary extra information). Therefore, as mentioned before, the Serre spectral sequence is not an algorithm that allows us to compute $H_*(E)$ in terms of $H_*(B)$ and $H_*(G)$.

On the other hand, if the base and fiber spaces are objects with effective homology, then it is possible to determine the effective homology of the total space E. We obtain in this way a real algorithm allowing us to compute the looked-for homology groups $H_*(E)$, replacing the Serre spectral sequence technique. In particular, some interesting homology groups related to complex Postnikov towers have been determined using this method (see, for instance, [RS05b] or [RS06]). In the next section we explain how the effective homology of the total space of a fibration is obtained.

3.1.2 Effective homology of a twisted product

One of the typical examples of application of the effective homology method is the computation of the homology groups of a twisted Cartesian product $G \times_\tau B$, which provides a *constructive version* of the Serre spectral sequence. Details of this construction can be found in [RS06].

First of all, we must point out that from now on in this section, all the chain complexes canonically associated with simplicial sets are *normalized*, that is to say, only the non-degenerate n-simplices are considered to be generators of the group of n-chains.

Given a fibration $G \hookrightarrow E \to B$ (defined by a twisting operator $\tau : B \to G$) where the fiber G and the base B are objects with effective homology and B is a 1-reduced simplicial set (which means $B_0 = B_1 = \{\star\}$), we want to obtain the effective homology of the total space $E = G \times_\tau B$.

Let us suppose there exist two homotopy equivalences

$$C_*(G) \Lleftarrow DG_* \Rrightarrow HG_* \qquad C_*(B) \Lleftarrow DB_* \Rrightarrow HB_*$$

with HG_* and HB_* effective chain complexes. How can we obtain a new equivalence between $C_*(G \times_\tau B)$ and an effective chain complex?

The starting point is the Eilenberg-Zilber reduction, which relates the Cartesian product of two simplicial sets and the tensor product of the associated chain complexes.

Theorem 3.2 (Eilenberg-Zilber reduction). [EZ53] Given two simplicial sets G and B, there exists a reduction

$$\rho = (f, g, h) : C_*(G \times B) \Rrightarrow C_*(G) \otimes C_*(B)$$

with the maps f, g, and h defined as:

$$
\begin{aligned}
f(x_n, y_n) &= \sum_{i=0}^{n} \partial_{i+1} \ldots \partial_n x_n \otimes \partial_0 \ldots \partial_{i-1} y_n \\
g(x_p \otimes y_q) &= \sum_{(\alpha,\beta)\in\{(p,q)\text{-shuffles}\}} (-1)^{sg(\alpha,\beta)} (\eta_{\beta_q} \ldots \eta_{\beta_1} x_p, \eta_{\alpha_p} \ldots \eta_{\alpha_1} y_q) \\
h(x_n, y_n) &= \sum (-1)^{n-p-q+sg(\alpha,\beta)} (\eta_{\beta_q+n-p-q} \ldots \eta_{\beta_1+n-p-q} \eta_{n-p-q-1} \partial_{n-q+1} \ldots \partial_n x_n, \\
&\qquad \eta_{\alpha_{p+1}+n-p-q} \ldots \eta_{\alpha_1+n-p-q} \partial_{n-p-q} \ldots \partial_{n-q-1} y_n)
\end{aligned}
$$

where a (p,q)-shuffle $(\alpha,\beta) = (\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q)$ is a permutation of the set $\{0, 1, \ldots, p+q-1\}$ such that $\alpha_i < \alpha_{i+1}$ and $\beta_j < \beta_{j+1}$, $sg(\alpha,\beta) = \sum_{i=1}^{p}(\alpha_i - i - 1)$, and the third sum (which defines the homotopy operator h) is taken over all the indices $0 \leq q \leq n-1, 0 \leq p \leq n-q-1$ and $(\alpha,\beta) \in \{(p+1,q)\text{-shuffles}\}$.

The maps f, g, and h are known respectively as the *Alexander-Whitney*, *Eilenberg-MacLane*, and *Shih operators*.

In our case, we have a twisted product $E = G \times_\tau B$. We must take account of the torsion τ, that does not change the underlying graded group of the chain complex $C_*(G \times B)$, only the differential map is modified by a perturbation

$$\delta(f, b) = (\partial_0 f \cdot \tau(b), \partial_0 b) - (\partial_0 f, \partial_0 b)$$

This is a typical situation where the Basic Perturbation Lemma (Theorem 1.70) could be applied. In order to guarantee the necessary local nilpotency condition, we consider in both chain complexes $C_*(G \times B)$ and $C_*(G) \otimes C_*(B)$ the canonical filtrations of the Cartesian product of two simplicial sets and of the tensor product of two chain complexes respectively, defined as follows.

Definition 3.3. Let X and Y be two simplicial sets, and $C_* = C_*(X \times Y)$ the chain complex associated with the Cartesian product $X \times Y$. We define the filtration $F_\times$ of C_* through the degeneracy degree with respect to the second space: a generator (x_n, y_n) of $C_n = C_n(X \times Y)$ has a filtration degree less than or equal to p if $\exists \bar{y}_p \in Y_p$ such that $y_n = \eta_{i_{n-p}} \ldots \eta_{i_1} \bar{y}_p$.

Definition 3.4. Given two chain complexes C_* and D_*, we define the filtration $F_\otimes$ of the tensor product $C_* \otimes D_*$ through the dimension of the second component:

$$F_{\otimes_p}(C_* \otimes D_*)_n = \bigoplus_{h \leq p} C_{n-h} \otimes D_h$$

It is not difficult to prove that $F_\otimes$ and $F_\times$ are filtrations of the corresponding chain complexes. In addition, it is clear that both of them are canonically bounded.

In our particular case, $F_\times$ is a filtration of $C_*(G \times B)$ and $F_\otimes$ is a filtration of the bottom chain complex $C_*(G) \otimes C_*(B)$. Furthermore, one can prove that the three operators f, g, and h involved in the Eilenberg-Zilber reduction are compatible with these filtrations and the perturbation δ decreases the filtration degree of $C_*(G \times B)$ by one unit. Since the filtration $F_\times$ is bounded, it can easily be deduced that the composition $h \circ \delta$ is locally nilpotent, so that the hypotheses of the BPL are satisfied. In this way, a new reduction $\rho_1 = (f_1, g_1, h_1) : C_*(G \times_\tau B) \Rrightarrow C_*(G) \otimes_t C_*(B)$ is obtained, where the symbol $\otimes_t$ represents a twisted (perturbed) tensor product, induced by τ. This result is known as the "twisted" Eilenberg-Zilber theorem, due to Edgard Brown [Bro59], and put under its modern form by Shih Weishu [Shi62] and Ronnie Brown [Bro67].

Theorem 3.5 (Twisted Eilenberg-Zilber Theorem). [Bro59] Given two simplicial sets G and B and a twisting operator $\tau : B \to G$, it is possible to construct a reduction

$$\rho = (f, g, h) : C_*(G \times_\tau B) \Rrightarrow C_*(G) \otimes_t C_*(B)$$

where $C_*(G) \otimes_t C_*(B)$ is a chain complex with the same underlying graded module as the tensor product $C_*(G) \otimes C_*(B)$, but the differential is modified to take account of the twisting operator τ.

On the other hand, making use of the tensor product of two reductions (Proposition 1.61) and of the effective homologies of G and B, we obtain a new equivalence

$$\begin{array}{ccc} & DG_* \otimes DB_* & \\ \swarrow\!\!\swarrow & & \searrow\!\!\searrow \\ C_*(G) \otimes C_*(B) & & HG_* \otimes HB_* \end{array}$$

and it is clear that the right chain complex $HG_* \otimes HB_*$ is effective. We remark also that the canonical filtration $F_\otimes$ of tensor products can also be considered on the chain complexes $DG_* \otimes DB_*$ and $HG_* \otimes HB_*$.

Let us consider now the necessary perturbation $\bar{\delta}$ of $C_*(G) \otimes C_*(B)$ to set the twisted product $C_*(G) \otimes_t C_*(B)$ (this perturbation $\bar{\delta}$ has been obtained when applying the BPL to the Eilenberg-Zilber reduction). As far as the base space B is 1-reduced, it can be proved (see [RS06]) that $\bar{\delta}$ decreases the filtration degree of $C_*(G) \otimes C_*(B)$ at least by 2 units. On the other hand, both homotopy operators of the reductions $DG_* \otimes DB_* \Rrightarrow C_*(G) \otimes C_*(B)$ and $DG_* \otimes DB_* \Rrightarrow HG_* \otimes HB_*$ increase the filtration degree at most by one unit. These facts will play an important role in the computation of the effective homology of the twisted tensor product $C_*(G) \otimes_t C_*(B)$.

First, we must apply the Trivial Perturbation Lemma to the left reduction $DG_* \otimes DB_* \Rrightarrow C_*(G) \otimes C_*(B)$, with the perturbation $\bar{\delta}$ of $C_*(G) \otimes C_*(B)$. We obtain in this way a reduction $\rho_2 = (f_2, g_2, h_2) : DG_* \otimes_t DB_* \Rrightarrow C_*(G) \otimes_t C_*(B)$ where $DG_* \otimes_t DB_*$ is a twisted tensor product defined by a perturbation $\bar{\delta}'$ of $DG_* \otimes DB_*$, which also decreases the filtration degree at least by 2 units.

Then, we consider the right reduction $DG_* \otimes DB_* \Rrightarrow HG_* \otimes HB_*$ and the perturbation $\bar{\delta}'$ of $DG_* \otimes DB_*$. The hypotheses of the Basic Perturbation Lemma are again satisfied and therefore a reduction $\rho_3 = (f_3, g_3, h_3) : DG_* \otimes_t DB_* \Rrightarrow HG_* \otimes_t HB_*$ is obtained, where $HG_* \otimes_t HB_*$ is also a perturbed tensor product, which in this case is an effective chain complex.

Therefore one has the following equivalence:

$$\begin{array}{ccc} & DG_* \otimes_t DB_* & \\ {\scriptstyle \rho_2} \swarrow\!\!\swarrow & & \searrow\!\!\searrow {\scriptstyle \rho_3} \\ C_*(G) \otimes_t C_*(B) & & HG_* \otimes_t HB_* \end{array}$$

We consider again the reduction $\rho_1 : C_*(G \times_\tau B) \Rrightarrow C_*(G) \otimes_t C_*(B)$ (given by the twisted Eilenberg-Zilber theorem), and then the composition of the two equivalences

$$\begin{array}{ccccccc} & C_*(G \times_\tau B) & & & & DG_* \otimes_t DB_* & \\ {\scriptstyle \mathrm{Id}} \swarrow\!\!\swarrow & & \searrow\!\!\searrow {\scriptstyle \rho_1} & & {\scriptstyle \rho_2} \swarrow\!\!\swarrow & & \searrow\!\!\searrow {\scriptstyle \rho_3} \\ C_*(G \times_\tau B) & & & C_*(G) \otimes_t C_*(B) & & & HG_* \otimes_t HB_* \end{array}$$

gives us the effective homology of the twisted Cartesian product $E = G \times_\tau B$.

This effective homology provides us in particular an algorithm to compute the homology groups of the total space of a fibration $G \hookrightarrow E \to B$, replacing in this way the Serre spectral sequence technique. But the spectral sequence itself can also give useful information about the construction, so that its computation is also interesting even if the homology groups of $E = G \times_\tau B$ are already known. As we will see in the next section, using the theoretical results and the algorithms presented in Chapter 2, we have developed a new algorithm that makes it possible to compute the Serre spectral sequence associated with a fibration whenever the base and fiber spaces are objects with effective homology.

3.1.3 An algorithm computing the Serre spectral sequence

Let us consider a fibration

$$G \hookrightarrow E \to B$$

where the base space B is 1-reduced. The associated Serre spectral sequence gives

$$E^2_{p,q} = H_p(B; H_q(G))$$

In many cases, this property is not sufficient to compute all the components of the spectral sequence, since the differential maps $d^r_{p,q}$ are unknown and we do not always have the necessary *extra* information to compute them.

Nevertheless, this spectral sequence was in fact defined by Serre as the spectral sequence of the filtered chain complex $(C_*(E), F_\times)$, where $E = G \times_\tau B$ and $F_\times$ is the canonical filtration of the chain complex associated with a Cartesian product, introduced in Definition 3.3. Since the perturbation δ induced by the twisting operator τ decreases the filtration degree of $C_*(G \times B)$ by one unit, the filtration $F_\times$ is also a filtration of the *perturbed* chain complex $C_*(G \times_\tau B)$. It is worth noting that in general the filtered chain complex $(C_*(G \times_\tau B), F_\times)$ is not effective, and consequently its spectral sequence cannot directly be determined. The effective homology will be necessary.

Let us suppose that G and B are objects with effective homology, with equivalences

$$C_*(G) \overset{\rho_1^G}{\Lleftarrow} DG_* \overset{\rho_2^G}{\Rrightarrow} HG_* \qquad C_*(B) \overset{\rho_1^B}{\Lleftarrow} DB_* \overset{\rho_2^B}{\Rrightarrow} HB_*$$

We recall from Section 3.1.2 that the effective homology of the twisted product $E = G \times_\tau B$ is obtained as the composition of two equivalences:

$$C_*(G \times_\tau B) \overset{\mathrm{Id}}{\Lleftarrow} C_*(G \times_\tau B) \overset{\rho_1}{\Rrightarrow} C_*(G) \otimes_t C_*(B) \overset{\rho_2}{\Lleftarrow} DG_* \otimes_t DB_* \overset{\rho_3}{\Rrightarrow} HG_* \otimes_t HB_*$$

Making use of the theoretical results obtained in Chapter 2, we are going to prove that these equivalences allow us to deduce isomorphisms on the spectral sequences associated with the different spaces.

We consider first the reduction $\rho_1 = (f_1, g_1, h_1) : C_*(G \times_\tau B) \Rrightarrow C_*(G) \otimes_t C_*(B)$, which was determined by application of the BPL to the Eilenberg-Zilber reduction $\rho = (f, g, h) : C_*(G \times B) \Rrightarrow C_*(G) \otimes C_*(B)$. The chain complex $C_*(G \times_\tau B)$ is obtained from $C_*(G \times B)$ and a perturbation δ, and $C_*(G) \otimes_t C_*(B)$ is equal to the chain complex $C_*(G) \otimes C_*(B)$ replacing the old differential d by a perturbed differential $d + \bar{\delta}$. Let us recall the formulas of the different components of the reduction ρ_1, given by the Basic Perturbation Lemma:

$$\begin{aligned} \bar{\delta} &= f \circ \delta \circ \phi \circ g = f \circ \psi \circ \delta \circ g, \\ f_1 &= f \circ \psi = f \circ (\mathrm{Id} - \delta \circ \phi \circ h), \\ g_1 &= \phi \circ g, \\ h_1 &= \phi \circ h = h \circ \psi \end{aligned}$$

where the operators ϕ and ψ are defined by

$$\begin{aligned} \phi &= \sum_{i=0}^{\infty} (-1)^i (h \circ \delta)^i \\ \psi &= \sum_{i=0}^{\infty} (-1)^i (\delta \circ h)^i = \mathrm{Id} - \delta \circ \phi \circ h \end{aligned}$$

As mentioned in Section 3.1.2, the three components f, g, and h of the Eilenberg-Zilber reduction are known to be compatible with the filtrations $F_\times$ of $C_*(G \times B)$ and $F_\otimes$ of $C_*(G) \otimes C_*(B)$. Moreover, the perturbation δ decreases the filtration degree by one unit. In this way, the compositions $h \circ \delta$ and $\delta \circ h$, and therefore the series ϕ and ψ, decrease the filtration degree too. In particular, the new perturbation $\bar{\delta}$ of $C_*(G) \otimes C_*(B)$ is a filtered complex morphism, implying that $F_\otimes$ can also be considered as a filtration of the twisted (perturbed) tensor product $C_*(G) \otimes_t C_*(B)$. In addition, we have already said that the perturbation $\bar{\delta}$ decreases the filtration index at least by two units.

Consequently, we observe in the formulas given by the BPL that the three components of the reduction $\rho_1 = (f_1, g_1, h_1)$ are also compatible with the filtrations of $C_*(G \times_\tau B)$ and $C_*(G) \otimes_t C_*(B)$. In particular, h has order ≤ 0, and applying our Theorem 2.14, we obtain an isomorphism between the associated spectral sequences for every level $r > 0$:

$$E(C_*(G \times_\tau B), F_\times)^r_{p,q} \cong E(C_*(G) \otimes_t C_*(B), F_\otimes)^r_{p,q} \quad \text{for all } r \geq 1$$

Let us now turn to the right equivalence

$$\begin{array}{ccc} & DG_* \otimes_t DB_* & \\ \overset{\rho_2}{\swarrow} & & \overset{\rho_3}{\searrow} \\ C_*(G) \otimes_t C_*(B) & & HG_* \otimes_t HB_* \end{array}$$

The reductions ρ_2 and ρ_3 were determined applying the Trivial Perturbation Lemma (Theorem 1.69) and the BPL to two reductions $\rho_2' : DG_* \otimes DB_* \Rrightarrow C_*(G) \otimes C_*(B)$ and $\rho_3' : DG_* \otimes DB_* \Rrightarrow HG_* \otimes HB_*$ respectively. Besides, ρ_2' is obtained as the tensor product (given by Proposition 1.61) of the reductions $\rho_1^G : DG_* \Rrightarrow C_*(G)$ and $\rho_1^B : DB_* \Rrightarrow C_*(B)$. Analogously, ρ_3' is the tensor product of the reductions $\rho_2^G : DG_* \Rrightarrow HG_*$ and $\rho_2^B : DB_* \Rrightarrow HB_*$. Let us prove in general that the tensor product of two reductions has a good behavior with respect to the canonical filtration $F_\otimes$.

Lemma 3.6. Let $\rho = (f, g, h) : C_* \Rrightarrow D_*$ and $\rho' = (f', g', h') : C'_* \Rrightarrow D'_*$ be two reductions, and $\rho'' = (f'', g'', h'') : C_* \otimes C'_* \Rrightarrow D_* \otimes D'_*$ the tensor product of the reductions ρ and ρ', defined by

$$\begin{aligned} f'' &= f \otimes f' \\ g'' &= g \otimes g' \\ h'' &= h \otimes \mathrm{Id}_{C'_*} + (g \circ f) \otimes h' \end{aligned}$$

Then the maps f'' and g'' are compatible with the filtrations $F_\otimes$ defined on $C_* \otimes C'_*$ and $D_* \otimes D'_*$, and the homotopy operator h'' has order ≤ 1.

Proof. Let $c \otimes c'$ be a generator of $F_{\otimes_p}(C_* \otimes C'_*)_n$. Then $c' \in C'_h$ with $h \leq p, c \in C_m$ with $m = n - h$. One has $f''(c \otimes c') = f(c) \otimes f'(c')$ and $f'(c') \in D'_h$ with $h \leq p$, so that

$$f''(c \otimes c') \in F_{\otimes_p}(D_* \otimes D'_*)_n$$

Similarly, if $d \otimes d'$ is a generator of $F_{\otimes_p}(D_* \otimes D'_*)$, then

$$g''(d \otimes d') = g(d) \otimes g'(d') \in F_{\otimes_p}(C_* \otimes C'_*)$$

Therefore f'' and g'' are morphisms of filtered chain complexes.

Finally, given $c \otimes c' \in F_{\otimes_p}(C_* \otimes C'_*)_n$ with $c' \in C'_h$, $h \leq p$, and $c \in C_m$ with $m = n - h$, then $h''(c \otimes c') = h(c) \otimes c' + g \circ f(c) \otimes h'(c')$. One has $c' \in C'_h$ with $h \leq p$ and therefore $h(c) \otimes c' \in F_{\otimes_p}(C_* \otimes C'_*)_{n+1}$. On the other hand $h'(c') \in C'_{h+1}$, so that $g \circ f(c) \otimes h'(c') \in F_{\otimes_{p+1}}(C_* \otimes C'_*)_{n+1}$. In this way

$$h''(c \otimes c') = h(c) \otimes c' + g \circ f(c) \otimes h'(c') \in F_{\otimes_{p+1}}(C_* \otimes C'_*)_{n+1}$$

which implies that h'' has order ≤ 1. □

We come back now to our particular case. We consider first the left reduction $\rho'_2 = (f'_2, g'_2, h'_2) : DG_* \otimes DB_* \Rrightarrow C_*(G) \otimes C_*(B)$, which is given by the tensor product of the reductions $\rho_1^G : DG_* \Rrightarrow C_*(G)$ and $\rho_1^B : DB_* \Rrightarrow C_*(B)$. Making use of the previous lemma, one has that f'_2 and g'_2 are filtered complex morphisms, and h'_2 has order ≤ 1. When applying the TPL with the perturbation $\bar{\delta}$ of $C_*(G) \otimes_t C_*(B)$ (that decreases the filtration index by 2 units), we obtain a perturbation $\bar{\delta}' = g'_2 \circ \bar{\delta} \circ f'_2$ of $DG_* \otimes DB_*$ which also decreases the filtration index by 2. Furthermore, the *perturbed* reduction $\rho_2 = (f_2, g_2, h_2) : DG_* \otimes_t DB_* \Rrightarrow C_*(G) \otimes_t C_*(B)$ is given by $f_2 = f'_2, g_2 = g'_2$, and $h_2 = h'_2$, so that f_2 and g_2 are also filtered chain complex morphisms and h_2 has order ≤ 1. In this way the hypotheses of Theorem 2.14 are satisfied again and one has isomorphisms on the corresponding spectral sequences, in this case for $r \geq 2$:

$$E(C_*(G) \otimes_t C_*(B), F_\otimes)^r_{p,q} \cong E(DG_* \otimes_t DB_*, F_\otimes)^r_{p,q} \quad \text{for } r \geq 2$$

Similar arguments as before on the third reduction $\rho_3 : DG_* \otimes_t DB_* \Rrightarrow HG_* \otimes_t HB_*$ produce isomorphisms:

$$E(DG_* \otimes_t DB_*, F_\otimes)^r_{p,q} \cong E(HG_* \otimes_t HB_*, F_\otimes)^r_{p,q} \quad \text{for } r \geq 2$$

and composing with the previous results one has

$$E(C_*(G \times_\tau B), F_\times)^r_{p,q} \cong E(HG_* \otimes_t HB_*, F_\otimes)^r_{p,q} \quad \text{for all } r \geq 2$$

The inverse isomorphisms between these groups are the maps induced by the compositions $f_3 \circ g_2 \circ f_1 : C_*(G \times_\tau B) \to HG_* \otimes_t HB_*$ and $g_1 \circ f_2 \circ g_3 : HG_* \otimes_t HB_* \to C_*(G \times_\tau B)$.

We note now that the filtered chain complex $(HG_* \otimes_t HB_*, F_\otimes)$ is effective, so that the associated spectral sequence can easily be computed using the algorithms presented in Section 2.3.1. Thanks to our isomorphism, we can also determine the Serre spectral sequence of the fibration $G \hookrightarrow E \to B$ (which is the spectral sequence associated with the filtered chain complex $(C_*(G \times_\tau B), F_\times)$) by means of the spectral sequence of $(HG_* \otimes_t HB_*, F_\otimes)$. We obtain in this way the following algorithm.

Algorithm 6.
Input:

- a fibration $G \hookrightarrow E \to B$ defined by a twisting operator $\tau : B \to G$, with B a 1-reduced simplicial set,
- equivalences $C_*(G) \Lleftarrow DG_* \Rrightarrow HG_*$ and $C_*(B) \Lleftarrow DB_* \Rrightarrow HB_*$, where HG_* and HB_* are effective chain complexes.

Output: all the components of the Serre spectral sequence associated with the fibration, that is to say:

- the groups $E^r_{p,q}$ for each $p, q \in \mathbb{Z}$ and $r \geq 2$, with a basis-divisors description,
- the differential maps $d^r_{p,q}$ for every $p, q \in \mathbb{Z}$ and $r \geq 2$,
- the convergence level for each degree $n \in \mathbb{N}$,
- the filtration of the homology groups $H_*(E) = H_*(G \times_\tau B)$, in other words, the groups $F_{H_p}H_n(E)$ for each degree $n \in \mathbb{N}$ and filtration index $p \in \mathbb{Z}$.

3.1.4 Implementation and examples

As explained in Section 2.5, we have developed a set of programs in Common Lisp allowing us to compute spectral sequences of filtered complexes with effective homology, even if the complexes are not of finite type. In particular, this new module for the Kenzo system can be used for the computation of the Serre spectral sequence associated with a fibration when the effective homologies of the base and the fiber spaces are known.

In this section we present two examples of calculation of the Serre spectral sequence. First, a twisted product $K(\mathbb{Z}, 1) \times_\tau S^2$ is considered. In this case, the spectral sequence is well-known and can be obtained without using a computer; we propose it as a didactic example for a better understanding of the new programs. The second example corresponds to one of the first stages of a Postnikov tower. We will not give so many details as in the preceding one and it is perhaps not so easy to understand, but it has a higher interest because its spectral sequence seems difficult to be studied by the theoretical methods documented in the literature.

3.1.4.1 $K(\mathbb{Z}, 1) \times_\tau S^2$

We consider the twisted Cartesian product $K(\mathbb{Z}, 1) \times_\tau S^2$ for a twisting operator $\tau : S^2 \to K(\mathbb{Z}, 1)$ with $\tau(\mathtt{s2}) = [1]$. We use here the standard simplicial description of the 2-sphere, with a unique non-degenerate simplex `s2` in dimension 2. A principal fibration is then defined by a unique 1-simplex of the fiber space $K(\mathbb{Z}, 1)$, which has the same homotopy type as the 1-sphere S^1. The result in this case is the Hopf fibration, the total space $K(\mathbb{Z}, 1) \times_\tau S^2$ being a simplicial model for the 3-sphere S^3. If we define $\tau(\mathtt{s2}) = [2]$ (with the same base and fiber spaces), the total space is then the real projectif space $P^3\mathbb{R}$. Let us remark that, since $K(\mathbb{Z}, 1)$ is not effective, the space $K(\mathbb{Z}, 1) \times_\tau S^2$

is not effective either, and therefore the effective homology is necessary to determine its spectral sequence.

The Kenzo program has a file that allows us to work with twisted Cartesian products. For example, the twisted product $K(\mathbb{Z},1) \times_\tau S^2$ is built in Kenzo by means of the following statements.

```
>(setf s2 (sphere 2))
[K208 Simplicial-Set]
>(setf kz1 (k-z 1))
[K1 Abelian-Simplicial-Group]
> (setf tau (build-smmr
          :sorc s2
          :trgt kz1
          :degr -1
          :sintr #'(lambda (dmns gmsm) (absm 0 '(1)))
          :orgn '(kz1-tw-s2)))
[K213 Fibration K208 -> K1]
> (setf kz1-twcp-s2 (fibration-total tau))
[K219 Simplicial-Set]
```

The object `tau` implements the twisting operator $\tau : S^2 \to K(\mathbb{Z},1)$ as a morphism of degree -1 that sends the unique non-degenerate simplex `s2` of dimension 2 to the 1-simplex `(1)` of the simplicial set `kz1` (if we changed the list `'(1)`, that represents this 1-simplex, by the list `'(2)`, we would obtain the Hopf fibration of the real projectif space $P^3\mathbb{R}$). The function `fibration-total` builds the total space of the fibration defined by the twisting operator `tau` (this operator contains as source and target spaces the base and the fiber spaces of the fibration respectively), which is a twisted product of the base and the fiber.

Since the effective complex of $K(\mathbb{Z},1)$ is $C_*(S^1)$, the effective complex of $K(\mathbb{Z},1)\times_\tau S^2$ will be $C_*(S^1)\otimes C_*(S^2)$, with an appropriate perturbation of the differential map. We can inspect it by applying the function `rbcc` (right bottom chain complex) to the effective homology of the space `kz1-twcp-s2`.

```
> (setf s1-twtp-s2 (rbcc (efhm kz1-twcp-s2)))
[K279 Chain-Complex]
```

What is this chain complex `K279`?

```
> (orgn s1-twtp-s2)
(add [K259 Chain-Complex] [K277 Morphism (degree -1): K259 -> K259])
```

This origin means that the complex `s1-twtp-s2` has been obtained by application of the BPL, “adding” a perturbation (the morphism `K277`, of degree -1) to the initial chain complex `K259`. We want to know now what `K259` is:

```
> (orgn (k 259))
(tnsr-prdc [K208 Simplicial-Set] [K16 Chain-Complex])
```

As expected, we have a tensor product of two chain complexes, which are the (normalized) chain complexes associated with the simplicial sets K208 and K16. And finally, what about these simplicial sets?

```
> (orgn (k 208))
(sphere 2)
> (orgn (k 16))
(circle)
```

In this way we can state that $\texttt{K208} = S^2$ and $\texttt{K16} = S^1$, and therefore the effective complex of $\texttt{kz1-twcp-s2} = K(\mathbb{Z},1) \times_\tau S^2$ is $C_*(S^1) \otimes C_*(S^2)$ with a perturbation of the differential.

In order to compute the Serre spectral sequence of this twisted product, it is necessary to change it into a filtered complex. The filtration of this chain complex is the canonical filtration of (twisted) Cartesian products, $F_\times$ given by: a generator (g_n, b_n) of $C_n(G \times_\tau B)$ has a filtration degree less than or equal to p if $\exists \bar{b}_p \in B_p$ such that $b_n = \eta_{i_{n-p}} \dots \eta_{i_1} \bar{b}_p$. Such a filtration can be implemented by means of the following function **crpr-flin**.

```
>(setf crpr-flin
      #'(lambda (degr crpr)
          (declare
           (type fixnum degr)
           (type crpr crpr))
          (let* ((b (cadr crpr))
                 (dgop (car b)))
             (declare
              (type iabsm b)
              (type fixnum dgop))
             (the fixnum
               (- degr (length (dgop-int-ext dgop)))))))
#<Interpreted Function (unnamed) @ #x20e5e53a>
> (change-chcm-to-flcc kz1-twcp-s2 crpr-flin '(crpr-flin))
[K219 Filtered-Simplicial-Set]
```

A filtration is also needed on the effective chain complex, $C_*(S^1) \otimes_t C_*(S^2)$. In this case $F_\otimes$ is the canonical filtration of tensor products:

$$F_{\otimes_p}(C_*(G) \otimes_t C_*(B))_n = \bigoplus_{h \leq p} C_{n-h}(G) \otimes C_h(B)$$

The implementation in Common Lisp is done in the following way.

```
>(setf tnpr-flin
      #'(lambda (degr tnpr)
       (declare
         (type fixnum degr)
         (type tnpr tnpr))
       (the fixnum
          (degr1 tnpr))))
#<Interpreted Function (unnamed) @ #x20cfd50a>
> (change-chcm-to-flcc s1-twtp-s2 tnpr-flin '(tnpr-flin))
[K279 Filtered-Chain-Complex]
```

Once the filtrations have been defined, the new programs can be used to compute the spectral sequence of the twisted product $K(\mathbb{Z},1) \times_\tau S^2$. In this specific case it is isomorphic in every level $r \geq 1$ to that of the effective complex $C_*(S^1) \otimes_t C_*(S^2)$, provided that both homotopies in the equivalence have order ≤ 0 because the base space S^2 has trivial effective homology. For instance, the groups $E^2_{2,0}$ and $E^2_{0,1}$ are equal to $\mathbb{Z}$.

```
> (spsq-group kz1-twcp-s2 2 2 0)
Spectral sequence E^2_{2,0}
Component Z
> (spsq-group kz1-twcp-s2 2 0 1)
Spectral sequence E^2_{0,1}
Component Z
```

These groups can be recognized as the elements of the Serre spectral sequence of the Hopf fibration.

It is also possible to find the *basis-divisors* representation of the groups $E^r_{p,q}$. We recall that this representation shows a list of combinations which generate the group, as well as the Betti number and the torsion coefficients (which are the coefficients of the elements of the denominator with regard to the list of combinations). For the groups $E^2_{2,0}$ and $E^2_{0,1}$ that have been computed above, we obtain the following basis-divisors descriptions.

```
>(spsq-basis-dvs kz1-twcp-s2 2 2 0)
((
  ----------------------------------------------------------------------{CMBN 2}
  <-1 *   <CrPr - s2 1-0 nil>>
  ------------------------------------------------------------------------------
    )
   (0))
> (spsq-basis-dvs kz1-twcp-s2 2 0 1)
((
  ----------------------------------------------------------------------{CMBN 1}
  <-1 *   <CrPr 0 * - (1)>>
  ------------------------------------------------------------------------------
    )
   (0))
```

In both cases, the "basis" (list of combinations) has a unique element and the list of divisors is the list `(0)`, which means that both groups are isomorphic to $\mathbb{Z}$. For $E^2_{2,0}$, the generator is the element $-1 * (\eta_1\eta_0[\,], \texttt{s2}) \in C_2(K(\mathbb{Z},1) \times_\tau S^2)$, which is not a torsion element. Let us remark that Kenzo uses the inverse order for the elements of twisted products: the first component corresponds to the base space, and the second one to the fiber. In a similar way, the unique generator of $E^2_{0,1}$ is $-1 * ([1], \eta_0\star)$, which again is not a torsion element.

The differential map on a group $E^r_{p,q}$ can be computed making use of the function `spsq-dffr`. The last argument must be a list that represents the coordinates of the element we want to apply the differential to. In the following example, the differential $d^2_{2,0}$ is applied to the generator of the group $E^2_{2,0} \cong \mathbb{Z}$ (that, as we have seen, is the following combination of degree 2: $-1*(\eta_1\eta_0[\,], \texttt{s2})$), and therefore the list of coordinates must be `(1)`.

```
> (spsq-dffr kz1-twcp-s2 2 2 0 '(1))
(1)
```

The obtained list `(1)` shows that the result of applying $d^2_{2,0}$ to the generator of the group $E^2_{2,0} \cong \mathbb{Z}$ is the combination $1 * g^2_{0,1}$, where $g^2_{0,1}$ is the generator of $E^2_{0,1} \cong \mathbb{Z}$ (that is, the combination of degree 1: $-1 * ([1], \eta_0\star)$). This last result means that the differential map $d^2_{2,0} : E^2_{2,0} \to E^2_{0,1}$ maps $(\eta_1\eta_0[\,], \texttt{s2})$ to $([1], \eta_0\star)$. Since the next stage in the spectral sequence E^3 is isomorphic to the bigraded homology group of E^2 (in other words, $E^3_{p,q} \cong H_{p,q}(E^2) = \operatorname{Ker} d^2_{p,q}/\operatorname{Im} d^2_{p+2,q-1}$), it is clear that the groups $E^3_{0,1}$ and $E^3_{2,0}$ must be null.

```
> (spsq-group kz1-twcp-s2 3 0 1)
Spectral sequence E^3_{0,1}
nil
> (spsq-group kz1-twcp-s2 3 2 0)
Spectral sequence E^3_{2,0}
nil
```

Our programs also allow us to obtain, for each degree n, the level r at which the convergence of the spectral sequence has been reached, that is, the smallest r such that $E^\infty_{p,q} = E^r_{p,q}$ for all p, q with $p+q = n$. For instance, for $n = 0$ and $n = 1$ the convergence levels are 1 and 3 respectively.

```
>(spsq-cnvg kz1-twcp-s2 0)
1
>(spsq-cnvg kz1-twcp-s2 1)
3
```

Thus, we can obtain the groups $E^\infty_{p,q}$ with $p + q = 0$ or $p + q = 1$ by computing the corresponding groups $E^1_{0,0}$, $E^3_{0,1}$, and $E^3_{1,0}$.

```
> (spsq-group kz1-twcp-s2 1 0 0)
Spectral sequence E^1_{0,0}
Component Z
> (spsq-group kz1-twcp-s2 3 0 1)
Spectral sequence E^3_{0,1}
nil
> (spsq-group kz1-twcp-s2 3 1 0)
Spectral sequence E^3_{1,0}
nil
```

To finish with this example, we can also determine the filtration of the homology groups $H_*(G \times_\tau B)$ induced by the filtration $F_\times$ of the chain complex $C_*(G \times_\tau B)$. For instance, for $H_3(G \times_\tau B) \equiv H_3 \cong \mathbb{Z}$ we obtain $F_0H_3 = F_1H_3 = 0 \subset F_2H_3 = F_3H_3 = H_3 \cong \mathbb{Z}$.

```
> (hmlg-fltr kz1-twcp-s2 3 0)
Filtration F_0 H_3
nil
> (hmlg-fltr kz1-twcp-s2 3 1)
Filtration F_1 H_3
nil
> (hmlg-fltr kz1-twcp-s2 3 2)
Filtration F_2 H_3
Component Z
> (hmlg-fltr kz1-twcp-s2 3 3)
Filtration F_3 H_3
Component Z
```

3.1.4.2 Postnikov tower

We have already mentioned that the fibration $K(\mathbb{Z},1) \hookrightarrow K(\mathbb{Z},1) \times_\tau S^2 \to S^2$ is elementary and the computation of the associated spectral sequence can be done without any special difficulty. In this section we introduce a more complicated example, whose spectral sequence cannot be determined by hand. Our programs compute the groups $E^r_{p,q}$ in a short time for $p+q \leq 6$. For degree $p+q \geq 7$, several hours of computation were necessary.

We consider the space X_3 of a Postnikov tower [May67] with $\pi_i = \mathbb{Z}_2$ at each stage and the "simplest" non-trivial Postnikov invariant. The theoretical details of the construction of this space are not included here, they can be found in [RS05b]. This complex can be built by Kenzo by means of the following statements:

```
> (setf X2 (k-z2 2))
[K133 Abelian-Simplicial-Group]
> (setf k3 (chml-clss X2 4))
[K245 Cohomology-Class on K150 of degree 4]
> (setf tau3 (z2-whitehead X2 k3))
[K260 Fibration K133 -> K246]
```

```
> (setf X3 (fibration-total tau3))
[K266 Kan-Simplicial-Set]
```

The space X_3 is a twisted Cartesian product $X_3 = K(\mathbb{Z}_2, 3) \times_{k_3} K(\mathbb{Z}_2, 2)$, total space of a fibration $K(\mathbb{Z}_2, 3) \hookrightarrow X_3 \to K(\mathbb{Z}_2, 2)$. The twisting operator k_3 is called a *k-invariant* of the Postnikov tower. The object `K266` is already of finite type, but its effective homology gives us an associated effective chain complex which is much smaller. For instance, `X3` has $1,043,600$ generators in dimension 5 and the small chain complex `effX3` has only 6.

```
> (setf effX3 (rbcc (efhm X3)))
[K641 Chain-Complex]
```

In order to compute the Serre spectral sequence of our fibration, we must define filtrations on `X3` and `effX3` as done in Section 3.1.4.1.

```
> (change-chcm-to-flcc X3 crpr-flin '(crpr-flin))
[K443 Filtered-Kan-Simplicial-Set]
> (change-chcm-to-flcc effX3 tnpr-flin '(tnpr-flin))
[K641 Filtered-Chain-Complex]
```

As far as several examples of use of our new programs have already been explained, in this case we do not consider necessary to give many details about the calculations. In the following lines we include the computation of a few elements of the spectral sequence, and at the end of the section we will show all the obtained results by means of two diagrams.

Some groups $E^r_{p,q}$ at the stage $r = 2$ are:

```
> (spsq-group X3 2 4 0)
Spectral sequence E^2_{4,0}
Component Z/4Z
> (spsq-group X3 2 6 0)
Spectral sequence E^2_{6,0}
Component Z/2Z
> (spsq-group X3 2 5 3)
Spectral sequence E^2_{5,3}
Component Z/2Z
Component Z/2Z
```

For $p + q = 4$ or 5, the spectral sequence converges at the level $r = 5$.

```
> (spsq-cnvg X3 4)
5
> (spsq-cnvg X3 5)
5
```

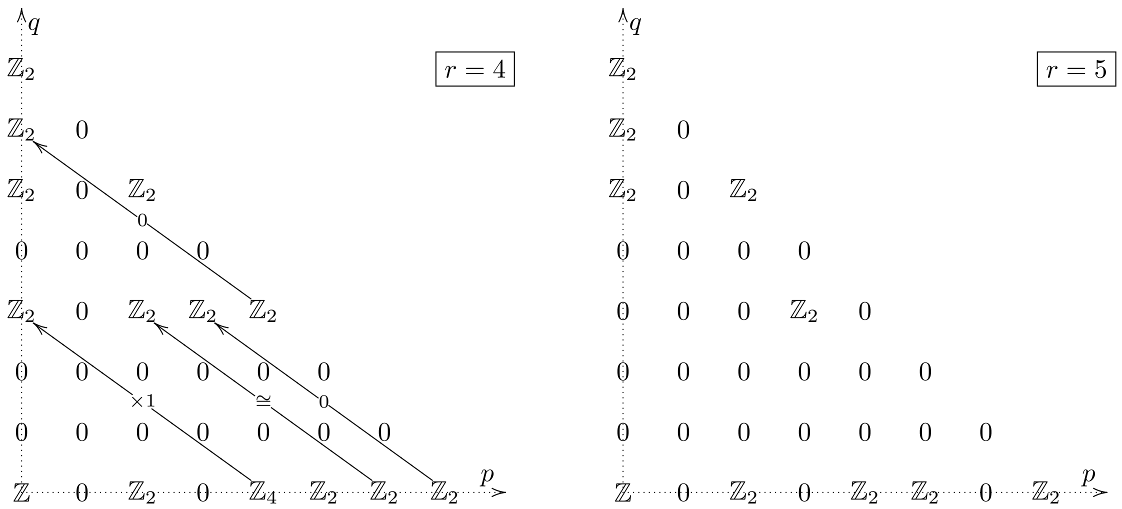
This means that there are some differential maps d^4 which are not null. To be precise, the programs compute $d^4_{4,0}$ and $d^4_{6,0}$ that map the unique generators of $E^4_{4,0} \cong \mathbb{Z}_4$ and $E^4_{6,0} \cong \mathbb{Z}_2$ to the unique generators of $E^4_{0,3}$ and $E^4_{2,3}$ (both isomorphic to $\mathbb{Z}_2$) respectively.

```
>  (spsq-dffr X3 4 4 0 '(1))
(1)
>  (spsq-dffr X3 4 6 0 '(1))
(1)
```

Finally, we can compute the filtration of the graded homology group. For degree $n = 6$ one has $H_6(X_3) \cong \mathbb{Z}_2^2$ and the filtration is given by:

```
> (dotimes (i 7)
     (hmlg-fltr X3 6 i))
Filtration F_0 H_6
Component Z/2Z
Filtration F_1 H_6
Component Z/2Z
Filtration F_2 H_6
Component Z/2Z
Filtration F_3 H_6
Component Z/2Z
Component Z/2Z
Filtration F_4 H_6
Component Z/2Z
Component Z/2Z
Filtration F_5 H_6
Component Z/2Z
Component Z/2Z
Filtration F_6 H_6
Component Z/2Z
Component Z/2Z
nil
```

To finish the study of this example, we show two figures which include all the results that our programs have determined. The two diagrams correspond to the *critical* levels $r = 4$ and $r = 5$ of the spectral sequence (only the groups $E^r_{p,q}$ with $p+q < 8$ are drawn).



The groups $E^4_{p,q}$ are the same as the corresponding $E^2_{p,q}$ and $E^3_{p,q}$, which means that the first non-null differential maps appear at stage $r = 4$. For $p + q \leq 6$, the spectral sequence converges at level $r = 5$, that is to say, $E^5_{p,q} = E^\infty_{p,q}$. For $p + q = 7$, the convergence is reached at the stage $r = 9$; the groups $E^5_{0,7} \cong \mathbb{Z}_2$ and $E^5_{2,5} \cong \mathbb{Z}_2$ die at levels 9 and 7 respectively.

In this case, from the final groups of the spectral sequence we can deduce the homology groups $H_0(X_3) \cong \mathbb{Z}, H_1(X_3) \cong H_3(X_3) = 0$, and $H_2(X_3) \cong H_4(X_3) \cong H_7(X_3) \cong \mathbb{Z}_2$. However, for $H_5(X_3)$ and $H_6(X_3)$ we find an extension problem with two possible solutions. The effective homology method, on the contrary, solves this problem and gives the correct answers $H_5(X_3) \cong \mathbb{Z}_4$ and $H_6(X_3) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

3.2 Eilenberg-Moore spectral sequence

3.2.1 Introduction

Let $G \hookrightarrow E \to B$ be a fibration, where the total space E can be seen as a twisted product of the fiber and the base spaces, $E = G \times_\tau B$. In the first part of this chapter we have introduced the Serre spectral sequence that, from knowledge of the homology groups of G and B, converges to the homology groups of the total space E. We can also consider the inverse problems: given the homology groups of E and B, how can we compute the homology groups of the fiber, $H_*(G)$? Or similarly, supposing that $H_*(E)$ and $H_*(G)$ are known, is it possible to determine $H_*(B)$?

The Eilenberg-Moore spectral sequence was introduced in [EM65b], trying to give a solution to these questions. In fact there are two Eilenberg-Moore spectral sequences, corresponding to each one of the two different problems exposed before. In this section we will explain the *Cotor* spectral sequence, which expresses the homology of the fiber space G as a "Cotor" operation between the homologies of the base space and the total space. The symmetric *Tor* spectral sequence describes the homology of the base space as a "Tor" involving the homologies of the total space, the fiber space and the structural group. The algebraic functors $\mathrm{Tor}^{H_*(B)}(-,-)$ and $\mathrm{Tor}^{C_*(B)}(-,-)$ were introduced by John C. Moore in [Moo59] and generalize the derived functor $\mathrm{Tor}^{R}(-,-)$ of the tensor product of the category of differential graded modules over a differential graded algebra. Dualizing these notions, we obtain the *Cotor* construction that appears in the definition of the Eilenberg-Moore spectral sequence. For details, see [EM65b].

Theorem 3.7 (Eilenberg-Moore spectral sequence). [EM65b] Let $G \hookrightarrow E \to B$ be a fibration with B a 1-reduced simplicial set. There exists a second quadrant spectral sequence $(E^r, d^r)_{r\geq 1}$ such that

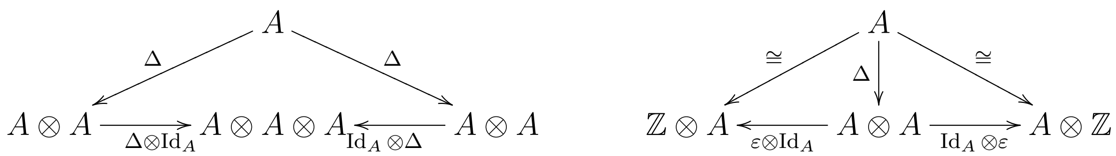
$$E^2_{p,q} = \mathrm{Cotor}^{H_*(B)}(H_*(E), \mathbb{Z})$$

that converges to the graded group $H_*(G)$.

As in the case of the Serre spectral sequence, one could think that this spectral sequence is an algorithm that allows us to compute the homology groups of the fiber space G when $H_*(B)$ and $H_*(E)$ are known, but this is false. And as we will see, the effective homology method will again give a solution to this problem.

The Eilenberg-Moore spectral sequence is built in fact as the spectral sequence of a second quadrant bicomplex. In order to explain this construction, the following definitions are necessary.

Definition 3.8. A $\mathbb{Z}$*-coalgebra* A is a $\mathbb{Z}$-module with two morphisms, an associative *coproduct* $\Delta : A \to A \otimes A$ and a *counit* $\varepsilon : A \to \mathbb{Z}$, such that the following diagrams are commutative:



Definition 3.9. A *differential (graded) coalgebra* is a chain complex $A_* = (A_n, d_n)_{n\in\mathbb{N}}$ together with a $\mathbb{Z}$-module morphism $\varepsilon : A_0 \to \mathbb{Z}$ (the *counit*, which induces a chain complex morphism $\varepsilon : A_* \to \mathbb{Z}$ where $\mathbb{Z}$ denotes the chain complex $C_*(\mathbb{Z}, 0)$ with only one non-null group $C_0(\mathbb{Z}, 0) = \mathbb{Z}$) and a chain complex morphism $\Delta : A_* \to A_* \otimes A_*$ (the *coproduct*), satisfying the commutativity properties of a coalgebra.

In particular, the chain complex associated with a simplicial set can be seen as a differential coalgebra as follows.

Definition 3.10. Let K be a simplicial set. The *canonical coalgebra structure* of the chain complex $C_*(K)$ is defined by means of the *Alexander-Whitney coproduct* $\Delta : C_*(K) \to C_*(K) \otimes C_*(K)$ given by

$$\Delta(x_n) = \sum_{i=0}^{n} \partial_{i+1} \dots \partial_n x_n \otimes \partial_0 \dots \partial_{i-1} x_n, \quad \text{for } x_n \in K_n$$

and the counit $\varepsilon : C_0(K) \to \mathbb{Z}$ defined as $\varepsilon(x_0) = 1$ if $x_0 \in K_0$.

Definition 3.11. Let A_* be a differential coalgebra, a *differential right comodule* is a chain complex M_* provided with an external coproduct $\Delta_M : M_* \to M_* \otimes A_*$ such that $(\mathrm{Id}_{M_*} \otimes \varepsilon) \circ \Delta_M = \mathrm{Id}_{M_*}$ and $(\mathrm{Id}_{M_*} \otimes \Delta) \circ \Delta_M = (\Delta_M \otimes \mathrm{Id}_{A_*}) \circ \Delta_M$.
An analogous definition is given for a *differential left comodule.*

From now on in this section, our differential coalgebras A_* are assumed to be 1-reduced, that is to say, the component A_0 is isomorphic to $\mathbb{Z}$ by means of the counit $\varepsilon : A_0 \to \mathbb{Z}$, and the component A_1 is null. We denote $1 \equiv \varepsilon^{-1}(1) \in A_0$.

Given a 1-reduced differential coalgebra A_*, we define a new chain complex $\bar{A}_* = (\bar{A}_n, \bar{d}_n)_{n\in\mathbb{N}}$ with

$$\bar{A}_n = \begin{cases} 0 & \text{if } n = 0 \\ A_n & \text{if } n > 0 \end{cases}$$

and differential map $\bar{d}_n = d_n$ for all $n \in \mathbb{N}$.

One can also define a new chain complex morphism $\bar{\Delta} : \bar{A}_* \to \bar{A}_* \otimes \bar{A}_*$ given by $\bar{\Delta}(a) = \Delta(a) - 1 \otimes a - a \otimes 1$. Furthermore, if M_* is a differential right comodule, then it is possible to construct a morphism $\bar{\Delta}_M : M_* \to M_* \otimes \bar{A}_*$ where $\bar{\Delta}_M(x) = \Delta_M(x) - x \otimes 1$ (and similarly for a differential left comodule). The "coproducts" $\bar{\Delta}$ and $\bar{\Delta}_M$ are also associative.

We denote by $\bar{A}_*^{\otimes p}$ the iterated tensor product $\bar{A}_* \otimes \overset{p}{\cdots} \otimes \bar{A}_*$.

Definition 3.12. Let A_* be a differential coalgebra, M_* a differential right comodule, and N_* a differential left comodule. The chain complex $\mathrm{Cobar}^{A_*}(M_*, N_*)$ is defined by

$$\mathrm{Cobar}^{A_*}(M_*, N_*)_n = \bigoplus_{p\geq 0} (M_* \otimes \bar{A}_*^{\otimes p} \otimes N_*)_{n+p}$$

with differential map given by two components: the *tensorial differential* d_t is deduced from the differential maps of M_*, A_*, and N_*, and the *cosimplicial differential* d_c is defined by means of the various coproducts. More specifically:

$$\begin{aligned}
d_t(x \otimes a_1 \otimes \cdots \otimes a_p \otimes y) =& (-1)^p d_M(x) \otimes a_1 \otimes \cdots \otimes a_p \otimes y \\
&+ (-1)^{p+|x|} x \otimes d_A(a_1) \otimes \cdots \otimes a_p \otimes y \\
&+ \dots \\
&+ (-1)^{p+|x|\cdot|a_1|\cdots|a_{p-1}|} x \otimes a_1 \otimes \cdots \otimes d_A(a_p) \otimes y \\
&+ (-1)^{p+|x|\cdot|a_1|\cdots|a_p|} x \otimes a_1 \otimes \cdots \otimes a_p \otimes d_N(y) \\
d_c(x \otimes a_1 \otimes \cdots \otimes a_p \otimes y) =& \bar{\Delta}_M(x) \otimes a_1 \otimes \cdots \otimes a_p \otimes y \\
&- x \otimes \bar{\Delta}(a_1) \otimes \cdots \otimes a_p \otimes y \\
&\pm \dots \\
&+ (-1)^p x \otimes a_1 \otimes \cdots \otimes \bar{\Delta}(a_p) \otimes y \\
&+ (-1)^{p+1} x \otimes a_1 \otimes \cdots \otimes a_p \otimes \bar{\Delta}_N(y)
\end{aligned}$$

where $|x|$ and $|a_j|$ denote the degrees of $x \in M_*$ and $a_j \in A_*$.

The number n is the total degree, and in this case the complementary degree is given by $q = n + p$. We denote $C_{p,q} \equiv (M_* \otimes \bar{A}_*^{\otimes p} \otimes N_*)_q$ and we observe that if $z = x \otimes a_1 \otimes \cdots \otimes a_p \otimes y \in C_{p,q}$, then $d_t(z) \in C_{p,q-1}$ and $d_c(z) \in C_{p+1,q}$. If we *change* the sign of the index p (in other words, we consider p with negative sign), then $\mathrm{Cobar}^{A_*}(M_*, N_*)$ can be seen as a second quadrant bicomplex with horizontal differential $d' = d_c$ and vertical differential map $d'' = d_t$.

The chain complex $\mathrm{Cobar}^{A_*}(M_*, N_*)$ is a generalization of the Cobar construction introduced by J. Frank Adams in [Ada56]. This particular case is obtained when $M_* = N_* = \mathbb{Z}$, a chain complex with a unique non-null group in degree 0. The group of n-chains of $\mathrm{Cobar}^{A_*}(\mathbb{Z}, \mathbb{Z}) \equiv \mathrm{Cobar}(A_*)$ is then given by

$$\mathrm{Cobar}(A_*)_n = \bigoplus_{p \geq 0} (\bar{A}_{i_1} \otimes \cdots \otimes \bar{A}_{i_p})_{n+p}$$

with $i_1, \ldots, i_p \geq 0$ and $i_1 + \cdots + i_p - p = n$. Let us remark that $\bar{A}_0 = \bar{A}_1 = 0$, which implies that $\bar{A}_{i_1} \otimes \cdots \otimes \bar{A}_{i_p} = 0$ if $i_1 + \cdots + i_p = q < 2p$. We can represent this second quadrant bicomplex (in the first quadrant) as follows:

$$\begin{array}{c|cccc}
q & & & & \\
0 & \bar{A}_6 & \xrightarrow{d_c} (\bar{A}_* \otimes \bar{A}_*)_6 & \xrightarrow{d_c} & (\bar{A}_* \otimes \bar{A}_* \otimes \bar{A}_*)_6 \\
 & \downarrow d_t & \downarrow d_t & & \\
0 & \bar{A}_5 & \xrightarrow{d_c} (\bar{A}_* \otimes \bar{A}_*)_5 & & 0 \\
 & \downarrow d_t & \downarrow d_t & & \\
0 & \bar{A}_4 & \xrightarrow{d_c} (\bar{A}_* \otimes \bar{A}_*)_4 & & 0 \\
 & \downarrow d_t & & & \\
0 & \bar{A}_3 & 0 & & 0 \\
 & \downarrow d_t & & & \\
0 & \bar{A}_2 & 0 & & 0 \\
0 & 0 & 0 & & 0 \\
\hline
\mathbb{Z} & 0 & 0 & & 0 \quad p
\end{array}$$

It is worth emphasizing that the second quadrant bicomplex $\mathrm{Cobar}(A_*)$ is *tapered*, that is to say, $C_{p,q} = 0$ if $q < 2p$, and in particular it is bounded. This property is also satisfied in the general case $\mathrm{Cobar}^{A_*}(M_*, N_*)$ whenever A_* is a 1-reduced coalgebra, and will be relevant when computing its effective homology.

On the other hand, one can see that the chain complex $\mathrm{Cobar}^{A_*}(M_*, N_*)$ is canonically isomorphic to the bicomplex used by Eilenberg and Moore in [EM65b] in order to define the Cotor functor and the corresponding spectral sequence. In particular, this implies that the Eilenberg-Moore spectral sequence associated with a fibration $G \hookrightarrow E \to B$ (with 1-reduced base space B), which is known to converge to the homology groups of the fiber space G, can be computed as the spectral sequence of the bicomplex

$$\mathrm{Cobar}^{C_*(B)}(C_*(E), \mathbb{Z})$$

Nevertheless, this filtered complex can be very complicated and the computation of the associated spectral sequence cannot always be done in a direct way. The effective homology will help us to complete the calculations.

3.2.2 Effective homology of the fiber space of a fibration

As in the computation of the effective homology of a twisted product explained in Section 3.1.2, all the chain complexes associated with simplicial sets which appear in this section are supposed to be normalized. Let us notice that if B is a 1-reduced simplicial set (that is to say, $B_0 = B_1 = \{\star\}$), then the normalized chain complex $C_*^N(B) \equiv C_*(B)$ is a 1-reduced differential coalgebra.

Let $G \hookrightarrow E \rightarrow B$ be a fibration where the base space B is 1-reduced. The simplicial sets B and $E = G \times_\tau B$ are supposed to be objects with effective homology, in other words, two equivalences

$$C_*(B) \Lleftarrow\!\!\!\!\!\Lleftarrow DB_* \Rrightarrow HB_* \qquad C_*(E) \Lleftarrow\!\!\!\!\!\Lleftarrow DE_* \Rrightarrow HE_*$$

are given, where HB_* and HE_* are effective chain complexes. In the following lines we give an overview of the construction of the effective homology of the fiber space G. Details are explained in [Rub91] or [RS06].

On the one hand, and thanks to the effective homologies of the simplicial sets B and $E = G \times_\tau B$, it is not difficult to build the effective homology of the chain complex $\mathrm{Cobar}^{C_*(B)}(C_*(G) \otimes_t C_*(B), \mathbb{Z})$, where $\otimes_t$ denotes the twisted tensor product obtained by applying the twisted Eilenberg-Zilber theorem. It is in fact a particular application of the computation of the effective homology of a bicomplex explained in Section 2.4.1; in this case one has a second quadrant bicomplex but, provided that $C_*(B)$ is a 1-reduced coalgebra, the bicomplex is tapered (and therefore bounded), which guarantees the local nilpotency condition necessary for the application of the Basic Perturbation Lemma.

The effective homology of the columns is given simply by some iterated tensor products of the effective homologies of the chain complexes $C_*(B)$ and $C_*(G) \otimes_t C_*(B)$ (the last one is obtained as the composition of the given effective homology of $E = G \times_\tau B$ and the twisted Eilenberg-Zilber reduction $C_*(G \times_\tau B) \Rrightarrow C_*(G) \otimes_t C_*(B)$). In a first step, we cancel the horizontal differential of $\mathrm{Cobar}^{C_*(B)}(C_*(G) \otimes_t C_*(B), \mathbb{Z})$, which is nothing but replacing the $C_*(B)$-coproduct by $\Delta_0(x) = 1 \otimes x + x \otimes 1$. Then the horizontal differential is reinstalled as a perturbation, and applying the Trivial Perturbation Lemma on the left reduction and the Basic Perturbation Lemma on the right one we obtain an equivalence

$$
\begin{array}{ccc}
& \widetilde{\mathrm{Cobar}}^{DB_*}(DE_*, \mathbb{Z}) & \\
\swarrow\!\!\!\swarrow & & \searrow\!\!\!\searrow \\
\mathrm{Cobar}^{C_*(B)}(C_*(E), \mathbb{Z}) & & \widetilde{\mathrm{Cobar}}^{HB_*}(HE_*, \mathbb{Z})
\end{array}
$$

where the $\widetilde{\mathrm{Cobar}}$'s are second quadrant multicomplexes that are somehow similar to the corresponding Cobar constructions, but now new differentials appear (the notion of A^∞-structure was designed by James Stasheff to handle such a situation [Sta63]). Since HB_* and HE_* are effective chain complexes, the right chain complex $\widetilde{\mathrm{Cobar}}^{HB_*}(HE_*, \mathbb{Z})$ is effective too, and therefore this equivalence determines the effective homology of $\mathrm{Cobar}^{C_*(B)}(C_*(E), \mathbb{Z})$.

On the other hand, although we are not going to give details about this fact (see [RS06] if necessary), it is possible to construct a reduction

$$\mathrm{Cobar}^{C_*(B)}(C_*(G) \otimes_t C_*(B), \mathbb{Z}) \Rrightarrow C_*(G)$$

And finally, the composition of the two equivalences

$$
\begin{array}{ccccc}
& \mathrm{Cobar}^{C_*(B)}(C_*(G) \otimes_t C_*(B), \mathbb{Z}) & & \widetilde{\mathrm{Cobar}}^{DB_*}(DE_*, \mathbb{Z}) & \\
\swarrow\!\!\!\swarrow & & \overset{\mathrm{Id}}{\searrow\!\!\!\searrow} \quad \swarrow\!\!\!\swarrow & & \searrow\!\!\!\searrow \\
C_*(G) & & \mathrm{Cobar}^{C_*(B)}(C_*(G) \otimes_t C_*(B), \mathbb{Z}) & & \widetilde{\mathrm{Cobar}}^{HB_*}(HE_*, \mathbb{Z})
\end{array}
$$

provides us the looked-for effective homology of the fiber space G.

In this way, the effective homology method gives an algorithm for the computation of the homology groups of the fiber space G of our fibration $G \hookrightarrow E \to B$ whenever B and E are objects with effective homology (and B is 1-reduced). As for the Serre spectral sequence, this effective homology can also be used to determine all the components of the associated Eilenberg-Moore spectral sequence, as we explain in the following section.

3.2.3 An algorithm computing the Eilenberg-Moore spectral sequence

Given a fibration $G \hookrightarrow E \to B$ with a 1-reduced base space B, the associated Eilenberg-Moore spectral sequence can be defined as the spectral sequence of the second quadrant bicomplex

$$\mathrm{Cobar}^{C_*(B)}(C_*(E), \mathbb{Z})$$

Thanks to the reduction $C_*(G \times_\tau B) = C_*(E) \Rrightarrow C_*(G) \otimes_t C_*(B)$ (given by the twisted Eilenberg-Zilber theorem), it can be seen that this spectral sequence is isomorphic to that of the new bicomplex

$$\mathrm{Cobar}^{C_*(B)}(C_*(G) \otimes_t C_*(B), \mathbb{Z})$$

The effective homology of this chain complex has been determined in Section 3.2.2 as part of the effective homology of the fiber space G, following the general method for the computation of the effective homology of a bicomplex introduced in Section 2.4.1. We remark again that our bicomplex is tapered and therefore the local nilpotency condition which is necessary for the application of the BPL is satisfied.

In a similar way, our general Algorithm 5 for the computation of the spectral sequence associated with a bicomplex can also be applied in this particular case. As explained in Section 2.4.2, the spectral sequence of a bounded bicomplex is isomorphic at every level $r \geq 1$ to the spectral sequence of the associated effective multicomplex, which in our case is $\widetilde{\text{Cobar}}^{HB_*}(HE_*, \mathbb{Z})$. In this way, it is possible to determine the groups and differential maps for every stage $r \geq 1$, the convergence level, and the filtration induced on the homology groups of $\text{Cobar}^{C_*(B)}(C_*(E), \mathbb{Z})$, which are in fact isomorphic to $H_*(G)$. This produces a new algorithm that allows us to determine the Eilenberg-Moore spectral sequence associated with a fibration $G \hookrightarrow E \to B$ when the base and total spaces are objects with effective homology.

Algorithm 7.
Input:

- a fibration $G \hookrightarrow E \to B$ defined by a twisting operator $\tau : B \to G$, with B a 1-reduced simplicial set,
- equivalences $C_*(B) \twoheadleftarrow\!\!\Leftarrow DB_* \Rightarrow\!\!\!\!\!\twoheadrightarrow HB_*$ and $C_*(E) \twoheadleftarrow\!\!\Leftarrow DE_* \Rightarrow\!\!\!\!\!\twoheadrightarrow HE_*$, where HB_* and HE_* are effective chain complexes.

Output: all the components of the associated Eilenberg-Moore spectral sequence:

- the groups $E^r_{p,q}$ for every $p, q \in \mathbb{Z}$ and $r \geq 1$, with their basis-divisors representation,
- the differential maps $d^r_{p,q}$ for all $p, q \in \mathbb{Z}$ and $r \geq 1$,
- the convergence level for each degree $n \in \mathbb{N}$,
- the filtration of the homology groups $H_*(G)$, that is to say, the groups $F_{H_p}H_n(G)$ for each degree $n \in \mathbb{N}$ and filtration index $p \in \mathbb{Z}$.

3.2.4 Loop spaces

We consider a particular case of fibration $G \hookrightarrow E \to B$ where G is the "inverse" of B, in other words, the total space E is contractible. For a fibration of topological spaces, the inverse of B is given by the *loop space* $\Omega(B)$, which is the space of all the continuous maps $f : I = [0, 1] \to B$ such that $f(0) = f(1) = \star$. The *path space* $P(B)$ is the space of all the continuous maps $f : I = [0, 1] \to B$ such that $f(0) = \star$. Then, a canonical fibration $\Omega(B) \hookrightarrow P(B) \to B$ is defined, and it can be seen that the total space $P(B)$ is contractible.

The analogous construction in simplicial topology was introduced by Daniel Kan [Kan58] and works as follows.

Definition 3.13. Let X be a reduced simplicial set. Define $G_n(X)$ as the free group generated by the set $X_{n+1} - \eta_0 X_n$. A map $\tau : X_{n+1} \to G_n(X)$ is given by

$$\begin{aligned} \tau(x_{n+1}) &= x_{n+1} \quad \text{if } x_{n+1} \notin \eta_0 X_n \\ \tau(x_{n+1}) &= e_n \qquad \text{if } x_{n+1} \in \eta_0 X_n \end{aligned}$$

where e_n is the null element of $G_n(X)$. If $x_{n+1} \in X_{n+1} - \eta_0 X_n$, let us denote by $\tau(x_{n+1})$ the corresponding generator of $G_n(X)$. Face and degeneracy operators are defined on the generators of $G_n(X)$ as follows:

$$\begin{aligned} \partial_0 \tau(x_{n+1}) &= \tau(\partial_1 x_{n+1}) \cdot \tau(\partial_0 x_{n+1})^{-1} & \\ \partial_i \tau(x_{n+1}) &= \tau(\partial_{i+1} x_{n+1}) & \text{if } 1 \leq i \leq n \\ \eta_i \tau(x_{n+1}) &= \tau(\eta_{i+1} x_{n+1}) & \text{if } 0 \leq i \leq n \end{aligned}$$

These definitions can be extended to group morphisms $\partial_i : G_n(X) \to G_{n-1}(X)$ and $\eta_i : G_n(X) \to G_{n+1}(X)$, for $0 \leq i \leq n$, so that $G(X) = \{G_n(X), \partial_i, \eta_i\}_{n \geq 0}$ is a simplicial group.

The map $\tau : X \to G(X)$ is clearly a twisting operator, which defines a fibration $G(X) \hookrightarrow G(X) \times_\tau X \to X$. Furthermore, it can be proved (see [May67]) that the twisted product $G(X) \times_\tau X$ is contractible, so that $G(X)$ can be seen as the *inverse* of X. Since $G(X)$ is a combinatorial model for the loop space construction, in the sequel we will denote it by $G(X) \equiv \Omega(X)$.

The problem now is: given a simplicial set X (whose homology groups are known), is it possible to compute the homology groups of its loop space, $H_*(\Omega(X))$? More generally, can we determine the homology groups of the iterated loop space $\Omega^k(X)$ for $k \geq 1$? This question is known as the *Adams' problem*, and it has been solved only for some particular cases. In 1956, Frank Adams (see [Ada56] and [AH56]) constructed an algorithm computing $H_*(\Omega(X))$, based on his famous Cobar construction (previously mentioned in the introduction of this section), valid when X is a simplicial set of finite type. Nevertheless, it was not possible to extend it to the second loop space $\Omega^2(X)$. Eighteen years later, Hans Baues [Bau80] gave a solution for the case $k = 2$, but again it was not valid for the third loop space $\Omega^3(X)$. For X arbitrary and $k \geq 2$, the problem has not yet been solved by traditional methods. On the contrary, the effective homology method gives a solution to the Adams' problem.

Let X be a 1-reduced simplicial set with effective homology. The effective homology of the loop space $\Omega(X)$ is computed as a particular case of the process explained in Section 3.2.2 for the computation of the fiber space of a fibration. In this case, one has a fibration $\Omega(X) \hookrightarrow \Omega(X) \times_\tau X \to X$ where the base space X is given with its effective homology and the total space $E = \Omega(X) \times_\tau X$ is contractible, such that a reduction

$C_*(\Omega(X) \times_\tau X) \Rrightarrow \mathbb{Z}$ can be built. The associated effective chain complex of $\Omega(X)$ is then the second quadrant multicomplex $\widetilde{\text{Cobar}}^{HX_*}(\mathbb{Z},\mathbb{Z})$.

In this way, the effective homology method gives an algorithm for the computation of the homology groups of $\Omega(X)$: if X is a 1-reduced simplicial set with effective homology, then it is possible to compute the effective homology of $\Omega(X)$ and in particular we can compute the groups $H_*(\Omega(X))$. Furthermore, if X is k-reduced, the process may be iterated k times, producing an effective homology version of $\Omega^m(X)$, for $m \leq k$. This provides a solution to the Adams' problem.

On the other hand, our Algorithm 7 allows us to compute the Eilenberg-Moore spectral sequence associated with the fibration $\Omega(X) \hookrightarrow \Omega(X) \times_\tau X \to X$, which is known to converge to the homology groups $H_*(\Omega(X))$. In particular, we know that this spectral sequence is isomorphic to the spectral sequence associated with the effective multicomplex $\widetilde{\text{Cobar}}^{HX_*}(\mathbb{Z},\mathbb{Z})$. In the following section, two examples of calculations are included.

3.2.5 Implementation and examples

The new module for the Kenzo system presented in Section 2.5, which allows computations of spectral sequences associated with filtered complexes, makes it possible to determine the Eilenberg-Moore spectral sequence between a simplicial set X and its loop space $\Omega(X)$ when X is an object with effective homology. In particular, our programs have determined the different elements of the spectral sequences of some spaces that, up to now, have not appeared in the literature.

In this section we focus our attention on the study of the Eilenberg-Moore spectral sequence of the spaces $X = \Omega(S^3)$ and $Y = \Omega(S^3) \cup_2 D^3$. The first space and its loop space have been extensively considered by theoretical methods and a lot of results about them are known. However, for our second example, the attachment of the 3-disk increases the difficulty of the computation of the Eilenberg-Moore spectral sequence between $\Omega(S^3) \cup_2 D^3$ and its loop space which, up to our knowledge, had not been determined before.

3.2.5.1 $\Omega(S^3)$

Let us consider the simplicial set $X = \Omega(S^3)$. It is built by Kenzo my means of the following statements:

```
> (setf s3 (sphere 3))
[K660 Simplicial-Set]
> (setf X (loop-space s3))
[K665 Simplicial-Group]
```

The loop space of X is located through the symbol `OX`, and it is a simplicial group.

```
> (setf OX (loop-space X))
[K677 Simplicial-Group]
```

We can see that this simplicial group is not effective: if we ask for its basis in dimension 4, we obtain an error.

```
> (basis OX 4)
Error: The object [K677 Simplicial-Group] is locally-effective.
```

Now we consider the effective complex of $\Omega(X)$, in this case $\widetilde{\mathrm{Cobar}}^{C_*(S^3)}(\mathbb{Z}, \mathbb{Z})$, which is the right chain complex of the equivalence providing its effective homology.

```
> (setf effOX (rbcc (efhm OX)))
[K906 Chain-Complex]
```

Obviously this complex is effective, and therefore we can obtain its basis, which in dimension 4 is a list of 3 elements.

```
> (basis effOX 4)
(<<AlLp[1 <<AlLp[2 s3]>>][3 <<AlLp[2 s3][2 s3]>>]>> <<AlLp[3 <<AlLp[2 s3][2 s3]>>]
[1 <<AlLp[2 s3]>>]>> <<AlLp[1 <<AlLp[2 s3]>>][1 <<AlLp[2 s3]>>][1 <<AlLp[2 s3]>>]
[1 <<AlLp[2 s3]>>]>>)
```

And after these first instructions, we are going to compute the Eilenberg-Moore spectral sequence associated with the simplicial set X and its loop space $\Omega(X)$, which, as we know, is the spectral sequence of the effective complex of $\Omega(X)$.

Previously, we must define a filtration on this effective complex, which is the natural filtration of the Cobar construction (given by the column number). It is implemented by means of the function `cobar-flin`. Let us emphasize that $\widetilde{\mathrm{Cobar}}^{C_*(S^3)}(\mathbb{Z}, \mathbb{Z})$ is a second quadrant multicomplex, and therefore the first index p has negative sign.

```
> (change-chcm-to-flcc effOX cobar-flin '(cobar-flin))
[K906 Filtered-Chain-Complex]
```

Once the filtration has been defined, we can compute the spectral sequence. We can obtain some groups, for instance $E^1_{-2,6} \cong \mathbb{Z}^2$, $E^1_{-2,8} \cong \mathbb{Z}^3$, and $E^1_{-3,10} \cong \mathbb{Z}^6$.

```
> (spsq-group effOX 1 -2 6)
Spectral sequence E^1_{-2,6}
Component Z
Component Z
> (spsq-group effOX 1 -2 8)
Spectral sequence E^1_{-2,8}
Component Z
Component Z
Component Z
> (spsq-group effOX 1 -3 10)
Spectral sequence E^1_{-3,10}
Component Z
Component Z
Component Z
Component Z
Component Z
Component Z
```

It is also possible to compute the differential maps, specifying in a list the coefficients of the element we want to apply the differential to, with respect to the generators of the group. For example, $d^1_{-2,8} : E^1_{-2,8} \cong \mathbb{Z}^3 \to E^1_{-3,8} \cong \mathbb{Z}^3$ applied to the three generators of the group $E^1_{-2,8}$:

```
> (spsq-dffr effOX 1 -2 8 '(1 0 0))
(3 3 0)
> (spsq-dffr effOX 1 -2 8 '(0 1 0))
(-2 0 2)
> (spsq-dffr effOX 1 -2 8 '(0 0 1))
(0 -3 -3)
```

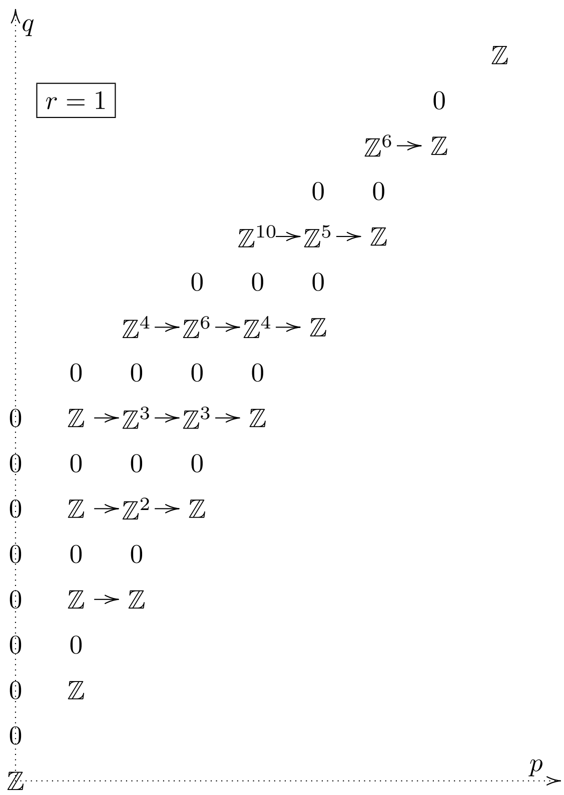
In this case the convergence level is $r = 1$ for $n = 0$ and 1, and $r = 2$ for every $2 \leq n \leq 8$.

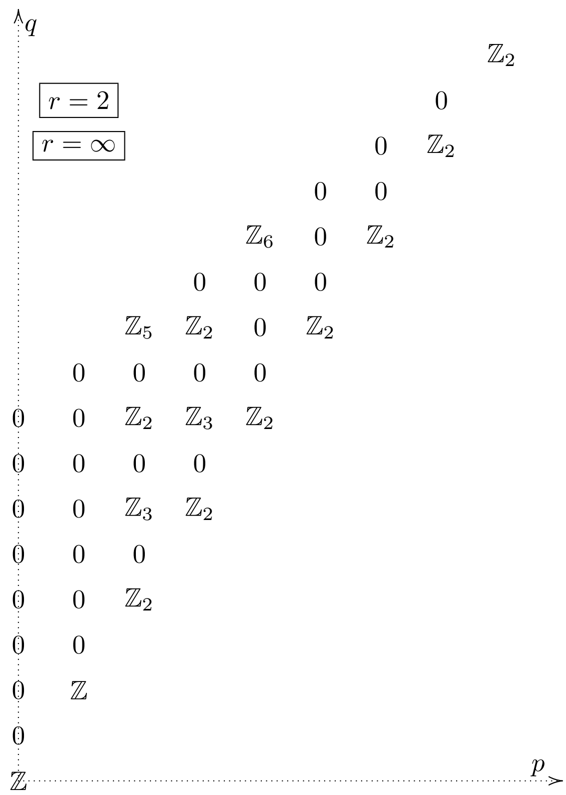
```
> (dotimes (n 9)
     (format t "~1%Convergence level for n=~D: ~D" n (spsq-cnvg effOX n)))
Convergence level for n=0: 1
Convergence level for n=1: 1
Convergence level for n=2: 2
Convergence level for n=3: 2
Convergence level for n=4: 2
Convergence level for n=5: 2
Convergence level for n=6: 2
Convergence level for n=7: 2
Convergence level for n=8: 2
nil
```

And finally, the filtration of the homology groups. For instance, the homology group in dimension 5 is $H_5(\Omega(X)) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3 \cong \mathbb{Z}_6$. We observe the different groups for each filtration index.

```
> (homology effOX 5)
Homology in dimension 5 :
Component Z/3Z
Component Z/2Z
---done---
> (dotimes (i 7)
    (hmlg-fltr effOX 5 (- i)))
Filtration F_0 H_5
Component Z/6Z
Filtration F_-1 H_5
Component Z/6Z
Filtration F_-2 H_5
Component Z/6Z
Filtration F_-3 H_5
Component Z/6Z
Filtration F_-4 H_5
Component Z/2Z
Filtration F_-5 H_5
Component Z/2Z
Filtration F_-6 H_5
nil
```

In the following figures we include the *critical* stages $r = 1$ and $r = 2$, for degrees $p + q \leq 8$. The groups $E^2_{p,q}$ are in fact the final groups $E^\infty_{p,q}$. Although it is a second quadrant spectral sequence, we represent it in the first quadrant.





3.2.5.2 $\Omega(S^3) \cup_2 D^3$

The second example considered in this section is the simplicial set $Y = \Omega(S^3) \cup_2 D^3$, obtained from $\Omega(S^3)$ by attaching a 3-disk by a map $\gamma : S^2 \to \Omega(S^3)$ of degree 2. The Eilenberg-Moore spectral sequence associated with its loop space $\Omega(Y)$ can be computed as follows.

We begin by constructing the spaces Y, $\Omega(Y)$, and the associated filtered (effective) chain complex which defines the spectral sequence.

```
> (setf Y
        (disk-pasting X 3 'new
          (list (loop3 0 's3 1)
            (absm 3 +null-loop+)
            (loop3 0 's3 1)
            (absm 3 +null-loop+))))
[K925 Simplicial-Set]
> (setf OY (loop-space Y))
[K930 Simplicial-Group]
> (setf effOY (rbcc (efhm OY)))
[K1071 Chain-Complex]
> (change-chcm-to-flcc effOY cobar-flin '(cobar-flin))
[K1071 Filtered-Chain-Complex]
```

Some groups $E^r_{p,q}$:

```
> (spsq-group effOY 1 -2 8)
Spectral sequence E^1_{-2,8}
Component Z/2Z
Component Z/2Z
Component Z
> (spsq-group effOY 1 -3 8)
Spectral sequence E^1_{-3,8}
Component Z/2Z
Component Z/2Z
Component Z/2Z
Component Z/2Z
```

Differential map $d^1_{-2,8} : E^1_{-2,8} = \mathbb{Z}_2^2 \oplus \mathbb{Z} \to E^1_{-3,8} = \mathbb{Z}_2^4$:

```
> (spsq-dffr effOY 1 -2 8 '(1 0 0))
(1 0 1 0)
> (spsq-dffr effOY 1 -2 8 '(0 1 0))
(0 0 1 1)
> (spsq-dffr effOY 1 -2 8 '(0 0 1))
(0 0 0 0)
```

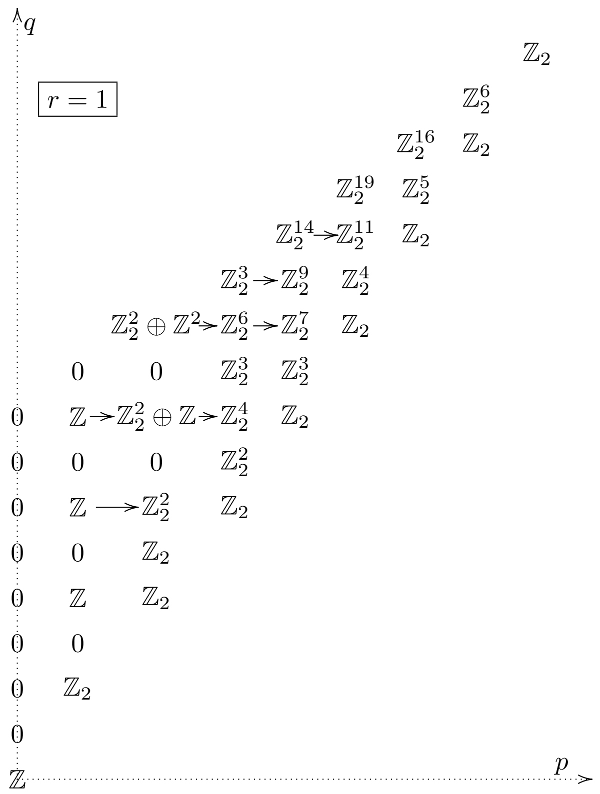
The convergence level for $n \leq 8$ is again 1 or 2:

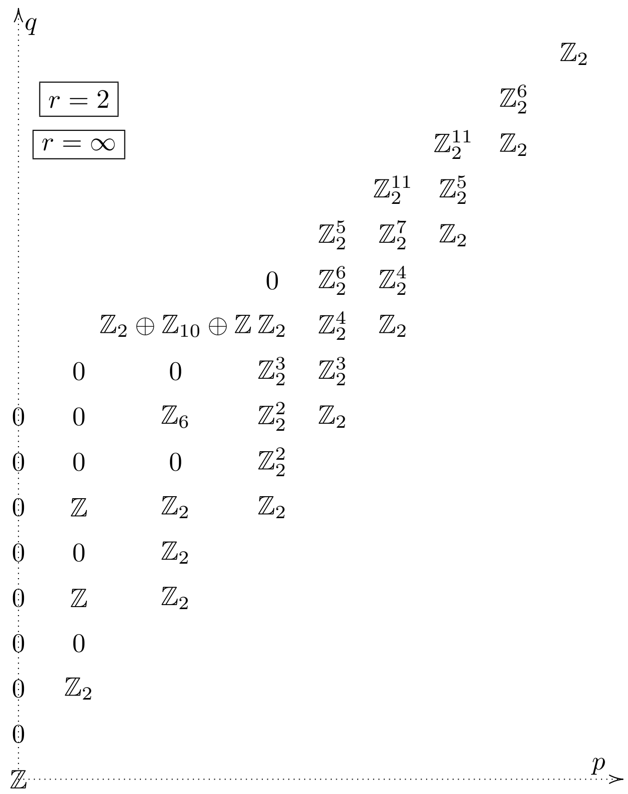
```
> (dotimes (n 5)
     (format t "~1%Convergence level for n=~D: ~D" n (spsq-cnvg effOY n)))

Convergence level for n=0: 1
Convergence level for n=1: 1
Convergence level for n=2: 1
Convergence level for n=3: 1
Convergence level for n=4: 2
Convergence level for n=5: 2
Convergence level for n=6: 2
Convergence level for n=7: 2
Convergence level for n=8: 2
nil
```

Filtration of the homology group $H_4(\Omega(Y)) = \mathbb{Z}_2^4$.

```
> (homology OY 4)
Homology in dimension 4 :
Component Z/2Z
Component Z/2Z
Component Z/2Z
Component Z/2Z
---done---
> (hmlg-fltr effOY 4 -4)
Filtration F_-4 H_4
Component Z/2Z
> (hmlg-fltr effOY 4 -3)
Filtration F_-3 H_4
Component Z/2Z
Component Z/2Z
Component Z/2Z
> (hmlg-fltr effOY 4 -2)
Filtration F_-2 H_4
Component Z/2Z
Component Z/2Z
Component Z/2Z
Component Z/2Z
```

The groups $E^r_{p,q}$ for $r = 1$ and $r = 2$ (which is in fact the level $r = \infty$) for degree $p+q \leq 8$ are represented in the following figures. All the non-null arrows $d^1_{p,q}$ are drawn.


q
$r = 1$
$\mathbb{Z}_2$
$\mathbb{Z}_2^6$
$\mathbb{Z}_2^{16}$ $\mathbb{Z}_2$
$\mathbb{Z}_2^{19}$ $\mathbb{Z}_2^5$
$\mathbb{Z}_2^{14} \to \mathbb{Z}_2^{11}$ $\mathbb{Z}_2$
$\mathbb{Z}_2^3 \to \mathbb{Z}_2^9$ $\mathbb{Z}_2^4$
$\mathbb{Z}_2^2 \oplus \mathbb{Z}^2 \to \mathbb{Z}_2^6 \to \mathbb{Z}_2^7$ $\mathbb{Z}_2$
0 0 $\mathbb{Z}_2^3$ $\mathbb{Z}_2^3$
0 $\mathbb{Z} \to \mathbb{Z}_2^2 \oplus \mathbb{Z} \to \mathbb{Z}_2^4$ $\mathbb{Z}_2$
0 0 0 $\mathbb{Z}_2^2$
0 $\mathbb{Z} \longrightarrow \mathbb{Z}_2^2$ $\mathbb{Z}_2$
0 0 $\mathbb{Z}_2$
0 $\mathbb{Z}$ $\mathbb{Z}_2$
0 0
0 $\mathbb{Z}_2$
0
$\mathbb{Z}$
p


q
$r = 2$
$r = \infty$
$\mathbb{Z}_2$
$\mathbb{Z}_2^6$
$\mathbb{Z}_2^{11}$ $\mathbb{Z}_2$
$\mathbb{Z}_2^{11}$ $\mathbb{Z}_2^5$
$\mathbb{Z}_2^5$ $\mathbb{Z}_2^7$ $\mathbb{Z}_2$
0 $\mathbb{Z}_2^6$ $\mathbb{Z}_2^4$
$\mathbb{Z}_2 \oplus \mathbb{Z}_{10} \oplus \mathbb{Z}$ $\mathbb{Z}_2$ $\mathbb{Z}_2^4$ $\mathbb{Z}_2$
0 0 $\mathbb{Z}_2^3$ $\mathbb{Z}_2^3$
0 0 $\mathbb{Z}_6$ $\mathbb{Z}_2^2$ $\mathbb{Z}_2$
0 0 0 $\mathbb{Z}_2^2$
0 $\mathbb{Z}$ $\mathbb{Z}_2$ $\mathbb{Z}_2$
0 0 $\mathbb{Z}_2$
0 $\mathbb{Z}$ $\mathbb{Z}_2$
0 0
0 $\mathbb{Z}_2$
0
$\mathbb{Z}$
p

Chapter 4

Effective homology of free simplicial Abelian groups

In the previous chapters of this memoir we have only dealt with spectral sequences which are defined by means of filtered chain complexes and which converge to the homology groups of some associated spaces. This kind of spectral sequences includes some classical examples, namely those of Serre, Eilenberg-Moore, or bicomplexes. On the other hand, we can also find in the literature other spectral sequences that do not appear naturally associated with filtered complexes and can be useful for the computation of homotopy groups, which is one of the most challenging problems in the field of Algebraic Topology. This is the case of the Bousfield-Kan spectral sequence, that appeared first in [BK72a] trying to generalize the Adams' spectral sequence [Ada60] (which can be used for the computation of the homotopy groups of a simplicial set X).

As we will see in Chapter 5, one of the main ingredients in the Bousfield-Kan spectral sequence is the constructor associating to a simplicial set X the $\mathbb{Z}$-free simplicial Abelian group RX generated by X; more specifically, in the computation of this spectral sequence, the *effective* homology of the iterated groups R^kX is required (the *ordinary* homology of R^kX can easily be determined using Cartan's algorithm [Car55]).

This chapter is devoted to a version *with effective homology* of the constructor R. If a 1-reduced simplicial set X with effective homology is given, this version of the constructor R computes a version with effective homology of the result RX. An iterative application of this constructor computes therefore a version with effective homology of R^kX for k a positive integer.

If the simplicial set X is contractible, a different *specific* algorithm can also be constructed allowing one to compute the effective homology of RX. We have not introduced it in this memoir, but it can be found in [Rom06a].

4.1 Previous definitions and results

We include here some information which is necessary for the construction of the effective homology of RX.

4.1.1 The Dold-Kan correspondence

In this section, the functors N_* and Γ are presented, providing an equivalence between the categories of chain complexes and simplicial Abelian groups. Most of these definitions and results can be found in [May67] and [GJ99].

We begin by introducing the *normalization* functor N_* from the category $\mathcal{A}$ of simplicial Abelian groups to the category $\mathcal{C}$ of chain complexes.

Definition 4.1. The functor $N_* : \mathcal{A} \to \mathcal{C}$ is defined as follows.

1. Let G be a simplicial Abelian group, then $N_*(G) = (N_n(G), d_n)_{n\in\mathbb{N}}$ is the chain complex given by:
   - the group of n-chains is
   $$N_n(G) = G_n \cap \operatorname{Ker} \partial_0 \cap \ldots \cap \operatorname{Ker} \partial_{n-1},$$
   - the differential map $d_n : N_n(G) \to N_{n-1}(G)$ is defined as $d_n = (-1)^n \partial_n$.
2. Given two simplicial Abelian groups G and F and a simplicial Abelian group morphism $f : G \to F$, the chain complex morphism $N_*(f) : N_*(G) \to N_*(F)$ (which will be denoted by f^N) is defined as follows. Given $x \in N_n(G) \subseteq G_n$, then
   $$N_*(f)(x) = f(x)$$

We recall that, if G is a simplicial Abelian group, G_* denotes G regarded as a chain complex: the group of n-chains is G_n, with differential map $d_n = \sum_{i=0}^{n}(-1)^i\partial_i$. Then it is not difficult to see that $N_*(G)$ is a chain subcomplex of G_*. On the other hand, let us notice that given a simplicial Abelian group morphism $f : G \to F$ and $x \in N_n(G)$, then $f(x) \in N_n(F)$ since $\partial_i f(x) = f(\partial_i x) = 0 \quad \text{for all} \;\; 0 \leq i < n$, so that $N_*(f)$ is well-defined. Similarly one can observe that $N_*(f)$ is in fact a chain complex morphism.

The inclusion inc : $N_*(G) \hookrightarrow G_*$ (which is a chain complex morphism) induces a morphism on the corresponding homology groups,

$$H_*(\mathrm{inc}) : H_*(N_*(G)) \longrightarrow H_*(G_*)$$

Theorem 4.2. [May67] Let G be a simplicial Abelian group. Then

$$H_n(\mathrm{inc}) : H_n(N_*(G)) \longrightarrow H_n(G_*)$$

is an isomorphism for each $n \in \mathbb{N}$.

Proof. We define a decreasing filtration of the chain complex $G_* = (G_n, d_n)_{n\in\mathbb{N}}$ by

$$x \in F^pG_n \text{ if } x \in G_n \text{ and } \partial_i x = 0 \text{ for } 0 \leq i < \min(n,p)$$

Then $F^{p+1}G_* = (F^{p+1}G_n, d_n)_{n\in\mathbb{N}}$ is a chain subcomplex of $F^pG_* = (F^pG_n, d_n)_{n\in\mathbb{N}}$; let $i^p : F^{p+1}G_* \hookrightarrow F^pG_*$ denote the inclusion (which is a chain complex morphism). One can also observe that $F^pG_n = G_n$ if $p \leq 0$ and $F^pG_n = N_n(G)$ if $p \geq n$.

An epimorphism of chain complexes $f^p : F^pG_* \to F^{p+1}G_*$ can be built in the following way. If $x \in F^pG_n$, then $f^p(x)$ is given by

$$f^p(x) = \begin{cases} x & \text{if } n \leq p \\ x - \eta_p\partial_p x & \text{if } n > p \end{cases}$$

A simple calculation proves that f^p is a chain complex morphism and it is clear that $f^p \circ i^p$ is the identity map of $F^{p+1}G_*$.

Then, we define a chain homotopy $h^p : F^pG_* \to F^pG_{*+1}$. Let $x \in F^pG_n$,

$$h^p(x) = \begin{cases} 0 & \text{if } n < p \\ (-1)^p\eta_p x & \text{if } n \geq p \end{cases}$$

We see that $h^p(x) \in F^pG_{n+1}$, and a little more calculating shows that

$$d \circ h^p(x) + h^p \circ d(x) = x - (i^p \circ f^p)(x)$$

in all degrees n, and both sides of the equation are 0 in degrees $n < p$.

One can observe that the maps f^p, i^p, and h^p satisfy the *important* properties of a reduction (Definition 1.58). In other words, the hypotheses of Remark 1.59 hold, and therefore a reduction $\rho^p = (f^p, i^p, h'^p) : F^pG_* \Rightarrow\!\!\!\!\Rightarrow F^{p+1}G_*$ can be built.

A reduction $\rho = (f, i, h) : G_* \Rightarrow\!\!\!\!\Rightarrow N_*(G)$ is obtained then as the composition of the reductions ρ^p. The chain complex morphisms $i : N_*(G) \hookrightarrow G_*$ and $f : G_* \to N_*(G)$ are given in degree n by the compositions

$$\begin{aligned} i^0 \circ \cdots \circ i^{n-1} &: F^nG_n = N_n(G) \longrightarrow F^0G_n = G_n \\ f^{n-1} \circ \cdots \circ f^0 &: F^0G_n = G_n \longrightarrow F^nG_n = N_n(G) \end{aligned}$$

and the homotopy operator $h : G_* \to G_{*+1}$ is defined in degree n by

$$i^0 \cdots i^{n-1} \circ h'^n \circ f^{n-1} \cdots f^0 + i^0 \cdots i^{n-2} \circ h'^{n-1} \circ f^{n-2} \cdots f^0 + \cdots + i^0 \circ h'^1 \circ f^0 + h'^0$$

From the reduction $\rho = (f, i, h) : G_* \Rightarrow\!\!\!\!\Rightarrow N_*(G)$ it follows in particular that the inclusion $i : N_*(G) \hookrightarrow G_*$ induces an isomorphism on the homology groups,

$$H_n(i) = H_n(\text{inc}) : H_n(N_*(G)) \cong H_n(G_*) \quad \text{for } n \geq 0$$

□

Corollary 4.3. Let G be a simplicial Abelian group, then one can construct a reduction

$$\rho : G_* \Rrightarrow N_*(G)$$

Definition 4.4. Let G be a simplicial Abelian group. For each degree n, we define $D_n(G)$ as the subgroup of G_n generated by all the degenerate simplices:

$$\begin{aligned} &D_0(G) = 0 \\ &D_n(G) = \eta_0(D_{n-1}(G)) + \eta_1(D_{n-1}(G)) + \cdots + \eta_{n-1}(D_{n-1}(G)) \end{aligned}$$

On account of the simplicial identities, the graded group $D_*(G) = (D_n(G))_{n\in\mathbb{N}}$ is closed under the differential $d = \sum(-1)^i\partial_i$, which means that $D_*(G) = (D_n(G), d_n)_{n\in\mathbb{N}}$ is a chain subcomplex of G_*.

Corollary 4.5. [May67] Given a simplicial Abelian group G,

$$G_* = N_*(G) \oplus D_*(G)$$

Proof. Provided that $f \circ i$ is the identity map of $N_*(G)$, one has $G_* = N_*(G) \oplus \operatorname{Ker} f$. Moreover, it is not difficult to prove that $\operatorname{Ker} f = D_*(G)$, and in this way we obtain the searched equation. □

After this corollary, we find it convenient to include here the following remarks.

Remark 4.6. Given $x \in G_*$, its unique decomposition as a direct sum $x = y + z$, with $y \in N_*(G)$ and $z \in D_*(G)$, is the following:

$$\begin{aligned} y &= i \circ f(x) = f(x) \in N_*(G) \\ z &= x - y = x - f(x) \in D_*(G) \end{aligned}$$

Remark 4.7. Let G be a simplicial Abelian group. Making use of the equation $G_* = N_*(G) \oplus D_*(G)$, we can identify $N_*(G)$ with the quotient $G_*/D_*(G)$ which will be denoted by G_*^N. The isomorphisms between both chain complexes are

$$\begin{aligned} &\varphi : N_*(G) \longrightarrow G_*^N, \qquad && \varphi(x) = [x] \\ &\psi : G_*^N \longrightarrow N_*(G), && \psi([x]) = f(x) \end{aligned}$$

Once we have introduced the functor $N_* : \mathcal{A} \to \mathcal{C}$ and its main properties, we include now the definition of the functor $\Gamma : \mathcal{C} \to \mathcal{A}$. We will see later that $\Gamma \circ N_*$ and $N_* \circ \Gamma$ are the identity functors of the categories $\mathcal{A}$ and $\mathcal{C}$ respectively.

Definition 4.8. The functor Γ from the category $\mathcal{C}$ of chain complexes to the category $\mathcal{A}$ of simplicial Abelian groups is defined as follows.

1. Let $C_* = (C_n, d_n)_{n\in\mathbb{N}}$ be a chain complex, the simplicial Abelian group $\Gamma(C_*)$ is built in the following way.

- The set of n-simplices is

$$\Gamma_n(C_*) = C_n \oplus \left(\bigoplus_{r=0}^{n-1} \bigoplus_{0 \leq j_1 < \cdots < j_{n-r} < n} \sigma_{j_{n-r}} \ldots \sigma_{j_1} C_r \right)$$

where $\sigma_{j_{n-r}} \ldots \sigma_{j_1} C_r$ is the Abelian group whose elements are symbols $\sigma_{j_{n-r}} \ldots \sigma_{j_1} x$ with $x \in C_r$, and the group addition is defined by

$$\sigma_{j_{n-r}} \ldots \sigma_{j_1} x + \sigma_{j_{n-r}} \ldots \sigma_{j_1} y = \sigma_{j_{n-r}} \ldots \sigma_{j_1}(x+y)$$

- We define the faces $\partial_i : \Gamma_n(C_*) \to \Gamma_{n-1}(C_*)$ as
  i. given $x \in C_n$, then

$$\partial_i(x) = \begin{cases} 0 & \text{if } i < n \\ d_n(x) & \text{if } i = n \end{cases}$$

  ii. if $x = \sigma_{j_k} \ldots \sigma_{j_1} y$ with $y \in C_r$, $k = n - r$, and $0 \leq j_1 < \cdots < j_k < n$,

$$\partial_i(x) = \partial_i(\sigma_{j_k} \ldots \sigma_{j_1} y) = \begin{cases} \sigma_{h_{k-1}} \ldots \sigma_{h_1} y \\ \sigma_{h_k} \ldots \sigma_{h_1} d_r(y) \\ 0 \end{cases}$$

when $\partial_i \eta_{j_k} \ldots \eta_{j_1}$ is expressed in the canonical form of Property 1.29 of simplicial sets as

$$\begin{cases} \eta_{h_{k-1}} \ldots \eta_{h_1} \\ \eta_{h_k} \ldots \eta_{h_1} \partial_r \\ \eta_{h_k} \ldots \eta_{h_1} \partial_j, \text{ with } j < r \end{cases} \qquad \text{respectively.}$$

- The degeneracy operators $\eta_i : \Gamma_n(C_*) \to \Gamma_{n+1}(C_*)$ are given by
  i. if $x \in C_n$,

$$\eta_i(x) = \sigma_i x$$

  ii. if $x = \sigma_{j_k} \ldots \sigma_{j_1} y$ with $y \in C_r$, $k = n - r$, and $0 \leq j_1 < \cdots < j_k < n$,

$$\eta_i(x) = \eta_i(\sigma_{j_k} \ldots \sigma_{j_1} y) = \sigma_{h_{k+1}} \ldots \sigma_{h_1} y$$

when $\eta_i \eta_{j_k} \ldots \eta_{j_1}$ is expressed in canonical form as $\eta_{h_{k+1}} \ldots \eta_{h_1}$.

2. Given two chain complexes $C_* = (C_n, d_{C_n})_{n \in \mathbb{N}}$ and $D_* = (D_n, d_{D_n})_{n \in \mathbb{N}}$ and a chain complex morphism $f : C_* \to D_*$, the corresponding simplicial Abelian group morphism $\Gamma(f) : \Gamma(C_*) \to \Gamma(D_*)$ will be denoted by f^Γ and is defined as follows.

   i. If $x \in C_n$, then

$$\Gamma(f)(x) = f(x) \in D_n \subseteq \Gamma_n(D_*)$$

ii. If $x = \sigma_{j_k} \dots \sigma_{j_1} y$ with $y \in C_r$, $k = n - r$, and $0 \leq j_1 < \cdots < j_k < n$, then

$$\Gamma(f)(x) = \Gamma(f)(\sigma_{j_k} \dots \sigma_{j_1} y) = \sigma_{j_k} \dots \sigma_{j_1} f(y) \in \Gamma_n(D_*)$$

We include now two useful remarks about the functor Γ.

Remark 4.9. Let $x \in \Gamma_n(C_*)$ such that $x = \sigma_{j_k} \dots \sigma_{j_1} y$ with $y \in C_r$, $k = n - r$, and $0 \leq j_1 < \cdots < j_k < n$. Given i such that $0 \leq i \leq n$, it can be proved that $\partial_i \eta_{j_k} \dots \eta_{j_1}$ is expressed in canonical form as $\eta_{h_k} \dots \eta_{h_1} \partial_r$ if and only if $i = n$ and $j_k < n - 1$. This implies that the only case for which $\partial_i(x)$ is defined as $\sigma_{h_k} \dots \sigma_{h_1} d_r(y)$ is the case $i = n$ and $j_k < n - 1$.

Remark 4.10. The functor Γ has a good behavior with respect to the direct sum of two chain complexes. Let C_* and D_* be chain complexes, there exists a canonical isomorphism

$$\Gamma(C_* \oplus D_*) \cong \Gamma(C_*) \oplus \Gamma(D_*) \cong \Gamma(C_*) \times \Gamma(D_*)$$

This result can be iterated and it is also valid for infinite direct sums, in other words, given $\{C_*^k\}_k$ a family of chain complexes, then

$$\Gamma(\bigoplus_k C_*^k) \cong \bigoplus_k \Gamma(C_*^k)$$

Finally, one can see that the functors N_* and Γ form an equivalence between the categories $\mathcal{C}$ and $\mathcal{A}$, which is called the *Dold-Kan correspondence*. The three following theorems explain concretely the relation between these two functors.

Theorem 4.11. [May67] The functors $N_* : \mathcal{A} \to \mathcal{C}$ and $\Gamma : \mathcal{C} \to \mathcal{A}$ form an equivalence of categories, that is to say, $\Gamma \circ N_* \equiv \mathrm{Id}_{\mathcal{A}}$ and $N_* \circ \Gamma \equiv \mathrm{Id}_{\mathcal{C}}$.

The proof of this theorem can be found in [GJ99]. We include here the explicit definition of the isomorphisms giving the first relation $\Gamma \circ N_* \equiv \mathrm{Id}_{\mathcal{A}}$, which will be used later. Let $G \in \mathcal{A}$ be a simplicial Abelian group, we want to prove $\Gamma(N_*(G)) \cong G$.

First, a morphism $\lambda : \Gamma(N_*(G)) \to G$ is defined by

i. given $x \in N_n(G) \subseteq G_n$, we consider

$$\lambda(x) = (-1)^{\lfloor \frac{n+1}{2} \rfloor} x \in G_n$$

ii. if $x = \sigma_{j_k} \dots \sigma_{j_1} y$ with $y \in N_r(G)$, $k = n - r$, and $0 \leq j_1 < \cdots < j_k < n$, then

$$\lambda(x) = \lambda(\sigma_{j_k} \dots \sigma_{j_1} y) = \eta_{j_k} \dots \eta_{j_1} \lambda(y) \in G_{r+k} = G_n$$

In order to define the inverse map, $\gamma : G \to \Gamma(N_*(G))$, we make use of the relation $G_* = N_*(G) \oplus D_*(G)$ of Corollary 4.5. As seen in Remark 4.6, every $x \in G_n$ can be split in a unique way as sum of two components, $x = y + z$ where $y \in N_n(G)$ and $z \in D_n(G)$. Then the definition of $\gamma(x)$ can be done in a recursive way as follows.

- If $n = 0$ (that is, $x \in G_0 = N_0(G)$), we define

$$\gamma(x) = x \in N_0(G) = \Gamma_0(N_*(G))$$

- Given $n > 0$, let us suppose that $\gamma(x)$ has been defined for every $\tilde{x} \in G_{n-1}$.

- Let $x \in G_n$, $x = y + z$ with $y \in N_n(G)$ and $z \in D_n(G)$. Since $z \in D_n(G)$, z can be expressed as a sum of degenerate elements of G_n, $z = \sum_{i=0}^{n-1} \eta_i x_i$ with $x_i \in G_{n-1}$. Then we define

$$\gamma(x) = (-1)^{\lfloor \frac{n+1}{2} \rfloor} y + \sum_{i=0}^{n-1} \eta_i \gamma(x_i)$$

  Let us notice that $y \in N_n(G) \subseteq \Gamma_n(N_*(G))$ and $\gamma(x_i) \in \Gamma_{n-1}(N_*(G))$, which implies that $\gamma(x) \in \Gamma_n(N_*(G))$.

One can easily observe that λ and γ are morphisms of simplicial Abelian groups and the equations $\gamma \circ \lambda = \mathrm{Id}_{\Gamma(N_*(G))}$ and $\lambda \circ \gamma = \mathrm{Id}_G$ hold.

Theorem 4.12. [May67] Let G and F be simplicial Abelian groups and $f, g : G \to F$ simplicial group morphisms between them. Let $h : G \to F$ be a simplicial homotopy $h : f \simeq g$, then there exists a chain homotopy

$$s : N_*(G) \longrightarrow N_{*+1}(F), \quad s : N_*(f) \simeq N_*(g)$$

Proof. Given the simplicial homotopy $h : f \simeq g$ (which consists of maps $h_i : G_n \to F_{n+1}$ for $0 \leq i \leq n$), we begin by considering the chain complexes $G_* = (G_n, d_{G_n})_{n \in \mathbb{N}}$ and $F_* = (F_n, d_{F_n})_{n \in \mathbb{N}}$, where the differential maps d_G and d_F are given by the alternate sum $\sum (-1)^i \partial_i$. Then we define

$$s' : G_n \longrightarrow F_{n+1} \quad \text{given by } s' = \sum_{i=0}^{n} (-1)^i h_i$$

It is not difficult to prove that s' is a chain homotopy, $s' : f' \simeq g'$, where $f', g' : G_* \to F_*$ are the chain complex morphisms induced by the simplicial maps f and g. Furthermore, we observe that $s'(D_*(G)) \subseteq D_{*+1}(F)$ so that it makes sense to consider $s' : G^N_* \to F^N_{*+1}$, recalling that $G^N_* = G_*/D_*(G)$ and $F^N_* = F_*/D_*(F)$ respectively. One immediately deduces that s' is a chain homotopy between the corresponding maps $f', g' : G^N_* \to F^N_*$. Finally, making use of Remark 4.7, we have isomorphisms $G^N_* \cong N_*(G)$ and $F^N_* \cong N_*(F)$, which provides us the chain homotopy $s : N_*(f) \simeq N_*(g)$. ☐

Theorem 4.13. [May67] Let C_* and D_* be chain complexes and $f, g : C_* \to D_*$ chain complex morphisms between them. Let $s : C_* \to D_{*+1}$ be a chain homotopy $s : f \simeq g$, then there exists a simplicial homotopy

$$h : \Gamma(C_*) \longrightarrow \Gamma(D_*), \quad h : \Gamma(f) \simeq \Gamma(g)$$

Proof. We define $h_i : \Gamma_n(C_*) \to \Gamma_{n+1}(D_*)$ as follows.

i. If $x \in C_n$, then

$$\begin{aligned} h_n(x) &= \sigma_n(f(x)) - \sigma_n(s \circ d_n(x)) - s(x) \\ h_{n-1}(x) &= \sigma_{n-1}(f(x)) - \sigma_n(s \circ d_n(x)) \\ h_i(x) &= \sigma_i(f(x)) \quad \text{if } i < n-1 \end{aligned}$$

ii. If $x = \sigma_{j_k} \dots \sigma_{j_1} y$ with $y \in C_r$, $k = n - r$, and $0 \leq j_1 < \dots < j_k < n$, then $h_i(x)$ is built inductively:

$$\begin{aligned} h_i(x) = h_i(\sigma_{j_k} \dots \sigma_{j_1} y) &= \eta_{j_k} h_{i-1}(\sigma_{j_{k-1}} \dots \sigma_{j_1} y) \quad \text{if } j_k \leq i-1 \\ h_i(x) = h_i(\sigma_{j_k} \dots \sigma_{j_1} y) &= \eta_{j_k+1} h_i(\sigma_{j_{k-1}} \dots \sigma_{j_1} y) \quad \text{if } j_k > i-1 \end{aligned}$$

It is easy to prove, by means of a simple calculation, that h is a simplicial homotopy $h : \Gamma(f) \simeq \Gamma(g)$.

□

4.1.2 Some remarks about Eilenberg-MacLane spaces

In Section 1.2.3 we have introduced the definition and some basic properties of Eilenberg-MacLane spaces, which have also been used in other parts of this memoir as examples of our computations. These particular simplicial Abelian groups will appear again in Section 4.3.2 as an ingredient in the computation of the effective homology of RX. We find it convenient to present here some useful remarks.

4.1.2.1 Effective homology of Eilenberg-MacLane spaces

We recall that an Eilenberg-MacLane space of type (π, n) is a simplicial group K such that $\pi_n(K) = \pi$ and $\pi_i(K) = 0$ if $i \neq n$. The simplicial group K is called a $K(\pi, n)$ if it is an Eilenberg-MacLane space of type (π, n) and it is minimal.

The *ordinary* homology groups of a $K(\pi, n)$ can easily be determined making use of Cartan's algorithm [Car55], but the computation of an *effective* version of these groups is known to be a difficult problem, especially regarding the algorithmic complexity. This effective version is necessary, for instance, in the construction of the Postnikov tower [May67], and will also be needed in our work dealing with the Bousfield-Kan spectral sequence.

For $\pi = \mathbb{Z}$, the effective homology of $K(\mathbb{Z}, n)$ can be computed for every $n \geq 1$, in the following way. Let us recall that we can recursively build $K(\mathbb{Z}, n)$ by means of the classifying space constructor: $K(\mathbb{Z}, 0)$ is given by $K(\mathbb{Z}, 0)_m = \mathbb{Z}$ for all $m \geq 0$, with face and degeneracy operators equal to the identity map; for $n \geq 1$, $K(\mathbb{Z}, n) = \overline{\mathcal{W}}^n(K(\mathbb{Z}, 0))$.

In the case $n = 1$, $K(\mathbb{Z}, 1) = \overline{\mathcal{W}}(K(\mathbb{Z}, 0))$ has the homotopy type of the sphere S^1, and a reduction $C_*(K(\mathbb{Z}, 1)) \Rrightarrow C_*(S^1)$ can be built. This provides the effective homology of the simplicial Abelian group $K(\mathbb{Z}, 1)$. On the other hand, given G a simplicial group with effective homology, there exists a general algorithm which computes the effective homology of the classifying space $\overline{\mathcal{W}}(G)$. This algorithm is similar to the one computing the effective homology of the loop space of a simplicial set, based this time on the Bar construction. For details, see [Rea93].

In this way, for every $n \geq 1$ it is possible to construct recursively the effective homology of $K(\mathbb{Z}, n) = \overline{\mathcal{W}}^n(K(\mathbb{Z}, 0)) = \overline{\mathcal{W}}^{n-1}(K(\mathbb{Z}, 1))$, in other words, we can build an equivalence

$$\begin{array}{ccc} & DK_*^n & \\ \swarrow\!\!\swarrow & & \searrow\!\!\searrow \\ C_*(K(\mathbb{Z}, n)) & & HK_*^n \end{array}$$

where HK_*^n is an effective chain complex. This construction is also implemented in the Kenzo system.

4.1.2.2 Another model for Eilenberg-MacLane spaces

As mentioned before, there exist different models for the spaces $K(\pi, n)$'s, although in fact all of them are isomorphic. Up to now in this memoir, we have considered the classifying space model. However, in this chapter we find it more convenient to introduce the following one.

Definition 4.14. Given an Abelian group π and a non-negative integer n, we define $C_*(\pi, n) = (C_m(\pi, n), d_m)_{m \in \mathbb{N}}$ as the chain complex given by

$$C_m(\pi, n) = \begin{cases} \pi & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

with all the differential maps d_m equal to zero.

One can immediately observe that $H_n(C_*(\pi, n)) = \pi$ and $H_m(C_*(\pi, n)) = 0$ if $m \neq n$. If we apply to $C_*(\pi, n)$ the functor Γ introduced in Section 4.1.1, we obtain the simplicial Abelian group $\Gamma(C_*(\pi, n))$. On account of Proposition 1.52 and Theorem 4.11, one has

$$\pi_m(\Gamma(C_*(\pi, n)) = H_m(N_*(\Gamma(C_*(\pi, n)))) \cong H_m(C_*(\pi, n)) = \begin{cases} \pi & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

which implies that $\Gamma(C_*(\pi, n))$ is an Eilenberg-MacLane space of type (π, n). Furthermore it can be proved that this simplicial Abelian group is minimal, and therefore it is a $K(\pi, n)$, isomorphic to the model previously considered.

4.2 The free simplicial Abelian group RX

As said in the introduction of this chapter, the simplicial Abelian group RX plays an essential role in the Bousfield-Kan spectral sequence, which will be studied in Chapter 5. In this section we include the definition and some relevant information about RX which has mostly been extracted from [BK72b].

Definition 4.15. Let X be a simplicial set with a base point $\star \in X_0$ and R a commutative ring (with unit). Then RX is defined as the simplicial R-module

$$RX = \frac{R[X]}{R[\star]}$$

where $R[X]$ denotes the simplicial R-module freely generated by the simplices of X, and $R[\star]$ is the simplicial submodule generated by the base point $\star$ and its degeneracies (which are also represented by $\star$).

More concretely, for each degree n the set of n-simplices is

$$RX_n = R[X_n - \{\star\}]$$

that is, the free R-module generated by the set $X_n - \{\star\}$. The face and degeneracy operators $\partial_i : RX_n \to RX_{n-1}$ and $\eta_i : RX_n \to RX_{n+1}$ are defined for a generator of RX_n, $x \in X_n - \{\star\}$, as the classes in the quotient $R[X]/R[\star]$ of the elements $\partial_i x \in X_{n-1} \subset RX_{n-1}$ and $\eta_i x \in X_{n+1} \subset RX_{n+1}$ respectively. The base point of RX is the null combination and will also be denoted by $\star$.

If we apply the constructor R to the simplicial set RX, we obtain a new simplicial Abelian group

$$R^2X = R(RX) = \frac{R[R[X]/R[\star]]}{R[R[\star]]}$$

and in a recursive way we can define

$$R^kX = R(R^{k-1}X) \quad \text{for all } k \in \mathbb{N}$$

The constructor R defines a functor $R : \mathcal{S} \to \mathcal{A}$ from the category $\mathcal{S}$ of simplicial sets to the category $\mathcal{A}$ of simplicial Abelian groups. Furthermore, there exist an $\mathcal{S}$-morphism $\Phi : X \to RX$ and an $\mathcal{A}$-morphism $\Psi : R^2X \to RX$, given by $\Phi(x) = 1 * x$ for all $x \in X$ and $\Psi(1 * y) = y$ for all $y \in RX$, which induce natural transformations $\Phi : \mathrm{Id} \to R$ and $\Psi : R^2 \to R$. It is easy to see that $\{R, \Phi, \Psi\}$ is a triple on the category $\mathcal{S}$ in the sense of [EM65a].

The most important property which RX satisfies is the following one:

Property 4.16. Given X a pointed simplicial set and R a commutative ring, there exists a canonical isomorphism

$$\pi_*(RX, \star) \cong \widetilde{H}_*(X; R)$$

where $\widetilde{H}_*(X;R)$ denotes the reduced homology groups of X with coefficients in R, in other words, the homology groups of the chain complex $\widetilde{C}_*(X;R) = (R[X]/R[\star])_*$.

Proof. Provided that RX is a simplicial Abelian group, and making use of Proposition 1.52, one has $\pi_*(RX,\star) = H_*(N_*(RX))$. Furthermore, from Theorem 4.2 we know that $H_*(N_*(RX)) \cong H_*(RX_*)$, where RX_* is RX regarded as a chain complex. It only remains to observe that RX_* is in fact the chain complex $\widetilde{C}_*(X;R) = (R[X]/R[\star])_*$, and therefore we obtain $\pi_*(RX,\star) \cong H_*(RX_*) = \widetilde{H}_*(X;R)$. □

On the other hand, the simplicial map $\Phi : X \rightarrow RX$ induces a morphism $\pi_*(\Phi) : \pi_*(X,\star) \rightarrow \pi_*(RX,\star)$ between the corresponding homotopy groups and it can be seen that the composition

$$\pi_*(X,\star) \xrightarrow{\pi_*(\Phi)} \pi_*(RX,\star) \cong \widetilde{H}_*(X;R)$$

is in fact the Hurewicz homomorphism.

From now on in this work we will consider integer coefficients, in other words, we choose the case $R = \mathbb{Z}$. Therefore RX is the free simplicial Abelian group generated by X, where the base point and all its degeneracies are put equal to zero.

As we have seen, the homotopy groups of RX are closely connected with the homology groups of X. In particular, if X is a simplicial set with effective homology, then it is possible to compute $\pi_*(RX,\star) \cong \widetilde{H}_*(X;\mathbb{Z}) = \widetilde{H}_*(X)$, but the computation of the *effective* homology groups $H_*(RX)$ is much more complicated. As mentioned in the introduction of this chapter, the effective homology method can be used to solve this problem: given a (1-reduced) simplicial set X with effective homology, we have developed an algorithm that constructs the effective homology of the simplicial Abelian group RX, which will be necessary for the computation of the Bousfield-Kan spectral sequence associated with X.

4.3 Effective homology of RX

Unless otherwise stated, in this section all the chain complexes associated with simplicial sets are normalized. In particular, $\widetilde{C}_*(X)$ will denote the reduced normalized chain complex $\widetilde{C}^N_*(X)$ associated with a simplicial set X.

Let X be a 1-reduced pointed simplicial set with effective homology, such that an equivalence is given:

$$C_*(X) \Lleftarrow DX_* \Rrightarrow HX_*$$

where HX_* is an effective chain complex. Our goal in this section is the computation of the effective homology of the free simplicial Abelian group RX. This effective homology will be obtained as the composition of two equivalences, μ_L (the left equivalence) and μ_R (the right one), which are built in Sections 4.3.1 and 4.3.2 respectively.

4.3.1 Left equivalence

The main ingredients for the left equivalence in the effective homology of RX are the functors Γ and N_* introduced in Section 4.1.1.

We begin by applying the functor N_* to RX. Making use of Remark 4.7, the chain complex $N_*(RX)$ satisfies

$$N_*(RX) \cong RX_*^N = \frac{RX_*}{D_*(RX)}$$

where $RX_* = (RX_n, d_n)_{n\in\mathbb{N}}$ is the chain complex whose group of n-chains is $RX_n = \mathbb{Z}[X_n]/\mathbb{Z}[\star]$, with differential map $d_n = \sum_{i=0}^{n}(-1)^i\partial_i$, and $D_*(RX)$ is the subcomplex of RX_* generated by the degenerate simplices of RX.

We observe that the chain complex RX_* is in fact $C_*(X)/C_*(\star) = \widetilde{C}_*(X)$ (in this case $C_*(X)$ is the non-normalized chain complex), and in the same way RX_*^N is equal to $\widetilde{C}_*^N(X)$, from now on denoted by $\widetilde{C}_*(X)$. This implies

$$N_*(RX) \cong \widetilde{C}_*(X)$$

The isomorphisms are those of Remark 4.7:

$$\begin{aligned} \varphi :\ & N_*(RX) \longrightarrow \widetilde{C}_*(X) = RX_*^N, \quad \varphi(x) = [x] \\ \psi :\ & \widetilde{C}_*(X) = RX_*^N \longrightarrow N_*(RX), \quad \psi([x]) = f(x) \end{aligned}$$

where the function $f : RX_* \to N_*(RX)$ was defined in the proof of Theorem 4.2.

If we apply now the functor Γ to the relation $N_*(RX) \cong \widetilde{C}_*(X)$, we obtain

$$\Gamma(N_*(RX)) \cong \Gamma(\widetilde{C}_*(X))$$

and composing with the isomorphisms $\lambda : \Gamma(N_*(RX)) \to RX$ and $\gamma : RX \to \Gamma(N_*(RX))$ introduced in Theorem 4.11, one has the isomorphism

$$RX \cong \Gamma(\widetilde{C}_*(X))$$

which is given by the compositions

$$\begin{aligned} & RX \xrightarrow{\gamma} \Gamma(N_*(RX)) \xrightarrow{\Gamma(\varphi)} \Gamma(\widetilde{C}_*(X)) \\ & \Gamma(\widetilde{C}_*(X)) \xrightarrow{\Gamma(\psi)} \Gamma(N_*(RX)) \xrightarrow{\lambda} RX \end{aligned}$$

Proposition 4.17. Given a simplicial set X, there exists an explicit isomorphism

$$RX \cong \Gamma(\widetilde{C}_*(X))$$

On the other hand, one can see that, given a reduction ρ between two chain complexes C_* and D_*, it is possible to construct a new reduction, that we call $\Gamma(\rho)$, between the chain complexes associated with $\Gamma(C_*)$ and $\Gamma(D_*)$.

Proposition 4.18. Let C_* and D_* be chain complexes and $\rho = (f, g, h) : C_* \Rrightarrow D_*$ a reduction between them. Then a new reduction

$$\Gamma(\rho) : C_*(\Gamma(C_*)) \Rrightarrow C_*(\Gamma(D_*))$$

can be determined.

Proof. On the one hand, if we apply the functor Γ to the components f and g of the reduction, we obtain two simplicial Abelian group morphisms $\Gamma(f) \equiv f^\Gamma : \Gamma(C_*) \to \Gamma(D_*)$ and $\Gamma(g) \equiv g^\Gamma : \Gamma(D_*) \to \Gamma(C_*)$, and as far as $f \circ g = \mathrm{Id}_{D_*}$, one has

$$f^\Gamma \circ g^\Gamma = \Gamma(f) \circ \Gamma(g) = \Gamma(f \circ g) = \Gamma(\mathrm{Id}_{D_*}) = \mathrm{Id}_{\Gamma(D_*)}$$

On the other hand, what happens with the component h in the reduction? Since the map $h : C_* \to C_{*+1}$ is a chain homotopy $h : \mathrm{Id}_{C_*} \simeq g \circ f$, as seen in Theorem 4.13 we can construct a simplicial homotopy, that we will denote by h^Γ,

$$h^\Gamma : \Gamma(\mathrm{Id}_{C_*}) = \mathrm{Id}_{\Gamma(C_*)} \simeq \Gamma(g \circ f) = g^\Gamma \circ f^\Gamma$$

We can consider now the (normalized) chain complexes associated with $\Gamma(C_*)$ and $\Gamma(D_*)$, which are respectively $C_*(\Gamma(C_*))$ and $C_*(\Gamma(D_*))$. The simplicial maps $f^\Gamma : \Gamma(C_*) \to \Gamma(D_*)$ and $g^\Gamma : \Gamma(D_*) \to \Gamma(C_*)$ induce chain complex morphisms $\bar{f}^\Gamma : C_*(\Gamma(C_*)) \to C_*(\Gamma(D_*))$ and $\bar{g}^\Gamma : C_*(\Gamma(D_*)) \to C_*(\Gamma(C_*))$. To simplify the notation, we call them $\bar{f}$ and $\bar{g}$. Since $f^\Gamma \circ g^\Gamma = \mathrm{Id}_{\Gamma(D_*)}$, one has $\bar{f} \circ \bar{g} = \mathrm{Id}_{C_*(\Gamma(D_*))}$.

Then, as seen in Remark 1.40, using the simplicial homotopy $h^\Gamma : \mathrm{Id}_{\Gamma(C_*)} \simeq g^\Gamma \circ f^\Gamma$, we can construct a chain homotopy $\bar{h}^\Gamma \equiv \bar{h} : C_*(\Gamma(C_*)) \to C_{*+1}(\Gamma(C_*))$ given by the alternate sum of the components h_i^Γ, such that $\bar{h} : \mathrm{Id}_{C_*(\Gamma(C_*))} \simeq \bar{g} \circ \bar{f}$.

In this way, we obtain the chain complex morphisms $\bar{f} : C_*(\Gamma(C_*)) \to C_*(\Gamma(D_*))$ and $\bar{g} : C_*(\Gamma(D_*)) \to C_*(\Gamma(C_*))$, and the homotopy operator $\bar{h} : C_*(\Gamma(C_*)) \to C_{*+1}(\Gamma(C_*))$ which satisfy the equations

1) $\bar{f} \circ \bar{g} = \mathrm{Id}_{C_*(\Gamma(D_*))}$;

2) $d_{C_*(\Gamma(C_*))} \circ \bar{h} + \bar{h} \circ d_{C_*(\Gamma(C_*))} = \mathrm{Id}_{C_*(\Gamma(C_*))} - \bar{g} \circ \bar{f}$;

These are the *important* equations in the definition of reduction; as seen in Remark 1.59, it is possible to define a chain homotopy $\bar{h}'$ on $C_*(\Gamma(C_*))$ such that $\rho' = (\bar{f}, \bar{g}, \bar{h}')$ is a reduction $\rho' : C_*(\Gamma(C_*)) \Rrightarrow C_*(\Gamma(D_*))$. In fact, this step is not necessary because it can be proved that, for any generators $x \in \Gamma_n(C_*)$ and $y \in \Gamma_n(D_*)$, the compositions $\bar{f} \circ \bar{h}(x)$, $\bar{h} \circ \bar{g}(y)$, and $\bar{h} \circ \bar{h}(x)$ are combinations of degenerate simplices of $\Gamma_{n+1}(D_*)$, $\Gamma_{n+1}(C_*)$, and $\Gamma_{n+2}(C_*)$ respectively. Since we are working with normalized chain complexes, one has directly the equations $\bar{f} \circ \bar{h} = 0, \bar{h} \circ \bar{g} = 0$, and $\bar{h} \circ \bar{h} = 0$, and we have obtained the searched reduction $\Gamma(\rho) = (\bar{f}, \bar{g}, \bar{h}) : C_*(\Gamma(C_*)) \Rrightarrow C_*(\Gamma(D_*))$. □

Algorithm 8.
Input: a reduction $\rho : C_* \Rrightarrow D_*$ between two chain complexes C_* and D_*.
Output: a reduction $\Gamma(\rho) : C_*(\Gamma(C_*)) \Rrightarrow C_*(\Gamma(D_*))$.

Let us consider again our 1-reduced pointed simplicial set X with effective homology

$$\begin{array}{ccc} & DX_* & \\ \swarrow\!\!\swarrow & & \searrow\!\!\searrow \\ C_*(X) & & HX_* \end{array}$$

Using this equivalence, it is not difficult to construct a new equivalence

$$\begin{array}{ccc} & \widetilde{DX}_* & \\ \swarrow\!\!\swarrow & & \searrow\!\!\searrow \\ \widetilde{C}_*(X) & & \widetilde{HX}_* \end{array}$$

where $\widetilde{C}_*(X)$ is the reduced (normalized) chain complex $\widetilde{C}_*(X) = C_*^N(X)/C_*[\star]$, $\widetilde{DX}_*$ and $\widetilde{HX}_*$ are easily deduced from DX_* and HX_* respectively, and $\widetilde{HX}_*$ is also an effective chain complex. This new equivalence provides us the effective homology of the chain complex $\widetilde{C}_*(X)$.

Provided that X is 1-reduced, $C_*(X)$ satisfies $C_0(X) = \mathbb{Z}[\star] \cong \mathbb{Z}$ and $C_1(X) = 0$. Then we can choose HX_* such that $HX_0 \cong \mathbb{Z}$ and $HX_1 = 0$, and then $\widetilde{HX}_0 = \widetilde{HX}_1 = 0$.

The next step consists in applying the functor Γ to both reductions (using our Proposition 4.18), so that one has the following equivalence:

$$\begin{array}{ccc} & C_*(\Gamma(\widetilde{DX}_*)) & \\ \swarrow\!\!\swarrow & & \searrow\!\!\searrow \\ C_*(\Gamma(\widetilde{C}_*(X))) & & C_*(\Gamma(\widetilde{HX}_*)) \end{array}$$

In this way we have obtained the next result.

Proposition 4.19. Let X be a 1-reduced pointed simplicial set with effective homology $C_*(X) \Lleftarrow DX_* \Rrightarrow HX_*$. Then an equivalence

$$C_*(\Gamma(\widetilde{C}_*(X))) \Lleftarrow C_*(\Gamma(\widetilde{DX}_*)) \Rrightarrow C_*(\Gamma(\widetilde{HX}_*))$$

can be determined, where $\widetilde{DX}_*$ and $\widetilde{HX}_*$ are chain complexes deduced from DX_* and HX_* respectively, and $\widetilde{HX}_*$ is effective and satisfies $\widetilde{HX}_0 = \widetilde{HX}_1 = 0$.

Finally, composing the results of Propositions 4.17 and 4.19, we obtain an equivalence

$$C_*(RX) \Lleftarrow C_*(\Gamma(\widetilde{DX}_*)) \Rrightarrow C_*(\Gamma(\widetilde{HX}_*))$$

which will be the left equivalence μ_L in the effective homology of the simplicial Abelian group RX.

Algorithm 9.
Input:

- a 1-reduced pointed simplicial set X,
- an equivalence $C_*(X) \Lleftarrow DX_* \Rrightarrow HX_*$, where HX_* is an effective chain complex.

Output: an equivalence $\mu_L : C_*(RX) \Lleftarrow C_*(\Gamma(\widetilde{DX}_*)) \Rrightarrow C_*(\Gamma(\widetilde{HX}_*))$, where $\widetilde{DX}_*$ and $\widetilde{HX}_*$ are chain complexes obtained respectively from DX_* and HX_*, $\widetilde{HX}_*$ is effective and $\widetilde{HX}_0 = \widetilde{HX}_1 = 0$.

In order to determine the effective homology of RX, a second (right) equivalence $\mu_R : C_*(\Gamma(\widetilde{HX}_*)) \Longleftrightarrow HR_*$ is necessary, HR_* being an effective chain complex. In other words, we need to compute the effective homology of the simplicial Abelian group $\Gamma(\widetilde{HX}_*)$; this will be the goal of the following section.

4.3.2 Right equivalence

Given a simplicial set X with effective homology $C_*(X) \Longleftrightarrow HX_*$, in the previous section we have developed an algorithm for computing an equivalence $\mu_L : C_*(RX) \Longleftrightarrow C_*(\Gamma(\widetilde{HX}_*))$, where $\widetilde{HX}_*$ is an effective chain complex obtained from HX_* such that $\widetilde{HX}_0 = \widetilde{HX}_1 = 0$. The next step now is to determine the effective homology of $\Gamma(\widetilde{HX}_*)$. More generally, this section is devoted to the computation of the effective homology of the simplicial Abelian group $\Gamma(E_*)$ for a *general* effective chain complex E_* which is null in degrees 0 and 1.

Let E_* be an effective chain complex such that $E_0 = E_1 = 0$. As shown in Theorem 1.19, E_* can be seen as a direct sum of elementary complexes, that is to say,

$$E_* = \bigoplus_k C_*^k$$

where each C_*^k is elementary. As far as $E_0 = E_1 = 0$, each C_*^k is also null in degrees 0 and 1.

We recall from Definition 1.18 that a chain complex $C_* = (C_n, d_n)_{n\in\mathbb{N}}$ is called elementary if there exists $m \in \mathbb{N}$ (in our case, $m \geq 2$) such that $C_n = 0$ for $n \neq m, m+1$, $C_m \cong \mathbb{Z}$, and $d_{m+1} : C_{m+1} \to C_m$ is monomorphic (which implies that $C_{m+1} \cong \mathbb{Z}$ or $C_{m+1} = 0$). If $C_{m+1} = 0$, then C_* is the chain complex $C_*(\mathbb{Z}, m)$ introduced in Definition 4.14. In the case $C_{m+1} \cong \mathbb{Z}$, C_* can be seen as the chain complex

$$0 \longleftarrow 0 \longleftarrow \cdots \longleftarrow 0 \longleftarrow \mathbb{Z} \overset{d_{m+1}}{\longleftarrow} \mathbb{Z} \longleftarrow 0 \longleftarrow 0 \longleftarrow \cdots$$

where the differential map $d_{m+1} : \mathbb{Z} \to \mathbb{Z}$ is the multiplication by some $t \in \mathbb{Z}$.

It is worth emphasizing that for each degree $n \in \mathbb{N}$, E_n is a finite type group and therefore only a finite number of C_n^k's are relevant. In other words, for each $n \in \mathbb{N}$ there exist $p_n \geq 0$ and $k_1^n, \ldots, k_{p_n}^n$ such that

$$E_n = \bigoplus_{i=1}^{p_n} C_n^{k_i^n}$$

As far as the functor Γ has a good behavior with respect to the direct sum of chain complexes (as explained in Remark 4.10), when applying it to the chain complex E_* we obtain

$$\Gamma(E_*) = \Gamma(\bigoplus_k C_*^k) \cong \bigoplus_k \Gamma(C_*^k)$$

In general the infinite direct sum of a family of simplicial Abelian groups does not coincide with the corresponding Cartesian product, but one must bear in mind that in this case we have special properties. Since each group E_n is a sum of a finite number of components $C_n^{k_i^n}$, the set of n-simplices $\Gamma_n(E_*)$ can be expressed as

$$\Gamma_n(E_*) \cong \bigoplus_{2\leq j\leq n} \left(\bigoplus_{1\leq i\leq p_j} \Gamma_n(C_*^{k_i^j}) \right) \cong \prod_{2\leq j\leq n} \left(\prod_{1\leq i\leq p_j} \Gamma_n(C_*^{k_i^j}) \right)$$

Provided that the face and degeneracy operators of direct sums and Cartesian products are also the same, one has

$$\Gamma(E_*) \cong \bigoplus_k \Gamma(C_*^k) \cong \prod_k \Gamma(C_*^k)$$

The effective homology of this Cartesian product can be computed when the effective homologies of the different components $\Gamma(C_*^k)$ are known, by simple iteration of the method explained in Section 3.1.2 (for the computation of the effective homology of a Cartesian product of two simplicial sets). This method cannot always be generalized to an infinite Cartesian product, but in this case the result holds since the product is finite in each degree. In this way, in order to compute the effective homology of $\Gamma(E_*)$, we need to determine the effective homology of $\Gamma(C_*)$, for C_* an elementary chain complex. Two different cases have to be considered.

First of all, let $C_* = C_*(\mathbb{Z}, m)$ for some $m \geq 2$. Then

$$\Gamma(C_*) = \Gamma(C_*(\mathbb{Z}, m))$$

and this is in fact one of the possible models for the Eilenberg-MacLane space $K(\mathbb{Z}, m)$, as seen in Section 4.1.2. In the same section we have also explained that the space $K(\mathbb{Z}, m)$ is known to be an object with effective homology for every $m \geq 1$, and therefore we can suppose that an equivalence $C_*(K(\mathbb{Z}, m)) \Longleftrightarrow\!\!\!\!\!\Rightarrow HK^m_*$ is available, HK^m_* being an effective chain complex.

In the second case to be considered, the elementary chain complex C_* is of the form

$$0 \longleftarrow 0 \longleftarrow \cdots \longleftarrow 0 \longleftarrow \mathbb{Z} \overset{d_{m+1}}{\longleftarrow} \mathbb{Z} \longleftarrow 0 \longleftarrow 0 \longleftarrow \cdots$$

where the only non-null differential map $d_{m+1} : \mathbb{Z} \to \mathbb{Z}$ is given by $d_{m+1}(1) = t \in \mathbb{Z}$.

One can easily observe that this chain complex can be expressed as the Cone of the morphism

$$f : C_*(\mathbb{Z}, m+1) \longrightarrow C_*(\mathbb{Z}, m+1)$$

defined by $f(1) = d_{m+1}(1) = t$.

Now, what happens when we apply the functor Γ to the Cone of a morphism? We will study this problem as a particular case of a more general situation, the application of this functor to a *short exact sequence*.

Definition 4.20. A *short exact sequence* of chain complexes is a sequence of chain complex morphisms

$$0 \longrightarrow A_* \overset{i}{\longrightarrow} B_* \overset{j}{\longrightarrow} C_* \longrightarrow 0$$

which is exact. In this case this means that the morphism i is injective, the morphism j is surjective, and $\operatorname{Im} i = \operatorname{Ker} j$.

When applying the functor Γ we obtain

$$0 \longrightarrow \Gamma(A_*) \overset{\Gamma(i)}{\longrightarrow} \Gamma(B_*) \overset{\Gamma(j)}{\longrightarrow} \Gamma(C_*) \longrightarrow 0$$

where $\Gamma(i)$ and $\Gamma(j)$ are simplicial Abelian group morphisms. Since $j : B_* \to C_*$ is surjective, it is known (see [GJ99, p. 155]) that $\Gamma(j)$ is a fibration with fiber

$$(\Gamma(j))^{-1}(0) = \operatorname{Ker}(\Gamma(j)) = \operatorname{Im}(\Gamma(i)) \cong \Gamma(A_*)$$

In our context, we need the short exact sequences to be *effective*.

Definition 4.21. An *effective short exact sequence* of chain complexes is a diagram

$$0 \longrightarrow A_* \underset{r}{\overset{i}{\rightleftarrows}} B_* \underset{s}{\overset{j}{\rightleftarrows}} C_* \longrightarrow 0$$

where i and j are chain complex morphisms and r (the *retraction*) and s (the *section*) are graded group morphisms (which in general are not compatible with the differential maps) satisfying

1) $r \circ i = \mathrm{Id}_{A_*}$;
2) $i \circ r + s \circ j = \mathrm{Id}_{B_*}$;
3) $j \circ s = \mathrm{Id}_{C_*}$.

One can easily observe that these three equations imply in particular that i is injective, j is surjective, and $\mathrm{Im}\, i = \mathrm{Ker}\, j$, so that the morphisms i and j define a short exact sequence of chain complexes. As a result, $\Gamma(j) : \Gamma(B_*) \to \Gamma(C_*)$ is a fibration with fiber space isomorphic to $\Gamma(A_*)$.

On the other hand, let us note that the graded group morphisms $r : B_* \to A_*$ and $s : C_* \to B_*$ induce maps $\Gamma_n(r) : \Gamma_n(B_*) \to \Gamma_n(A_*)$ and $\Gamma_n(s) : \Gamma_n(C_*) \to \Gamma_n(B_*)$ for each $n \in \mathbb{N}$, which are group morphisms and are compatible with all the degeneracies η_i (for $0 \leq i \leq n$) and with the faces ∂_i for $0 \leq i < n$, although they are not necessarily compatible with the last face ∂_n (the only one where the differential maps of the chain complexes A_*, B_*, and C_* appear, as explained in Remark 4.9). Furthermore, it is clear that the following equations hold:

1) $\Gamma(r) \circ \Gamma(i) = \mathrm{Id}_{\Gamma(A_*)}$;
2) $\Gamma(i) \circ \Gamma(r) + \Gamma(s) \circ \Gamma(j) = \mathrm{Id}_{\Gamma(B_*)}$;
3) $\Gamma(j) \circ \Gamma(s) = \mathrm{Id}_{\Gamma(C_*)}$.

These identities allow us to see that $\Gamma(B_*)$ can be expressed as a *symmetric* twisted Cartesian product with fiber space $\Gamma(A_*)$ and base space $\Gamma(C_*)$.

Definition 4.22. Let G be a simplicial group, B a simplicial set, and $\tau = \{\tau_n : B_n \to G_{n-1}\}_{n \geq 1}$ (that we call in this case the *symmetric twisting operator*) satisfying

$$\begin{aligned} \partial_{n-1}\tau(b) &= \tau(\partial_{n-1}b) \cdot \tau(\partial_n b)^{-1} \\ \partial_i\tau(b) &= \tau(\partial_i b), \quad \text{if } 0 \leq i < n-1 \\ \eta_i\tau(b) &= \tau(\eta_i b), \quad \text{for all } \ 0 \leq i \leq n-1 \\ \tau(\eta_n b) &= e_n \end{aligned}$$

where b is an n-simplex of B and e_n is the null element of the group G_n.

We define the *symmetric twisted (Cartesian) product* (with *fiber space* G and *base space* B) as the simplicial set, denoted again by $E(\tau)$ or $G \times_\tau B$, which is given by

$$\begin{aligned} E(\tau)_n &= G_n \times B_n \\ \partial_i(g, b) &= (\partial_i g, \partial_i b), \quad \text{if } 0 \leq i < n \\ \partial_n(g, b) &= (\partial_n g \cdot \tau(b), \partial_n b) \\ \eta_i(g, b) &= (\eta_i g, \eta_i b), \quad \text{for all } \ 0 \leq i \leq n \end{aligned}$$

for $(g, b) \in G_n \times B_n$.

In fact, this was the initial definition of twisted product given by Daniel Kan [Kan58], and is the one used by the Kenzo system. It is not difficult to see that both definitions are closely connected and in particular a similar method to the one introduced in Section 3.1.2 allows us to determine the effective homology of a symmetric twisted product when the effective homologies of the base and the fiber spaces are known and B is 1-reduced. From now on in this section, we will work with this *new* definition of twisted product.

Proposition 4.23. Given an effective short exact sequence

$$0 \longrightarrow A_* \underset{r}{\overset{i}{\rightleftarrows}} B_* \underset{s}{\overset{j}{\rightleftarrows}} C_* \longrightarrow 0$$

there exists an explicit isomorphism between the simplicial Abelian group $\Gamma(B_*)$ and a twisted product $\Gamma(A_*) \times_\tau \Gamma(C_*)$.

Proof. To simplify the notation, we denote also by i, j, r, and s the corresponding maps induced on the simplicial Abelian groups $\Gamma(A_*)$, $\Gamma(B_*)$, and $\Gamma(C_*)$.

We define a map

$$\tau_n : \Gamma_n(C_*) \longrightarrow \Gamma_{n-1}(A_*)$$

given by $\tau_n = r \circ \partial_n \circ s - r \circ s \circ \partial_n$, which is clearly a group morphism. It is not hard to prove that it satisfies the conditions of a (symmetric) twisting operator, and therefore it defines a (symmetric) twisted Cartesian product $\Gamma(A_*) \times_\tau \Gamma(C_*)$.

Then, the equations $r \circ i = \mathrm{Id}_{\Gamma(A_*)}$, $i \circ r + s \circ j = \mathrm{Id}_{\Gamma(B_*)}$, and $j \circ s = \mathrm{Id}_{\Gamma(C_*)}$ allow us to construct the maps

$$\phi_n : \Gamma_n(A_*) \times \Gamma_n(C_*) \longrightarrow \Gamma_n(B_*)$$
$$\psi_n : \Gamma_n(B_*) \longrightarrow \Gamma_n(A_*) \times \Gamma_n(C_*)$$

given by $\phi_n(a, c) = i(a) + s(c)$ and $\psi_n(b) = (r(b), j(b))$. It is clear that they are group morphisms and $\psi_n \circ \phi_n = \mathrm{Id}_{\Gamma_n(A_*) \times \Gamma_n(C_*)}$ and $\phi_n \circ \psi_n = \mathrm{Id}_{\Gamma_n(B_*)}$. Furthermore, one can prove that ϕ and ψ are compatible with the faces and degeneracies of $\Gamma(B_*)$ and $\Gamma(A_*) \times_\tau \Gamma(C_*)$, so that we obtain the searched isomorphism

$$\Gamma(B_*) \cong \Gamma(A_*) \times_\tau \Gamma(C_*)$$

□

Let us suppose now that $\Gamma(A_*)$ and $\Gamma(C_*)$ are objects with effective homology, and $\Gamma(C_*)$ is 1-reduced. Thanks to this proposition, it is possible to compute the effective homology of $\Gamma(B_*)$, following the ideas of Section 3.1.2 for the computation of the effective homology of a twisted product.

On the other hand, one can see that given a chain complex morphism $f : A_* \to B_*$ there exists an effective short exact sequence

$$0 \longrightarrow \mathrm{Desusp}_*(B_*) \underset{r}{\overset{i}{\rightleftarrows}} \mathrm{Cone}(f)_* \underset{s}{\overset{j}{\rightleftarrows}} A_* \longrightarrow 0$$

where the chain complex $\mathrm{Desusp}_*(B_*)$ is the *Desuspension* chain complex of B_*, given by $\mathrm{Desusp}_n(B_*) = B_{n+1}$. The maps i and s are defined in each degree n by the canonical inclusions of A_n and B_{n+1} in $\mathrm{Cone}(f)_n = A_n \oplus B_{n+1}$, while r and j are the corresponding projections. One can observe that i and j are chain complex morphisms, but r and s are not compatible with the differential maps unless the morphism f being null.

Coming back now to our particular situation, let us recall that we have an elementary chain complex C_* of the form

$$0 \longleftarrow 0 \longleftarrow \cdots \longleftarrow 0 \longleftarrow \mathbb{Z} \overset{d_{m+1}}{\longleftarrow} \mathbb{Z} \longleftarrow 0 \longleftarrow 0 \longleftarrow \cdots$$

which can be expressed as the Cone of the morphism

$$f : C_*(\mathbb{Z}, m+1) \longrightarrow C_*(\mathbb{Z}, m+1)$$

given by $f(1) = d_{m+1}(1) = t$.

The following short exact sequence is then obtained

$$0 \longrightarrow \mathrm{Desusp}_*(C_*(\mathbb{Z}, m+1)) \underset{r}{\overset{i}{\rightleftarrows}} \mathrm{Cone}(f)_* \underset{s}{\overset{j}{\rightleftarrows}} C_*(\mathbb{Z}, m+1) \longrightarrow 0$$

and in this case the chain complex $\mathrm{Desusp}_*(C_*(\mathbb{Z}, m+1))$ is equal to $C_*(\mathbb{Z}, m)$.

If we apply the functor Γ, using Proposition 4.23, we have a fibration

$$\Gamma(C_*(\mathbb{Z}, m)) \hookrightarrow \Gamma(\mathrm{Cone}(f)_*) \to \Gamma(C_*(\mathbb{Z}, m+1))$$

where the total space $\Gamma(\mathrm{Cone}(f)_*) \cong \Gamma(C_*)$ can be expressed as a (symmetric) twisted product $\Gamma(C_*(\mathbb{Z}, m)) \times_\tau \Gamma(C_*(\mathbb{Z}, m+1))$. Recalling now that $\Gamma(C_*(\mathbb{Z}, m)) = K(\mathbb{Z}, m)$ and $\Gamma(C_*(\mathbb{Z}, m+1)) = K(\mathbb{Z}, m+1)$, one has

$$\Gamma(C_*) \cong K(\mathbb{Z}, m) \times_\tau K(\mathbb{Z}, m+1)$$

Provided that $K(\mathbb{Z}, m)$ and $K(\mathbb{Z}, m+1)$ are objects with effective homology (and $K(\mathbb{Z}, m+1)$ is 1-reduced since $m \geq 2$), the effective homology of $\Gamma(C_*)$ can be computed, and in this way the effective homology of $\Gamma(C_*)$ has been determined for the two different types of elementary chain complexes C_*.

Proposition 4.24. Given an elementary chain complex C_* such that $C_0 = C_1 = 0$, it is possible to construct an equivalence $C_*(\Gamma(C_*)) \Lleftarrow D\Gamma C_* \Rrightarrow H\Gamma C_*$, where $H\Gamma C_*$ is an effective chain complex.

Let us now turn to the general case, where E_* is an effective chain complex such that $E_0 = E_1 = 0$. We recall that E_* can be expressed as

$$E_* = \bigoplus_k C_*^k$$

where each C_*^k is elementary, and

$$\Gamma(E_*) = \Gamma(\bigoplus_k C_*^k) \cong \prod_k \Gamma(C_*^k)$$

We have proved that the simplicial Abelian groups $\Gamma(C_*^k)$ are objects with effective homology, such that there exist equivalences $C_*(\Gamma(C_*^k)) \Lleftarrow D\Gamma C_*^k \Rrightarrow H\Gamma C_*^k$ for each k. If we iterate the process explained in Section 3.1.2 for the computation of the effective homology of the Cartesian product of two simplicial sets, we obtain an equivalence

$$C_*(\prod_k \Gamma(C_*^k)) \Longleftrightarrow \bigotimes_k H\Gamma C_*^k$$

As far as for each degree n the group $\Gamma_n(E_*)$ is a finite product of components

$$\Gamma_n(E_*) \cong \prod_{2\leq j\leq n} \left(\prod_{1\leq i\leq p_j} \Gamma_n(C_*^{k_i^j}) \right)$$

the right chain complex $H\Gamma E_* = \bigotimes_k H\Gamma C_*^k$ satisfies the same property and since each $\Gamma(C_*^{k_i})$ is effective $H\Gamma E_*$ is effective too. In this way, we have obtained the effective homology of the simplicial Abelian group $\Gamma(E_*)$.

Algorithm 10.
Input: an effective chain complex E_* such that $E_0 = E_1 = 0$.
Output: an equivalence $C_*(\Gamma(E_*)) \Lleftarrow D\Gamma E_* \Rrightarrow H\Gamma E_*$, where $H\Gamma E_*$ is an effective chain complex.

Finally, this algorithm can be applied in particular to the effective chain complex $\widetilde{HX}_*$ deduced from HX_* in Section 4.3.1 (which satisfies $\widetilde{HX}_0 = \widetilde{HX}_1 = 0$), obtaining in this way the looked-for right equivalence for the effective homology of RX, $\mu_R : C_*(\Gamma(\widetilde{HX}_*)) \Longleftrightarrow HR_*$.

4.3.3 Final result

Once we have developed algorithms for the left and right equivalences, the last step for the construction of the effective homology of RX consists simply in assembling the puzzle.

Let X be a pointed simplicial set with effective homology given by

$$C_*(X) \Lleftarrow DX_* \Rrightarrow HX_*$$

We begin by making use of Algorithm 9 to obtain the left equivalence μ_L:

$$C_*(RX) \Lleftarrow C_*(\Gamma(\widetilde{DX}_*)) \Rrightarrow C_*(\Gamma(\widetilde{HX}_*))$$

where $\widetilde{HX}_*$ is an effective chain complex deduced from HX_* which is null in degrees 0 and 1.

Then, we apply Algorithm 10 to the effective chain complex $\widetilde{HX}_*$, which leads to an equivalence μ_R:

$$C_*(\Gamma(\widetilde{HX}_*)) \Lleftarrow \widetilde{DR}_* \Rrightarrow HR_*$$

with HR_* an effective chain complex.

Finally, in order to determine the effective homology of RX, it only remains to compose both equivalences μ_L and μ_R.

Algorithm 11.
Input:

- a 1-reduced pointed simplicial set X,
- an equivalence $C_*(X) \Lleftarrow DX_* \Rrightarrow HX_*$, where HX_* is an effective chain complex.

Output: an equivalence $C_*(RX) \Lleftarrow DR_* \Rrightarrow HR_*$, where HR_* is effective.

The effective homology of RX allows us in particular to compute the homology groups $H_*(RX)$, which play an important role in the construction of the Bousfield-Kan spectral sequence, as we will see in Chapter 5.

4.4 Implementation

For the implementation of the algorithms explained in this chapter, we try to develop new programs (in Common Lisp) enhancing the Kenzo system, in a similar way to the module for the computation of spectral sequences associated with filtered complexes presented in Section 2.5.1. For the moment, the implementation of the effective homology of RX has not been finished, although we have already written several functions which are necessary for the final construction.

4.4.1 The simplicial Abelian group RX

First of all, a set of programs implementing the definition of RX has been developed. The main function is the following one:

`zx` *smst*
The returned value is an object of type `Abelian-Simplicial-Group`, which is the free simplicial Abelian group RX generated by the simplicial set $X = smst$.

As an example, the case $X = S^2$ is considered.

```
> (setf zs2 (zx (sphere 2)))
[K1090 Abelian-Simplicial-Group]
> (orgn zs2)
(free-abelian-simplicial-group sphere 2)
```

An n-simplex of RX is a combination of n-simplices of $X = S^2$. For instance, the combination $s = 5 * \eta_1\eta_0\texttt{s2} - 2 * \eta_3\eta_1\texttt{s2} + 3 * \eta_3\eta_2\texttt{s2}$ is a 4-simplex.

```
> (setf s (cmbn 4 5 (absm 3 's2) -2 (absm 10 's2) 3 (absm 12 's2)))
----------------------------------------------------------------------{CMBN 4}
<5 * <AbSm 1-0 s2>>
<-2 * <AbSm 3-1 s2>>
<3 * <AbSm 3-2 s2>>
------------------------------------------------------------------------------
```

Its faces can be computed as follows.

```
> (dotimes (i 5)
     (format t "~2%d_~D s" i)
     (print (face zs2 i 4 s)))

d_0 s
<AbSm 0
----------------------------------------------------------------------{CMBN 2}
<5 * <AbSm - s2>>
------------------------------------------------------------------------------
>

d_1 s
<AbSm -
----------------------------------------------------------------------{CMBN 3}
<5 * <AbSm 0 s2>>
<-2 * <AbSm 2 s2>>
------------------------------------------------------------------------------
>
```

```
d_2 s
<AbSm -
----------------------------------------------------------------------{CMBN 3}
<5 * <AbSm 0 s2>>
<1 * <AbSm 2 s2>>
-----------------------------------------------------------------------------
>

d_3 s
<AbSm -
----------------------------------------------------------------------{CMBN 3}
<-2 * <AbSm 1 s2>>
<3 * <AbSm 2 s2>>
-----------------------------------------------------------------------------
>

d_4 s
<AbSm -
----------------------------------------------------------------------{CMBN 3}
<-2 * <AbSm 1 s2>>
<3 * <AbSm 2 s2>>
-----------------------------------------------------------------------------
>
nil
```

We observe that $\partial_0 s$ is a degenerate simplex, the element $\eta_0(5 * \texttt{s2})$. The other four faces are the non-degenerate simplices $\partial_1 s = 5 * \eta_0\texttt{s2} - 2 * \eta_2\texttt{s2}$, $\partial_2 s = 5 * \eta_0\texttt{s2} + \eta_2\texttt{s2}$, $\partial_3 s = -2 * \eta_1\texttt{s2} + 3 * \eta_2\texttt{s2}$, and $\partial_4 s = -2 * \eta_1\texttt{s2} + 3 * \eta_2\texttt{s2}$.

When we compute the boundary, $\partial_0 s$ is not considered because it is degenerate. In addition, $\partial_3 s = \partial_4 s$ which implies they are canceled each other out. In this way, we obtain $d(s) = -1 * \partial_1 s + \partial_2 s = -1 * (5 * \eta_0\texttt{s2} - 2 * \eta_2\texttt{s2}) + (5 * \eta_0\texttt{s2} + \eta_2\texttt{s2}) \in C_3^N(RX)$.

```
> (dffr zs2 4 s)

----------------------------------------------------------------------{CMBN 3}
<-1 *
----------------------------------------------------------------------{CMBN 3}
<5 * <AbSm 0 s2>>
<-2 * <AbSm 2 s2>>
-----------------------------------------------------------------------------
>
<1 *
----------------------------------------------------------------------{CMBN 3}
<5 * <AbSm 0 s2>>
<1 * <AbSm 2 s2>>
-----------------------------------------------------------------------------
>
-----------------------------------------------------------------------------
```

We can iterate the construction and build $R(RX) = R^2S^2$, which is again a simplicial Abelian group.

```
> (setf zzs2 (zx zs2))
[K1102 Abelian-Simplicial-Group]
> (orgn zzs2)
(free-abelian-simplicial-group free-abelian-simplicial-group sphere 2)
```

In this case the simplices are more complicated, for instance the combination $ds = -1 * (5 * \eta_0 \mathtt{s2} - 2 * \eta_2 \mathtt{s2}) + (5 * \eta_0 \mathtt{s2} + \eta_2 \mathtt{s2})$ is a 3-simplex of R^2X. It is the boundary of $s = 5 * \eta_1\eta_0 \mathtt{s2} - 2 * \eta_3\eta_1 \mathtt{s2} + 3 * \eta_3\eta_2 \mathtt{s2} \in RX \subseteq R^2X$ and therefore when we apply the differential map we obtain the null combination.

```
> (setf ds (cmbn 3 -1 (absm 0 (cmbn 3 5 (absm 1 's2) -2 (absm 4 's2)))
                    1 (absm 0 (cmbn 3 5 (absm 1 's2) 1 (absm 4 's2)))))

----------------------------------------------------------------------{CMBN 3}
<-1 * <AbSm -
----------------------------------------------------------------------{CMBN 3}
<5 * <AbSm 0 s2>>
<-2 * <AbSm 2 s2>>
------------------------------------------------------------------------------
>>
<1 * <AbSm -
----------------------------------------------------------------------{CMBN 3}
<5 * <AbSm 0 s2>>
<1 * <AbSm 2 s2>>
------------------------------------------------------------------------------
>>
------------------------------------------------------------------------------

> (dffr zzs2 3 ds)

----------------------------------------------------------------------{CMBN 2}
------------------------------------------------------------------------------
```

4.4.2 Effective homology of RX

As mentioned before, the implementation of the construction of the effective homology of RX is not finished. In our particular example $X = S^2$, both RX and R^2X are simplicial Abelian groups which have an infinite number of simplices in each dimension, they are not effective. Moreover, the slots `efhm` are empty because Kenzo is not yet taught how to obtain the effective homology of these objects.

```
> (efhm zs2)
Error: I don't know how to determine the effective homology of:
[K1090 Abelian-Simplicial-Group] (Origin: (free-abelian-simplicial-group sphere 2)).
> (efhm zzs2)
Error: I don't know how to determine the effective homology of:
[K1102 Abelian-Simplicial-Group]
(Origin: (free-abelian-simplicial-group free-abelian-simplicial-group sphere 2)).
```

These slots should be set when our Algorithm 11 (which allows us to construct the effective homology of RX when X is a simplicial set with effective homology) will be completely implemented. Once the effective homologies of RS^2 and R^2S^2 will be available, Kenzo will be able to compute their homology groups.

As a first step toward the implementation of Algorithm 11 we have written the functions corresponding to Algorithm 8. Given a reduction $\rho : C_* \Rrightarrow D_*$, it is possible to build a reduction $\Gamma(\rho) : C_*(\Gamma(C_*)) \Rrightarrow C_*(\Gamma(D_*))$ by means of the following method:

`gamma` *rdct*
: The returned value is an object of type `Reduction`, obtained by applying the functor Γ to the reduction *rdct*, following our Algorithm 8.

For instance, let us consider the Eilenberg-MacLane space $K(\mathbb{Z}, 1)$. We have already said that there exists a reduction $C_*(K(\mathbb{Z}, 1)) \Rrightarrow C_*(S^1)$, which is the right reduction in the effective homology of $K(\mathbb{Z}, 1)$.

```
> (setf kz1 (k-z 1))
[K1 Abelian-Simplicial-Group]
> (efhm kz1)
[K22 Homotopy-Equivalence K1 <= K1 => K16]
> (setf rho (rrdct (efhm kz1)))
[K21 Reduction K1 => K16]
> (orgn (k 16))
(circle)
```

If we apply the function `gamma` to the reduction `rho` : `K1` $\Rrightarrow$ `K16`, we obtain a new reduction $\Gamma($`K1`$) \Rrightarrow \Gamma($`K16`$)$.

```
> (gamma rho)
[K1141 Reduction K1114 => K1126]
> (orgn (k 1114))
(gamma [K1 Abelian-Simplicial-Group])
> (orgn (k 1126))
(gamma [K16 Chain-Complex])
```

As mentioned before, it is only one necessary step in the implementation of the effective homology of RX, and several functions involved in the construction are not yet written. Concretely, the maps giving the isomorphism $RX \cong \Gamma(\widetilde{C}_*(X))$ of Proposition 4.17 must be written in Common Lisp, and it is also necessary to implement Algorithm 10 which gives us the effective homology of $\Gamma(E_*)$ for E_* an effective chain complex.

Chapter 5

Effective homology and Bousfield-Kan spectral sequence

The Bousfield-Kan spectral sequence first appeared in [BK72a], designed to present the Adams spectral sequence [Ada60] in a different way, in the framework of *combinatorial topology*, to make easier the study of its algebraic properties. The Adams spectral sequence and its satellite spectral sequences are the main tools to compute homotopy groups, in particular stable and unstable sphere homotopy groups. The Adams spectral sequence and the others did allow topologists to *compute* many homotopy groups, but no *constructive* version of this spectral sequence is yet available; in other words no routine translation work allows a programmer to implement this spectral sequence on a theoretical or concrete machine to produce an *algorithm* computing homotopy groups (such an algorithm should compute all the homotopy groups of spaces, the only final unavoidable restriction being time and space complexity). Note the current situation does not prevent topologists from using specific programs and computers for *auxiliary partial* computations, see for example [Tan85] or [Rav86].

Another point must be noted about the present work: usually the research work around this spectral sequence is mainly devoted to the particular situation of spheres; also the stable situation, significantly easier, is firstly considered. On the contrary the challenge here consists in studying systematically the *general case*: the *unstable* spectral sequence for *arbitrary* simply connected spaces.

As said in the introduction of the previous chapter, the Bousfield-Kan spectral sequence is not directly defined by means of filtered complexes, and therefore our algorithms of Chapter 2 cannot be applied to compute it. In this chapter, we try to develop a new algorithm, based again on the effective homology technique, allowing one to compute the Bousfield-Kan spectral sequence associated with a simplicial set X. As announced before, we make use intensively of our construction of the effective homology of the simplicial Abelian group RX, explained in Chapter 4. This constructive version of the Bousfield-Kan spectral sequence is not finished yet; we present here the general ideas that we hope will allow its construction. Furthermore, we present (complete) algorithms which construct the first two levels of the spectral sequence.

This chapter is divided into two different parts. The first one is focused on cosimplicial objects, which play an essential role in the construction of the Bousfield-Kan spectral sequence, including some new results and algorithms we have developed. The second part (Section 5.2) contains the definition of the spectral sequence, a proof of its convergence, algorithms computing the $E^1_{p,q}$ and $E^2_{p,q}$ terms, and the sketch of a new algorithm for its *complete* computation.

5.1 Some algorithms for cosimplicial structures

Cosimplicial structures are one of the main ingredients in the construction of the Bousfield-Kan spectral sequence. We include in this section some definitions, results, and algorithms dealing with them.

5.1.1 Cosimplicial objects

The notion of cosimplicial object is dual to that of simplicial object (see Definition 1.27), although much less work about them has appeared in the literature. Basic definitions and results about cosimplicial objects can be found, for instance, in [GJ99] or [BK72b].

Definition 5.1. Given a category $\mathcal{D}$, the category $c\mathcal{D}$ of *cosimplicial objects over* $\mathcal{D}$ is defined as follows.

- An object $X \in c\mathcal{D}$ consists of
  - for every integer $n \geq 0$, an object $X^n \in \mathcal{D}$;
  - for every pair of integers (i, n) such that $0 \leq i \leq n$, *coface* and *codegeneracy* operators $\partial^i : X^{n-1} \to X^n$ and $\eta^i : X^{n+1} \to X^n$ (both of them morphisms in the category $\mathcal{D}$) satisfying the *cosimplicial identities*:

$$\begin{aligned}
\partial^j \partial^i &= \partial^i \partial^{j-1} && \text{if } i < j \\
\eta^j \eta^i &= \eta^{i-1} \eta^j && \text{if } i > j \\
\eta^j \partial^i &= \partial^i \eta^{j-1} && \text{if } i < j \\
\eta^j \partial^i &= \mathrm{Id} && \text{if } i = j, j+1 \\
\eta^j \partial^i &= \partial^{i-1} \eta^j && \text{if } i > j+1
\end{aligned}$$

- A *cosimplicial map* $f : X \to Y$ in $c\mathcal{D}$ comprises maps $f^n : X^n \to Y^n$ (which are morphisms in $\mathcal{D}$) which commute with coface and codegeneracy operators.

Definition 5.2. An augmentation of a cosimplicial object $X \in c\mathcal{D}$ consists of an object $X^{-1} \in \mathcal{D}$ and a morphism $\partial^0 : X^{-1} \to X^0$ such that $\partial^1 \partial^0 = \partial^0 \partial^0 : X^{-1} \to X^1$.

5.1.2 Cosimplicial Abelian groups

As a first particular case of cosimplicial object, we consider $\mathcal{D}$ the category of Abelian groups.

Definition 5.3. A *cosimplicial Abelian group* is a cosimplicial object over the category of Abelian groups.

A functor associates to every cosimplicial Abelian group G a cochain complex $G^* = (G^n, \delta^n)_{n\in\mathbb{N}}$: the group of n-cochains is G^n, and the coboundary map $\delta^n : G^{n-1} \to G^n$ is given by $\delta^n = \sum_{i=0}^{n}(-1)^i\partial^i$. Moreover, it is possible to construct the *cosimplicial normalization*, which is dual to the normalized chain complex associated with a simplicial Abelian group, explained in Section 4.1.1.

Definition 5.4. Let G be a cosimplicial Abelian group, then the cochain complex $N^*(G) = (N^n(G), \delta^n)_{n\in\mathbb{N}}$ is defined by

$$N^n(G) = G^n \cap \operatorname{Ker}\eta^0 \cap \ldots \cap \operatorname{Ker}\eta^{n-1}$$

with coboundary map $\delta^n : N^{n-1}(G) \to N^n(G)$ given by the alternate sum

$$\delta^n = \sum_{i=0}^{n}(-1)^i\partial^i$$

It turns out that $N^*(G)$ is a cochain subcomplex of G^*, and therefore the inclusion $\mathrm{inc} : N^*(G) \hookrightarrow G^*$ induces a morphism on the cohomology groups,

$$H^*(\mathrm{inc}) : H^*(N^*(G)) \longrightarrow H^*(G^*)$$

As in the simplicial case, it can be proved that $H^n(\mathrm{inc})$ is an isomorphism for all n. We include here the proof of this result, which up to our knowledge cannot be found in the literature. The general scheme is similar to the one of Theorem 4.2, but here some more calculating is necessary.

Theorem 5.5. Let G be a cosimplicial Abelian group. Then

$$H^n(\mathrm{inc}) : H^n(N^*(G)) \longrightarrow H^n(G^*)$$

is an isomorphism for each codimension n.

Proof. We begin by defining the following decreasing filtration of the cochain complex $G^* = (G^n, \delta^n)_{n\in\mathbb{N}}$:

$$x \in F^pG^n \text{ if } x \in G^n \text{ and } \eta^i x = 0 \text{ for all } 0 \leq i < \min(n,p)$$

It is clear that $F^{p+1}G^* \subseteq F^pG^*$ and moreover given $x \in F^pG^n$ then one has $\delta^{n+1}(x) = \sum_{i=0}^{n+1}(-1)^i\partial^i x \in F^pG^{n+1}$, so that $F^{p+1}G^*$ is a cochain subcomplex of F^pG^*.

The inclusion $i^p : F^{p+1}G^* \hookrightarrow F^pG^*$ is a cochain complex morphism. Furthermore, we observe that $F^pG^n = G^n$ if $p \leq 0$ and $F^pG^n = N^n(G)$ if $p \geq n$.

Then we construct an epimorphism of cochain complexes $f^p : F^pG^* \to F^{p+1}G^*$ as follows. Given $x \in F^pG^n$, $f^p(x)$ is defined as

$$f^p(x) = \begin{cases} x & \text{if } n \leq p \\ x - \sum_{i=0}^{p}(-1)^{i+p}\partial^i\eta^p x & \text{if } n > p \end{cases}$$

It is not difficult to prove that the map f^p is well-defined (that is to say, if $x \in F^pG^n$ then $f^p(x) \in F^{p+1}G^n$) and it is a cochain complex morphism. Furthermore, it is clear that $f^p \circ i^p$ is the identity map of the cochain complex $F^{p+1}G^*$.

The next step consists in defining a cochain homotopy $h^p : F^pG^* \to F^pG^{*-1}$. Given an element $x \in F^pG^n$,

$$h^p(x) = \begin{cases} 0 & \text{if } n \leq p \\ (-1)^p\eta^p x & \text{if } n > p \end{cases}$$

We observe $h^p(x) \in F^pG^{n-1}$ and one can also prove that

$$\delta \circ h^p(x) + h^p \circ \delta(x) = x - (i^p \circ f^p)(x)$$

Following Remark 1.59 (which is also valid for cochain complexes), the maps f^p, i^p, and h^p allow us to construct a reduction $\rho^p = (f^p, i^p, h'^p) : F^pG^* \Rrightarrow F^{p+1}G^*$. These reductions can be composed in order to determine a reduction of cochain complexes

$$\rho = (f, g, h) : G^* \Rrightarrow N^*(G)$$

The cochain complex morphisms $i = \text{inc} : N^*(G) \hookrightarrow G^*$ and $f : G^* \to N^*(G)$ are given in dimension n by the compositions

$$\begin{aligned} i^0 \circ \cdots \circ i^{n-1} : &\quad F^nG^n = N^n(G) \longrightarrow F^0G^n = G^n \\ f^{n-1} \circ \cdots \circ f^0 : &\quad F^0G^n = G^n \longrightarrow F^nG^n = N^n(G) \end{aligned}$$

and the homotopy operator $h : G^* \to G^{*-1}$ is defined in dodimension n as

$$i^0 \cdots i^{n-1} \circ h'^n \circ f^{n-1} \cdots f^0 + i^0 \cdots i^{n-2} \circ h'^{n-1} \circ f^{n-2} \cdots f^0 + \cdots + i^0 \circ h'^1 \circ f^0 + h'^0$$

This implies in particular that the inclusion $i = \text{inc} : N^*(G) \hookrightarrow G^*$ induces an isomorphism on the cohomology groups

$$H^n(i) = H^n(\text{inc}) : H^n(N^*(G)) \cong H^n(G^*) \quad \text{for all } n \geq 0$$

□

Algorithm 12.
Input: a cosimplicial Abelian group G.
Output: a reduction of cochain complexes $\rho : G^* \Rrightarrow N^*(G)$.

Let us denote by $D^n(G)$ the subgroup of G^n of all the elements $x \in G^n$ of the form $x = \sum_{i=0}^{n-1} \partial^i y_i$, with $y_i \in G^{n-1}$ for all i. It is not hard to prove that, given $x \in D^n(G)$, then $\delta^{n+1}(x) \in D^{n+1}(G)$, and therefore $D^*(G) = (D^n(G), \delta^n)_{n\in\mathbb{N}}$ is a cochain subcomplex of G^*.

Corollary 5.6. Let G be a cosimplicial Abelian group. Then

$$G^* = N^*(G) \oplus D^*(G)$$

which implies that $N^*(G)$ is isomorphic to the quotient $G^*/D^*(G)$.

Proof. As in the simplicial case, from the identity $f \circ i = \mathrm{Id}_{N^*(G)}$ it follows that $G^* = N^*(G) \oplus \mathrm{Ker}\, f$. Moreover, a little calculation shows that $\mathrm{Ker}\, f = D^*(G)$. □

On the other hand, it is known that the functor N^* is an equivalence between the category of cosimplicial Abelian groups and the category of (positive) cochain complexes.

5.1.3 Cosimplicial simplicial Abelian groups

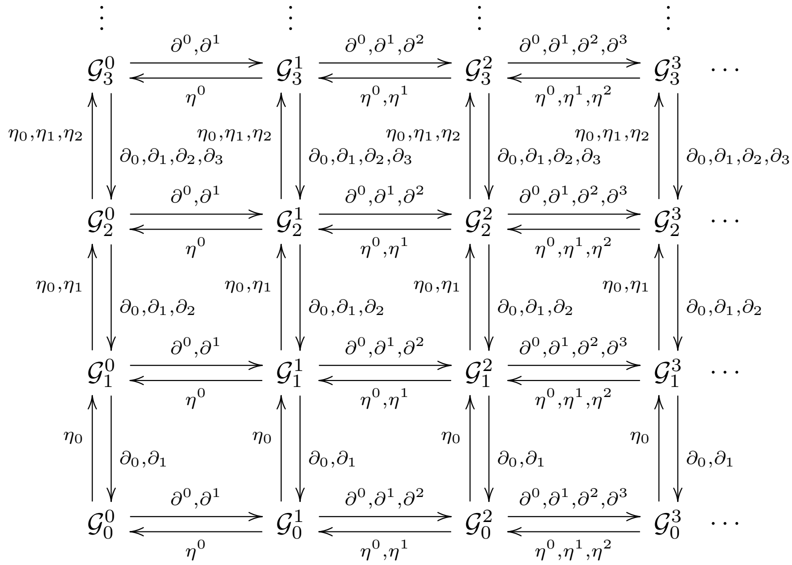
Let us consider now a more *complicated* case of cosimplicial object, choosing $\mathcal{D}$ the category of simplicial Abelian groups. As we explain in this section, we can combine both simplicial and cosimplicial normalizations to construct a reduction from the initial space over the double normalization.

5.1.3.1 Definitions and fundamental results

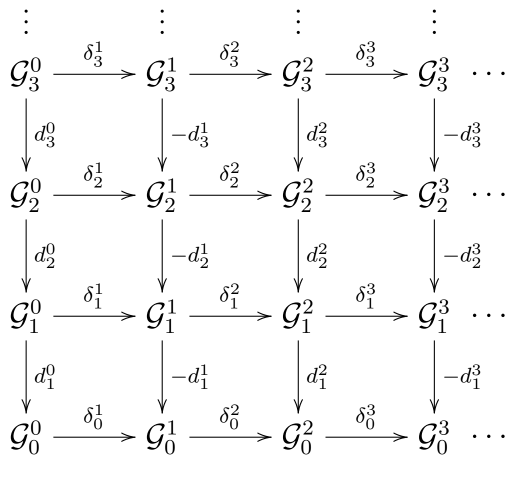
Definition 5.7. A *cosimplicial simplicial Abelian group* is a cosimplicial object over the category $\mathcal{A}$ of simplicial Abelian groups.

A cosimplicial simplicial Abelian group $\mathcal{G}$ is therefore a bigraded family $\mathcal{G} = \{\mathcal{G}_q^p\}_{p,q\in\mathbb{N}}$ of Abelian groups, together with face, coface, degeneracy and codegeneracy operators $\partial_i : \mathcal{G}_q^p \to \mathcal{G}_{q-1}^p$, $\partial^j : \mathcal{G}_q^{p-1} \to \mathcal{G}_q^p$, $\eta_i : \mathcal{G}_q^p \to \mathcal{G}_{q+1}^p$, and $\eta^j : \mathcal{G}_q^{p+1} \to \mathcal{G}_q^p$, for $0 \le i \le q$ and $0 \le j \le p$, all of them group morphisms. The face and degeneracy operators ∂_i and η_i must satisfy the simplicial identities, while for ∂^j and η^j the cosimplicial identities hold. Furthermore, ∂_i and η_i commute with both coface and codegeneracy maps ∂^j and η^j.

The degree q is the *simplicial degree*, and p is the *cosimplicial degree*. The *total degree* is in this case $n = q - p$. The following graphical representation of $\mathcal{G}$ can be helpful.



Each column $\mathcal{G}^p = \{\mathcal{G}_q^p\}_{q\in\mathbb{N}}$ (with the maps $\partial_i : \mathcal{G}_q^p \to \mathcal{G}_{q-1}^p$ and $\eta_i : \mathcal{G}_q^p \to \mathcal{G}_{q+1}^p$) is a simplicial Abelian group, and therefore we can consider the associated chain complex $\mathcal{G}_*^p = (\mathcal{G}_q^p, d_q^p)_{q\in\mathbb{N}}$, where the differential $d_q^p : \mathcal{G}_q^p \to \mathcal{G}_{q-1}^p$ is given by $d_q^p = \sum_{i=0}^q (-1)^i \partial_i$. Similarly, the rows $\mathcal{G}_q = \{\mathcal{G}_q^p\}_{p\in\mathbb{N}}$ are cosimplicial Abelian groups and can be seen as cochain complexes $\mathcal{G}_q^* = (\mathcal{G}_q^p, \delta_q^p)_{p\in\mathbb{N}}$ with coboundary maps $\delta_q^p : \mathcal{G}_q^{p-1} \to \mathcal{G}_q^p$ defined as $\delta_q^p = \sum_{j=0}^p (-1)^j \partial^j$. This allows us to consider the second quadrant bicomplex $\mathcal{G}_{*,*} = \{\mathcal{G}_q^p\}_{p,q\in\mathbb{N}}$, with horizontal differential maps $d'_{p,q} = \delta_q^{p+1} = \sum_{j=0}^{p+1} (-1)^j \partial^j$ and vertical arrows $d''_{p,q} = (-1)^p d_q^p = (-1)^p \sum_{i=0}^q (-1)^i \partial_i$. The factor $(-1)^p$ in the vertical map is necessary to guarantee the equation $d' \circ d'' + d'' \circ d' = 0$ is satisfied.



On the other hand, it is also possible to construct the *double normalization*. First of all, we can apply to each column the simplicial normalization functor N_* (with the necessary sign in each column), obtaining in this way a cosimplicial object over the category of chain complexes. If we consider then the cosimplicial normalization N^*, a second quadrant bicomplex $N^*(N_*(\mathcal{G}))$ is obtained. Conversely, we can first apply cosimplicial and then simplicial normalization, obtaining the bicomplex $N_*(N^*(\mathcal{G}))$. We observe that the order does not change the result, $N^*(N_*(\mathcal{G})) = N_*(N^*(\mathcal{G}))$.

Definition 5.8. Let $\mathcal{G}$ be a cosimplicial simplicial Abelian group, the *double normalization* $N^*N_*(\mathcal{G})$ is the second quadrant bicomplex $N^*N_*(\mathcal{G}) \equiv N^*(N_*(\mathcal{G})) = N_*(N^*(\mathcal{G}))$. In other words,

$$N^pN_q(\mathcal{G}) = \mathcal{G}^p_q \cap \operatorname{Ker} \partial_0 \cap \ldots \cap \operatorname{Ker} \partial_{q-1} \cap \operatorname{Ker} \eta^0 \cap \ldots \cap \operatorname{Ker} \eta^{p-1}$$

with horizontal and vertical differential morphisms $d'_{p,q} = \delta^{p+1}_q = \sum_{j=0}^{p+1}(-1)^j\partial^j$ and $d''_{p,q} = (-1)^p d^p_q = (-1)^{p+q}\partial_q$.

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \\
N^0N_3(\mathcal{G}) & \xrightarrow{\delta^1_3} & N^1N_3(\mathcal{G}) & \xrightarrow{\delta^2_3} & N^2N_3(\mathcal{G}) & \xrightarrow{\delta^3_3} & N^3N_3(\mathcal{G}) & \cdots \\
\downarrow{\scriptstyle d^0_3} & & \downarrow{\scriptstyle -d^1_3} & & \downarrow{\scriptstyle d^2_3} & & \downarrow{\scriptstyle -d^3_3} & & \\
N^0N_2(\mathcal{G}) & \xrightarrow{\delta^1_2} & N^1N_2(\mathcal{G}) & \xrightarrow{\delta^2_2} & N^2N_2(\mathcal{G}) & \xrightarrow{\delta^3_2} & N^3N_2(\mathcal{G}) & \cdots \\
\downarrow{\scriptstyle d^0_2} & & \downarrow{\scriptstyle -d^1_2} & & \downarrow{\scriptstyle d^2_2} & & \downarrow{\scriptstyle -d^3_2} & & \\
N^0N_1(\mathcal{G}) & \xrightarrow{\delta^1_1} & N^1N_1(\mathcal{G}) & \xrightarrow{\delta^2_1} & N^2N_1(\mathcal{G}) & \xrightarrow{\delta^3_1} & N^3N_1(\mathcal{G}) & \cdots \\
\downarrow{\scriptstyle d^0_1} & & \downarrow{\scriptstyle -d^1_1} & & \downarrow{\scriptstyle d^2_1} & & \downarrow{\scriptstyle -d^3_1} & & \\
N^0N_0(\mathcal{G}) & \xrightarrow{\delta^1_0} & N^1N_0(\mathcal{G}) & \xrightarrow{\delta^2_0} & N^2N_0(\mathcal{G}) & \xrightarrow{\delta^3_0} & N^3N_0(\mathcal{G}) & \cdots
\end{array}$$

One can easily observe that the associated total complex $T_*(N^*N_*(\mathcal{G}))$ is a chain subcomplex of $T_*(\mathcal{G}_{*,*})$. Furthermore, in the next theorem we prove that there exists a reduction between them.

Theorem 5.9. Given a cosimplicial simplicial Abelian group $\mathcal{G}$, one can build a chain complex reduction

$$\rho : T_*(\mathcal{G}_{*,*}) \Rrightarrow T_*(N^*N_*(\mathcal{G}))$$

Proof. Let us consider an intermediate bicomplex, $N_*(\mathcal{G}^*)$, where only the simplicial normalization is taken:

$$\begin{aligned} N_q(\mathcal{G}^p) &= \mathcal{G}^p_q \cap \operatorname{Ker} \partial_0 \cap \ldots \cap \operatorname{Ker} \partial_{q-1} \\ &\cong \mathcal{G}^p_q/(\operatorname{Im} \eta_0 + \cdots + \operatorname{Im} \eta_{q-1}) \end{aligned}$$

with horizontal and vertical differential maps given by $d'_{p,q} = \delta^{p+1}_q = \sum_{j=0}^{p+1}(-1)^j\partial^j$ and $d''_{p,q} = (-1)^p d^p_q = (-1)^{p+q}\partial_q$.

$$
\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \\
N_3(\mathcal{G}^0) & \xrightarrow{\delta_3^1} & N_3(\mathcal{G}^1) & \xrightarrow{\delta_3^2} & N_3(\mathcal{G}^2) & \xrightarrow{\delta_3^3} & N_3(\mathcal{G}^3) & \cdots \\
\downarrow {\scriptstyle d_3^0} & & \downarrow {\scriptstyle -d_3^1} & & \downarrow {\scriptstyle d_3^2} & & \downarrow {\scriptstyle -d_3^3} & \\
N_2(\mathcal{G}^0) & \xrightarrow{\delta_2^1} & N_2(\mathcal{G}^1) & \xrightarrow{\delta_2^2} & N_2(\mathcal{G}^2) & \xrightarrow{\delta_2^3} & N_2(\mathcal{G}^3) & \cdots \\
\downarrow {\scriptstyle d_2^0} & & \downarrow {\scriptstyle -d_2^1} & & \downarrow {\scriptstyle d_2^2} & & \downarrow {\scriptstyle -d_2^3} & \\
N_1(\mathcal{G}^0) & \xrightarrow{\delta_1^1} & N_1(\mathcal{G}^1) & \xrightarrow{\delta_1^2} & N_1(\mathcal{G}^2) & \xrightarrow{\delta_1^3} & N_1(\mathcal{G}^3) & \cdots \\
\downarrow {\scriptstyle d_1^0} & & \downarrow {\scriptstyle -d_1^1} & & \downarrow {\scriptstyle d_1^2} & & \downarrow {\scriptstyle -d_1^3} & \\
N_0(\mathcal{G}^0) & \xrightarrow{\delta_0^1} & N_0(\mathcal{G}^1) & \xrightarrow{\delta_0^2} & N_0(\mathcal{G}^2) & \xrightarrow{\delta_0^3} & N_0(\mathcal{G}^3) & \cdots
\end{array}
$$

It follows that the total complex $T_*(N_*(\mathcal{G}^*))$ is a chain subcomplex of $T_*(\mathcal{G}_{*,*})$. In addition, each column $\mathcal{G}^p$ of $\mathcal{G}$ is a simplicial Abelian group, and therefore (thanks to Corollary 4.3) one has reductions

$$\rho^p = (f^p, i^p, h^p) : \mathcal{G}^p_* \Rrightarrow N_*(\mathcal{G}^p)$$

where $i^p : N_*(\mathcal{G}^p) \hookrightarrow \mathcal{G}^p_*$ is the chain complex inclusion.

A reduction $\rho' = (f', i', h') : T_*(\mathcal{G}_{*,*}) \Rrightarrow T_*(N_*(\mathcal{G}^*))$ is then constructed, where $i' : T_*(N_*(\mathcal{G}^*)) \hookrightarrow T_*(\mathcal{G}_{*,*})$ is the inclusion, and f' and h' are defined, for $x \in \mathcal{G}^p_q$, as

$$
\begin{aligned}
f'(x) &= f^p(x) \in N_q(\mathcal{G}^p) \\
h'(x) &= (-1)^p h^p(x) \in \mathcal{G}^p_{q+1}
\end{aligned}
$$

Provided that the morphisms f^p and h^p are defined by means of the face and degeneracy operators, in the same way for every column $\mathcal{G}^p$ (the definition is included in the proof of Theorem 4.2), one can easily observe that

$$\delta^{p+1} \circ f^p = f^{p+1} \circ \delta^{p+1}; \quad \delta^{p+1} \circ h^p = h^{p+1} \circ \delta^{p+1}$$

Then, on account of the equations satisfied by each reduction $\rho^p = (f^p, i^p, h^p)$, it is not hard to verify that $\rho' = (f', i', h') : T_*(\mathcal{G}_{*,*}) \Rrightarrow T_*(N_*(\mathcal{G}^*))$ is in fact a reduction.

In order to build a second reduction $\rho'' = (f'', i'', h'') : T_*(N_*(\mathcal{G}^*)) \Rrightarrow T_*(N^*N_*(\mathcal{G}))$, let us observe that each row q of the bicomplex $N_*(\mathcal{G}^*)$, $N_q(\mathcal{G}^*)$, can be seen as the cochain complex associated with the cosimplicial Abelian group $N_q(\mathcal{G}) = \{N_q(\mathcal{G}^p)\}_{p\in\mathbb{N}}$, while each row q of $N^*N_*(\mathcal{G})$, $N^*(N_q(\mathcal{G}))$, is its cosimplicial normalization. Therefore (making use of Algorithm 12), we can build reductions

$$\rho_q = (f_q, i_q, h_q) : N_q(\mathcal{G}^*) \Rrightarrow N^*(N_q(\mathcal{G}))$$

where i_q is the inclusion $i_q : N^*(N_q(\mathcal{G})) \hookrightarrow N_q(\mathcal{G}^*)$. Then we consider the maps $f'' : T_*(N_*(\mathcal{G}^*)) \to T_*(N^*N_*(\mathcal{G}))$ and $h'' : T_*(N_*(\mathcal{G}^*)) \to T_{*+1}(N_*(\mathcal{G}^*))$ given by

$$
\begin{aligned}
f''(y) &= f_q(y) \in N^p N_q(\mathcal{G}) \\
h''(y) &= h_q(y) \in N_q(\mathcal{G}^{p+1})
\end{aligned}
$$

if $y \in N^pN_q(\mathcal{G})$, and $i'' = \text{inc} : T_*(N^*N_*(\mathcal{G})) \hookrightarrow T_*(N_*(\mathcal{G}^*))$. Then one can easily prove that $\rho'' = (f'', i'', h'') : T_*(N_*(\mathcal{G}^*)) \Rrightarrow T_*(N^*N_*(\mathcal{G}))$ is a reduction.

Finally, the composition of the reductions $\rho' : T_*(\mathcal{G}_{*,*}) \Rrightarrow T_*(N_*(\mathcal{G}^*))$ and $\rho'' : T_*(N_*(\mathcal{G}^*)) \Rrightarrow T_*(N^*N_*(\mathcal{G}))$ provides us the searched reduction

$$\rho : T_*(\mathcal{G}_{*,*}) \Rrightarrow T_*(N^*N_*(\mathcal{G}))$$

□

Algorithm 13.
Input: a cosimplicial simplicial Abelian group $\mathcal{G}$.
Output: a reduction $\rho : T_*(\mathcal{G}_{*,*}) \Rrightarrow T_*(N^*N_*(\mathcal{G}))$.

The following corollaries are immediate consequences of our Theorem 5.9.

Corollary 5.10. Let $\mathcal{G}$ be a cosimplicial simplicial Abelian group. Then the inclusion $\text{inc} : T_*(N^*N_*(\mathcal{G})) \hookrightarrow T_*(\mathcal{G}_{*,*})$ induces an isomorphism between the graded homology groups of both bicomplexes:

$$H_n(\text{inc}) : H_n(N^*N_*(\mathcal{G})) \cong H_n(\mathcal{G}_{*,*}) \quad \text{for all } n \in \mathbb{N}$$

Corollary 5.11. Given $\mathcal{G}$ a cosimplicial simplicial Abelian group, then

$$\mathcal{G}_{*,*} = N^*N_*(\mathcal{G}) \oplus D^*D_*(\mathcal{G})$$

where $D^*D_*(\mathcal{G})$ is the sub(bi)complex of $\mathcal{G}_{*,*}$ given by

$$D^pD_q(\mathcal{G}) = \{x \in \mathcal{G}^p_q, x = \sum_{i=0}^{q-1} \eta_i y_i + \sum_{j=0}^{p-1} \partial^j z_j, \text{ with } y_i \in \mathcal{G}^p_{q-1}, z_j \in \mathcal{G}^{p-1}_q\}$$

As a result, the double normalization $N^*N_*(\mathcal{G})$ is isomorphic to the quotient $\mathcal{G}_{*,*}/D^*D_*(\mathcal{G})$.

5.1.3.2 Homotopy spectral sequence of a cosimplicial simplicial Abelian group

Let $\mathcal{G}$ be a cosimplicial simplicial Abelian group. The double normalization $N^*N_*(\mathcal{G})$ is a second quadrant bicomplex and therefore we can consider the associated (second quadrant) spectral sequence which converges to the homology groups $H_*(N^*N_*(\mathcal{G}))$. In this way, we have a particular case of spectral sequence associated with a filtered chain complex, studied in Chapter 2.

This spectral sequence can also be built in a more explicit way by means of additive relations, as explained in [BK73a]. Let us consider, for each column $\mathcal{G}^p$ (which is a simplicial Abelian group), the homotopy groups $\pi_q(\mathcal{G}^p)$. The codegeneracy maps

$\eta^j : \mathcal{G}^p_q \to \mathcal{G}^{p-1}_q$ induce maps $\pi_q(\eta^j) \equiv \eta^j : \pi_q(\mathcal{G}^p) \to \pi_q(\mathcal{G}^{p-1})$ for $0 \leq j \leq p-1$. Then it makes sense to define, for each pair (p, q), the *normalized homotopy group*

$$\pi'_q(\mathcal{G}^p) = \pi_q(\mathcal{G}^p) \cap \operatorname{Ker} \eta^0 \cap \ldots \cap \operatorname{Ker} \eta^{p-1} \subseteq \pi_q(\mathcal{G}^p)$$

This group is in fact canonically isomorphic to the q-homotopy group of the simplicial Abelian group $N^p(\mathcal{G})$, given by

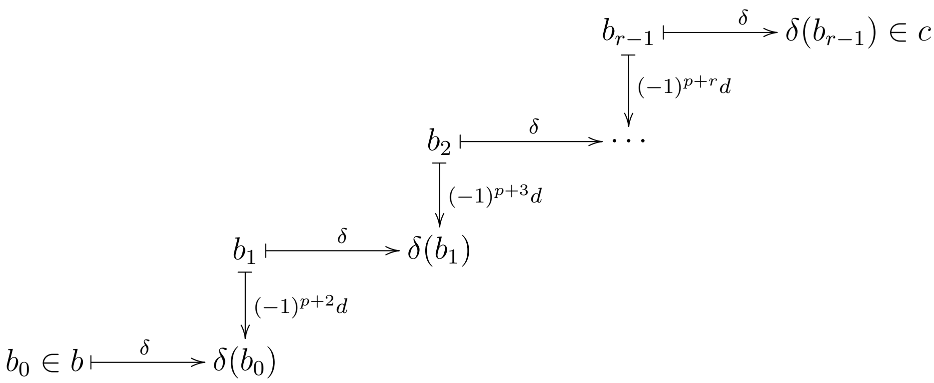
$$N^p(\mathcal{G})_q = \mathcal{G}^p_q \cap \operatorname{Ker} \eta^0 \cap \ldots \cap \operatorname{Ker} \eta^{p-1}$$

with the same face and degeneracy operators as $\mathcal{G}^p$ (which are well-defined since they are compatible with the codegeneracies η^j).

For each $r \geq 1$, we define the following *relations* (which are not always defined multivalued functions, see [Whi78] for details):

$$d^r : \pi'_q(\mathcal{G}^p) \rightharpoonup \pi'_{q+r-1}(\mathcal{G}^{p+r})$$

Given $b \in \pi'_q(\mathcal{G}^p)$ and $c \in \pi'_{q+r-1}(\mathcal{G}^{p+r})$, we put $d^r(b) = c$ whenever there exist elements $b_i \in N^{p+i} N_{q+i}(\mathcal{G})$ for $0 \leq i < r$ such that $b_0 \in b$, $\delta^{p+r}_{q+r-1}(b_{r-1}) \in c$ and $\delta^{p+i}_{q+i-1}(b_{i-1}) = (-1)^{p+i+1} d^{p+i}_{q+i}(b_i)$ for $0 < i < r$. That is to say, $d^r(b) = c$ whenever one can get from b to c by a diagram chasing of the form:



It is not difficult to see that these relations have the following properties.

(1) *Naturality*: let $b \in \pi'_q(\mathcal{G}^p)$ and $c \in \pi'_{q+r-1}(\mathcal{G}^{p+r})$ such that $d^r(b) = c$, $f : \mathcal{G} \to \mathcal{F}$ a cosimplicial map, and f_* the induced map on the corresponding homotopy groups. Then $d^r(f_*(b)) = f_*(c)$.

(2) *Additivity*: if $b, b' \in \pi'_q(\mathcal{G}^p)$ and $c, c' \in \pi'_{q+r-1}(\mathcal{G}^{p+r})$ satisfy $d^r(b) = c$ and $d^r(b') = c'$, then $d^r(b - b') = c - c'$.

(3) For $r = 1$, $d^1 : \pi'_q(\mathcal{G}^p) \to \pi'_q(\mathcal{G}^{p+1})$ is the function induced by the coboundary map, $\delta^{p+1}_q = \sum_{j=0}^{p+1} (-1)^j \partial^j$.

(4) If $r > 1$, the domain of definition of d^r is the kernel of d^{r-1}, that is to say, given $b \in \pi'_q(\mathcal{G}^p)$ then $d^r(b) = c$ for some $c \in \pi'_{q+r-1}(\mathcal{G}^{p+r})$ if and only if $d^{r-1}(b) = 0$.

(5) For $r > 1$, the indeterminacy of d^r is the image of d^{r-1}, in other words, if $b \in \pi'_q(\mathcal{G}^p)$ then $d^r(0) = b$ if and only if $d^{r-1}(a) = b$ for some $a \in \pi'_{q-r+2}(\mathcal{G}^{p-r+1})$.

(6) The relations d^r are "differentials", that is, if $b \in \pi'_q(\mathcal{G}^p)$ and $c \in \pi'_{q+r-1}(\mathcal{G}^{p+r})$ are such that $d^r(b) = c$, then $d^r(c) = 0$.

These properties produce a second quadrant spectral sequence $E = (E^r, d^r)_{r\geq 1}$ with

$$E^1_{p,q} = \pi'_q(\mathcal{G}^p)$$
$$E^r_{p,q} = \frac{\pi'_q(\mathcal{G}^p) \cap \operatorname{Ker} d^{r-1}}{\pi'_q(\mathcal{G}^p) \cap \operatorname{Im} d^{r-1}}, \qquad r > 1$$

where the differential maps $d^r : E^r_{p,q} \to E^r_{p+r,q+r-1}$ are the morphisms induced by the relations d^r on the corresponding quotients.

If $\mathcal{G}$ is augmented (that is, there exist a simplicial Abelian group $\mathcal{G}^{-1}$ and a morphism $\partial^0 : \mathcal{G}^{-1} \to \mathcal{G}^0$ such that $\partial^1\partial^0 = \partial^0\partial^0 : \mathcal{G}^{-1} \to \mathcal{G}^1$), then one can define a natural filtration of the graded group $\pi_*(\mathcal{G}^{-1})$

$$\cdots \subseteq F^{p+1}\pi_q(\mathcal{G}^{-1}) \subseteq F^p\pi_q(\mathcal{G}^{-1}) \subseteq \cdots \subseteq F^0\pi_q(\mathcal{G}^{-1}) = \pi_q(\mathcal{G}^{-1}) \qquad \text{for each } q \in \mathbb{N}$$

together with isomorphisms

$$E^\infty_{p,q} \cong \frac{F^p\pi_q(\mathcal{G}^{-1})}{F^{p+1}\pi_q(\mathcal{G}^{-1})}$$

which implies that the spectral sequence converges to the homotopy groups of the simplicial Abelian group $\mathcal{G}^{-1}$, $E^1 \Rightarrow \pi_*(\mathcal{G}^{-1})$.

Let us remark that if each group $\mathcal{G}^p_q$ of the cosimplicial simplicial Abelian group $\mathcal{G}$ is finitely generated, then the corresponding spectral sequence is easily computable by means of elementary operations with integer matrices (even if the total space $T_*(\mathcal{G}_{*,*})$ is not effective). Similarly, if each column $\mathcal{G}^p_*$ has effective homology, then one can compute the groups $E^r_{p,q}$ (which only depend on the columns $p-r+1, \ldots, p+r-1$ of the associated bicomplex) and the differential maps $d^r_{p,q}$ for every $p, q, r \in \mathbb{N}$, although it is not always possible to determine the final groups $E^\infty_{p,q}$.

Algorithm 14.
Input:

- a cosimplicial simplicial Abelian group $\mathcal{G}$,
- reductions $\rho^p = (f^p, g^p, h^p) : \mathcal{G}^p_* \Rightarrow\!\!\Rightarrow H\mathcal{G}^p_*$ for each column $p \geq 0$, where $H\mathcal{G}^p_* = (H\mathcal{G}^p_q, d^p_q)_{q\in\mathbb{N}}$ is an effective chain complex.

Output:

- the groups $E^r_{p,q}$ for each $p, q \in \mathbb{Z}$ and $r \geq 1$, with a basis-divisors description,
- the differential maps $d^r_{p,q}$ for every $p, q \in \mathbb{Z}$ and $r \geq 1$.

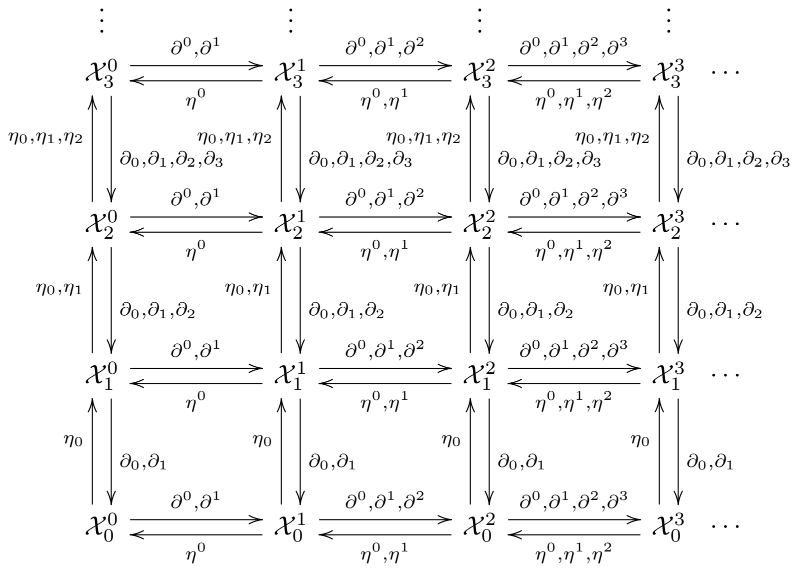
5.1.4 Cosimplicial simplicial sets

A cosimplicial simplicial Abelian group is a particular case of cosimplicial object over the category of simplicial sets.

5.1.4.1 Definitions and examples

Definition 5.12. A *cosimplicial simplicial set*, also called a *cosimplicial space*, is a cosimplicial object over the category $\mathcal{S}$ of simplicial sets.

In other words, a cosimplicial space $\mathcal{X}$ consists of a bigraded family $\mathcal{X} = \{\mathcal{X}_q^p\}_{p,q\in\mathbb{N}}$ with face, coface, degeneracy and codegeneracy maps $\partial_i : \mathcal{X}_q^p \to \mathcal{X}_{q-1}^p$, $\partial^j : \mathcal{X}_q^{p-1} \to \mathcal{X}_q^p$, $\eta_i : \mathcal{X}_q^p \to \mathcal{X}_{q+1}^p$, and $\eta^j : \mathcal{X}_q^{p+1} \to \mathcal{X}_q^p$, for $0 \leq i \leq q$ and $0 \leq j \leq p$, satisfying the same properties as seen for cosimplicial simplicial Abelian groups. A cosimplicial space $\mathcal{X}$ can be represented by a diagram as follows:



An initial example of cosimplicial space is the *cosimplicial standard simplex* Δ.

Definition 5.13. The *cosimplicial standard simplex* Δ consists in codimension n of the standard n-simplex $\Delta[n]$ (which is a simplicial set), and the coface and codegeneracy maps are the standard maps $\partial^j : \Delta[n-1] \to \Delta[n]$ and $\eta^j : \Delta[n+1] \to \Delta[n]$ introduced in Section 1.2.1.

Another important example of cosimplicial simplicial set is the cosimplicial resolution of a simplicial set. This cosimplicial space will be intensively used in the second part of this chapter, because it is the initial point for the definition of the Bousfield-Kan spectral sequence.

Definition 5.14. Let X be a pointed simplicial set and R a ring, the *cosimplicial resolution* of X with respect to R is the augmented cosimplicial space $\mathcal{R}X$ given by

- for each cosimplicial degree p, the column $\mathcal{R}X^p$ is the simplicial R-module $R^{p+1}X$ obtained by applying $p+1$ times the functor R (Definition 4.15) to the simplicial set X (with the corresponding face and degeneracy maps);
- the coface and codegeneracy operators are defined as

$$\begin{array}{llll} \partial^j: & \mathcal{R}X^{p-1} = R^pX \longrightarrow & \mathcal{R}X^p = R^{p+1}X, & \partial^j = R^j\Phi R^{p-j} \\ \eta^j: & \mathcal{R}X^{p+1} = R^{p+2}X \longrightarrow & \mathcal{R}X^p = R^{p+1}X, & \eta^j = R^j\Psi R^{p-j} \end{array}$$

  where the maps $\Phi : X \to RX$ and $\Psi : R^2X \to RX$ are given by $\Phi(x) = 1 * x$ for all $x \in X$ and $\Psi(1 * y) = y$ for all $y \in RX$, as defined in Section 4.2;
- the augmentation is given by the map $\Phi : X \to RX$.

We will usually work with $R = \mathbb{Z}$. In this case, it is worth emphasizing that each column $\mathcal{R}X^p = R^{p+1}X$ is a simplicial Abelian group, which implies that for each $q \geq 0$ the set $\mathcal{R}X^p_q$ is an Abelian group, and the face operators $\partial_i : \mathcal{R}X^p_q \to \mathcal{R}X^p_{q-1}$ and the degeneracies $\eta_i : \mathcal{R}X^p_q \to \mathcal{R}X^p_{q+1}$ are group morphisms. On the other hand, one can observe that the codegeneracy maps $\eta^j : \mathcal{R}X^{p+1}_q \to \mathcal{R}X^p_q$ are also morphisms of groups for all $j \geq 0$, but $\partial^j : \mathcal{R}X^{p-1}_q \to \mathcal{R}X^p_q$ is a group morphism only if $j \geq 1$. For $j = 0$, $\partial^0 : \mathcal{R}X^{p-1}_q \to \mathcal{R}X^p_q$ is not a morphism of groups. For this reason, the cosimplicial space $\mathcal{R}X$ is said to be *grouplike*.

The fact of ∂^0 not being a group morphism prevents the construction of the bicomplexes $\mathcal{R}X_{*,*}$ and $N^*N_*(\mathcal{R}X)$, and in this way it is not possible to define an associated spectral sequence as the spectral sequence of $N^*N_*(\mathcal{R}X)$, as done in the case of cosimplicial simplicial Abelian groups. Nevertheless, it can be seen that there also exists a spectral sequence associated with any cosimplicial space, which generalizes the one introduced in Section 5.1.3.2.

5.1.4.2 Homotopy spectral sequence of a cosimplicial space

Let $\mathcal{X} = \{\mathcal{X}^p_q\}_{p,q\in\mathbb{N}}$ be a cosimplicial space. Although it is not possible (in general) to construct directly a spectral sequence by means of an associated bicomplex (as done for cosimplicial simplicial Abelian groups), Bousfield and Kan proved in [BK73a] that there also exists a *unique* spectral sequence generalizing the previous one.

First of all, let us remark that the codegeneracy operators $\eta^j : \mathcal{X}^p_q \to \mathcal{X}^{p-1}_q$ induce maps $\pi_q(\eta^j) \equiv \eta^j : \pi_q(\mathcal{X}^p) \to \pi_q(\mathcal{X}^{p-1})$ for $0 \leq j \leq p-1$, which are morphisms of groups. Therefore, it makes sense to define the *normalized homotopy groups*

$$\pi'_q(\mathcal{X}^p) = \pi_q(\mathcal{X}^p) \cap \operatorname{Ker} \eta^0 \cap \ldots \cap \operatorname{Ker} \eta^{p-1} \subseteq \pi_q(\mathcal{X}^p)$$

Theorem 5.15. [BK73a] Let $\mathcal{X}$ be a cosimplicial space with base point. Then there are unique relations

$$d^r : \pi'_q(\mathcal{X}^p) \rightharpoonup \pi'_{q+r-1}(\mathcal{X}^{p+r}), \quad r \geq 1 \text{ and } q > p \geq 0$$

which satisfy properties (1) through (6) of Section 5.1.3.2, and if $\mathcal{X}$ is a cosimplicial simplicial Abelian group coincide with the relations defined there.

These relations, which are built by means of universal examples (see [BK73a]), involve the loop space and classifying space functors; they are rather sophisticated, so that their *constructive* construction, certainly possible, is postponed to future work. These relations produce the Bousfield-Kan spectral sequence for a cosimplicial space.

Theorem 5.16. [BK73a] Let $\mathcal{X}$ be a cosimplicial space with base point. Then there exists a second quadrant spectral sequence $E = (E^r, d^r)_{r\geq 1}$ with

$$E^1_{p,q} = \pi'_q(\mathcal{X}^p)$$

This spectral sequence is called the *(homotopy) spectral sequence of the cosimplicial space* $\mathcal{X}$. If $\mathcal{X}$ is augmented, and under some favorable conditions, it converges to the homotopy groups $\pi_*(\mathcal{X}^{-1})$. Furthermore, if $\mathcal{X}$ is a cosimplicial simplicial Abelian group, it coincides with the spectral sequence defined in Section 5.1.3.2.

We will explain in the next sections how the current tools provided by effective homology allow us to obtain easily the groups $E^1_{p,q}$ and $E^2_{p,q}$ of the Bousfield-Kan spectral sequence when every column of the cosimplicial space is provided with effective homology. As explained before, the effective construction of the additive relations which are necessary to obtain the $E^r_{p,q}$ for $r \geq 3$ seems an interesting challenge. It must be mentioned that our spectral sequence is also associated with a tower of fibrations coming from the *realization process* for the studied cosimplicial space; applying analogous methods of effective *homotopy* to these fibrations could also produce the same spectral sequence in a more intrinsic environment, see Section 5.2.3.3.

An important example of homotopy spectral sequence of a cosimplicial space is the one associated with the cosimplicial resolution $\mathcal{R}X$ of a 1-reduced simplicial set X with base point $\star \in X_0$. This will lead to the Bousfield-Kan spectral sequence of a simplicial set X, which if X is 1-reduced converges to the homotopy groups $\pi_*(X, \star)$. The second part of this chapter is focused on the study of this spectral sequence.

5.2 Construction of the Bousfield-Kan spectral sequence

The Bousfield-Kan spectral sequence was introduced in [BK72a], followed by the book [BK72b], which is the most complete reference about this rich mathematical object. One year later, Bousfield and Kan also published the papers [BK73a] and [BK73b], including some interesting properties about their famous spectral sequence.

Theorem 5.17 (Bousfield-Kan spectral sequence). [BK72a] Let X be a simplicial set with base point $\star \in X_0$, and R a ring. There exists a second quadrant spectral sequence $E = (E^r, d^r)_{r\geq 1}$, whose E^1 term is given by

$$E^1_{p,q} = \pi'_q(\mathcal{R}X^p) = \pi_q(R^{p+1}X) \cap \operatorname{Ker}\eta^0 \cap \ldots \cap \operatorname{Ker}\eta^{p-1}$$

which in the case $R = \mathbb{Z}$ and under suitable hypotheses (for instance, if X is 1-reduced) converges to the homotopy groups $\pi_*(X, \star)$.

In order to obtain this spectral sequence, Bousfield and Kan considered different constructions, the cosimplicial resolution of X introduced in Definition 5.14 playing always an essential role. The initial definition (that of [BK72a]) makes use of derived functors. In [BK72b], the spectral sequence is defined by means of the homotopy spectral sequence of a tower of fibrations. Finally, in [BK73a] we find the homotopy spectral sequence of the cosimplicial space $\mathcal{R}X$ as defined in Section 5.1.4.2, by means of additive relations and universal examples. In [BK72b] and [BK73b] it was proved that the different constructions lead to the same spectral sequence.

In the next section, the first stage E^1 of the Bousfield-Kan spectral sequence associated with a simplicial set X (given by Theorem 5.17) is studied, allowing us to deduce some interesting properties of the spectral sequence. In Section 5.2.2, we present two algorithms which compute the levels E^1 and E^2 when X is a simplicial set with effective homology. Finally, Section 5.2.3 includes the sketch of a new algorithm (which is yet not finished), based again on the effective homology technique, computing in this case the *whole* Bousfield-Kan spectral sequence. However, we must remark that this general algorithm would not be always sufficient to determine the "limit" groups $\pi_*(X)$, because of the possible extension problems.

5.2.1 Study of the first level of the spectral sequence

The initial page E^1 of the Bousfield-Kan spectral sequence is done with the homotopy groups of the columns of the cosimplicial space $\mathcal{R}X$, which are *normalized* in the horizontal sense:

$$E^1_{p,q} = \pi'_q(\mathcal{R}X^p) = \pi_q(R^{p+1}X) \cap \operatorname{Ker}\eta^0 \cap \ldots \cap \operatorname{Ker}\eta^{p-1}$$

Theorem 5.18. Let X be a 1-reduced pointed simplicial set, and $E = (E^r, d^r)_{r\geq 1}$ the associated Bousfield-Kan spectral sequence. Then E^1 satisfies

$$E^1_{p,q} = 0 \quad \text{if } q < 2p + 2$$

This theorem implies in particular that the bigraded module $E^1 = \{E^1_{p,q}\}_{p,q\in\mathbb{Z}}$ is *tapered*, which ensures the convergence of the spectral sequence under comfortable conditions. This property is probably already known, but we have not been able to find a reference. The proof that we propose here is completely elementary (it only depends on the Hurewicz theorem), but is relatively difficult. However, it has the advantage of describing in a detailed way the structure of the level E^1, having this description itself its own interest. We begin by explaining the general organization of this proof, the details are then a sequence of elementary lemmas.

Following the indexation (p, q) of a page E^r of the spectral sequence, we denote

$$\pi_{p,q} \equiv \pi_q(R^{p+1}X)$$

which is the *vertical* homotopy group in the position (p, q) *before* the horizontal normalization. In particular $\pi_{0,q} = \pi_q(RX) \cong H_q(X)$ for $q \geq 1$ is an *initial* group; we are going to show that these groups $H_q(X)$ are sufficient to produce the page E^1 of the spectral sequence, following a recursive process which must be well understood. We are going to give a description of $\pi_{p,q}$ of the following form

$$\pi_{p,q} \cong \bigoplus_{G \in Gen_q} (\pi_G)^{n_G(p)}$$

where

- Gen_q is the set of *genealogies* of degree q, a notion which will be defined later;
- π_G is the Abelian group canonically associated with a genealogy G;
- n_G is the *enumeration function* describing for each p how many components π_G take part of the description of $\pi_{p,q}$.

In other words, each group $\pi_{p,q}$ has an structure depending just on the ordinate q, since it is a sum of components π_G for $G \in Gen_q$ a genealogy of degree q; the unique factor which depends on p is the *number* of components, described by the enumeration function n_G.

The first "stage" in the normalization process replaces $\pi_{p,q}$ by

$$\pi^1_{p,q} \equiv \pi_{p,q} \cap \operatorname{Ker} \eta^{p-1}, \quad p \geq 1$$

The usual considerations for the decompositions in direct sums involved in the cosimplicial identity $\eta^{p-1}\partial^p = \mathrm{Id}$ give

$$\pi^1_{p,q} \cong \bigoplus_{G \in Gen_q} (\pi_G)^{n_G(p) - n_G(p-1)} \quad \text{for } p \geq 1$$

That is to say, a "stage" of normalization replaces the exponent n_G of π_G in the expression of $\pi_{p,q}$ by its *discrete derivative*.

We are going to show also that an enumeration function n_G, for G a genealogy of degree q, is in fact a polynomial of degree $< q/2$, a bound which only depends on the ordinate q. This degree is then decreased by one unit at each stage of normalization, and a sufficient number of such stages cancels the exponent. The number of stages of normalization which one must apply to $\pi_{p,q}$ in order to obtain $E^1_{p,q} = \pi'_q(R^{p+1}X) \equiv \pi'_{p,q}$ is exactly p; since the possible degree for n_G is bounded by $q/2$, it follows that $\pi'_{p,q} = 0$ for $q < 2p + 2$.

A genealogy $G \in Gen_q$ is the description of a recursive process which reproduces indefinitely the number $n_G(p)$ (which in general increases with p) of components π_G in $\pi_{p,q}$, the ordinate q being fixed. Each column $R^{p+1}X$ of our cosimplicial space $\mathcal{R}X$ can be written:

$$R^{p+1}X \cong \prod_{q \geq 2} K(\pi_{p,q}, q)$$

The next column can be described in an analogous way, although now a little more complex:

$$R^{p+2}X \cong \prod_{q' \geq 2} K(H_{q'}(R^{p+1}X), q') \cong \prod_{q' \geq 2} K(H_{q'}(\prod_{q \geq 2} K(\pi_{p,q}, q)), q') \cong \prod_{q \geq 2} K(\pi_{p+1,q}, q)$$

This expression allows us to describe the collection $\{\pi_{p+1,q}\}_{q \geq 2}$ with the help of the previous collection $\{\pi_{p,q}\}_{q \geq 2}$ by means of homology groups of Eilenberg-MacLane spaces. To this aim one must apply the Künneth formula; each Eilenberg-MacLane space is going to *produce* (new genealogies of) homology groups which are going to combine among them following Künneth. The relation $H_q(K(\pi, q)) = \pi$ implies the *repetition* in each $\pi_{p+1,q}$ of every occurrences of π_G in $\pi_{p,q}$; since each occurrence of π_G in $\pi_{p+1,q}$ is going to repeat itself, the process of generation between the columns p and $p + 1$ is going to be repeated again between the columns $p + 1$ and $p + 2$, etc. In this process one must control the *history* of each component π_G in $\pi_{p,q}$ by going back until the starting point, the column 0 which is simply RX. This history of π_G is described in the genealogy G and it allows us to control the degree of the polynomial $n_G(p)$. We are going to show that in fact the process of normalization kills *all* these repetitions, except on the left of the line $q = 2p + 2$ where the situation remains more complicated; in particular this is the only part of the page where there are new appearances.

Once we have given the general ideas, we start now with the detailed proof of Theorem 5.18, which makes use of several lemmas.

First of all, for $p = 0$ one has

$$E^1_{0,q} = \pi'_{0,q} = \pi_{0,q} = \pi_q(RX)$$

And provided that X is 1-reduced, $RX = \mathbb{Z}[X]/\mathbb{Z}[\star]$ is also 1-reduced and it follows that $\pi_0(RX) \cong \pi_1(RX) = 0$. Therefore

$$E^1_{0,q} = 0 \quad \text{for } 0 \leq q < 2$$

Then, what happens for $p = 1$?

Lemma 5.19. Let X be a 1-reduced pointed simplicial set and $\mathcal{R}X$ its cosimplicial resolution. Then

$$\pi'_{1,q} = \pi'_q(\mathcal{R}X^1) = 0 \quad \text{for } 0 \leq q < 4$$

Proof. We consider the columns 0 and 1 of the cosimplicial space $\mathcal{R}X$:

$$RX \underset{\eta^0}{\overset{\partial^0,\partial^1}{\rightleftarrows}} R^2X$$

and the induced maps on the homotopy groups

$$\pi_q(RX) \underset{\eta^0}{\overset{\partial^0,\partial^1}{\rightleftarrows}} \pi_q(R^2X)$$

As far as R^2X is also 1-reduced, one has $\pi_0(R^2X) \cong \pi_1(R^2X) = 0$. As a result

$$\pi'_{1,q} = \pi'_q(R^2X) = \pi_q(R^2X) \cap \operatorname{Ker} \eta^0 = 0 \qquad \text{for } q = 0, 1$$

On the other hand, the cosimplicial identity $\eta^0\partial^0 = \mathrm{Id}_{RX}$ induces $\eta^0\partial^0 = \mathrm{Id}_{\pi_q(RX)}$ for each $q \geq 0$. Moreover, the coface $\partial^0 : RX \to R^2X$ is given by $\partial^0 = \Phi R$, so that, as stated in Section 4.2, the induced map on the homotopy groups is the Hurewicz homomorphism

$$\pi_q(RX) \longrightarrow \pi_q(R^2X) \cong \widetilde{H}_q(RX)$$

Then, the Hurewicz Theorem 1.50 implies that, for dimensions $q = 2$ and 3, the map $\partial^0 : \pi_2(RX) \to \pi_2(R^2X)$ is an isomorphism, and $\partial^0 : \pi_3(RX) \to \pi_3(R^2X)$ is an epimorphism. As far as $\eta^0\partial^0 = \mathrm{Id}_{\pi_q(RX)}$ for all q, ∂^0 is always injective and therefore in dimension 3 it is also an isomorphism. Making use again of the equation $\eta^0\partial^0 = \mathrm{Id}_{\pi_q(RX)}$, one has that $\eta^0 : \pi_2(R^2X) \to \pi_2(RX)$ and $\eta^0 : \pi_3(R^2X) \to \pi_3(RX)$ are isomorphisms, so that

$$\pi'_{1,q} = \pi'_q(R^2X) = \pi_q(R^2X) \cap \operatorname{Ker} \eta^0 = 0 \qquad \text{if } q = 2, 3$$

In this way, we have proved $\pi'_{1,q} = \pi'_q(\mathcal{R}X^1) = 0$ if $0 \leq q < 4$. □

For $p > 1$, our proof becomes more complicated, and several previous ideas and definitions are necessary. It is clear that the result is true when $q = 0$ or 1 (since $R^{p+1}X$ is 1-reduced for all $p \geq 0$) and therefore in the sequel we will consider $q \geq 2$.

We begin by recalling that the cosimplicial space $\mathcal{R}X$ is *grouplike*: each column $\mathcal{R}X^p = R^{p+1}X$ is a simplicial Abelian group (and therefore the face and degeneracy operators $\partial_i : R^{p+1}X_q \to R^{p+1}X_{q-1}$ and $\eta_i : R^{p+1}X_q \to R^{p+1}X_{q+1}$, $0 \leq i \leq q$, are group morphisms), the codegeneracy maps $\eta^j : \mathcal{R}X^{p+1}_q = R^{p+2}X_q \to \mathcal{R}X^p_q = R^{p+1}X_q$ are also group morphisms for all $0 \leq j \leq p$, and $\partial^j : \mathcal{R}X^{p-1}_q = R^pX_q \to \mathcal{R}X^p_q = R^{p+1}X_q$ is a group morphism if $1 \leq j \leq p$. In other words, there is a unique operator which is not compatible with the group addition, which is the first coface

$$\partial^0 : \mathcal{R}X^{p-1}_q = R^p X_q \longrightarrow \mathcal{R}X^p_q = R^{p+1} X_q \quad \text{for each } p, q \in \mathbb{N}$$

The fact of η^j being a group morphism for all j (and compatible with the face and degeneracy operators) allows us to define the simplicial Abelian group $N^p(\mathcal{R}X)$, whose set of q-simplices is given by

$$N^p(\mathcal{R}X)_q = R^{p+1} X_q \cap \operatorname{Ker} \eta^0 \cap \ldots \cap \operatorname{Ker} \eta^{p-1}$$

Taking into account that the codegeneracy maps η^j commute with ∂_i and η_i, it is not hard to prove that the homotopy groups of $N^p(\mathcal{R}X)$ are in fact the normalized homotopy groups $\pi'_q(\mathcal{R}X^p) \equiv \pi'_{p,q}$ (which, as seen before, define the level E^1 of the Bousfield-Kan spectral sequence of the simplicial set X), that is to say,

$$\begin{aligned}\pi_q(N^p(\mathcal{R}X)) &= \pi_q(R^{p+1}X \cap \operatorname{Ker} \eta^0 \cap \ldots \cap \operatorname{Ker} \eta^{p-1}) \\ &\cong \pi_q(R^{p+1}X) \cap \operatorname{Ker} \eta^0 \cap \ldots \cap \operatorname{Ker} \eta^{p-1} \equiv \pi'_{p,q}\end{aligned}$$

where the maps $\eta^j \equiv \pi_q(\eta^j) : \pi_{p,q} = \pi_q(R^{p+1}X) \to \pi_{p-1,q} = \pi_q(R^p X)$ are induced by the codegeneracy operators $\eta^j : \mathcal{R}X^p = R^{p+1}X \to \mathcal{R}X^{p-1} = R^p X$.

If we consider only the last codegeneracy and coface operators

$$R^p X \underset{\eta^{p-1}}{\overset{\partial^p}{\rightleftarrows}} R^{p+1} X$$

then the kernel $R^{p+1}X \cap \operatorname{Ker} \eta^{p-1}$ is also a simplicial Abelian group and the cosimplicial identity $\eta^{p-1}\partial^p = \operatorname{Id}_{R^p X}$ gives a decomposition

$$R^{p+1}X \cong (R^{p+1}X \cap \operatorname{Ker} \eta^{p-1}) \oplus (R^p X)$$

which is preserved in the homotopy groups:

$$\pi_q(R^{p+1}X) \cong \pi_q(R^{p+1}X \cap \operatorname{Ker} \eta^{p-1}) \oplus \pi_q(R^p X) \quad \text{for all } q$$

Equivalently, if we denote $\pi^1_{p,q} \equiv \pi_q(R^{p+1}X \cap \operatorname{Ker} \eta^{p-1}) \cong \pi_q(R^{p+1}X) \cap \operatorname{Ker} \eta^{p-1}$, then we have

$$\pi_{p,q} \cong \pi^1_{p,q} \oplus \pi_{p-1,q} \quad \text{for all } q$$

In view of the cosimplicial identities, given an element $x \in R^{p+1}X \cap \operatorname{Ker} \eta^{p-1}$, one has $\eta^{p-2}\eta^j x = \eta^j \eta^{p-1} x = 0$ for $0 \leq j \leq p-2$, which implies that $\eta^j x \in R^p X \cap \operatorname{Ker} \eta^{p-2}$. Therefore it is possible to define the simplicial group morphisms

$$\eta^j : R^{p+1}X \cap \operatorname{Ker} \eta^{p-1} \longrightarrow R^p X \cap \operatorname{Ker} \eta^{p-2}, \quad 0 \leq j \leq p-2$$

Similarly, if $x \in R^{p+1}X \cap \operatorname{Ker} \eta^{p-1}$, then $\eta^p \partial^j x = \partial^j \eta^{p-1} x = 0$ for $0 \leq j \leq p-1$, so that it makes sense to define

$$\partial^j : R^{p+1}X \cap \operatorname{Ker} \eta^{p-1} \longrightarrow R^{p+2}X \cap \operatorname{Ker} \eta^p, \quad 0 \leq j \leq p-1$$

which is a simplicial group morphism if $j \geq 1$. Let us recall that ∂^0 is not a morphism of groups.

Furthermore, since $\eta^p \partial^p = \eta^p \partial^{p+1} = \operatorname{Id}_{R^{p+1}X}$, we can also define

$$\tilde{\partial}^p = \partial^p - \partial^{p+1} : R^{p+1}X \cap \operatorname{Ker} \eta^{p-1} \longrightarrow R^{p+2}X \cap \operatorname{Ker} \eta^p$$

It is worth emphasizing that ∂^p and ∂^{p+1} are group morphisms (note that $p > 0$), so that the last definition makes sense and $\tilde{\partial}^p$ is again a morphism of simplicial Abelian groups.

In this way, we have built

$$\begin{array}{rll} \eta^j : R^{p+1}X \cap \operatorname{Ker} \eta^{p-1} \longrightarrow R^pX \cap \operatorname{Ker} \eta^{p-2} & & 0 \leq j \leq p-2 \\ \partial^j, \tilde{\partial}^p : R^{p+1}X \cap \operatorname{Ker} \eta^{p-1} \longrightarrow R^{p+2}X \cap \operatorname{Ker} \eta^p & & 0 \leq j < p \end{array}$$

and it is not hard to prove that they satisfy the cosimplicial identities of Definition 5.1. This gives rise to a new cosimplicial space, that we call $\mathcal{N}^1(\mathcal{R}X)$, which in codimension p is the simplicial Abelian group

$$\mathcal{N}^1(\mathcal{R}X)^p = \mathcal{R}X^{p+1} \cap \operatorname{Ker} \eta^p = R^{p+2}X \cap \operatorname{Ker} \eta^p$$

In particular, for the columns $p-2$ and $p-1$, one has the operators

$$\mathcal{N}^1(\mathcal{R}X)^{p-2} = R^pX \cap \operatorname{Ker} \eta^{p-2} \underset{\eta^{p-2}}{\overset{\tilde{\partial}^{p-1}}{\rightleftarrows}} \mathcal{N}^1(\mathcal{R}X)^{p-1} = R^{p+1}X \cap \operatorname{Ker} \eta^{p-1}$$

which satisfy $\eta^{p-2}\tilde{\partial}^{p-1} = \operatorname{Id}_{R^pX \cap \operatorname{Ker} \eta^{p-2}}$. The decomposition

$$R^{p+1}X \cap \operatorname{Ker} \eta^{p-1} \cong (R^{p+1}X \cap \operatorname{Ker} \eta^{p-2} \cap \operatorname{Ker} \eta^{p-1}) \oplus (R^pX \cap \operatorname{Ker} \eta^{p-2})$$

is then deduced, and for the homotopy groups one has

$$\pi_q(R^{p+1}X \cap \operatorname{Ker} \eta^{p-1}) \cong \pi_q(R^{p+1}X \cap \operatorname{Ker} \eta^{p-2} \cap \operatorname{Ker} \eta^{p-1}) \oplus \pi_q(R^pX \cap \operatorname{Ker} \eta^{p-2}) \quad \text{for all } q$$

In other words, if we denote $\pi^2_{p,q} \equiv \pi_q(R^{p+1}X \cap \operatorname{Ker} \eta^{p-2} \cap \operatorname{Ker} \eta^{p-1}) \cong \pi_q(R^{p+1}X) \cap \operatorname{Ker} \eta^{p-2} \cap \operatorname{Ker} \eta^{p-1}$, then

$$\pi^1_{p,q} \cong \pi^2_{p,q} \oplus \pi^1_{p-1,q} \quad \text{for all } q$$

Taking into account that $\mathcal{N}^1(\mathcal{R}X)$ is also grouplike, one can iterate the process, obtaining in a recursive way (grouplike) cosimplicial spaces

$$\mathcal{N}^s(\mathcal{R}X) = \mathcal{N}^1(\mathcal{N}^{s-1}(\mathcal{R}X)), \quad \text{for } s \geq 2$$

whose p-column $\mathcal{N}^s(\mathcal{R}X)^p$ is the simplicial Abelian group

$$\mathcal{N}^s(\mathcal{R}X)^p = R^{p+s+1}X \cap \operatorname{Ker}\eta^p \cap \ldots \cap \operatorname{Ker}\eta^{p+s-1}$$

For the columns $p-s-1$ and $p-s$, the last codegeneracy and coface operators η^{p-s-1} and $\tilde{\partial}^{p-s} = \partial^{p-s} - \partial^{p-s+1} + \cdots + (-1)^s\partial^p$

$$R^pX \cap \operatorname{Ker}\eta^{p-s-1} \cap \ldots \cap \operatorname{Ker}\eta^{p-2} \quad \underset{\eta^{p-s-1}}{\overset{\tilde{\partial}^{p-s}}{\rightleftarrows}} \quad R^{p+1}X \cap \operatorname{Ker}\eta^{p-s} \cap \ldots \cap \operatorname{Ker}\eta^{p-1}$$

provide, as in the previous case, the following decomposition:

$$\begin{aligned}\pi_q(R^{p+1}X \cap \operatorname{Ker}\eta^{p-s} \cap \ldots \cap \operatorname{Ker}\eta^{p-1}) &\cong \pi_q(R^{p+1}X \cap \operatorname{Ker}\eta^{p-s-1} \cap \ldots \cap \operatorname{Ker}\eta^{p-1}) \\ &\oplus \pi_q(R^pX \cap \operatorname{Ker}\eta^{p-s-1} \cap \ldots \cap \operatorname{Ker}\eta^{p-2})\end{aligned}$$

that is to say,

$$\pi^s_{p,q} \cong \pi^{s+1}_{p,q} \oplus \pi^s_{p-1,q} \qquad \text{for all } q$$

where $\pi^s_{p,q} \equiv \pi_q(R^{p+1}X \cap \operatorname{Ker}\eta^{p-s} \cap \ldots \cap \operatorname{Ker}\eta^{p-1})$.

We include these results in the next lemma, which will be used later.

Lemma 5.20. Given a simplicial set X, and non-negative integers $p, q \in \mathbb{N}$ and $0 \leq s \leq p-1$, there exists an isomorphism

$$\begin{aligned}\pi_q(R^{p+1}X \cap \operatorname{Ker}\eta^{p-s} \cap \ldots \cap \operatorname{Ker}\eta^{p-1}) &\cong \pi_q(R^{p+1}X \cap \operatorname{Ker}\eta^{p-s-1} \cap \ldots \cap \operatorname{Ker}\eta^{p-1}) \\ &\oplus \pi_q(R^pX \cap \operatorname{Ker}\eta^{p-s-1} \cap \ldots \cap \operatorname{Ker}\eta^{p-2})\end{aligned}$$

We recall now that we are interested in the normalized groups

$$\pi'_{p,q} = \pi_q(R^{p+1}X) \cap \operatorname{Ker}\eta^0 \cap \ldots \cap \operatorname{Ker}\eta^{p-1} \cong \pi_q(R^{p+1}X \cap \operatorname{Ker}\eta^0 \cap \ldots \cap \operatorname{Ker}\eta^{p-1}) = \pi^p_{p,q}$$

In order to *determine* these groups, we are going to describe $\pi_{p,q} = \pi_q(R^{p+1}X)$ as a direct sum of different components. First of all, we remark that the homotopy groups of $R^{p+1}X$ are isomorphic to the (reduced) homology groups of the previous column R^pX, in other words

$$\pi_{p,q} \cong \widetilde{H}_q(R^pX) \cong H_q(R^pX), \quad q \geq 2$$

Furthermore, R^pX is a simplicial Abelian group and therefore, as stated in Theorem 1.56, it is isomorphic to a product of Eilenberg-MacLane spaces of the form

$$R^pX \cong \prod_{n\geq 0} K(\pi_n(R^pX), n) = \prod_{n\geq 2} K(\pi_{p-1,n}, n)$$

If we apply (in a recursive way) the Eilenberg-Zilber Theorem 3.2, we obtain an isomorphism

$$\pi_{p,q} \cong H_q(R^pX) \cong H_q\left(\bigotimes_{n\geq 2} C_*(K(\pi_{p-1,n}, n))\right)$$

Then, we can make use of the Künneth formula, which relates the homology groups of the tensor product of two chain complexes with the homologies of the components.

Theorem 5.21 (Künneth formula). [Dol72] Let C_* and D_* be (free) chain complexes. Then there exists a natural exact sequence

$$0 \longrightarrow (H_*(C_*) \otimes H_*(D_*))_n \longrightarrow H_n(C_* \otimes D_*) \longrightarrow (H_*(C_*) * H_*(D_*))_{n-1} \longrightarrow 0$$

where $*$ denotes the *torsion product* of two groups (see [Dol72] for details). This exact sequence provides an isomorphism

$$H_n(C_* \otimes D_*) \cong \left(\bigoplus_{m+r=n} (H_m(C_*) \otimes H_r(D_*)) \right) \oplus \left(\bigoplus_{m+r=n-1} (H_m(C_*) * H_r(D_*)) \right)$$

In our case, this formula gives a decomposition of the group $\pi_{p,q} \cong H_q(R^p X)$ in terms of $H_i(K(\pi_{p-1,n}, n)) \cong H_i(K(H_n(R^{p-1}X), n))$. Taking into account that given a group π and a positive integer m one has $H_m(K(\pi, m)) = \pi$ and $H_{m+1}(K(\pi, m)) = 0$ (a property of Eilenberg-MacLane spaces which can easily be proved making use of the Hurewicz Theorem 1.50), for instance for $q = 6$ we obtain

$$\begin{aligned} \pi_{p,6} \cong H_6(R^p X) \cong \pi_{p-1,6} \oplus H_6(K(\pi_{p-1,2}, 2)) \oplus H_6(K(\pi_{p-1,3}, 3)) \oplus H_6(K(\pi_{p-1,4}, 4)) \\ \oplus (\pi_{p-1,2} \otimes \pi_{p-1,4}) \oplus (\pi_{p-1,2} * \pi_{p-1,3}) \end{aligned}$$

Iterating the process for the groups

$$\pi_{p-1,n} \cong H_n(R^{p-1}X) \cong H_n \left(\prod_{m \geq 2} K(\pi_{p-2,m}, m) \right)$$

we obtain an expression for the group $\pi_{p,q} \cong H_q(R^p X)$ based on the initial groups $\pi_{0,n} \cong H_n(X) \equiv H_n$. As an example, we show in the following lines the decomposition of the groups $\pi_{1,6} \cong H_6(R^2 X)$ and $\pi_{2,7} \cong H_7(R^3 X)$.

$$\begin{aligned} H_6(R^2 X) \cong\ & H_6(K(H_2, 2) \times K(H_3, 3) \times K(H_4, 4) \times K(H_4(K(H_2, 2)), 4) \\ & \times K(H_5, 5) \times K(H_5(K(H_2, 2)), 5) \times K(H_5(K(H_3, 3)), 5) \\ & \times K(H_2 \otimes H_3, 5) \times K(H_6, 6)) \\ \cong\ & [H_6(K(H_6, 6))] \oplus [H_6(K(H_2, 2))]^2 \oplus [H_6(K(H_3, 3))]^2 \oplus [H_6(K(H_4, 4))]^2 \\ & \oplus [H_6(K(H_4(K(H_2, 2)), 4))] \oplus [H_2(K(H_2, 2)) \otimes H_4(K(H_4, 4))]^2 \\ & \oplus [H_2(K(H_2, 2)) \otimes H_4(K(H_4(K(H_2, 2)), 4))] \oplus [H_2(K(H_2, 2)) * H_3(K(H_3, 3))]^2 \\ \cong\ & H_6 \oplus [H_6(K(H_2, 2))]^2 \oplus [H_6(K(H_3, 3))]^2 \oplus [H_6(K(H_4, 4))]^2 \\ & \oplus [H_6(K(H_4(K(H_2, 2)), 4))] \oplus [H_2 \otimes H_4]^2 \oplus [H_2 \otimes H_4(K(H_2, 2))] \oplus [H_2 * H_3]^2 \end{aligned}$$

$$\begin{aligned}
H_7(R^3X) \cong\ & H_7(K(H_2,2)\times K(H_3,3)\times K(H_4,4)\times K([H_4(K(H_2,2))]^2,4)\\
&\times K(H_5,5)\times K([H_5(K(H_2,2))]^2,5)\times K([H_5(K(H_3,3))]^2,5)\\
&\times K([H_2\otimes H_3]^2,5)\times K(H_6,6)\times\cdots)\\
\cong\ & H_7\oplus[H_7(K(H_2,2))]^3\oplus[H_7(K(H_3,3))]^3\oplus[H_7(K(H_4,4))]^3\\
&\oplus[H_7(K(H_4(K(H_2,2)),4))]^3\oplus[H_7(K(H_5,5))]^3\oplus[H_7(K(H_5(K(H_2,2)),5))]^3\\
&\oplus[H_7(K(H_5(K(H_3,3)),5))]^3\oplus[H_7(K(H_2\otimes H_3,5))]^3\oplus[H_4(K(H_2,2))\otimes H_3]^3\\
&\oplus[H_3\otimes H_4]^3\oplus[H_3\otimes H_4(K(H_2,2))]^3\oplus[H_2\otimes H_5(K(H_3,3))]^3\\
&\oplus[H_2\otimes H_5]^3\oplus[H_2\otimes H_5(K(H_2,2))]^3\oplus[H_2\otimes H_5(K(H_3,3))]^3\\
&\oplus[H_2\otimes(H_2\otimes H_3)]^3\oplus[H_2 * H_4]^3\oplus[H_2 * H_4(K(H_2,2))]^3
\end{aligned}$$

For higher p and q this decomposition becomes more complicated, but one can observe that all the groups $\pi_{p,q}\cong H_q(R^pX)$ are isomorphic to a direct sum of different components (which may occur several times) obtained recursively from the initial groups $H_n\equiv H_n(X)$.

We can formalize this construction defining the notion of *genealogy*. The set of genealogies *Gen* is a totally ordered graded set defined as the disjoint union of a family of totally ordered graded sets $\{GG_n\}_{n\in\mathbb{N}}$, which are built in a recursive way. The starting point is

$$GG_0=\{H_i\}_{i\geq 2}$$

where each H_i is a symbol. The *degree* of H_i is i, and the *ordering* in GG_0 is defined by $H_i<H_j$ if $i<j$.

Let us suppose now that we have built GG_i for $i<n$. We consider the set GGG_{n-1} given by the disjoint union

$$GGG_{n-1}=\coprod_{i<n} GG_i$$

which becomes totally ordered simply by making use of the ordering of each GG_i and considering that if $i<j$ then each element of GG_i is less than every element of GG_j.

We define then GG_n as the set of expressions G of the form

$$G=[(d_1\ G_1)c_1(d_2\ G_2)c_2\cdots c_{k-1}(d_k\ G_k)]$$

where

- k is a positive integer ($k\geq 1$);
- $G_j\in GGG_{n-1}$ for all $1\leq j\leq k$;
- $G_{j-1}\leq G_j$ for $1<j\leq k$;
- $G_k\in GG_{n-1}$;

- if each G_j has degree $\tilde{d}_j$, then $d_j \geq \tilde{d}_j$ and $d_j \neq \tilde{d}_j + 1$. Moreover, if $k = 1$, then $d_j \neq \tilde{d}_j$;
- each c_j is a *connector* (a symbol) in the set $\{\otimes, *\}$.

The degree of G is defined as

$$d = d_1 + \cdots + d_k + n_*$$

where n_* is the number of connectors $*$ which appear in G.

The ordering of GG_n is the lexicographical ordering obtained by considering in a successive way, for instance,

$$k, G_1, \ldots, G_k, d_1, \ldots, d_k, c_1, \ldots, c_{k-1}$$

Once we have built recursively GG_n for all $n \in \mathbb{N}$, we define the *set of genealogies*

$$Gen = \coprod_{n \in \mathbb{N}} GG_n = \bigcup_{n \in \mathbb{N}} GGG_n$$

It is worth remarking that given $q \geq 2$ it is possible to construct (in a recursive way, starting with the base symbols H_i) the set Gen_q of genealogies of degree q, which is a finite set. For instance, for $q = 4$ one has two genealogies $G_4^1 = H_4$ and $G_4^2 = [(4\ H_2)]$. For degree 5 we obtain $G_5^1 = H_5$, $G_5^2 = [(5\ H_2)]$, $G_5^3 = [(5\ H_3)]$, and $G_5^4 = [(2\ H_2) \otimes (3\ H_3)]$. For $q = 6, 7$, and 8 there exist 8, 19, and 45 genealogies respectively.

Let us suppose now that each symbol H_i represents the group

$$H_i \equiv H_i(X) \cong \pi_i(RX), \quad i \geq 2$$

Then one can associate to every genealogy $G \in Gen_q$ of degree $q \geq 2$ an Eilenberg-MacLane space $K(G)$ built as follows:

- if $G = H_q$, then $K(G) = K(H_q(X), q)$;
- if $G = [(d_1\ G_1)c_1(d_2\ G_2)c_2 \cdots c_{k-1}(d_k\ G_k)]$, then

$$K(G) = K(H_{d_1}(K(G_1))c_1(\cdots c_{k-1}H_{d_k}(K(G_k))\cdots), q)$$

where each c_j represents a tensor product $\otimes$ or a torsion product $*$. We define then the group π_G associated with G by

$$\pi_G = H_q(K(G))$$

which is given in fact by

- if $G = H_q$, then $\pi_G = H_q = H_q(X)$;

- if $G = [(d_1\ G_1)c_1(d_2\ G_2)c_2 \cdots c_{k-1}(d_k\ G_k)]$, then

$$\pi_G = H_{d_1}(K(G_1))c_1(\cdots c_{k-1}H_{d_k}(K(G_k))\cdots)$$

We consider again the groups

$$\pi_{p,q} = \pi_q(R^{p+1}X) \cong H_q(R^pX), \quad q \geq 2$$

On account of the isomorphisms

$$H_q(R^pX) \cong H_q\left(\prod_{n\geq 2} K(\pi_{p-1,n}, n)\right) \cong H_q\left(\bigotimes_{n\geq 2} C_*(K(\pi_{p-1,n}, n))\right)$$

and applying the Künneth formula in a recursive way, one obtains that $\pi_{p,q}$ is a direct sum of different *components*

$$\pi_{p,q} \cong (C_1)^{m_1} \oplus \cdots \oplus (C_r)^{m_r}$$

where each component C_i (which appears m_i times) is directly related to some genealogy of degree q, $G_i \in Gen_q$, such that $C_i = \pi_{G_i}$.

In other words, $\pi_{p,q}$ can be described as

$$\pi_{p,q} \cong \bigoplus_{G\in Gen_q} (\pi_G)^{n_G(p)}$$

where the function n_G counts the number of times the group π_G appears as a component of $\pi_{p,q} \cong H_q(R^{p+1}X)$.

We note that different genealogies can lead to isomorphic groups, for instance

$$\begin{aligned} G &= [(4\ H_2) \otimes (7\ [(4\ H_2) \otimes (3\ H_3)])] \in GG_2 \\ G' &= [(4\ [(4\ H_2)]) \otimes (7\ [(4\ H_2) \otimes (3\ H_3)])] \in GG_2 \\ \pi_G &\cong \pi_{G'} \cong H_4(K(H_2, 2)) \otimes H_4(K(H_2, 2)) \otimes H_3 \end{aligned}$$

Since the isomorphisms between the groups associated with different genealogies are not always easy to deal with, we will consider that π_G and $\pi_{G'}$ are different components.

To simplify the notation, we identify each genealogy G with the associated group π_G. The problem is: how can we count the number of times a genealogy $G \in Gen_q$ occurs in the group $H_q(R^pX) \cong \pi_q(R^{p+1}X) = \pi_{p,q}$? In other words, how can we determine the exponent $n_G(p)$ for each genealogy G?

Given a genealogy $G \in Gen_q$ of degree $q \geq 2$, the *enumeration function*

$$\begin{aligned} n_G : \mathbb{N} &\longrightarrow \mathbb{N} \\ p &\longmapsto n_G(p) \end{aligned}$$

which calculates how many times $G \equiv \pi_G$ appears as a component of the group $\pi_{p,q} \cong H_q(R^pX)$ can be built recursively as follows.

- If $G = H_q$, then it is not difficult to observe that

$$n_G(p) = 1 \quad \text{for all } p \geq 0$$

- If $G = [(d_1\ G_1)c_1(d_2\ G_2)c_2 \cdots c_{k-1}(d_k\ G_k)]$, then G can be present in $\pi_{p,q}$ in two different ways:
  - if G takes part of $H_q(R^{p-1}X) \cong \pi_{p-1,q}$, this provides *directly* an appearance of G in $\pi_{p,q} \cong H_q(R^pX)$, corresponding to a factor $H_q(K(G))$ obtained when applying the Künneth formula. In other words, G appears in $\pi_{p,q}$ so many times as G takes part already of $\pi_{p-1,q}$, that is, $n_G(p-1)$ times;
  - G also occurs in $\pi_{p,q} \cong H_q(R^pX)$ so many times as G can be obtained in terms of the genealogies $G_1, \ldots, G_k$ which take part of the groups $H_i(R^{p-1}X) \cong \pi_{p-1,i}$ with $2 \leq i \leq q-2$. In this case we regroup the factors in groups of equal genealogies:

$$G_{1,1}, \ldots, G_{1,k_1}, G_{2,1}, \ldots, G_{2,k_2}, \ldots, G_{r,1}, \ldots, G_{r,k_r}$$

    such that $k_1 + \cdots + k_r = k$, $G_{j,1} = \cdots = G_{j,k_j} \equiv G'_j$ for each $1 \leq j \leq r$. Then one can observe that the number of times the genealogy G is obtained in this way is given by

$$\mathrm{Bin}(n_{G'_1}(p-1), k_1) \cdots \mathrm{Bin}(n_{G'_r}(p-1), k_r)$$

    where for each $p \geq 1$

$$\mathrm{Bin}(n_{G'_j}(p-1), k_j) = \begin{cases} \frac{n_{G'_j}(p-1)\cdots(n_{G'_j}(p-1)-k_j+1)}{k_j!} & \text{if } n_{G'_j}(p-1) \geq k_j \\ 0 & \text{if } n_{G'_j}(p-1) < k_j \end{cases}$$

One has therefore

$$n_G(p) = n_G(p-1) + \mathrm{Bin}(n_{G'_1}(p-1), k_1) \cdots \mathrm{Bin}(n_{G'_r}(p-1), k_r)$$

This means that the function $n_G(p)$ is obtained by *discrete integration* of the function

$$\mathrm{Bin}(n_{G'_1}(p-1), k_1) \cdots \mathrm{Bin}(n_{G'_r}(p-1), k_r)$$

Lemma 5.22. Given a genealogy $G \in Gen_q$ with $q \geq 2$, the number of times G appears in the group $\pi_{p,q} \cong H_q(R^pX)$ for each $p \geq 0$, $n_G(p)$, is given by a polynomial in the variable p of degree $< q/2$.

Proof. We apply induction on the degree q.

If $q = 2$ or 3, then it is not hard to observe that $H_q(R^pX) \cong H_q(X) \equiv H_q$ for all $p \geq 0$. It is clear then that the unique genealogy which takes part of $\pi_{p,q}$ is $G = H_q$, and

$$n_{H_q}(p) = 1 \quad \text{for all } p \geq 0$$

Therefore if $q = 2$ or $q = 3$ then $n_G(p)$ is a polynomial of degree $0 < q/2$ for every genealogy $G \in Gen_q$.

Let us consider now $q > 3$, and let us suppose that for any genealogy $\widetilde{G} \in Gen_{\tilde{q}}$ such that $\tilde{q} < q$ the function $n_{\widetilde{G}}(p)$ is a polynomial in p of degree $< \tilde{q}/2$. Let G be a genealogy of degree q.

- If $G = H_q$, then
$$n_G(p) = 1 \quad \text{for all } p \geq 0$$
so that we have a polynomial of degree $0 < q/2$.

- If $G = [(d_1\ G_1)c_1(d_2\ G_2)c_2 \cdots c_{k-1}(d_k\ G_k)]$, we must distinguish two cases:
  - if $k = 1$, then $G = [(d_1\ G_1)]$, with $d_1 = q$ and G_1 a genealogy of degree $\tilde{d}_1 \leq d_1 - 2 = q - 2$. One has therefore
$$n_G(p) = n_G(p-1) + n_{G_1}(p-1)$$
where $n_{G_1}(p-1)$ is a polynomial in p of degree $t < \tilde{d}_1/2$. In this way $n_G(p)$ is the *discrete primitive* of a polynomial of degree t, so that it must be a polynomial of degree $t+1$, with
$$t + 1 < \frac{\tilde{d}_1}{2} + 1 \leq \frac{q-2}{2} + 1 = \frac{q}{2}$$
  - if $k \geq 2$, then
$$n_G(p) = n_G(p-1) + \operatorname{Bin}(n_{G'_1}(p-1), k_1) \cdots \operatorname{Bin}(n_{G'_r}(p-1), k_r)$$
where the family of genealogies $G_1, \ldots, G_k$ has been regrouped as
$$G_{1,1}, \ldots, G_{1,k_1}, G_{2,1}, \ldots, G_{2,k_2}, \ldots, G_{r,1}, \ldots, G_{r,k_r}$$
with $k_1 + \cdots + k_r = k$, and $G_{j,1} = \cdots = G_{j,k_j} \equiv G'_j$ for all $1 \leq j \leq r$.
If $k_j > n_{G'_j}(p-1)$ for some $1 \leq j \leq r$, then
$$\operatorname{Bin}(n_{G'_1}(p-1), k_1) \cdots \operatorname{Bin}(n_{G'_r}(p-1), k_r) = 0$$
and the result holds. We suppose therefore $k_j \leq n_{G'_j}(p-1)$ for all j.
If each G'_j has degree d'_j, making use of the induction hypothesis, $n_{G'_j}(p-1)$ is a polynomial in p of degree $t_j < d'_j/2$, which implies that
$$\operatorname{Bin}(n_{G'_j}(p-1), k_j) = \frac{n_{G'_j}(p-1) \cdots (n_{G'_j}(p-1) - k_j + 1)}{k_j!}$$
is a polynomial in p of degree $k_j \cdot t_j \leq k_j \cdot (d'_j/2 - 1) = k_j \cdot d'_j/2 - k_j$. As a result, the polynomial
$$\operatorname{Bin}(n_{G'_1}(p-1), k_1) \cdots \operatorname{Bin}(n_{G'_r}(p-1), k_r)$$

has degree

$$\begin{aligned} t &= k_1 \cdot t_1 + \cdots + k_r \cdot t_r \le k_1 \cdot \frac{d'_1}{2} + \cdots + k_r \cdot \frac{d'_r}{2} - k_1 - \cdots - k_r \\ &= \frac{k_1 \cdot d'_1 + \cdots + k_r \cdot d'_r}{2} - k \le \frac{q}{2} - k \le \frac{q}{2} - 2 \end{aligned}$$

Since $n_G(p)$ is the primitive of $\operatorname{Bin}(n_{G'_1}(p-1), k_1) \cdots \operatorname{Bin}(n_{G'_r}(p-1), k_r)$, $n_G(p)$ must be a polynomial in p of degree

$$t + 1 \le \frac{q}{2} - 2 + 1 = \frac{q}{2} - 1 < \frac{q}{2}$$

□

We have seen therefore that for each pair (p,q) with $p \ge 0$ and $q \ge 2$, the group $\pi_{p,q} = \pi_q(R^{p+1}X) \cong H_q(R^pX)$ can be expressed as a direct sum of (groups associated with) genealogies of degree q, each of them appearing several times:

$$\pi_{p,q} \cong \bigoplus_{G \in Gen_q} (\pi_G)^{n_G(p)}$$

where $n_G(p)$ is a polynomial in the variable p of degree $< q/2$.

On the other hand, for the previous column, the group $\pi_{p-1,q} \cong H_q(R^{p-1}X)$ will be given by

$$\pi_{p-1,q} \cong \bigoplus_{G \in Gen_q} (\pi_G)^{n_G(p-1)}$$

with $n_G(p-1) \le n_G(p)$ for each genealogy $G \in Gen_q$.

We must focus again on the normalized homotopy groups

$$\pi'_{p,q} = \pi'_q(R^{p+1}X) \cong \pi_q(R^{p+1}X \cap \operatorname{Ker} \eta^0 \cap \ldots \cap \operatorname{Ker} \eta^{p-1})$$

We begin by considering only the last codegeneracy map η^{p-1} and the isomorphism

$$\pi_q(R^{p+1}X) \cong \pi_q(R^{p+1}X \cap \operatorname{Ker} \eta^{p-1}) \oplus \pi_q(R^pX)$$

which implies that the group $\pi^1_{p,q} = \pi_q(R^{p+1}X \cap \operatorname{Ker} \eta^{p-1})$ can be described as

$$\pi^1_{p,q} \cong \bigoplus_{G \in Gen_q} (\pi_G)^{n_G(p) - n_G(p-1)}$$

where the exponent $n_G(p) - n_G(p-1)$ is the *discrete derivative* of $n_G(p)$, denoted by $n^1_G(p)$, which is also a polynomial in the variable p.

We consider now the group $\pi^2_{p,q} = \pi_q(R^{p+1}X \cap \operatorname{Ker} \eta^{p-2} \cap \operatorname{Ker} \eta^{p-1})$ and the isomorphism

$$\pi^1_{p,q} \cong \pi^2_{p,q} \oplus \pi^1_{p-1,q}$$

It follows that

$$\pi^2_{p,q} \cong \bigoplus_{G \in Gen_q} (\pi_G)^{n^2_G(p)}$$

where $n^2_G(p) \equiv n^1_G(p) - n^1_G(p-1)$ is the discrete derivative of $n^1_G(p)$.

Then, iterating the process, for the group $\pi^s_{p,q} = \pi_q(R^{p+1}X \cap \mathrm{Ker}\,\eta^{p-s} \cap \ldots \cap \mathrm{Ker}\,\eta^{p-1})$ we have the isomorphism (see Lemma 5.20)

$$\pi^{s-1}_{p,q} \cong \pi^s_{p,q} \oplus \pi^{s-1}_{p-1,q}$$

which implies that $\pi^s_{p,q}$ is given by a direct sum of genealogies $G \in Gen_q$, where now each G appears

$$n^s_G(p) \equiv n^{s-1}_G(p) - n^{s-1}_G(p-1) \text{ times}$$

Finally, for $s = p$, the group $\pi'_{p,q} \cong \pi_q(R^{p+1}X \cap \mathrm{Ker}\,\eta^0 \cap \ldots \cap \mathrm{Ker}\,\eta^{p-1})$ will be also isomorphic to a direct sum of the form

$$\pi'_{p,q} \cong \bigoplus_{G \in Gen_q} (\pi_G)^{n^p_G(p)}$$

where for each genealogy G, $n^p_G(p)$ is constructed recursively from $n^{p-1}_G(p)$ as

$$n^p_G(p) \equiv n^{p-1}_G(p) - n^{p-1}_G(p-1)$$

Let us remark that if the initial function $n_G(p)$ is a polynomial of degree $t > 0$ then the discrete derivative $n^1_G(p)$ has degree $t-1$; if $n_G(p)$ has degree 0 (it is a constant), then $n^1_G(p) = 0$. In this way, the degree of $n^p_G(p)$ when $t - p$ if $t \geq p$, and $n^p_G(p) = 0$ if $t < p$.

Let us suppose now that $q < 2p + 2$. Then, making use of Lemma 5.22, for each genealogy G of degree q one has that $n_G(p)$ is a polynomial of degree $t < q/2$, and therefore

$$t \leq \frac{q}{2} - 1 < \frac{2p+2}{2} - 1 = p + 1 - 1 = p$$

It follows then that $n^p_G(p) = 0$ for every genealogy G of degree q, and therefore

$$\pi'_{p,q} = 0 \quad \text{if } q < 2p + 2$$

We have completed in this way the proof of Theorem 5.18: given a 1-reduced pointed simplicial set X then the associated Bousfield-Kan spectral sequence $E = (E^r, d^r)_{r \geq 1}$ satisfies

$$E^1_{p,q} = 0 \quad \text{if } q < 2p + 2$$

This important property implies in particular that the bigraded module $E^1 = \{E^1_{p,q}\}_{p,q \in \mathbb{Z}}$ is tapered, which guarantees the convergence of the spectral sequence:

given $p, q \in \mathbb{Z}$ then $d^r_{p,q} = 0$ whenever $r > q - 2p - 3$, and if $r > p$ then $d^r_{p-r,q-r+1} = 0$, which implies that $E^\infty_{p,q} = E^r_{p,q}$ for $r > \max\{p, q - 2p - 3\}$.

The convergence of the Bousfield-Kan spectral sequence is in fact already known, and was proved by other means, for instance in [BK72a] and [BK73a]. Our proof is more elementary but it could be useful. Furthermore, this reasoning will be used in the next section to compute the first level of the spectral sequence in the case $X = S^2$.

5.2.2 Algorithms computing E^1 and E^2

Let X be a 1-reduced pointed simplicial set. As said in the introduction of the second part of this chapter, the associated Bousfield-Kan spectral sequence (which converges to the homotopy groups $\pi_*(X)$) can be constructed by different means. We begin by considering the one of [BK73a], where this spectral sequence is built as the homotopy spectral sequence of the cosimplicial space $\mathcal{R}X$, making use of additive relations and universal examples as mentioned in Section 5.1.4.2.

The first level of this spectral sequence is therefore given by the normalized homotopy groups:

$$E^1_{p,q} = \pi'_q(\mathcal{R}X^p) = \pi_q(R^{p+1}X) \cap \operatorname{Ker}\eta^0 \cap \ldots \cap \operatorname{Ker}\eta^{p-1} \cong \pi_q(R^{p+1}X \cap \operatorname{Ker}\eta^0 \cap \ldots \cap \operatorname{Ker}\eta^{p-1})$$

and the first differential map $d^1 : E^1_{p,q} \to E^1_{p+1,q}$ is the morphism induced by the alternate sum $\delta^{p+1}_q = \sum_{j=0}^{p+1} (-1)^j \partial^j$.

As seen in the previous section, the group $\pi'_q(\mathcal{R}X^p) = \pi'_q(R^{p+1}X) \equiv \pi'_{p,q}$ is given by a direct sum of (groups associated with) genealogies

$$\pi'_{p,q} \cong \bigoplus_{G \in Gen_q} (\pi_G)^{n^p_G(p)}$$

We have already mentioned that it is possible to construct all the genealogies G of a given degree q, and one can also determine the corresponding enumeration functions $n_G(p)$, and then $n^p_G(p)$. Furthermore, if the initial groups $H_*(X)$ are known (and they are finitely generated), then the Betti number and the torsion coefficients of each genealogy G (which is constructed by means of tensor and torsion products of finitely generated groups) can also be determined, and in this way we can *compute* the normalized groups $\pi'_q(R^{p+1}X) = E^1_{p,q}$ for every $p, q \in \mathbb{N}$.

Nevertheless, in order to compute the groups $E^2_{p,q} \cong \operatorname{Ker} d^1_{p,q} / \operatorname{Im} d^1_{p-1,q}$ the previous information is not sufficient; an *effective* version of the groups $E^1_{p,q}$, with the corresponding generators, is necessary to determine the subgroups $\operatorname{Ker} d^1_{p,q}$ and $\operatorname{Im} d^1_{p-1,q}$. At this point, the effective homology method appears again.

We recall now the isomorphism $\pi_*(RX) \cong \widetilde{H}_*(X)$, satisfied by any simplicial set X, which implies $\pi_*(R^{p+1}X) \cong \widetilde{H}_*(R^pX)$ for every $p > 1$. Hence one has

$$E^1_{p,q} = \pi_q(R^{p+1}X) \cap \operatorname{Ker}\eta^0 \cap \ldots \cap \operatorname{Ker}\eta^{p-1} \cong \widetilde{H}_q(R^pX) \cap \operatorname{Ker}\eta^0 \cap \ldots \cap \operatorname{Ker}\eta^{p-1}$$

If X is a 1-reduced simplicial set with effective homology, then our Algorithm 11 (the fundamental result of Chapter 4) provides us the effective homology of the simplicial Abelian group RX. Iterating the process (taking into account that RX is also 1-reduced), it is possible to obtain the effective homology of R^pX for every $p \geq 1$. In this way, if X is an object with effective homology, then R^pX is also an object with effective homology, and this implies the groups $\widetilde{H}_q(R^pX) \cong \pi_q(R^{p+1}X)$ (with the corresponding generators) are computable.

The codegeneracy maps η^j are well-defined on these homotopy groups:

$$\pi_q(\eta^j) \equiv \eta^j : \pi_q(R^{p+1}X) \longrightarrow \pi_q(R^pX) \quad 0 \leq j \leq p-1$$

and as far as $\pi_q(R^{p+1}X)$ and $\pi_q(R^pX)$ are groups of finite type, these maps can be expressed as finite integer matrices. Therefore, the kernels $\ker \eta^j$ can be computed by means of elementary operations for each $0 \leq j \leq p-1$, and in this way one can determine the normalized groups $\pi'_{p,q} = E^1_{p,q}$ (with a basis-divisors description).

The differential map $d^1_{p,q} : E^1_{p,q} \to E^1_{p+1,q}$ is induced by $\delta^{p+1}_q = \sum_{j=0}^{p+1}(-1)^j\partial^j$. Then, for a class $[x] \in E^1_{p,q}$ (given by means of the coefficients with respect to the generators of the group) it is possible to compute the image $d^1_{p,q}([x]) = [\delta^{p+1}_q(x)] \in E^1_{p+1,q}$. This implies that, if X is an object with effective homology, the first level of the Bousfield-Kan spectral sequence is computable.

Algorithm 15.
Input:

- a 1-reduced pointed simplicial set X,
- an equivalence $C_*(X) \Lleftarrow DX_* \Rrightarrow HX_*$, where HX_* is an effective chain complex.

Output:

- the groups $E^1_{p,q} = \pi'_q(R^{p+1}X)$ for each $p, q \in \mathbb{Z}$, represented by means of their basis-divisors description,
- the differential maps $d^1_{p,q}$ for all $p, q \in \mathbb{Z}$.

Let us remark now that the differential maps $d^1_{p,q} : E^1_{p,q} \to E^1_{p+1,q}$ can also be expressed as finite integer matrices. Therefore it is possible to determine their kernel and their image, and using the Smith Normal Form technique we can easily compute the quotient groups

$$E^2_{p,q} = \frac{\operatorname{Ker} d^1_{p,q}}{\operatorname{Im} d^1_{p-1,q}}$$

We obtain therefore the following algorithm which determines the groups of the second level of the Bousfield-Kan spectral sequence. However, in this case the differential maps cannot be computed.

Algorithm 16.
Input:

- a 1-reduced pointed simplicial set X,

- an equivalence $C_*(X) \Lleftarrow DX_* \Rrightarrow HX_*$, where the chain complex HX_* is effective.

Output: the groups $E^2_{p,q}$ for every $p, q \in \mathbb{Z}$, with a basis-divisors representation.

We have seen in this way that the effective homology method makes it possible to compute the first two stages of the Bousfield-Kan spectral sequence of a pointed simplicial set X (when it is an object with effective homology). In this construction, our Algorithm 11 plays an essential role, providing us the effective homology of the simplicial Abelian groups R^pX for every $p \geq 1$, which in particular allow us to determine the effective version of the groups $\widetilde{H}_*(R^pX) \cong \pi_*(R^{p+1}X)$.

For instance, let us consider the case where the simplicial set X is the 2-sphere S^2, which is obviously an object with (trivial) effective homology. The first non-normalized homotopy groups of the columns $\mathcal{R}X^p$, $\pi_q(R^{p+1}S^2) \cong \widetilde{H}_q(R^pS^2)$, are given by the following figure.

q

0	$\mathbb{Z}$	$\mathbb{Z}^5 \oplus \mathbb{Z}_2^2 \oplus \mathbb{Z}_3$	$\mathbb{Z}^{14} \oplus \mathbb{Z}_2^9 \oplus \mathbb{Z}_3^3$	$\mathbb{Z}^{30} \oplus \mathbb{Z}_2^{24} \oplus \mathbb{Z}_3^6$
0	0	0	0	0
0	$\mathbb{Z}$	$\mathbb{Z}^3 \oplus \mathbb{Z}_2$	$\mathbb{Z}^6 \oplus \mathbb{Z}_2^3$	$\mathbb{Z}^{10} \oplus \mathbb{Z}_2^6$
0	0	0	0	0
0	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}^3$	$\mathbb{Z}^4$
0	0	0	0	0
$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
0	0	0	0	0
0	0	0	0	0

p

A little calculation with the genealogies of order ≤ 8 and the corresponding enumeration functions makes it possible to compute the normalized groups $\pi'_q(R^{p+1}S^2)$, which satisfy $\pi'_q(R^{p+1}S^2) = 0$ whenever $q < 2p + 2$.

q					
	0	$\mathbb{Z}$	$\mathbb{Z}^3 \oplus \mathbb{Z}_2^2 \oplus \mathbb{Z}_3$	$\mathbb{Z}^2 \oplus \mathbb{Z}_2^3$	0
	0	0	0	0	0
	0	$\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}_2$	0	0
	0	0	0	0	0
	0	$\mathbb{Z}$	0	0	0
	0	0	0	0	0
	$\mathbb{Z}$	0	0	0	0
	0	0	0	0	0
	0	0	0	0	0
					p

We must remark that, since our Algorithm 11 for the computation of the effective homology of RX is not yet implemented, we cannot obtain the generators of the groups $\pi_q(R^{p+1}S^2)$, which are necessary to determine the *effective* version of the normalized homotopy groups $\pi'_q(R^{p+1}S^2) = \pi_q(R^{p+1}S^2) \cap \operatorname{Ker} \eta^0 \cap \ldots \cap \operatorname{Ker} \eta^{p-1}$. Once Algorithm 11 will be written in Common Lisp, we will be able to obtain the generators of the groups $\pi_q(R^{p+1}S^2) \cong \widetilde{H}_q(R^pS^2)$, and in this way the codegeneracy maps

$$\pi_q(\eta^j) \equiv \eta^j : \pi_q(R^{p+1}S^2) \longrightarrow \pi_q(R^pS^2) \quad 0 \leq j \leq p-1$$

will be given by finite integer matrices, so that $E^1_{p,q} = \pi'_q(R^{p+1}S^2)$ will be determined by means of a basis-divisors description.

Then, applying the coboundary map $\delta^{p+1}_q = \sum_{j=0}^{p+1}(-1)^j\partial^j$ to the generators of the groups $\pi'_q(R^{p+1}S^2)$, we will obtain the matrix representation of the differential maps $d^1 : E^1_{p,q} \to E^1_{p+1,q}$, and computing their kernel and their image we will obtain the second level $E^2_{p,q}$ of the Bousfield-Kan spectral sequence.

In view of the groups $\pi'_q(R^{p+1}S^2)$ (and taking into account that $\pi'_q(R^{p+1}S^2) = E^1_{p,q} = 0$ if $q < 2p+2$), for $q \leq 8$ there exist only three possibly non-null differential maps at level $r = 1$, namely $d^1_{1,6}$, $d^1_{1,8}$ and $d^1_{2,8}$. All the differentials d^2 are equal to zero, and for $r = 3$, only $d^3_{1,8} : E^3_{1,8} \to E^3_{4,10}$ could be non-null. This implies all the groups $E^2_{p,q}$ for $q \leq 8$ will be already the final groups $E^\infty_{p,q}$, except possibly $E^2_{1,8}$. In this way, our Algorithm 16 makes it possible to compute many final groups $E^\infty_{p,q}$ of the Bousfield-Kan spectral sequence associated with the 2-sphere S^2. To be precise, one has the following diagram, where the groups denoted by $\star$ would be computable by means of Algorithms 15 and 16.

q ↑ $\boxed{r = \infty}$

0	$??$	$\star$	$\star$	0
0	0	0	0	0
0	$\star$	$\star$	0	0
0	0	0	0	0
0	$\mathbb{Z}$	0	0	0
0	0	0	0	0
$\mathbb{Z}$	0	0	0	0
0	0	0	0	0
0	0	0	0	0

→ p

Then, knowing that this spectral sequence converges to the homotopy groups $\pi_*(S^2)$ and that in this case we do not find extension problems, these groups would allow us to determine the homotopy groups $\pi_n(S^2)$ for $n \leq 5$. In particular, one has directly the well-known results $\pi_2(S^2) = \pi_3(S^2) = \mathbb{Z}$. For the the groups $\pi_4(S^2)$ and $\pi_5(S^2)$, one can claim $\pi_4(S^2) \subseteq \mathbb{Z} \oplus \mathbb{Z}_2$ and $\pi_5(S^2) \subseteq \mathbb{Z} \oplus \mathbb{Z}^2 \oplus \mathbb{Z}_2^3 = \mathbb{Z}^3 \oplus \mathbb{Z}_2^3$, but to compute them an implementation of Algorithms 11, 15 and 16 would be necessary.

When trying to compute $\pi_6(S^2)$, we need the group $E^\infty_{1,8}$, but as we have said, it cannot be determined by means of Algorithms 15 and 16 because the differential map $d^3_{1,8} : E^3_{1,8} \to E^3_{4,10}$ could be non-null.

In order to compute the differentials d^r with $r \geq 2$, the definition of the spectral sequence of a cosimplicial space does not seem to be sufficient. For this reason, in the following section we consider an equivalent definition of the Bousfield-Kan spectral sequence by means of towers of fibrations, which would be used to sketch a new algorithm computing the higher levels.

5.2.3 Toward an algorithm computing the Bousfield-Kan spectral sequence

As mentioned before, the construction of the Bousfield-Kan spectral sequence of a simplicial set X as the homotopy spectral sequence of the cosimplicial space $\mathcal{R}X$ (introduced in [BK73a]) does not provide us the necessary information to obtain an *algorithm* computing every level of the spectral sequence. In this section, we deal with a different version of the Bousfield-Kan spectral sequence, the one explained in [BK72b], which is isomorphic to the previous construction. Before focusing on our particular situation, we include some general definitions and results about the homotopy spectral sequence of a tower of fibrations.

5.2.3.1 Homotopy spectral sequence of a tower of fibrations

Let $(Y_n, f_n)_{n\geq 0}$ be a *tower of fibrations*, that is to say, a family of pointed simplicial sets $\{Y_n\}_{n\geq 0}$ with fibrations

$$\cdots \xrightarrow{f_{n+1}} Y_n \xrightarrow{f_n} Y_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} Y_0 \xrightarrow{f_0} Y_{-1} = \star$$

where $\star$ denotes the simplicial set with only one simplex $\star$ in each dimension.

Its *inverse limit* is a simplicial set

$$\varprojlim Y_n = Y$$

with projections $p_n : Y \to Y_n$ such that $f_n \circ p_n = p_{n-1}$, satisfying the corresponding universal property for inverse limits.

Let F_n be the fiber of $f_n : Y_n \to Y_{n-1}$ for each $n \geq 0$. The long exact sequence of homotopy [May67] provides us the following diagram:

$$\begin{array}{ccccccc}
\Big\downarrow{\scriptstyle f} & & & & \Big\downarrow{\scriptstyle f} & & \\
\pi_{q-p+1}(Y_p) & & & & \pi_{q-p}(Y_{p+1}) & \xrightarrow{\partial} & \pi_{q-p-1}(F_{p+2}) \\
\Big\downarrow{\scriptstyle f} & & & & \Big\downarrow{\scriptstyle f} & & \\
\pi_{q-p+1}(Y_{p-1}) & \xrightarrow{\partial} & \pi_{q-p}(F_p) & \xrightarrow{i} & \pi_{q-p}(Y_p) & & \\
\Big\downarrow{\scriptstyle f} & & & & \Big\downarrow{\scriptstyle f} & & \\
\pi_{q-p+1}(Y_{p-2}) & & & & \pi_{q-p}(Y_{p-1}) & \xrightarrow{\partial} & \pi_{q-p-1}(F_p) \\
\Big\downarrow{\scriptstyle f} & & & & \Big\downarrow{\scriptstyle f} & &
\end{array}$$

where $\partial : \pi_*(Y_{n-1}) \to \pi_{*-1}(F_n)$ is the connecting morphism and $i : \pi_*(F_n) \to \pi_*(Y_n)$ is induced by the inclusion $\mathrm{inc} : F_n \hookrightarrow Y_n$. We denote by f^r the composition $f \circ \overset{r}{\cdots} \circ f$. The picture leads to the following spectral sequence.

Theorem 5.23. [BK72b] Given a tower of fibrations $(Y_n, f_n)_{n\geq 0}$, there exists a second quadrant spectral sequence $E = (E^r, d^r)_{r\geq 1}$ given by

$$\begin{aligned}
E^r_{p,q} &= \frac{i^{-1}(\mathrm{Im}\, f^{r-1})}{\partial(\mathrm{Ker}\, f^{r-1})} &&\text{for } q \geq p \\
E^r_{p,q} &= 0 &&\text{otherwise}
\end{aligned}$$

(if $q = p$, then $E^r_{p,q}$ is the set of orbits of the action of $\mathrm{Ker}\, f^{r-1}$) which under some *good* conditions (see [BK72b] for details) converges to the homotopy groups of the inverse limit Y.

It is clear that, if the homotopy groups $\pi_*(Y_n)$ and $\pi_*(F_n)$ are finitely generated groups and they are explicitly known (with the corresponding generators) for all n, then the groups $E^r_{p,q}$ are computable because the involved maps f, i and ∂ can be expressed as finite integer matrices. In this way, as we want to develop an algorithm computing the spectral sequence associated with a tower of fibrations, we will try to construct first algorithms which determine the homotopy groups of the simplicial sets Y_n and of the fiber spaces F_n.

A particular example of tower of fibration comes associated with a cosimplicial space, as we explain in the following section.

5.2.3.2 Another definition for the homotopy spectral sequence of a cosimplicial space

Given a cosimplicial space $\mathcal{X}$, in Section 5.1.4.2 we have mentioned the construction of the associated homotopy spectral sequence, based on additive relations and universal examples. The same spectral sequence can also be built making use of towers of fibrations.

We begin by introducing some necessary definitions.

Definition 5.24. Let $\mathcal{X}$ be a cosimplicial space and K a simplicial set. The cosimplicial space $K \times \mathcal{X}$ is given in codimension n by the simplicial set $K \times \mathcal{X}^n$. The coface and codegeneracy operators are defined as the maps

$$\begin{array}{ll} \mathrm{Id}_K \times \partial^j : K \times \mathcal{X}^{n-1} \longrightarrow K \times \mathcal{X}^n & \text{for } 0 \leq j \leq n \\ \mathrm{Id}_K \times \eta^j : K \times \mathcal{X}^{n+1} \longrightarrow K \times \mathcal{X}^n & \text{for } 0 \leq j \leq n \end{array}$$

Definition 5.25. Let $\mathcal{X}$ and $\mathcal{Y}$ be cosimplicial spaces, the *function space* $\hom(\mathcal{X}, \mathcal{Y})$ is the simplicial set whose n-simplices are the cosimplicial morphisms

$$\Delta[n] \times \mathcal{X} \longrightarrow \mathcal{Y}$$

with faces ∂_i and degeneracies η_i given by the compositions

$$\begin{array}{l} \Delta[n-1] \times \mathcal{X} \stackrel{\partial^i \times \mathrm{Id}_{\mathcal{X}}}{\longrightarrow} \Delta[n] \times \mathcal{X} \longrightarrow \mathcal{Y} \\ \Delta[n+1] \times \mathcal{X} \stackrel{\eta^i \times \mathrm{Id}_{\mathcal{X}}}{\longrightarrow} \Delta[n] \times \mathcal{X} \longrightarrow \mathcal{Y} \end{array}$$

where $\Delta[n]$ is the standard n-simplex, and $\partial^i : \Delta[n-1] \to \Delta[n]$ and $\eta^i : \Delta[n+1] \to \Delta[n]$ are the standard maps introduced in Section 1.2.1.

Definition 5.26. Let $\mathcal{X}$ be a cosimplicial space, the *total space* $\operatorname{Tot} \mathcal{X}$ is the simplicial set defined as the function space

$$\operatorname{Tot} \mathcal{X} = \hom(\Delta, \mathcal{X})$$

The total space of a cosimplicial space $\mathcal{X}$ can be seen as an inverse limit

$$\operatorname{Tot} \mathcal{X} = \varprojlim \operatorname{Tot}_n \mathcal{X}$$

where $\operatorname{Tot}_n \mathcal{X}$ is the simplicial set

$$\operatorname{Tot}_n \mathcal{X} = \hom(\Delta^{[n]}, \mathcal{X}) \quad \text{for } n \geq -1$$

and $\Delta^{[n]} \subset \Delta$ is the *simplicial n-skeleton* of the cosimplicial space Δ, in other words, $\Delta^{[n]}$ consists in codimension m of the n-skeleton of the simplicial set $\Delta[m]$. The map $f_n : \operatorname{Tot}_n \mathcal{X} \to \operatorname{Tot}_{n-1} \mathcal{X}$ is induced by the inclusion $\Delta^{[n-1]} \subset \Delta^{[n]}$. Furthermore, one can observe that $\operatorname{Tot}_{-1} \mathcal{X} = \star$ and $\operatorname{Tot}_0 \mathcal{X} \cong \mathcal{X}^0$.

If $\mathcal{X}$ is augmented (that is, a simplicial set $\mathcal{X}^{-1}$ and a simplicial morphism $\partial^0 : \mathcal{X}^{-1} \to \mathcal{X}^0$ are given such that $\partial^1\partial^0 = \partial^0\partial^0 : \mathcal{X}^{-1} \to \mathcal{X}^1$), then ∂^0 induces morphisms

$$p_n : \mathcal{X}^{-1} \longrightarrow \operatorname{Tot}_n \mathcal{X}$$

which are compatible with the maps $f_n : \operatorname{Tot}_n \mathcal{X} \to \operatorname{Tot}_{n-1} \mathcal{X}$.

Whenever the cosimplicial space $\mathcal{X}$ is *fibrant* (see [BK72b] for details), it can be proved that $(\operatorname{Tot}_n \mathcal{X}, f_n)_{n\geq 0}$ is a tower of fibrations, and therefore we can consider the associated homotopy spectral sequence $E = (E^r, d^r)_{r\geq 1}$ which under favorable conditions [BK72b] converges to $\pi_*(\operatorname{Tot} \mathcal{X})$.

This new definition of the homotopy spectral sequence of a cosimplicial space *coincides* with the one explained in Section 5.1.4.2.

Let us consider now a pointed simplicial set X, and the associated augmented cosimplicial space $\mathcal{R}X$. Provided that it can be seen that every grouplike cosimplicial space is fibrant and $\mathcal{R}X$ is grouplike, the homotopy spectral sequence of $\mathcal{R}X$ can also be defined as the spectral sequence of the tower of fibrations

$$\left(\operatorname{Tot}_n \mathcal{R}X = \hom(\Delta^{[n]}, \mathcal{R}X), f_n\right)_{n\geq 0}$$

In this case, as stated in [BK72b, p. 282], the fiber space F_n of the fibration $f_n : \operatorname{Tot}_n \mathcal{R}X = \hom(\Delta^{[n]}, \mathcal{R}X) \to \operatorname{Tot}_{n-1} \mathcal{R}X = \hom(\Delta^{[n-1]}, \mathcal{R}X)$ is the function space $F_n = \hom(S^n, N^n(\mathcal{R}X))$, which can be represented as the iterated loop space

$$F_n = \Omega^n(N^n(\mathcal{R}X)) = \Omega^n(R^{n+1}X \cap \operatorname{Ker} \eta^0 \cap \ldots \cap \operatorname{Ker} \eta^{n-1})$$

One immediately observes that the first level of the spectral sequence is, as already known,

$$E^1_{p,q} = \pi_{q-p}(F_p) = \pi_{q-p}(\Omega^p(N^p(\mathcal{R}X))) \cong \pi_q(N^p(\mathcal{R}X)) \cong \pi'_q(\mathcal{R}X^p)$$

This is therefore a different way of constructing the Bousfield-Kan spectral sequence associated with the simplicial set X, introduced in Theorem 5.17. This new construction will be considered in the following section to explain some ideas to develop an algorithm computing every level of the Bousfield-Kan spectral sequence, using one more time the effective homology method.

5.2.3.3 Sketch of the algorithm

In Section 5.2.2 we have shown that the effective homology method can be used to compute E^1 and E^2 (and the differential map d^1) in the Bousfield-Kan spectral sequence associated with a pointed simplicial set X (with effective homology), obtaining in this way Algorithms 15 and 16.

The computation of the higher levels of the spectral sequence is more difficult, and we have not yet developed a complete algorithm allowing one to determine the groups $E^r_{p,q}$ for $r > 2$. However, we think that using the definition of the Bousfield-Kan spectral sequence by means of towers of fibrations it will be possible to construct the looked-for algorithm, following the sketch that we explain in this section. Several details must yet be verified but we hope our new algorithm could be finished in a not too far future.

Given X a 1-reduced simplicial set with effective homology, let us recall that the associated Bousfield-Kan spectral sequence can be defined as the spectral sequence of the tower of fibrations

$$\cdots \xrightarrow{f_{n+1}} \operatorname{Tot}_n \mathcal{R}X \xrightarrow{f_n} \operatorname{Tot}_{n-1} \mathcal{R}X \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} \operatorname{Tot}_0 \mathcal{R}X \cong RX \xrightarrow{f_0} \operatorname{Tot}_{-1} \mathcal{R}X = \star$$

where $\operatorname{Tot}_n \mathcal{R}X = \hom(\Delta^{[n]}, \mathcal{R}X)$, and $f_n : \operatorname{Tot}_n \mathcal{R}X \to \operatorname{Tot}_{n-1} \mathcal{R}X$ is induced by the inclusion $\Delta^{[n-1]} \subset \Delta^{[n]}$. The fibers can be expressed as

$$F_n = \Omega^n(N^n(\mathcal{R}X)) = \Omega^n(R^{n+1}X \cap \operatorname{Ker} \eta^0 \cap \ldots \cap \operatorname{Ker} \eta^{n-1})$$

In this way, the first fibration is

$$F_1 = \Omega(N^1(\mathcal{R}X)) \hookrightarrow \operatorname{Tot}_1 \mathcal{R}X \to RX$$

The base space RX satisfies

$$\pi_*(RX) \cong \widetilde{H}_*(X) = H_*(\widetilde{C}_*(X))$$

where $\widetilde{C}_*(X) = C_*(X)/C_*(\star)$. Since X is an object with effective homology, the homotopy groups of RX are computable, thanks to the previous isomorphism, in an explicit way (with the corresponding generators); we can say that RX has *effective homotopy*.

On the other hand, for the fiber $F_1 = \Omega(N^1(\mathcal{R}X)) = \Omega(R^2X \cap \operatorname{Ker} \eta^0)$ one has

$$\begin{aligned}\pi_*(F_1) &= \pi_*(\Omega(R^2X \cap \operatorname{Ker} \eta^0)) \cong \pi_{*+1}(R^2X \cap \operatorname{Ker} \eta^0) \cong \pi_{*+1}(R^2X) \cap \operatorname{Ker} \eta^0 \\ &= H_{*+1}(N_*(R^2X)) \cap \operatorname{Ker} \eta^0 \cong H_{*+1}(\widetilde{C}^N_*(RX)) \cap \operatorname{Ker} \eta^0 \cong H_{*+1}(\widetilde{C}^N_*(RX) \cap \operatorname{Ker} \eta^0) \\ &= H_*(\operatorname{Desusp}_*(\widetilde{C}_*(RX) \cap \operatorname{Ker} \eta^0))\end{aligned}$$

where the chain complex morphism

$$\eta^0 : \widetilde{C}^N_*(RX) = (R^2X)^N_* \cong N_*(R^2X) \to \widetilde{C}^N_*(X) = (RX)^N_* \cong N_*(RX)$$

is induced by the codegeneracy map $\eta^0 : \mathcal{R}X^1 = R^2X \to \mathcal{R}X^0 = RX$, and Desusp_* is the *Desuspension* constructor which, from a chain complex $C_* = (C_n, d_n)_{n\in\mathbb{N}}$, returns a new chain complex $\mathrm{Desusp}_*(C_*) = D_* = (D_n, d_{D_n})_{n\in\mathbb{N}}$ given by

$$\begin{aligned} D_n &= C_{n+1} \\ d_{D_n} &= d_{n+1} : D_n = C_{n+1} \longrightarrow D_{n-1} = C_n \end{aligned}$$

It could be a good idea to detail some of the isomorphisms appearing in the previous formula. First of all, the relation $\pi_{*+1}(R^2X \cap \mathrm{Ker}\,\eta^0) \cong \pi_{*+1}(R^2X) \cap \mathrm{Ker}\,\eta^0$ is given by the morphisms

$$\begin{aligned} [x] \in \pi_{*+1}(R^2X \cap \mathrm{Ker}\,\eta^0) &\longmapsto \qquad\qquad [x] \in \pi_{*+1}(R^2X) \cap \mathrm{Ker}\,\eta^0 \\ [x] \in \pi_{*+1}(R^2X) \cap \mathrm{Ker}\,\eta^0 &\longmapsto \quad [x - \partial^0\eta^0 x] \in \pi_{*+1}(R^2X \cap \mathrm{Ker}\,\eta^0) \end{aligned}$$

The *same* maps give the isomorphism

$$H_{*+1}(\widetilde{C}^N_*(RX)) \cap \mathrm{Ker}\,\eta^0 \cong H_{*+1}(\widetilde{C}^N_*(RX) \cap \mathrm{Ker}\,\eta^0)$$

Taking into account that X is a simplicial set with effective homology, and making use again of our Algorithm 11 presented in Chapter 4, one has that RX is also an object with effective homology. Furthermore, thanks to the short exact sequence

$$0 \longrightarrow \widetilde{C}^N_*(RX) \cap \mathrm{Ker}\,\eta^0 \xrightarrow{\text{inc}} \widetilde{C}^N_*(RX) \xrightarrow{\eta^0} \widetilde{C}^N_*(X) \longrightarrow 0$$

it is not difficult to prove that so is the chain complex $\mathrm{Desusp}_*(\widetilde{C}_*(RX) \cap \mathrm{Ker}\,\eta^0)$, which makes the fiber F_1 an object with *effective homotopy*. A similar reasoning would allow us to affirm that each fiber $F_n = \Omega^n(N^n(\mathcal{R}X))$ has also effective homotopy.

In this way, we have a fibration $F_1 = \Omega(N^1(\mathcal{R}X)) \hookrightarrow \mathrm{Tot}_1\,\mathcal{R}X \to RX$ where the fiber and base spaces are objects with effective homotopy. A similar method to the one for the effective homology should exist allowing one to compute the effective homotopy of the total space $\mathrm{Tot}_1\,\mathcal{R}X$. Iterating the process, we would obtain the effective homotopy of every space $\mathrm{Tot}_n\,\mathcal{R}X$.

Once we would have computed the effective homotopy of F_n and $\mathrm{Tot}_n\,\mathcal{R}X$ for all n, which would provide us the homotopy groups $\pi_*(F_n)$ and $\pi_*(\mathrm{Tot}_n\,\mathcal{R}X)$ (that are finite type groups) with the corresponding generators, it would be possible to compute the groups $E^r_{p,q}$ of the spectral sequence, which as seen in Section 5.2.3.1 are given by the formula

$$E^r_{p,q} = \frac{i^{-1}(\mathrm{Im}\,f^{r-1})}{\partial(\mathrm{Ker}\,f^{r-1})} \quad \text{for } q \geq p$$

Also the differential maps $d^r : E^r_{p,q} \to E^r_{p+r,q+r-1}$ could be determined, so that we would obtain the following algorithm.

"Possible" Algorithm 17.
Input:

- a 1-reduced pointed simplicial set X,
- an equivalence $C_*(X) \Lleftarrow DX_* \Rrightarrow HX_*$, where HX_* is an effective chain complex.

Output:

- the groups $E^r_{p,q}$ for every $r \geq 1$ and $p, q \in \mathbb{Z}$ of the Bousfield-Kan spectral sequence associated with X, with a basis-divisors description,
- the differential maps $d^r_{p,q}$ for all $p, q \in \mathbb{Z}$ and $r \geq 1$.

Let us emphasize that, in order to finish this algorithm, the following facts have to be verified.

- The fiber $F_n = \hom(S^n, N^n(\mathcal{R}X))$ can be expressed as the iterated loop space $\Omega^n(N^n(\mathcal{R}X))$, but we need an explicit isomorphism with the Kan construction for loop spaces, that of Definition 3.13.
- The fibrations $F_n \hookrightarrow \mathrm{Tot}_n \mathcal{R}X \to \mathrm{Tot}_{n-1} \mathcal{R}X$ are given by the projection $f_n : \mathrm{Tot}_n \mathcal{R}X \to \mathrm{Tot}_{n-1} \mathcal{R}X$. We need to construct the twisting operator $\tau : \mathrm{Tot}_{n-1} \mathcal{R}X \to F_n$, which will probably be induced by the first coface $\partial^0 : \mathcal{R}X^{n-1} \to \mathcal{R}X^n$.
- Finally, it is necessary to construct an *effective homotopy* version of a fibration. In other words, given a fibration

  $$G \hookrightarrow E \to B$$

  where B and G are objects with effective homotopy, an algorithm should determine the effective homotopy of the total space E.

Taking into account that the spectral sequence is convergent (as proved in Section 5.2.1), Algorithm 17 would allow us to determine also the final groups $E^\infty_{p,q}$ (which will be equal to some $E^r_{p,q}$ for r sufficiently large). But we must remark that the computation of all the groups $\{E^r_{p,q}\}_{1 \leq r \leq \infty}$, the level $r = \infty$ included, would not allow us in general to determine the "limit" groups π_{p+q}, because extension problems could be found.

Conclusions an further work

This work has been guided by the goal of relating effective homology and spectral sequences, two different techniques in Algebraic Topology for the computation of homology (and homotopy) groups.

On the one hand, spectral sequences have been a classical method used to approximate homology groups of some complicated spaces (see, for instance, [McC85]), but in many cases the available data do not allow the user to determine the higher differentials.

On the other hand, the effective homology method (introduced in [Ser87] and [Ser94]) provides real algorithms for the computation of the looked-for homology groups, replacing in this way the spectral sequence technique. This method has been implemented in the Kenzo system [DRSS99] and can be used, for instance, to compute homology groups of total spaces of fibrations, of iterated loop spaces, of classifying spaces, etc.

In this memoir we have shown that we can also make use of the effective homology technique to determine the different components of spectral sequences, focusing our attention on two particular situations. We have begun by studying spectral sequences associated with filtered complexes, which include many classical examples of spectral sequences, as those of Serre and Eilenberg-Moore. Afterward, the Bousfield-Kan spectral sequence, which is related with the computation of homotopy groups of spaces, has been considered.

The first part of this work (Chapters 2 and 3) is devoted to spectral sequences associated with filtered complexes [Mac63], constructing several algorithms which make it possible to compute spectral sequences of filtered complexes with effective homology. These algorithms (which have been fully implemented in Common Lisp as a new module for the Kenzo system) determine the groups $E^r_{p,q}$ and the differential maps $d^r_{p,q}$ for every stage r, the convergence level of the spectral sequence for each dimension n, and the filtration of the homology groups induced by the filtration of the chain complex.

A first example of application of our results are spectral sequences associated with bicomplexes. What is more interesting is that they also allow us to compute Serre [Ser51] and Eilenberg-Moore [EM65b] spectral sequences, as explained in Chapter 3, when the spaces involved in the constructions are objects with effective homology. Both situations have been illustrated by means of several examples implemented in Common Lisp.

Chapters 4 and 5 of this memoir are focused on the Bousfield-Kan spectral sequence associated with a simplicial set X, introduced in [BK72a] and [BK72b]. Making use again of the effective homology technique, we have tried to determine an algorithm computing the different parts of this spectral sequence. As a first necessary step, we have developed an algorithm which, given a 1-reduced simplicial set X with effective homology, computes the effective homology of the simplicial Abelian group RX. This construction has been described in Chapter 4, and is a relevant result for the computation of the first two levels of the Bousfield-Kan spectral sequence. It also plays an essential role to determine the higher stages, although, as explained in Chapter 5, for levels $r > 2$ the algorithm has not been fully constructed.

While the work dealing with spectral sequences of filtered complexes can be thought as *finished*, two different directions to continue the work presented in the second part of this memoir appear. First of all, it is necessary to finish the implementation of our algorithms for the computation of the effective homology of RX and of the first two stages of the Bousfield-Kan spectral sequence. In addition, the *theoretical* algorithm for the computation of the higher levels must be completed, and then, of course, implemented in Common Lisp.

For the first task, as mentioned in Section 4.4, some functions have been already written. To be precise, we have implemented Algorithm 8 (page 114), which allows us to apply the functor Γ introduced in Section 4.1.1 to a reduction. In order to finish the implementation of the effective homology of RX, we must also write in Common Lisp the functions giving the isomorphism $RX \cong \Gamma(\widetilde{C}_*(X))$ of Proposition 4.17 and the construction of the effective homology of $\Gamma(E_*)$ for E_* an effective chain complex. Once the effective homology of RX will be programmed, the implementation of Algorithms 15 and 16 which provide the levels 1 and 2 of the Bousfield-Kan spectral sequence does not seem a hard work.

On the other hand, we have already included in Section 5.2.3.3 the steps needed to complete an algorithm for the computation of every level of the Bousfield-Kan spectral sequence, whose implementation will probably be more difficult because new complicated structures such as function spaces appear there. As mentioned before, the construction of this new algorithm includes the study of the idea of *effective homotopy*, related with the notion of *solution for the homotopy problem*. According to the terminology introduced in [RS06, pp. 34-35] (in this case for the homology problem), given a Kan simplicial set X with base point $\star \in X_0$, a *solution for the homotopy problem* of X is a set $S = \{\sigma_i\}_{1\leq i\leq 5}$ of five *algorithms*:

1. $\sigma_1 : X \to \{\perp, \top\}$ ($\perp$ = false, $\top$ = true) is a predicate deciding for every $n \in \mathbb{N}$ and every n-simplex $x \in X_n$ whether x is an n-sphere or not, that is to say, whether $\partial_i x = \star$ for all $0 \leq i \leq n$ or $\partial_i x \neq \star$ for some i.

2. $\sigma_2 : \mathbb{N} \to \{\text{Abelian groups}\}$ associates to every integer $n \geq 0$ a group $\sigma_2(n)$ which must be isomorphic to $\pi_n(X, \star)$. The image $\sigma_2(n)$ will *represent* the isomorphism class of $\pi_n(X)$ in an effective way to be defined.

3. For each $n \in \mathbb{N}$, $\sigma_{3,n} : \sigma_2(n) \to S_n(X)$ associates to every n-homotopy class $\mathfrak{h}$ coded as an element $\mathfrak{h} \in \sigma_2(n)$ a sphere $\sigma_{3,n}(\mathfrak{h}) \in S_n(X)$ *representing* this homotopy class.

4. For each $n \in \mathbb{N}$, $\sigma_{4,n} : S_n(X) \to \sigma_2(n)$ associates to every n-sphere $x \in S_n(X)$ the homotopy class of x coded as an element of $\sigma_2(n)$.

5. For every $n \in \mathbb{N}$, $\sigma_{5,n} : \operatorname{Ker} \sigma_{4,n} \to X_{n+1}$ associates to every n-sphere $x \in S_n(X)$ whose homotopy class is known to be null by means of the previous algorithm (in other words, x is known to be homotopic to $\star \in X_n$) an element $y \in X_{n+1}$ such that $\partial_i y = \star$ for $0 \leq i \leq n$ and $\partial_{n+1} y = x$.

It is well-known the general problem of finding representants for elements of homotopy groups in a simplicial framework is surprisingly difficult, see for example [Ber95]. This is valid for arbitrary simplicial sets, not satisfying the Kan condition. On the other hand, when the Kan condition is satisfied, a *sphere* simplex can certainly be used as representant, only one simplex; usually the difficulty is transferred to the *algebraic* definition of this sphere, but our job is *Algebraic* Topology... The analogous situation when computing homology groups of iterated loop spaces through effective homology methods, using intensively the Kan model for iterated loop spaces, shows this challenge about effective homotopy is reasonable.

As it is the rule for every spectral sequence, determining the whole set $\{E^r_{p,q}\}_{1 \leq r \leq \infty}$, the level $r = \infty$ included, is not yet enough in general to determine the "limit" groups π_{p+q}, because of the possible extension problems. But again the methods of *effective* homotopy will obviously allow us to determine the homotopy groups of the cofiltration stages, in other words the elements of the fibration tower, which are underlying in the Bousfield-Kan spectral sequence. And the *tapered* property carefully studied in Chapter 5 shows this will be enough.

We must also remark that the Bousfield-Kan spectral sequence is a generalization of the Adams one [Ada60]. Therefore, our further work also includes the analysis of the exact relation between both spectral sequences, and in particular, we would like to study the role of the Steenrod operations [MT68] in the Bousfield-Kan spectral sequence. Again the rich algebraic structure underlying the E^1-page of this spectral sequence, through the innumerable copies of Eilenberg-Maclane spaces, is the natural framework to introduce an *effective* version of the unavoidable Steenrod operations. It is likely the appropriate intermediate tool is the nice E_∞-operad exhibited in [BF04]: it can be understood as a complete $\mathbb{Z}$-version of the Steenrod operations, and it is a reduction of the Barratt-Eccles E_∞-operad made of Eilenberg-MacLane spaces.

Bibliography

[Ada56] J. F. Adams. On the Cobar construction. *Proceedings of the National Academy of Sciences of USA*, 42, pp. 409–412, 1956.

[Ada60] J. F. Adams. On the non-existence of elements of Hopf invariant one. *Annals of Mathematics*, 72(1), pp. 20–104, 1960.

[AH56] J. F. Adams and P. J. Hilton. On the chain algebra of a loop space. *Commentarii Mathematici Helvetici*, 30(1), pp. 305–330, 1956.

[Bau80] H. J. Baues. *Geometry of loop spaces and the Cobar construction*, volume 230 of *Memoirs of the Americal Mathematical Society*. 1980.

[Ber95] C. Berger. Un groupoïde simplicial comme modèle de l'espace des chemins. *Bulletin de la Société Mathématique de France*, 123(1), pp. 1–32, 1995.

[BF04] C. Berger and B. Fresse. Combinatorial operad actions on cochains. *Mathematical Proceedings of the Cambridge Philosophical Society*, 137, pp. 135–174, 2004.

[BK72a] A. K. Bousfield and D. M. Kan. The homotopy spectral sequence of a space with coefficients in a ring. *Topology*, 11, pp. 79–106, 1972.

[BK72b] A. K. Bousfield and D. M. Kan. *Homotopy Limits, Completions and Localizations*, volume 304 of *Lecture Notes in Mathematics*. Springer-Verlag, 1972.

[BK73a] A. K. Bousfield and D. M. Kan. A second quadrant homotopy spectral sequence. *Transactions of the American Mathematical Society*, 177, pp. 305–318, 1973.

[BK73b] A. K. Bousfield and D. M. Kan. Pairings and products in the homotopy spectral sequence. *Transactions of the American Mathematical Society*, 177, pp. 319–343, 1973.

[BP52] M. G. Barratt and G. F. Paechter. A note on $\pi_r(V_{n,m})$. *Proceedings of the National Academy of Sciences of USA*, 38(2), pp. 119–121, 1952.

[Bro57] E. H. Brown, Jr. Finite computability of Postnikov complexes. *Annals of Mathematics*, 65(1), pp. 1–20, 1957.

[Bro59] E. H. Brown, Jr. Twisted tensor products, I. *Annals of Mathematics*, 69(1), pp. 223–246, 1959.

[Bro67] R. Brown. The twisted Eilenberg-Zilber theorem. *Celebrazioni Archimedi de Secolo XX, Simposio di Topologia*, pp. 34–37, 1967.

[Car55] H. Cartan. Algèbres d'Eilenberg-MacLane et homotopie. Séminaire H. Cartan, École Normal Supérieure, Paris, 1954-55. Exposés 2-11.

[Dol72] A. Dold. *Lectures on Algebraic Topology*. Springer-Verlag, 1972.

[DRSS99] X. Dousson, J. Rubio, F. Sergeraert, and Y. Siret. The Kenzo program. Institut Fourier, Grenoble, 1999. `http://www-fourier.ujf-grenoble.fr/~sergerar/Kenzo/`.

[EM53] S. Eilenberg and S. MacLane. On the groups $H(\pi, n)$, I. *Annals of Mathematics*, 58, pp. 55–106, 1953.

[EM65a] S. Eilenberg and J. C. Moore. Adjoint functors and triples. *Illinois Journal of Mathematics*, 9, pp. 381–398, 1965.

[EM65b] S. Eilenberg and J. C. Moore. Homology and fibrations, I: Coalgebras, cotensor product and its derived functors. *Commentarii Mathematici Helvetici*, 40, pp. 199–236, 1965.

[EZ50] S. Eilenberg and J. A. Zilber. Semi-simplicial complexes and singular homology. *Annals of Mathematics*, 51(3), pp. 499–513, 1950.

[EZ53] S. Eilenberg and J. A. Zilber. On products of complexes. *American Journal of Mathematics*, 75, pp. 200–204, 1953.

[Fre37] H. Freudenthal. Über die Klassen der Sphärenabbildungen. *Compositio Mathematica*, 5, pp. 299–314, 1937.

[GJ99] P. G. Goerss and J. F. Jardine. *Simplicial Homotopy Theory*, volume 174 of *Progress in Mathematics*. Birkhäuser, 1999.

[GL89] V. K. A. M. Gugenheim and L. Lambe. Perturbation theory in Differential Homological Algebra, I. *Illinois Journal of Mathematics*, 33, pp. 556–582, 1989.

[GLS91] V. K. A. M. Gugenheim, L. Lambe, and J. D. Stasheff. Perturbation theory in Differential Homological Algebra, II. *Illinois Journal of Mathematics*, 35, pp. 359–373, 1991.

[Gra96] P. Graham. *ANSI Common Lisp*. Prentice Hall, 1996.

[Hat04] A. Hatcher. *Spectral Sequences in Algebraic Topology*. 2004. `http://www.math.cornell.edu/~hatcher/SSAT/SSATpage.html`.

[Hop35] H. Hopf. Über die Abbildungen von Sphären auf Sphären niedrigerer Dimension. *Fundamenta Mathematicae*, 25, pp. 427–440, 1935.

[Hur35] W. Hurewicz. Beiträge zur Topologie der Deformationen I-II. *Proceedings of the Akademie van Wetenschappen*, 38, pp. 112–119, 521–528, 1935.

[Hur36] W. Hurewicz. Beiträge zur Topologie der Deformationen III-IV. *Proceedings of the Akademie van Wetenschappen*, 39, pp. 117–126, 215–224, 1936.

[Kan58] D. M. Kan. A combinatorial definition of homotopy groups. *Annals of Mathematics*, 67(2), pp. 282–312, 1958.

[KMM04] T. Kaczynski, K. Mischaikow, and M. Mrozek. *Computational Homology*, volume 157 of *Applied Mathematical Sciences*. Springer, 2004.

[Kos47] J. L. Koszul. Sur les opérateurs de dérivation dans un anneau. *Comptes Rendus des Séances de l'Academie des Sciences de Paris*, 225, pp. 217–219, 1947.

[Ler46] J. Leray. Structure de l'anneau d'homologie d'une représentation. *Comptes Rendus des Séances de l'Academie des Sciences de Paris*, 222, pp. 1419–1422, 1946.

[Ler50] J. Leray. L'anneau spectral et l'anneau filtré d'homologie d'un espace localement compact et d'une application continue. *Journal de Mathématiques pures et appliquées*, 29, pp. 1–139, 1950.

[Mac63] S. MacLane. *Homology*. Springer, 1963.

[Mah67] M. E. Mahowald. *The Metastable Homotopy of S^n*, volume 72 of *Memoirs of the American Mathematical Society*. 1967.

[May67] J. P. May. *Simplicial objects in Algebraic Topology*, volume 11 of *Van Nostrand Mathematical Studies*. 1967.

[McC85] J. McCleary. *User's guide to spectral sequences*. Publish or Perish, 1985.

[McC04] J. McCleary. *User's guide to spectral sequences*. 2004. `http://math.vassar.edu/faculty/mccleary/PDF.files/`.

[Moo59] J. C. Moore. Algèbre homologique et homologie des espaces classifiants. Séminaire H. Cartan, École Normal Supérieure, Paris, 1959. Exposé 7.

[MT68] R. E. Mosher and M. C. Tangora. *Cohomology operations and applications in homotopy theory*. Harper & Row, 1968.

[Poi95] H. Poincaré. Analysis Situs. *Journal de l'École Polytechnique*, 1, pp. 1–121, 1895.

[Rav86] D. C. Ravenel. *Complex cobordism and stable homotopy groups of spheres.* Academic Press, 1986.

[Rea93] P. Real. *Algoritmos de cálculo de homología efectiva de los espacios clasificantes.* PhD thesis, Universidad de Sevilla, 1993.

[Rom06a] A. Romero. Effective homology of free simplicial Abelian groups: the acyclic case. In *Proceedings of X Encuentro de Álgebra Computacional y Aplicaciones*, pp. 143–146. Universidad de Sevilla, 2006.

[Rom06b] A. Romero. From homological perturbation to spectral sequences: a case study. In *Global Integrability of Field Theories, Proceedings of GIFT 2006*, pp. 289–309. Universitätsverlag Karlsruhe, 2006.

[RRS06] A. Romero, J. Rubio, and F. Sergeraert. Computing spectral sequences. *Journal of Symbolic Computation*, 41(10), pp. 1059–1079, 2006.

[RS88] J. Rubio and F. Sergeraert. Homologie effective et suites spectrales d'Eilenberg-Moore. *Comptes Rendus des Séances de l'Academie des Sciences de Paris*, 306(17), pp. 723–726, 1988.

[RS97] J. Rubio and F. Sergeraert. Constructive Algebraic Topology, Lecture Notes Summer School on Fundamental Algebraic Topology. Institut Fourier, Grenoble, 1997. `http://www-fourier.ujf-grenoble.fr/~sergerar/Summer-School/`.

[RS02] J. Rubio and F. Sergeraert. Constructive Algebraic Topology. *Bulletin des Sciences Mathématiques*, 126(5), pp. 389–412, 2002.

[RS05a] J. Rubio and F. Sergeraert. Algebraic models for homotopy types. *Homology, Homotopy and Applications*, 7(2), pp. 139–160, 2005.

[RS05b] J. Rubio and F. Sergeraert. Postnikov "Invariants" in 2004. *Georgian Mathematical Journal*, 12, pp. 139–155, 2005.

[RS06] J. Rubio and F. Sergeraert. Constructive Homological Algebra and Applications, Lecture Notes Summer School on Mathematics, Algorithms, and Proofs. University of Genova, 2006. `http://www-fourier.ujf-grenoble.fr/~sergerar/Papers/Genova-Lecture-Notes.pdf`.

[RSS97] J. Rubio, F. Sergeraert, and Y. Siret. EAT: Symbolic Software for Effective Homology Computation. Institut Fourier, Grenoble, 1997. `ftp://fourier.ujf-grenoble.fr/pub/EAT`.

[Rub91] J. Rubio. *Homologie effective des espaces de lacets itérés: un logiciel.* PhD thesis, Institut Fourier, 1991.

[Ser51] J. P. Serre. Homologie singulière des espaces fibrés. *Annals of Mathematics*, 54(3), pp. 425–505, 1951.

[Ser53] J. P. Serre. Groupes d'homotopie et classes de groupes abéliens. *Annals of Mathematics*, 58(2), pp. 258–294, 1953.

[Ser87] F. Sergeraert. Homologie effective. *Comptes Rendus des Séances de l'Academie des Sciences de Paris*, 304(11 and 12), pp. 279–282 and 319–321, 1987.

[Ser94] F. Sergeraert. The computability problem in Algebraic Topology. *Advances in Mathematics*, 104(1), pp. 1–29, 1994.

[Shi62] W. Shih. Homologie des espaces fibrés. *Publications mathématiques de l'Institut des Hautes Études Scientifiques*, 13, 1962.

[Sta63] J. D. Stasheff. Homotopy associativity of H-spaces, I, II. *Transactions of the American Mathematical Society*, 108, pp. 275–292 and 293–312, 1963.

[Ste62] N. E. Steenrod. *Cohomology operations*, volume 50 of *Annals of Mathematics Studies*. Princeton University Press, 1962.

[Tan85] M. C. Tangora. *Computing the homology of the lambda algebra*, volume 337 of *Memoirs of the Americal Mathematical Society*. 1985.

[Tod62] H. Toda. *Compositional methods in homotopy groups of spheres*. Princeton University Press, 1962.

[Wei94] C. A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge studies in advanced mathematics*. Cambridge University Press, 1994.

[Whi78] G. W. Whitehead. *Elements of Homotopy Theory*, volume 61 of *Graduate texts in Mathematics*. Springer-Verlag, 1978.

Homología Efectiva y Sucesiones Espectrales

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Índice General

Introducción

La Topología Algebraica trata de utilizar métodos "algebraicos" para atacar problemas topológicos; en los casos más sencillos, consiste en asociar a un espacio topológico *invariantes algebraicos* que describan sus propiedades esenciales. Por ejemplo, podemos definir grupos especiales asociados a un espacio topológico de modo que se respete la relación de homeomorfismo de espacios. Esto nos permite estudiar algunas propiedades interesantes de espacios topológicos por medio de resultados sobre grupos, que son a menudo más fáciles de probar. De manera más general, existen varios *functores* que asignan a algunos espacios topológicos *objetos algebraicos*. En muchas ocasiones, si uno de estos functores se aplica a un espacio topológico de "tipo finito", entonces el resultado es también un objeto algebraico de tipo finito. Pero en general no existen *algoritmos* que sean capaces de calcular estos objetos algebraicos de tipo finito correspondientes a los distintos functores de la Topología Algebraica.

Dos ejemplos importantes de invariantes algebraicos son los grupos de homotopía y los grupos de homología (y de cohomología). El cálculo de los grupos de homología usuales (con coeficientes en $\mathbb{Z}$) de complejos simpliciales finitos es sencillo: un complejo simplicial determina un complejo de cadenas de tipo finito y sus grupos de homología se deducen de operaciones elementales con los operadores diferenciales (como se explica, por ejemplo, en [KMM04]), con lo que no es difícil construir este algoritmo. Más complicado es el problema del cálculo de los grupos de homotopía de un complejo simplicial finito X, que se denotan por $\pi_n(X)$.

La definición de grupo de homotopía fue dada por Hurewicz en [Hur35] y [Hur36] como una generalización de la noción de grupo fundamental, que se debe originalmente a Poincaré en [Poi95], un trabajo que puede ser considerado el origen de la Topología Algebraica. En un primer momento sólo se pudieron calcular algunos grupos del espacio no trivial más sencillo, la 2-esfera S^2; en concreto, Heinz Hopf [Hop35] calculó $\pi_2(S^2) = \mathbb{Z}$ y $\pi_3(S^2) = \mathbb{Z}$. El grupo $\pi_4(S^2) = \mathbb{Z}_2$ fue determinado por Hans Freudenthal en 1937 [Fre37], pero después no se obtuvieron nuevos resultados sobre grupos de homotopía de espacios hasta 1950. Los siguientes grupos $\pi_n(S^2)$ los calculó Jean Pierre Serre para $5 \leq n \leq 9$ [Ser51]. En realidad para $n = 6$ Serre probó únicamente que $\pi_6(S^2)$ tiene 12 elementos, pero no fue capaz de elegir entre las dos soluciones posibles $\mathbb{Z}_{12}$ y $\mathbb{Z}_2 \oplus \mathbb{Z}_6$; es el primer ejemplo de la historia en el que un topólogo se encontró con un *problema de extensión* serio. Dos años más tarde, Barratt y Paechter [BP52] probaron que existe un elemento de orden 4 en $\pi_6(S^2)$, con lo que finalmente $\pi_6(S^2) = \mathbb{Z}_{12}$. Otras referencias sobre el cálculo de grupos de homotopía de esferas son, por ejemplo, [Tod62], [Mah67] y [Rav86].

Serre también obtuvo un resultado general de *finitud* [Ser53] que afirma que, si X es un espacio simplemente conexo tal que los grupos de homología $H_n(X;\mathbb{Z})$ son de tipo finito, entonces los grupos de homotopía $\pi_n(X)$ son también grupos Abelianos de tipo finito. En 1957, Edgar Brown publicó en [Bro57] un algoritmo *teórico* para el cálculo de estos grupos, basado en la torre de Postnikov y utilizando aproximaciones finitas de conjuntos simpliciales infinitos, transformando de este modo los resultados de finitud de Serre en un resultado de calculabilidad. Sin embargo, el propio Edgar Brown señaló en su artículo que este algoritmo carece de uso práctico, incluso en el ordenador más potente que se pueda imaginar, como consecuencia de la complejidad hiper-exponencial del algoritmo.

El método de la homología efectiva apareció en los 80 tratando de proporcionar algoritmos *reales* para el cálculo de grupos de homología y de homotopía. Esta técnica fue introducida por Francis Sergeraert en [Ser87] y [Ser94], y su estado actual queda descrito en [RS97] y [RS06]. El método está basado en la noción de *objeto con homología efectiva*, que conecta un espacio con su *homología* por medio de equivalencias de cadenas, y que está fuertemente relacionado con la teoría de *perturbación homológica*, cuyas referencias fundamentales son los trabajos clásicos de Shi Weishu [Shi62] y Ronnie Brown [Bro67], y los de Victor Gugenheim, Larry Lambe y Jim Stasheff [GL89] [GLS91].

El método de la homología efectiva ha sido implementado mediante el sistema Kenzo [DRSS99] (cuya versión anterior se llama EAT [RSS97]), un programa en Common Lisp que ha hecho posible calcular algunos grupos de homología complicados que no habían sido determinados anteriormente. Kenzo puede calcular, entre otros, grupos de homología de espacios totales de fibrados, espacios de lazos iterados, espacios clasificantes, etc. Otras referencias útiles sobre el método de la homología efectiva y el sistema Kenzo son [RS88], [Rub91], [RS02] y [RS05a].

Las sucesiones espectrales son otra herramienta en Topología Algebraica que ha sido utilizada de manera tradicional para calcular grupos de homología y homotopía de espacios (véase, por ejemplo, [McC85] o [Hat04]). Un ejemplo clásico de sucesión espectral es la de Serre [Ser51], que proporciona información sobre los grupos de homología del espacio total de un fibrado a partir de los grupos de homología de la base y la fibra. La sucesión espectral de Eilenberg-Moore [EM65b] da información sobre los grupos de homología de la base (resp. la fibra) cuando se conocen los grupos de homología del espacio total y de la fibra (resp. base). Para el cálculo de grupos de homotopía podemos considerar las sucesiones espectrales de Adams [Ada60] o de Bousfield-Kan [BK72a].

Pero las sucesiones espectrales clásicas presentan un problema muy importante: no soy *algoritmos*. Una sucesión espectral es una familia de “páginas” $(E^r_{p,q}, d^r)_{r\geq 1}$ de módulos bigraduados con diferencial, donde cada página se obtiene como el módulo de homología de la anterior. Incluimos aquí una cita (en inglés) de John McCleary en su famoso libro [McC85], que explica este problema:

> *It is worth repeating the caveat about differentials mentioned in Chapter 1: knowledge of $E^r_{*,*}$ and d^r determines $E^{r+1}_{*,*}$ but not d^{r+1}. If we think of a spectral sequence as a black box, then the input is a differential bigraded module, usually $E^1_{*,*}$, and, with each turn of the handle, the machine computes a successive homology according to a sequence of differentials. If some differential is unknown, then some other (any other) principle is needed to proceed. From Chapter 1, the reader is acquainted with several algebraic tricks that allow further calculation. In the nontrivial cases, it is often a deep geometric idea that is caught up in the knowledge of a differential.*

En la mayoría de los casos se trata en realidad de un problema de calculabilidad: las diferenciales superiores de la sucesión espectral están definidas matemáticamente, pero su definición no es constructiva. En otras palabras, las diferenciales no son calculables con la información de la que se dispone habitualmente.

Otro problema distinto de las sucesiones espectrales es el problema de extensión en el límite. Una sucesión espectral nos proporciona una filtración de los grupos (de homología u homotopía) buscados, pero en algunos casos hay varias soluciones posibles. Éste es el problema que Jean Pierre Serre se encontró cuando trató de calcular $\pi_6(S^2)$, un problema que se puede resolver por medio de la técnica de la homología efectiva.

El objetivo de este trabajo ha sido relacionar las sucesiones espectrales y la homología efectiva, mostrando que el método de la homología efectiva puede ser utilizado para producir *algoritmos* que calculen las diversas componentes de algunas sucesiones espectrales, incluyendo las *diferenciales superiores*.

La organización de la memoria es la siguiente. El primer capítulo incluye algunas definiciones y resultados preliminares que serán usados en el resto del trabajo. En la primera sección introducimos los complejos de cadenas y las sucesiones espectrales, dos nociones fundamentales en Álgebra Homológica. La segunda sección está dedicada a la topología simplicial, centrándonos en conjuntos simpliciales, grupos de homotopía, y espacios de Eilenberg-MacLane. Finalmente, en la tercera sección, explicamos el método de la homología efectiva y el sistema Kenzo.

Después de este primer capítulo, la memoria se divide en dos partes diferentes. Los capítulos 2 y 3 están dedicados a la sucesiones espectrales asociadas a complejos filtrados, que bajo condiciones favorables convergen a sus grupos de homología. Por otro lado, los capítulos 4 y 5 se centran en la sucesión espectral de Bousfield-Kan, relacionada con el cálculo de grupos de homotopía.

El capítulo 2 contiene varios algoritmos para el cálculo de las diversas componentes de sucesiones espectrales de complejos filtrados con homología efectiva: los grupos $E^r_{p,q}$, las diferenciales $d^r_{p,q}$ para cada nivel r, así como el nivel r en el que se alcanza la convergencia para cada grado n y la filtración de los grupos de homología inducida por la filtración del complejo de cadenas. Nuestros resultados pueden ser aplicados, por ejemplo, para el cálculo de sucesiones espectrales asociadas a bicomplejos. Estos algoritmos han sido implementados como un nuevo módulo para el sistema Kenzo, que también queda explicado en este capítulo por medio de algunos ejemplos elementales.

Los resultados presentados en el capítulo 2 permiten también *calcular* dos de los ejemplos clásicos de sucesiones espectrales, las de Serre y Eilenberg-Moore, a las que se dedica el capítulo 3. Si los espacios que intervienen en los fibrados correspondientes son objetos con homología efectiva, podemos determinar las distintas componentes de las sucesiones espectrales asociadas por medio de nuestros resultados. De este modo hacemos *constructivas* estas sucesiones espectrales que hasta ahora no eran algoritmos. Ambas situaciones han sido ilustradas mediante varios ejemplos implementados en Common Lisp.

Otras sucesiones espectrales que tienen también un gran interés no vienen definidas por medio de ningún complejo filtrado. Es el caso de la sucesión espectral de Bousfield-Kan, que apareció por primera vez en [BK72a] tratando de generalizar la sucesión espectral de Adams [Ada60]. Aunque existe una definición formal de la sucesión espectral de Adams, no es sencillo *calcularla*, como se explica en la introducción de [Tan85]. En primer lugar, hay que determinar la cohomología del álgebra de Steenrod [Ste62]; después debemos encontrar las diferenciales superiores; finalmente, pueden aparecer problemas de extensión en el límite. En este caso, nuestros algoritmos para el cálculo de sucesiones espectrales de complejos filtrados no se pueden utilizar, pero el método de la homología efectiva puede ser útil para desarrollar una versión constructiva de la sucesión espectral de Bousfield-Kan.

Como un primer paso hacia esta versión *efectiva* de la sucesión espectral de Bousfield-Kan, el resultado principal del capítulo 4 es un algoritmo que calcula la homología efectiva del grupo Abeliano simplicial libre RX generado por un conjunto simplicial 1-reducido X. La homología "ordinaria" de RX puede ser deducida de una manera sencilla del trabajo de Cartan [Car55] sobre los espacios de Eilenberg-MacLane, pero esta información no es suficiente para nuestro objetivo: es necesaria la homología *efectiva*. Nuestro algoritmo utiliza varias construcciones de Topología Algebraica como la correspondencia de Dold-Kan entre las categorías de complejos de cadenas y grupos Abelianos simpliciales, fibrados o espacios de Eilenberg-MacLane. Algunas partes de este algoritmo han sido implementadas como un conjunto de programas en Common Lisp que también serán explicados en este capítulo.

El capítulo 5 está dedicado a la sucesión espectral de Bousfield-Kan asociada a un conjunto simplicial X, tratando de construir un algoritmo (basado en la técnica de la homología efectiva) que calcule todas sus componentes. La primera parte de este capítulo contiene varios algoritmos para tratar con estructuras cosimpliciales, que son uno de los ingredientes principales de esta sucesión espectral; la segunda está centrada en la *construcción* de la sucesión espectral de Bousfield-Kan. Comenzamos esta sección con una prueba de la convergencia, basada en cálculos elementales de Álgebra Homológica. A continuación utilizamos los resultados del capítulo 4 para *calcular* los dos primeros niveles. Para el cálculo del resto de "páginas", incluimos el esquema de un nuevo algoritmo que todavía no está terminado.

La memoria termina con un capítulo que contiene conclusiones y trabajo futuro, y finalmente incluimos la bibliografía.

Resumen de los capítulos

Presentamos a continuación un breve resumen de cada uno de los capítulos de esta memoria.

1 Preliminares

En el primer capítulo de la memoria incluimos las definiciones, notaciones y resultados básicos que serán utilizados en el resto del trabajo. Este capítulo está dividido en tres secciones. La primera está dedicada a dos nociones fundamentales en Álgebra Homológica: complejos de cadenas y sucesiones espectrales. La segunda sección contiene algunas definiciones y resultados de topología simplicial. Finalmente, presentamos las ideas fundamentales del método de la homología efectiva y del sistema Kenzo [DRSS99]. Las referencias fundamentales para cada una de las secciones son respectivamente [Mac63], [May67] y [RS97].

2 Homología efectiva y sucesiones espectrales de complejos filtrados: algoritmos y programas

El capítulo 2 se centra en las sucesiones espectrales asociadas a complejos filtrados [McC85], para las que existe una expresión formal que define los grupos $E^r_{p,q}$ y las diferenciales $d^r_{p,q}$, pero que en muchas ocasiones no se pueden *calcular*. Mostramos aquí que el método de la homología efectiva permite calcular este tipo de sucesiones espectrales cuando el complejo de cadenas inicial es un objeto con homología efectiva, obteniendo de este modo *algoritmos* reales.

Después de presentar algunas definiciones y resultados previos sobre filtraciones y sucesiones espectrales, en la sección 2.2 incluimos algunos resultados teóricos que relacionan los métodos de homología efectiva y sucesiones espectrales. Estos resultados son utilizados a continuación para desarrollar un algoritmo para calcular sucesiones espectrales de complejos filtrados con homología efectiva, que se explica en la sección 2.3. La sección 2.4 se centra en los bicomplejos, un caso particular de complejo de cadenas que tiene asociada una filtración canónica y por tanto una sucesión espectral, que puede ser calculada mediante nuestros resultados. Todos estos algoritmos han sido implementados en Common Lisp como un nuevo módulo para Kenzo, que presentamos en la última sección del capítulo por medio de algunos ejemplos elementales.

3 Homología efectiva y sucesiones espectrales de complejos filtrados: aplicaciones

El tercer capítulo de la memoria está dedicado a dos ejemplos clásicos de sucesiones espectrales, la sucesión espectral de Serre [Ser51] y la sucesión espectral de Eilenberg-Moore [EM65b]. Las dos fueron construidas por medio de complejos filtrados y han sido utilizadas para calcular grupos de homología de algunos espacios complicados, pero en muchos casos estas sucesiones espectrales no son algoritmos y no pueden ser determinadas completamente. Los resultados y algoritmos del capítulo anterior nos permiten sin embargo obtener todas las componentes de estas sucesiones espectrales cuando los espacios que forman parte de los fibrados son objetos con homología efectiva.

El capítulo está dividido en dos partes, la primera a la sucesión espectral de Serre y la segunda a la sucesión espectral de Eilenberg-Moore. La estructura de ambas es similar, con una breve introducción, el cálculo de la homología efectiva de los espacios correspondientes, los algoritmos obtenidos y finalmente algunos ejemplos que han sido implementados en Common Lisp.

4 Homología efectiva de grupos Abelianos simpliciales libres

Dejamos de lado aquí los complejos filtrados para centrarnos en la sucesión espectral de Boufield-Kan, que trata de calcular los grupos de homotopía de un conjunto simplicial. Uno de los ingredientes principales de esta sucesión espectral es el constructor que asocia a cada conjunto simplicial X el grupo Abeliano simplicial libre generado por X. De manera más específica, en el cálculo de la sucesión espectral de Bousfield-Kan se necesita la homología *efectiva* de los grupos R^kX (la homología *ordinaria* de R^kX se puede calcular fácilmente). El capítulo 4 de la memoria está dedicado al desarrollo de una versión con homología efectiva del constructor R. Dado un conjunto simplicial 1-reducido X con homología efectiva, esta versión del constructor R calcula una versión con homología efectiva del resultado RX; aplicando iterativamente este constructor obtenemos una versión con homología efectiva de R^kX para k un entero positivo.

El capítulo comienza con varias definiciones y resultados previos que son necesarios para los desarrollos presentados, y a continuación se incluye la definición y algunas propiedades básicas de la construcción R. La tercera sección es la más amplia y contiene nuestro resultado principal, el cálculo de la homología efectiva de RX, que se obtiene como composición de dos equivalencias. Por último, incluimos algunas consideraciones sobre la implementación de nuestros algoritmos, que todavía no ha sido completada.

5 Homología efectiva y sucesión espectral de Bousfield-Kan

En el último capítulo de la memoria tratamos de utilizar los resultados del capítulo anterior para construir un algoritmo que calcule la sucesión espectral de Bousfield-Kan asociada a un conjunto simplicial X. Esta versión constructiva de la sucesión espectral de Bousfield-Kan no está todavía terminada; presentamos aquí las ideas generales que esperamos nos permitan su construcción en un plazo no muy largo. Además, incluimos algoritmos (completos) que permiten determinar los dos primeros niveles de la sucesión espectral.

Este capítulo está dividido en dos partes diferentes. La primera está centrada en los objetos cosimpliciales, que desempeñan un papel esencial en la construcción de la sucesión espectral de Bousfield-Kan, incluyendo algunos resultados y algoritmos que hemos desarrollado. La segunda parte contiene la definición de la sucesión espectral, una prueba de su convergencia, el algoritmo que calcula los términos $E^1_{p,q}$ y $E^2_{p,q}$ y el esquema de un nuevo algoritmo para su cálculo *completo*.

Conclusiones y trabajo futuro

Este trabajo ha sido realizado con el objetivo de relacionar homología efectiva y sucesiones espectrales, dos técnicas diferentes en Topología Algebraica para el cálculo de grupos de homología (y de homotopía).

Por un lado, las sucesiones espectrales son un método clásico utilizado para aproximar grupos de homología de algunos espacios complicados (véase, por ejemplo, [McC85]), pero en muchos casos los datos disponibles no permiten al usuario determinar las diferenciales superiores.

Por otro lado, el método de la homología efectiva (introducido en [Ser87] y [Ser94]) proporciona algoritmos reales para el cálculo de los grupos de homología buscados, reemplazando de este modo la técnica de las sucesiones espectrales. Este método ha sido implementado en el sistema Kenzo [DRSS99] y puede ser utilizado, por ejemplo, para calcular grupos de homología de espacios totales de fibrados, de espacios de lazos iterados, de espacios clasificantes, etc.

En esta memoria hemos mostrado que también podemos utilizar la homología efectiva para determinar las distintas componentes de algunas sucesiones espectrales, centrándonos en dos situaciones particulares. Hemos comenzado estudiando las sucesiones espectrales asociadas a complejos filtrados, que incluyen varios ejemplos clásicos de sucesiones espectrales, como las de Serre y Eilenberg-Moore. A continuación se ha considerado la sucesión espectral de Bousfield-Kan, que está relacionada con el cálculo de grupos de homotopía de espacios.

La primera parte de este trabajo (los capítulos 2 y 3) está dedicada a las sucesiones espectrales asociadas a complejos filtrados [Mac63], construyendo varios algoritmos que permiten el cálculo de sucesiones espectrales de complejos filtrados con homología efectiva. Estos algoritmos (que han sido implementados en su totalidad en Common Lisp como un nuevo módulo para el sistema Kenzo) determinan los grupos $E^r_{p,q}$ y las diferenciales $d^r_{p,q}$ para todos los niveles $r \geq 1$, así como el nivel de convergencia de la sucesión espectral para cada dimensión n y la filtración de los grupos de homología inducida por la filtración del complejo de cadenas inicial.

Un primer ejemplo de uso de nuestros resultados son las sucesiones espectrales asociadas a bicomplejos. Pero sin duda las aplicaciones más interesantes vienen dadas por el cálculo de las sucesiones espectrales de Serre [Ser51] y Eilenberg-Moore [EM65b] cuando los espacios que aparecen en las construcciones son objetos con homología efectiva, como se explica en el capítulo 3. Ambas situaciones han sido ilustradas por medio de varios ejemplos implementados en Common Lisp.

Los capítulos 4 y 5 de esta memoria están centrados en la sucesión espectral de Bousfield-Kan asociada a un conjunto simplicial X, introducida en [BK72a] y [BK72b]. Utilizando de nuevo la técnica de la homología efectiva, hemos tratado de determinar un algoritmo que, dado un conjunto simplicial 1-reducido X con homología efectiva, calcule la homología efectiva del grupo Abeliano simplicial RX. Esta construcción queda descrita en el capítulo 4, y es un resultado relevante para el cálculo de los dos primeros niveles de la sucesión espectral de Bousfield-Kan. También desempeña un papel fundamental para determinar los niveles superiores aunque, como se explica en el capítulo 5, para $r > 2$ el algoritmo no ha sido construido completamente.

Mientras que el trabajo sobre las sucesiones espectrales de complejos filtrados puede considerarse *terminado*, aparecen dos nuevas líneas de investigación con las que continuar el trabajo presentado en la segunda parte de la memoria. En primer lugar, es necesario terminar la implementación de nuestros algoritmos para el cálculo de la homología efectiva de RX y los dos primeros niveles de la sucesión espectral de Bousfield-Kan. Además, debemos completar el algoritmo *teórico* para el cálculo de los niveles superiores, y después implementarlo.

Para la primera tarea, como se explica en la sección 4.4, ya hemos escrito algunas funciones. Concretamente, hemos implementado el algoritmo 8 (página 114), que nos permite aplicar el functor Γ introducido en la sección 4.1.1 a una reducción. Para finalizar la implementación de la homología efectiva de RX, tenemos que escribir también en Common Lisp las funciones correspondientes al isomorfismo $RX \cong \Gamma(\widetilde{C}_*(X))$ de la proposición 4.17 y la construcción de la homología efectiva de $\Gamma(E_*)$ para un complejo de cadenas efectivo E_*. Una vez que hayamos programado la homología efectiva de RX, la implementación de los algoritmos 15 y 16 que nos dan los niveles 1 y 2 de la sucesión espectral de Bousfield-Kan no parece una tarea demasiado complicada.

Por otro lado, ya hemos incluido en la sección 5.2.3.3 los pasos necesarios para completar el algoritmo para el cálculo de los niveles superiores de la sucesión espectral de Bousfield-Kan, cuya implementación será probablemente más difícil porque aparecen nuevas estructuras complicadas como los espacios de funciones. Como ya hemos mencionado anteriormente, la construcción de este nuevo algoritmo incluye el estudio de la idea de *homotopía efectiva*, relacionada con la noción de *solución para el problema homotópico*. Siguiendo la terminología introducida en [RS06, pp. 34-35] (en este caso para el problema homológico), dado un conjunto simplicial de Kan X con punto base $\star \in X_0$, una *solución para el problema homotópico* de X es un conjunto $S = \{\sigma_i\}_{1 \leq i \leq 5}$ de cinco *algoritmos*:

1. $\sigma_1 : X \to \{\bot, \top\}$ ($\bot$ = falso, $\top$ = verdadero) es un predicado que indica, para cada $n \in \mathbb{N}$ y cada n-símplice $x \in X_n$, si x es o no una n-esfera, es decir, si $\partial_i x = \star$ para todo $0 \leq i \leq n$ o bien $\partial_i x \neq \star$ para algún i.

2. $\sigma_2 : \mathbb{N} \to \{\text{grupos Abelianos}\}$ asocia a cada entero $n \geq 0$ un grupo $\sigma_2(n)$ que debe ser isomorfo a $\pi_n(X, \star)$. La imagen $\sigma_2(n)$ *representará* la clase de isomorfismo de $\pi_n(X)$ de un modo efectivo como se define a continuación.

3. Para cada $n \in \mathbb{N}$, $\sigma_{3,n} : \sigma_2(n) \to S_n(X)$ asocia a cada n-clase de homotopía $\mathfrak{h}$ codificada como un elemento $\mathfrak{h} \in \sigma_2(n)$ una esfera $\sigma_{3,n}(\mathfrak{h}) \in S_n(X)$ *representando* esta clase de homotopía.

4. Para cada $n \in \mathbb{N}$, $\sigma_{4,n} : S_n(X) \to \sigma_2(n)$ asocia a cada n-esfera $x \in S_n(X)$ la clase de homotopía de x codificada como un elemento de $\sigma_2(n)$.

5. Para cada $n \in \mathbb{N}$, $\sigma_{5,n} : \operatorname{Ker} \sigma_{4,n} \to X_{n+1}$ asocia a cada n-esfera $x \in S_n(X)$ cuya clase de homotopía se sabe que es nula gracias al algoritmo anterior (en otras palabras, se sabe que x es homótopo a $\star \in X_n$) un elemento $y \in X_{n+1}$ tal que $\partial_i y = \star$ para $0 \leq i \leq n$ y $\partial_{n+1} y = x$.

Es bien conocido que el problema general de encontrar representantes para elementos de grupos de homotopía en un marco simplicial es sorprendentemente difícil, véase por ejemplo [Ber95]. Esto es válido para conjuntos simpliciales arbitrarios, que no satisfagan la condición de Kan. Por el contrario, cuando se satisface la condición de Kan, una *esfera* puede ser usada como representante, simplemente un símplice; normalmente la dificultad se traslada a la definición *algebraica* de esta esfera, pero nuestro trabajo es precisamente la Topología *Algebraica*... La situación análoga en el cálculo de grupos de homología de espacios de lazos iterados a través de los métodos de la homología efectiva, usando intensivamente el modelo de Kan para los espacios de lazos iterados, muestra que este reto sobre la homotopía efectiva es razonable.

Siguiendo la regla de cualquier sucesión espectral, determinar la *totalidad* del conjunto $\{E^r_{p,q}\}_{1 \leq r \leq \infty}$, incluyendo el nivel $r = \infty$, no es suficiente en general para determinar los grupos “límite” π_{p+q}, debido a los problemas de extensión que puedan existir. Pero de nuevo los métodos de la homotopía *efectiva* nos permitirán determinar los grupos de homotopía de los distintos pasos de cofiltración, en otras palabras, los elementos de la torre de fibrados, que se encuentran subyacentes en la sucesión espectral de Bousfield-Kan. Y la propiedad de que E^1 es *afilado*, que hemos estudiando cuidadosamente en el capítulo 5, muestra que esto será suficiente.

Tenemos que remarcar también que la sucesión espectral de Bousfield-Kan es una generalización de la de Adams [Ada60]. Por lo tanto, nuestro trabajo futuro también incluye el análisis de la relación exacta entre ambas sucesiones espectrales, y en particular, nos gustaría estudiar el papel de las operaciones de Steenrod [MT68] en la sucesión espectral de Bousfield-Kan. De nuevo la rica estructura subyacente en la página E^1 de esta sucesión espectral, a través de las innumerables copias de espacios de Eilenberg-MacLane, es el marco natural para introducir una versión *efectiva* de las inevitables operaciones de Steenrod. Parece que la herramienta intermedia apropiada es la E_∞-operada expuesta en [BF04]: puede ser entendida como una versión completa de las operaciones de Steenrod, y es una reducción de la E_∞-operada de Barratt-Eccles construida con espacios de Eilenberg-MacLane.

Publicaciones

Incluimos a continuación los resúmenes de las publicaciones a las que ha dado lugar el trabajo presentado en esta memoria.

- **A. Romero. Effective homology of free simplicial Abelian groups: the acyclic case. En Actas del X Encuentro de Álgebra Computacional y Aplicaciones, pp. 143–146. Universidad de Sevilla, 2006.**

  En este trabajo se trata de aplicar el método de la homología efectiva para el cálculo de los grupos de homología del grupo Abeliano simplicial RX, que son necesarios para el cálculo de la sucesión espectral de Bousfield-Kan. Como un primer paso, se considera el caso en el que X es acíclico.

- **A. Romero. From homological perturbation to spectral sequences: a case study. In Global Integrability of Field Theories, Proceedings of GIFT 2006, pp. 289–309. Universitätsverlag Karlsruhe, 2006.**

  En este artículo se presenta un programa para calcular sucesiones espectrales. El algoritmo teórico correspondiente está basado en las técnicas de homología efectiva y de perturbación homológica. Se ilustran las ideas fundamentales de este algoritmo mediante un ejemplo relacionado con la famosa sucesión espectral de Serre.

- **A. Romero, J. Rubio, y F. Sergeraert. Computing spectral sequences. Journal of Symbolic Computation, 41(10), pp. 1059–1079, 2006.**

  John McCleary insiste en su interesante libro titulado "User's guide to spectral sequences" en el hecho de que la herramienta "sucesión espectral" *no* es en general un *algoritmo* que permita al usuario calcular los grupos de homología buscados. Este artículo explica cómo la noción de "objeto con homología efectiva" permite por el contrario obtener recursivamente todas las componentes de las sucesiones espectrales de Serre y de Eilenberg-Moore, cuando los datos iniciales son objetos con homología efectiva. En particular se resuelve el problema del cálculo de las diferenciales superiores, y también el problema de extensión en el límite. Además, estos métodos han sido implementados concretamente como una extensión del programa Kenzo. Se incluyen también dos ejemplos típicos de cálculo de sucesiones espectrales.