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Mathematical Connections in Preservice Secondary Mathematics Teachers' Solution Strategies to Algebra Problems

Conexiones matemáticas en las estrategias de solución de problemas de álgebra de profesores de matemáticas de secundaria en formación

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Abstract ∞ This study investigated the mathematical connections found in the solutions provided by 22 preservice secondary mathematics teachers to a set of algebra problems. Interest in, and research on, mathematical connections has gained prominence of the past decade. Here, we use the Extended Theory of Mathematical Connections or ETMC to explore the types of connections that this framework does and does not capture in the preservice teachers' solutions. The ETMC surfaced four types of mathematical connections across four problems: 'different representations', 'procedural', 'part-whole' and 'meaning'. The other types of connections defined in ETMC such as 'reversibility' or 'feature' were not found in our data, perhaps because of the specific problems that were used. Some mathematical connections were not highlighted when examining the solutions through the lens of ETMC ('meaning', 'implication or if/then' and modelling) addressing areas in which ETMC might be limited in its capacity to support researchers in identifying mathematical connections in different contexts.

Keywords ∞ Algebra; Mathematical connections; Solution strategies; Preservice secondary mathematics teachers

Resumen ∞ Este estudio investigó las conexiones matemáticas encontradas en las soluciones proporcionadas por 22 futuros profesores de matemáticas de secundaria a un conjunto de problemas de álgebra. El interés y la investigación sobre las conexiones matemáticas han ganado importancia en la última década. Se utiliza la Teoría Extendida de las Conexiones Matemáticas o ETMC para explorar los tipos de conexiones que este marco captura y no captura en las soluciones de los futuros profesores. El ETMC reveló cuatro tipos de conexiones matemáticas en cuatro problemas: "representaciones diferentes", "procedimiento", "parte-todo" y "significado". Los otros tipos de conexiones definidas en ETMC, como "reversibilidad" o "característica", no se encontraron en nuestros datos, quizás debido a los problemas específicos que se utilizaron. Algunas conexiones matemáticas no se pusieron de manifiesto al examinar las soluciones a través de ETMC ("significado", "implicación o si/entonces" y modelado), mostrando áreas en las que ETMC podría tener una capacidad limitada para ayudar a los investigadores a identificar conexiones matemáticas en diferentes contextos.

Palabras clave ∞ Álgebra; Conexiones matemáticas; Estrategias de solución; Futuros profesores de matemáticas de secundaria

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1. INTRODUCTION

Historically, the mathematics education community agrees on the role and importance of making connections in the teaching and learning of mathematics. National curricula in many countries including the USA (Common Core State Standards, 2022), Australia (Australian Curriculum, Assessment and Reporting Authority, 2022), and England (Department for Education, 2021) emphasise mathematical connections. The crux of this interest is identifying the connections that one establishes in doing mathematics. However, how exactly one can do this and what mathematical connections one establishes have been difficult to determine. Researchers interested in this question have pursued different paths. Businskas (2008) identified several types of mathematical connections based on the empirical data on the topic of quadratic functions and equations. Other researchers widely used these and have also identified additional types of mathematical connections. These efforts have resulted in the generation of several frameworks to help researchers to examine the proficiencies of students or teachers in making mathematical connections in the learning or teaching of mathematics.

One such specific framework is the Extended Theory of Mathematical Connections (ETMC; see Rodríguez-Nieto, Font et al., 2022). Any framework like ETMC, however, has a particular lens through which it views connections, and there are likely other connections in an individual’s mathematical work that are not captured by ETMC. This paper reports on part of a larger study wherein a sample of preservice secondary mathematics teachers were supported to enhance their mathematical proficiencies in teaching algebra. We aim to examine the mathematical connections apparent in the preservice secondary mathematics teachers’ solutions to a set of algebra problems by applying the ETMC to their solution strategies. By analysing these, we contextualise what types of connections ETMC makes salient, and by looking within and across problems, as well as at correct and incorrect responses, we illuminate types of connections that ETMC does not bring to light.

2. MATHEMATICAL CONNECTIONS

A mathematical connection can be defined as ‘a cognitive process through which a person relates two or more ideas, concepts, definitions, theorems, procedures, representations and meanings with each other, with other disciplines or with real life’ (García-García & Dolores-Flores, 2018, p. 229). Mathematical connections typically emerge when an individual develops written and/or oral arguments to mathematical tasks (García-García & Dolores-Flores, 2018).

The literature on mathematical connections is vast. For example, the research conducted by Eli et al. (2011) explored mathematical connections made by prospective middle-grades teachers. Their exploration highlighted the dominant presence of procedural and categorical connections within the teachers’ mathematical discourse. The findings emphasised the essential role of varied connections in shaping the teachers’ pedagogical approaches and underscored the significance of fostering a deeper comprehension of the underlying relationships among different mathematical concepts. Dolores-Flores et al. (2019) conducted an analysis of the mathematical connections forged by pre-university students while engaging in tasks related to rates of change. By exploring the dynamics of these connections, the researchers shed light on the underlying cognitive processes involved in the students' comprehension of mathematical concepts, emphasising the significance of procedural connections in the students' mathematical reasoning and problem-solving strategies.

García-García and Dolores-Flores (2021) delved into the mathematical connections established by pre-university students in solving challenging calculus application problems. The researchers identified five distinct types of intra-mathematical connections prevalent among the students. The study not only clarified the various categories of connections but also underscored the critical role of these connections in shaping the students' understanding and application of calculus concepts. The research emphasised the pressing need for targeted engagement aimed at enhancing the students' visualisation capabilities and fostering the development of their making connections skills. The findings offered crucial insights into the cognitive processes involved in mathematical problem-solving and highlighted the significance of promoting a holistic understanding of mathematical concepts among pre-university students.

By closely examining the nature of connections, Dogan et al. (2022) shed light on the intricate cognitive processes underlying students' comprehension and utilisation of linear algebra in various mathematical contexts. The authors highlighted the diverse categories of connections exhibited by students and emphasised the critical role played by these connections in shaping the students' overall understanding of complex mathematical concepts and mathematical proofs. The study offers insights for educators into how to enhance their teaching strategies to effectively convey mathematical concepts such as linear independence.

2.1. Extended Theory of Mathematical Connections or ETMC

The review of recent research highlights the necessity of the ETMC to comprehensively understand mathematical connections, the foundations of which are:

Different Representations: Connections established between different representations of mathematical concepts, fostering a deeper understanding of their interrelated nature.

Procedural: Connections formed based on the step-by-step procedures involved in mathematical operations or problem-solving.

If-Then: Connections derived from logical sequences and relationships, enabling one to draw conclusions based on conditional relationships between mathematical ideas.

Part-Whole Connections: Connections highlighting the relationship between different components or parts of mathematical concepts, fostering a comprehensive understanding of their interconnections.

Feature/Property: Connections centred around the essential features or properties of mathematical concepts, enabling one to grasp the fundamental characteristics and attributes that define these concepts.

Instruction-Oriented Connections: Connections emphasising the relationship between instructional strategies and the comprehension of mathematical concepts, highlighting the significance of effective teaching methodologies in facilitating students' understanding (for a review and the origins of these definitions see Hatisaru, 2023).

Several authors have contributed to the development of this extended theory, emphasising the need to refine existing models to accommodate new types of connections and deepen the study of mathematical connections in various educational contexts.

Rodríguez-Nieto, Rodríguez-Vásquez et al., (2022), for example, contributed to the advancement of the existing model for mathematical connections by introducing a novel category known as metaphorical connections (also found in Hatisaru, 2022). This addition to the model was instrumental in refining the conceptual framework, aiming to enhance the clarity and comprehensiveness of the categories within the model. The works by Rodríguez-Nieto, Font et al. (2022) and Rodríguez-Nieto et al. (2023) showcased the potential of a networking of theories between the ETMC and the onto-semiotic approach. Their combined approach provided a comprehensive analytical framework for understanding the intricate nature of mathematical connections.

In sum, several recent studies collectively stress the significance of ETMC, emphasising the need for a more comprehensive understanding of the intricate relationships between various mathematical concepts. As educators and researchers continue to explore and expand the boundaries of this extended theory or ETMC, it is expected to enhance the overall understanding of mathematical connections and their implications for effective teaching and learning of mathematics.

2.2. Research aims and questions

Previous research provides empirical support regarding the validity of ETMC (e.g., Rodríguez-Nieto, Font, et al., 2022; Rodríguez-Nieto et al., 2023) and its potential to investigate teachers' ability to establish connections in teaching mathematical concepts (Hatisaru, 2023). Previous research, however, has not intentionally examined the affordances and limitations of ETMC in capturing mathematical connections in solving mathematical problems. We use the ETMC to investigate the mathematical connections in a sample of preservice secondary mathematics teachers' solutions to four algebra problems and ask:

1) What are preservice secondary mathematics teachers' abilities for solving algebra problems? What solution strategies do they use?
2) What mathematical connections are highlighted by examining the solution strategies using the extended theory of mathematical connections? What mathematical connections are not captured?

It is inevitable that the features of the ETMC would be influenced by the mathematical domains and populations used in prior studies. Common mathematical domains for prior investigations of mathematical connections have included derivatives, linear algebra, rates of change, and calculus application problems. Here, we study mathematical connections that are visible in initial teacher education students' solutions to open-ended contextual problems within the domain of algebra.

3. METHODS

3.1. Context of the study and participants

The data presented in this paper come from a research study conducted by the first author at a metropolitan university in Western Australia, in 2022. Data were collected during classes within the Bachelor of Education (B.Ed.; lasts four years) and Master of Teaching (MTeach; takes two years) degrees units. Both degrees aim to equip students with the necessary knowledge and skills to teach Years 7 to 12 mathematics in secondary schools.

The second-year B.Ed. (5 Female and 6 Male) and the first-year MTeach (4 Female and 8 Male) students enrolled in the two mathematics education units offered in Semester 2 were invited to participate in the study; all agreed. Participation was purely voluntarily, and students' informed consent was obtained for their solutions to be used for research purposes. In both units, the teaching and learning activities are designed around the content strands of Number and Algebra, Statistics and Probability, and Measurement and Geometry, with special coverage of problem-solving.

3.2. Data generation

For six weeks during Semester 2, the students undertook problem-solving activities (one each week) as part of their Number and Algebra learning activities. We use four of the problems, those which are worded algebra problems. At the commencement of each class, the students were provided with a reflection form containing the problem and three prompt questions, which they completed in 20-25 minutes. Because the problems were done on different days, not all students did all four problems. The data for this paper is the responses to the first prompt question:

> Think and explain as many different possible solutions to the problem as you can. Name the solutions as Solution A, Solution B, Solution C and so on.

3.3. Problems used

The four problems are referred to in this paper as DICE, FARMER, ANY TWO NUMBERS, and BOOKS. All these problems can be solved in a variety of ways; they therefore allow us to examine the solution strategies generated, and what mathematical connections are manifested in them. They could be posed to students in lower secondary school. It is expected that preservice secondary mathematics teachers are familiar with multiple ways of solving these problems and recognise their differing efficiency.

> DICE: Die A and Die B have twelve sides each. Suppose that you roll die A and die B at the same time. When do the dice satisfy the following two conditions? The sum of 2 times A plus B equals 15. 3 times A minus B equals 5. (Ito-Hino, 1995)
>
> FARMER: A farmer had 19 animals on his farm — some chickens and some cows. He also knew that there was a total of 62 legs on the animals on the farm. How many of each kind of animal did he have? (Tripathi, 2008)
>
> ANY TWO NUMBERS: If you are given the sum and difference of any two numbers, show that you can always find out what the numbers are. (Kieran, 1992)
>
> BOOKS: You have some teen and young adult books. You gave one-half of the books plus one to a friend, one-half of the remaining books plus one to another friend, and one-half of the remaining books plus one to another friend. If you have one book left for you, how many books did you have at the start? (Adapted from Musser et al., 2008)

The statement of DICE comes very close to supplying the two simultaneous equations in two unknowns (*A* and *B*). FARMER can also be solved by identifying two unknowns (numbers of chickens and cows), then setting up and solving two simultaneous equations, or by noting immediately that the number of chickens is 19 minus the number of cows so that only one equation in one unknown (the number of cows) needs to be solved. Solution paths based on guess-check-and-improve or logical arithmetic reasoning (Stacey & MacGregor, 1999) are viable, too. A solution based on logical arithmetic reasoning can be:

> Give each of the 19 animals 2 legs. That requires $19 \times 2 = 38$ legs. You must give out more legs ($62 - 38 = 24$), so you need to give extra legs to $24 \div 2 = 12$ animals. That is, there are 12 cows, and hence 7 chickens.

ANY TWO NUMBERS can be set up as two simultaneous linear equations in two unknowns (the original numbers), but, in this case, there are additionally two parameters involved (the sum and difference) demanding higher algebraic competence than the other problems. Guess-check-and-improve solutions might be used as part of a 'look for a pattern' strategy — students may choose sum and difference pairs, find the numbers and then look for a pattern linking them. BOOKS can be solved algebraically, by writing five simple linear equations in five unknowns (original number of books, number after gift 1, etc.), or by writing just one equation in one variable by immediately building the stated relationships into the equation. Guess-check-and-improve is also a feasible method. The simplest solution is to work backwards from the one book remaining, through the various steps to the start, using logical arithmetic reasoning.

3.4. Data analysis approach

The participants' solutions to the relevant problems were content analysed by two authors of this paper (Hatisaru and Stacey), both of whom have extensive expertise in these types of data analyses (Hatisaru, 2023; Stacey & MacGregor, 1999). One of the participant's solutions to all four problems were weak; although this participant sometimes gave the correct answer, their working was chaotic, messy, and difficult to understand. This participant's data were therefore not included in data analysis. The remaining 22 participants' data were analysed; each participant was assigned a code: P1, P2, P3, P4, P5, ... to protect their anonymity. The total number of responses to the prompt over the 4 problems was 70, with 18 absences. In total, the responses of these 22 participants gave 128 solutions to the four problems — an average of 1.8 solutions per participant present per problem. We label these as 'solutions', whether they are correct or incorrect, completely, or incompletely implemented, or only suggested.

After examining the participants' solutions, the success rate for each problem was determined. Here, success on a problem means the participant presented at least one correct solution with a correct answer.

To find out how the participants approached to the problems, all 128 solutions were recorded. Based on extensive discussions and several iterations, the strategies emerging in participant solutions were identified and classified into the eleven categories presented in Table 1. Examples are presented in Figures 1 to 3 to illustrate this categorisation; strategies apparent in them are noted in UPPER CASE. Next, participants' strategies categorised into the groups presented in Table 1 were examined based on the mathematical connections defined in ETMC. Please note that within a single strategy, there can be multiple mathematical connections, as also suggested by Mhlolo (2013). Furthermore, our method examines the 'within strategy' connections made by a participant, rather than 'between strategy' connections that might be observed by studying all the solutions of one participant.

Table 1. Strategies emerging in participant solutions and their description

Strategy	Description
Algebraic	
equations, symbolic solving	Write algebraic equations and solve using standard algebraic method.
equations, numerical solving	Write algebraic equations and solve numerically.
equations, graphical solving	Write algebraic equations and plot to find the intersection point.
equations, using parameters, symbolic solving	Write algebraic equations with two unknowns and with two parameters and solve using standard algebraic methods.
equations, using parameters, numerical solving	Write algebraic equations with two unknowns and with two parameters and solve for a specific example.
equations, no parameters, symbolic solving	Write algebraic equations with two unknowns but with two specific numbers and solve using standard algebraic methods.
equations, pattern	Write algebraic equations with two unknowns and with selected specific sums and differences, solve using any method and look for a pattern linking solutions to sum and difference.
equations, vectors	Draw on ideas of vector algebra and change of basis (alternative axes).

Strategy	Description
Numerical	
numerical, systematic	Use a numerical path in a systematic way such as guess-check-and-improve or guess-and-check with tables.
numerical, unsystematic	Use apparently random guess-and-check
logical arithmetic reasoning	Think about the relations between the numbers/quantities involved and work from known numbers towards the solution.

4. Results

4.1. Success rate and solution strategies

Table 2 summarises the numbers of participants whose best solution was 'correct', 'partially correct', or 'incorrect' for each of the four problems, as well as the number of participants 'absent'. 'Partially correct' was only used as a code for ANY TWO NUMBERS problem to record when participants gave a correct solution to a specific case, without generalising the solution to *any* sum and difference. 61 of the 70 responses included a correct answer by at least one method (87%).

Table 2. The success rate for each problem among 22 participants

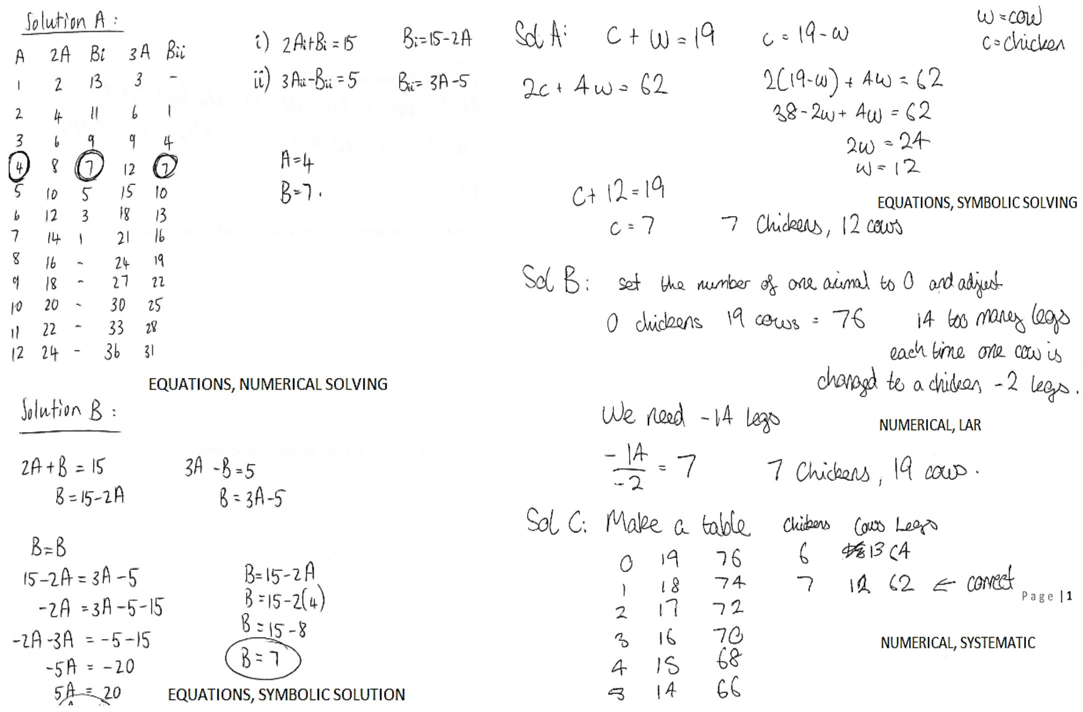
Problem	Correct	Partially correct	Incorrect	Absent	*Percent attempts correct
DICE	15	-	3	4	83%
FARMER	19	-	-	3	100%
ANY TWO NUMBERS	15	3	2	2	75%
BOOKS	12	-	1	9	92%

*Percentage of those students were present in the class who gave a correct answer.

Whilst most participants' answers to DICE were correct (P20, Figure 1), three participants' answers were incorrect. These participants failed to recognise the need for simultaneity, or perhaps they misinterpreted the question, thinking it had two separate parts. BOOK was solved incorrectly by only one participant; all others solved it correctly (P6, Figure 2). All participants present solved FARMER correctly (P2, Figure 1). ANY TWO NUMBERS was relatively difficult; out of 20 participants who undertook the problem, fifteen gave a complete solution to the problem (P14, Figure 2), three used single numerical examples rather than parameters (P17's Solution A in Figure 3, left); the remaining two participants' responses were incorrect (P10, Figure 3, right).

All but five participants gave at least one correct solution for every problem they attempted. This suggests that the algebraic competence of these five students was limited. We refer to these erroneous solutions where relevant to provide supporting evidence for the mathematical connections that were established or missed.

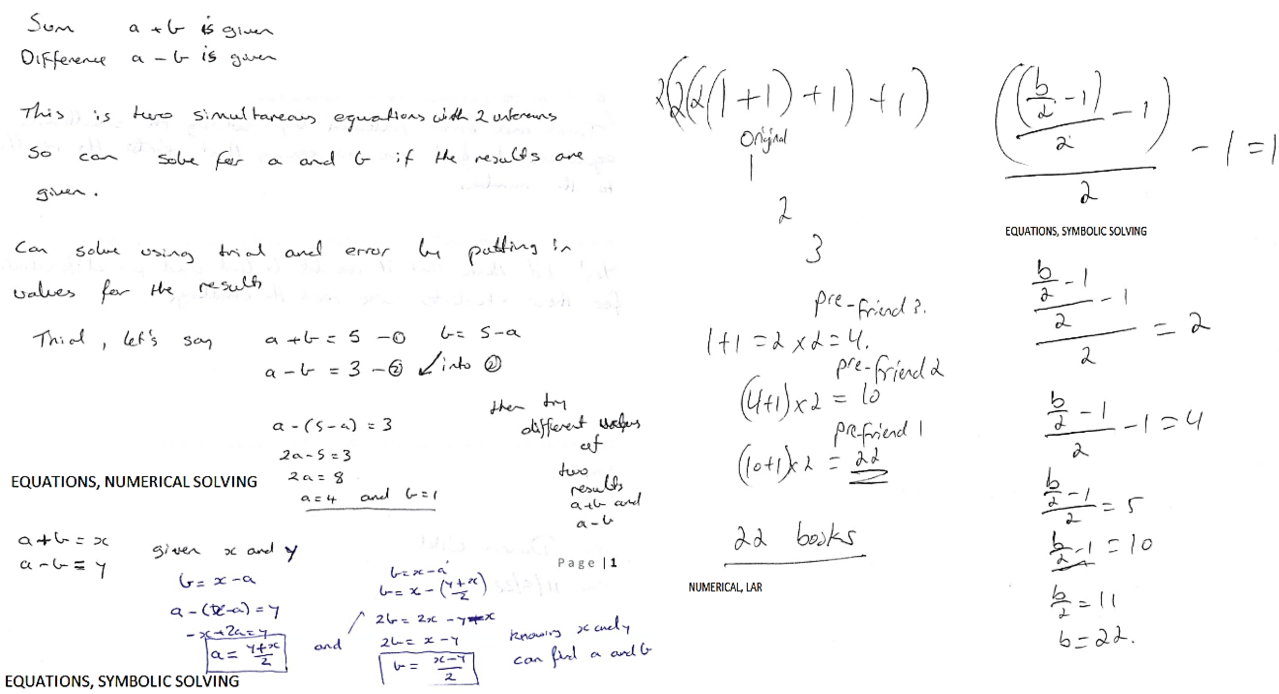
Figure 1. Sample solutions to DICE and FARMER



P20's 2 solutions to DICE P2's 3 solutions to FARMER*

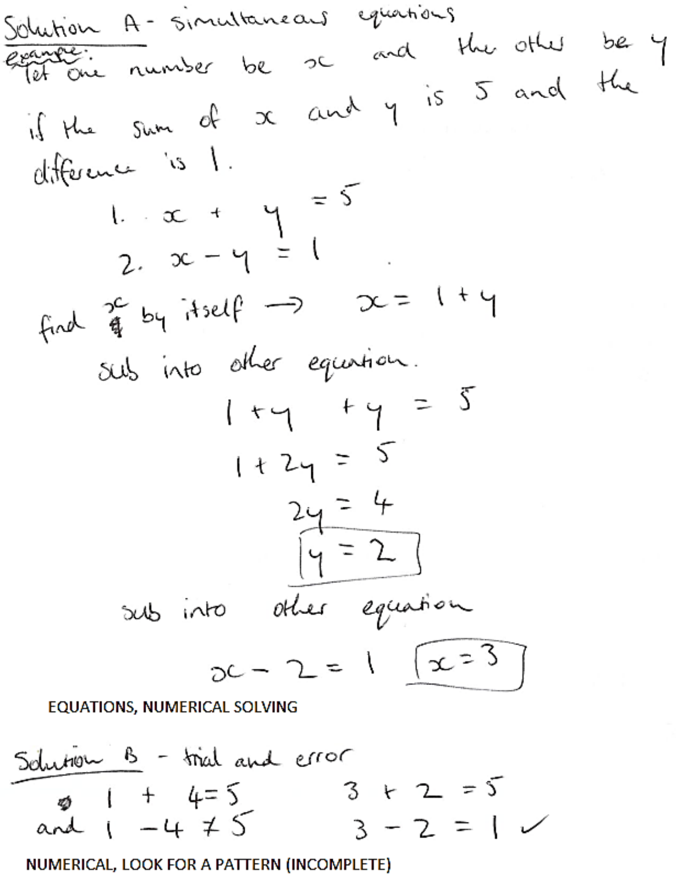
*We considered the incorrect answer in the final step of P2's solution B (i.e., 12 cows not 19) was a small slip, not detracting from the use of LAR or logical arithmetic reasoning.

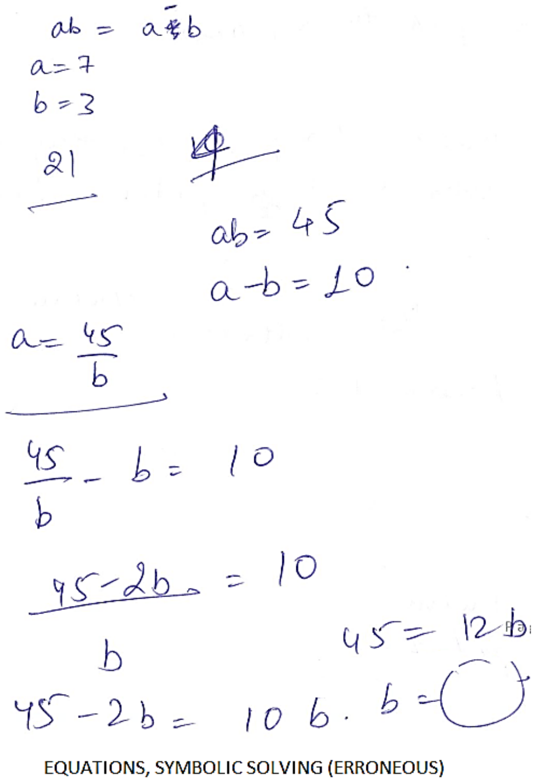
Figure 2. Sample solutions to ANY TWO NUMBERS and BOOK



P14's 2 solutions to ANY TWO NUMBERS P6's 2 solutions to BOOK

Figure 3. Further sample solutions to ANY TWO NUMBERS





P17's 2 solutions

P10's solution

The number of strategies that were apparent in the participants' solutions are presented in Table 3. Out of the 128 solutions generated, 13 are *suggested* without showing any implementation. Most of these are numerical approaches where students wrote 'guess-and-check' or 'table'. As no other detail is provided around how guess-and-check and table methods could be implemented, it is unknown whether they are systematic or not. Out of 115 solutions *implemented,* 11 gave incorrect answers, 7 were incomplete, and the remaining 97 were implemented correctly (84%).

The participants generated the same total number of solutions for DICE and FARMER (34 each, including similar numbers of incomplete/erroneous), 29 solutions for ANY TWO NUMBERS and 18 solutions for DICE. However, the percent of solutions correct for each problem is almost the same (82%, 85%, 86%, and 83%.). This indicates that, overall, the participants knew the different ways of solving these problems and were mostly able to execute them correctly.

Table 3. Strategies manifested in participant solutions

Problem	Implemented (115 solutions)			Suggested (13 solutions)
	complete (97 solutions)	incomplete (7 solutions)	erroneous (11 solutions)	
DICE	Equations, symbolic solving (17) Equations, numerical solving (7) Equations, graphical solving (1) Numerical, systematic (3)	Numerical, systematic (1)	Equations, numerical solving (2) Numerical, systematic (3)	Equations, symbolic solving (2) Numerical [suggested guess-and-check (4) table (2)]
FARMER	Equations, symbolic solving (20) Equations, numerical solving (1) Numerical, systematic (6) Logical arithmetic reasoning (2)	Logical arithmetic reasoning (1) Numerical, systematic (3)	Logical arithmetic reasoning (1)	Numerical [suggested guess-and-check (1)]
ANY TWO NUMBERS	Equations, using parameters, symbolic solving (16) Equations, no parameters, symbolic solving (5) Equations, using parameters, numerical solving (1) Equations, pattern (2) Equations, vectors (1)	Numerical, unsystematic (2)	Equations, no parameters, symbolic solving (1) Numerical, unsystematic (1)	Equations, symbolic solving (1) Numerical [suggested guess-and-check (2)]
BOOKS	Equations, symbolic solving (10) Logical arithmetic reasoning (4) Numerical, systematic (1)	-	Equations, symbolic solving (2) Equations, numerical solving (1)	Numerical [suggested guess-and-check (1)]

4.2. Mathematical connections apparent in participant strategies

4.2.1. Connections of 'different representations' type

To solve a problem by algebra, one makes connections with the problem statement to identify variables and relationships to formulate the equation(s) and then one uses procedural algebra, usually involving arithmetic, to get the solution. However, if one solves a problem numerically (e.g., guess-check-and-improve), one directly connects the relationships, stated verbally, to a sequence of numerical operations. So, the different solution strategies have different connections.

Relevant to this study, it was considered that both the verbal-algebraic connection when setting up equations, and the verbal-numerical connection when finding numerical relationship belong to the 'different representations' type of connection. For example, as shown in Figure 1, in solving DICE, P20 generates both 'equations, symbolic solving' and 'equations, numerical solving' strategies. In solving FARMER, P2 generates three strategies: 'equations, symbolic solving', 'logical arithmetic reasoning', and 'numerical, systematic' (Figure 1). The logical arithmetic reasoning also requires a verbal-numerical connection between the words and the numerical relationships. Most participants showed similar behaviour. Using the data in Table 3, we considered these strategies as involving 'verbal-algebraic' and 'verbal-numerical' connections, respectively (Table 4). As might be expected of student secondary mathematics teachers, verbal-algebraic connections were made more often (90 occurrences; 70%) than verbal-numerical (38 occurrences; 30%) connections.

Table 4. Connections of different representations type evident in the participants' solutions (total 128)

DICE	FARMER	ANY TWO NUMBERS	BOOKS
verbal-algebraic (29) verbal-numerical (13)	verbal-algebraic (21) verbal-numerical (14)	verbal-algebraic (27) verbal-numerical (5)	verbal-algebraic (13) verbal-numerical (6)

4.2.2. Connections of 'procedural' type

Some of the participants used the elimination and/or substitution method to solve the linear simultaneous equations that they set up in solving DICE, FARMER, or ANY TWO NUMBERS. It was considered that the procedural type of connection occurs when these standard procedures are employed. All 'equations' solutions for BOOKS used only one variable equations with standard solving methods. Table 5 presents the distribution of this type of connection across the four problems.

Table 5. Procedural type of connections evident for solving equations

DICE	FARMER	ANY TWO NUMBERS	BOOKS
elimination (10) substitution (9)	elimination (9) substitution (11)	elimination (9) substitution (9)	solving one variable equations (13)

4.2.3. Connections of 'part-whole' type

Part-whole connection is particularly relevant to ANY TWO NUMBERS which requires a formulation using parameters because both the two original numbers and their sum and difference are unspecified. In ANY TWO NUMBERS, sum and difference can be exemplified as *any* two numbers to find the original numbers, but this relationship must be generalised to *all* numbers as the chosen two numbers are *only* a particular case of the whole situation.

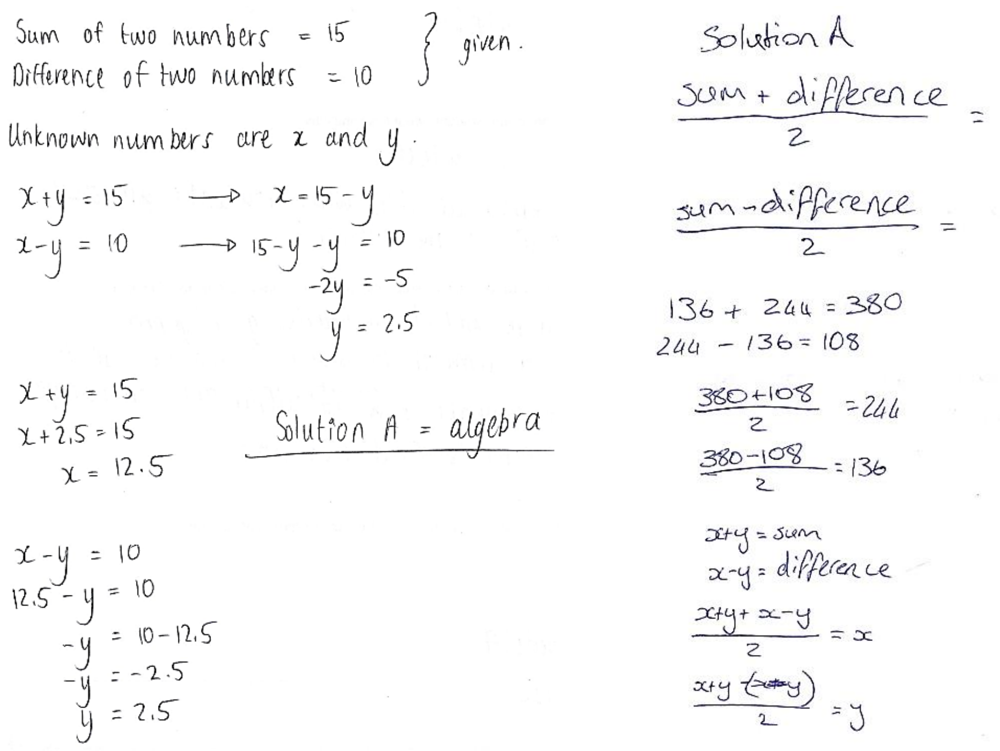
Table 6. Connections of part-whole type evident (or not) in solutions for ANY TWO NUMBERS (total 32)

Evident (20)	Not evident (9)	Unknown (3)
equations, using parameters equations, pattern equations, vectors	equations, no parameters, symbolic/numerical solving numerical, unsystematic	suggested strategies

Based on the data presented in Table 3, and as summarised in Table 6, out of 32 solutions for ANY TWO NUMBERS, part-whole connection was evident in total of 20 (63%) where a logical relationship was established between general and particular cases. It is unknown whether this connection would be made in 3 suggested solutions without implementation (9%). In the remaining 9 solutions (28%) this connection was missed. For example, P20 chose 15 and 10 as the sum and difference of two numbers and found the original numbers: 12.5 and 2.5 (Figure 4, left). They did not note that the solution based on a single example cannot be a generalisation of the situation.

P14 specified the sum and difference of any two numbers such as a and b being 5 and 3 ($a + b = 5$, $a - b = 3$), found the original numbers by solving the equations ($a = 4$, $b = 1$), but as opposed to P20, they recognised that 5 and 3 was only one example by stating: 'then try different values at two results $a + b$ and $a - b$'. Furthermore, they generalised the solution to all numbers in their second solution by using parameters for sum and difference: $a = \frac{y+x}{2}$, $b = \frac{x-y}{2}$, and concluded the solution as: 'knowing x and y can find a and b' (see Figure 2, left). Similarly, P7 and P8 made a relationship between a whole and its part. As shown in Figure 4 (right), P7 chose two numbers: 136 and 244, calculated their sum and difference: 380 and 108, and showed the relationship between them: half of the sum of sum and difference is 244; half of the difference between sum and difference is 136 — that is, the two original numbers. They next generalised this relationship to *any* numbers. In this way, they recognised that the pair 136 and 244 was only a particular example of the whole.

Figure 4. P20's (left) and P7's (right) solutions to ANY TWO NUMBERS

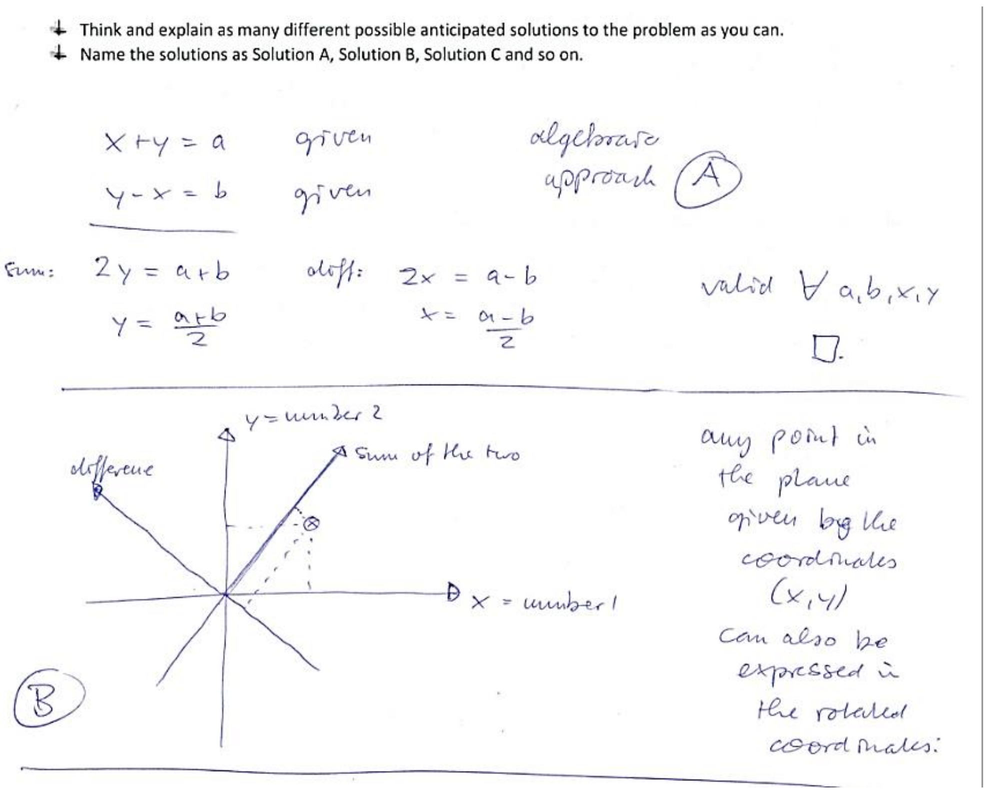


4.2.4. Connections of 'meaning' type

Within ETMC, there are two different types of meaning connections: (a) connections related to the meaning of a mathematical concept or definition and (b) connections related to different meanings of a mathematical concept that are useful for solving a given problem (see Rodríguez-Nieto et al., 2023).

In this study, one participant suggested a vector approach to solve ANY TWO NUMBERS (Solution B in Figure 5); by drawing on mathematics of change of basis and orthogonal projections of a point onto two different sets of axes. We find this as a unique meaning type of mathematical connection: connecting a problem to more advanced mathematical theory.

Figure 5. P17's 2 solutions to ANY TWO NUMBERS



4.3. Mathematical connections not captured by ETMC

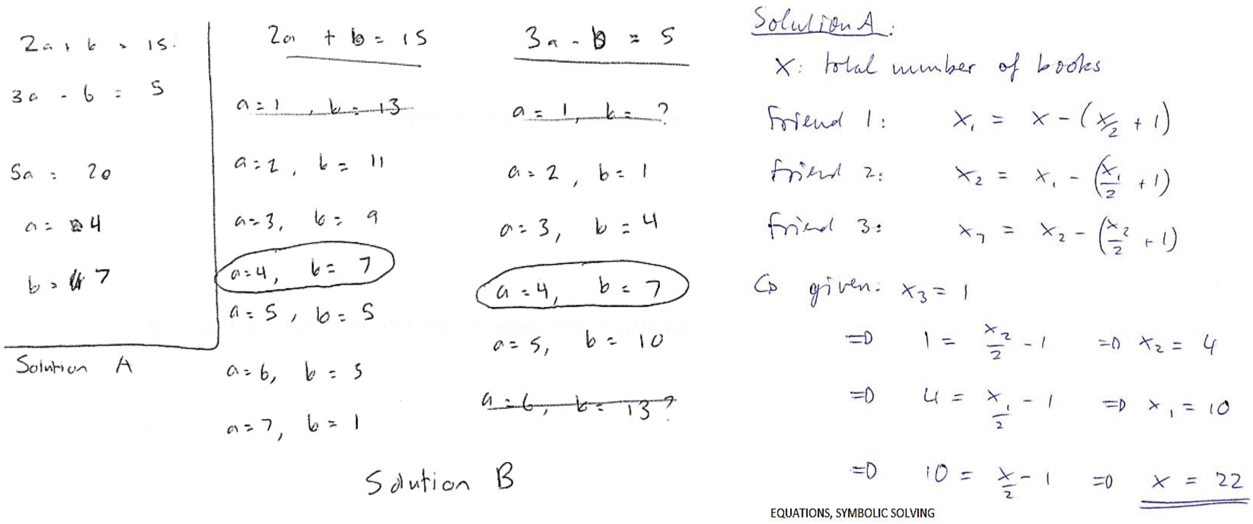
Through our analysis, we noted mathematical connections that do not readily fit into the ETMC types.

4.3.1. Connections of 'meaning' type

We believe there seem to be additional types of meaning connections. For example, meaning connections can be understanding the situation to representing it mathematically or including some feature of the situation into the solution. Furthermore, meaning connections can be intra-mathematical meaning, such as recognising the involvement of a mathematical principle or feature.

DICE, FARMER, and ANY TWO NUMBERS include the idea of simultaneity or constraints. DICE was used in a series of lessons where school students were introduced the 'meaning' of simultaneity, and they were gradually introduced simultaneous equations, along with formal algebraic procedures to solve them such as elimination and substitution methods (Ito-Hino, 1995). When solving these problems, the participants recognised the idea of simultaneity and used its meaning in their solutions (P12, Figure 6). As this would not fit in how connections of meaning type are defined in ETMC, we have not counted these situations as meaning connections. It means that such connections of meaning type are not brought out when the existing ETMC framework is applied.

Figure 6. P12's 2 solutions to DICE (left) and P18's one of the 3 solutions to BOOKS (right)



4.3.2. Connection of 'implication or if-then' type

In carrying out any mathematical work, there are line-by-line links that must be made. For example, quoting Mhlolo (2013): 'if $2x = 10$ (the premise) then $x = 5$ (the logical conclusion). Similarly, if A is a polygon whose interior angles add up to 180^o then A is a triangle' (p. 180). Are these small-scale implications properly called connections in ETMC and if not, why not?

Implication is a fundamental act of reasoning, involved in every step of a mathematical solution as we move from premise to conclusion. Whilst there was a wealth of implication or if-then statements in the participants' solutions (P18, Figure 6), in agreement with ETMC, our study had to choose a 'grain size' for implication connections to produce useful results. We decided, for example, that carrying out algebraic manipulation and doing arithmetic would not be classed as making connections. Other studies with other populations of participants would make other choices.

4.3.3. Connections of 'modelling' type

Modelling connections are not in the ETMC studies in general. We assume that it is because this framework is concerned only with intra-mathematical connections. However, in a much recent work employing ETMC, where mathematical connections made by one teacher and a sample of university students when solving a problem about launching a projectile were explored, modelling connection was defined as connections where a link is made between mathematics and the daily life of students. This connection is considered evident when one solves a non-mathematical (or application) problem where they need to pose a mathematical model (see Rodríguez-Nieto et al., 2023).

In solving DICE, there was one instance where a student made a different 'modelling' connection to the one intended. The question specified the dice had twelve sides, but the writer of the question and most students assumed that this meant that the numbers 1 to 12 appeared on the sides. The question did not specify the intended interpretation clearly. This student assumed the numbers were only {1, ..., 6} thus had made a different modelling connection — a different way of linking the real-world statement about a die to the mathematical world statement about the range of numbers to be considered. In a similar vein, another student noted that there would be no solution if there were pictures rather than numbers on the dice. In problem solving attempts, there can be missing modelling connections and incorrect connections; in this case, we think it is reasonable but different. Even if the ETMC framework is primarily intended to be for intra-mathematical work, modelling connections can influence any mathematical work related, even trivially, to any real-world context. The problems in this study had a real-world setting used simply as a 'border' (Stillman, 1998) and not intended to be taken seriously, but still the influence of connections to the real world was evident.

5. Discussion and conclusions

Our first research question concerned preservice secondary mathematics teachers' (hereafter PSMTs) strategic competence for solving the given open-ended algebra problems. Recall that PSMTs were asked to solve each of the four problems in as many ways as possible. Our results indicate that these PSMTs were generally able to solve all four problems successfully. 61 of the 70 responses included a correct answer using at least one method. 17 of the 22 participants gave at least one correct solution for every problem they attempted, and many were able to generate more than one correct solution for the problems. These findings indicate that the four problems used in the study were at an appropriate level for these PSMTs to successfully engage with, and thus their solutions provide us with reasonable data from which to analyse mathematical connections evident in solutions to algebra problems typically encountered in the secondary mathematics (algebra) curriculum.

Our second research question concerned the types of mathematical connections evident in PSMTs' solutions to these four problems, and we utilized the Extended Theory of Mathematical Connections or ETMC to seek out these types of connections. To a large degree, the ETMC framework was quite useful in analysing teachers' solutions. This framework highlighted 'different representations' and 'procedural' types of connections in teachers' strategies across all four problems, as well as 'part-whole' and 'meaning' connections for one of the problems. The other types of connections defined in ETMC including 'reversibility' or 'feature' (e.g., Rodríguez-Nieto, Font, et al., 2022) were not found in our data, perhaps because of the specific problems that we chose to use here.

We were also interested in types of connections present in PSMTs' strategies that did not fall within any of the existing ETMC connection types. In particular, we found types of 'meaning' connections that did not appear to be included in ETMC. Within ETMC, there are two different types of 'meaning' connections (see Section 4.2.4). In our data, we found previously-unaccounted-for 'meaning' connections related to the ways that a situation is represented mathematically, as well as connections involved in the recognition of a mathematical principle or feature within a problem context. In addition, we found connections where PSMTs made a link between mathematics and real life — a connection that was useful in helping them mathematise and solve these contextual problems. These types of modelling connections are also not currently present in ETMC (see an exception in Rodríguez-Nieto et al., 2023).

Our consideration of the affordances and constraints of ETMC for identifying connections also surfaced several other questions about this framework—and by extension other existing frameworks for identifying connections—that would be useful for researchers interested in mathematical connections to consider in future work. First, our analyses raised questions about 'meaning' connections. The ETMC framework identifies some types of connections related to meaning, but our data surfaced some additional 'meaning' connections. As currently conceived by the ETMC framework, 'meaning' connections take specific forms. But it is not difficult to imagine a much broader set of possible and additional 'meaning' connections, including the two documented in our data. In addition, 'meaning'—and meaning-making—seems to play an implicit role in other types of connections. 'Feature/property' connections occur when a learner makes meaning of a mathematical concept; 'analogy' connections involve meaning-making links between mathematical concepts and real-world situations (Hatisaru, 2022). Is the 'meaning' type of connections so broadly conceived that it functions as an 'other' code, capturing a very broad array of types of connections that do not otherwise fall into any existing ETMC category? Are 'meaning' connections a meaningful component of connections frameworks such as ETMC, or is this type of connections too broad and diverse such that it may be meaningless?

Second and related, our analyses made it apparent that different types of connections have different grain sizes. The 'meaning' type of connection, as noted above, had the potential to become quite broad and indicative of macro connections. But other connection types more naturally lend themselves to smaller grain sizes and micro connections. Extremes in either direction seem potentially problematic. For example, if a connection is too narrowly conceived, could this mean that every statement made by a learner, or even every word uttered, could be indicative of some type of connection? When too broadly conceived, might it be necessary to identify sub-connections within a single connection or type of connection? Furthermore, should all types of connections within a framework be approximately the same grain size? If so, how should this optimal grain size be determined?

Third, our investigation raised questions about whether it might be possible or useful to make qualitative determinations about the connections made by learners. Are all connections or types of connections equally valid or important, or are some connections more important or better than others, for a given problem, individual, or problem-solving circumstance or context? Should a framework for identifying connections note which connections are especially innovative or routine? We think, in particular, of P18's unique and unexpected vector solution to the ANY TWO NUMBERS problem (Figure 5). Perhaps all connections, even within the same type, are not equal; perhaps connection frameworks should incorporate qualitative features (and not exclusively types) that help distinguish among connections? Consistent with this idea, Mhlolo (2013) developed an analytical tool that can be used to determine the quality of following the connections made in practice. A Likert scale from Level 0 (connection made was mathematically erroneous) to Level 3 (connection made was mathematically acceptable and justified) was employed. Pilot testing of it on 20 lessons delivered by four Grade 11 teachers revealed that the analytical tool developed has potential to identify the strengths and weaknesses of mathematical connections made in a lesson.

Finally, given our focus here on an analysis of PSMTs' multiple solution strategies, we wonder about the ways that a strategy lens and a connection type lens might be in conversation with each other. A unique feature of the present study was the fact that teachers were asked to solve the same problem in multiple ways. Our method for looking at connections was to look within each strategy for the connections that were made. This means that connections (other than implications) were likely to be made between stages of a single solution strategy, such as moving from a verbal description to a mathematical description, then solving the intra-mathematical problem that arises. Or similarly, the different representations type of connection usually related to moving from the verbal statement to the mathematical formulation with a single strategy. This way of looking for connections does not identify those connections that one student knows between different ways of solving a problem. For example, many participants in the present study solved the simultaneous equations by substitution in one solution, and then by elimination in a second. The fact that the participants made an implicit connection between these two strategies (i.e., realising that they both accomplish the same goal) has not been identified here. That is, our method has drawn attention to 'within strategy' connections, rather 'between strategy' connections. Furthermore, our ability to elicit these types of connections is dependent on a particular artifact of this study's method, where students are asked to solve problems in multiple ways, but the multiple ways are not analysed together.

To conclude, examining PSMTs' solution strategies through the ETMC lens afforded insight into connections-related aspects of solving the algebra problems.

We noted the extent to which teacher solutions afforded mathematically rich productions for discovering the strategies used, the degree to which solutions included mathematical errors and imprecisions, the degree to which teachers made connections or missed making connections, and the extent to which the ETMC framework did or did not capture the connections available. We were not interested in using the framework to assess PSMTs' capability for making connections, and indeed this would be inappropriate given that we only had their responses to a written instrument. Rather, we believe that examining these solutions using ETMC provides rich information regarding making connections along the connection types defined in ETMC. We simultaneously acknowledge that there are aspects of making mathematical connections not highlighted through the choice of this particular study design or this particular framework. These insights are particularly important as one considers how this framework, or any similar instrument, can be used as a tool for working with teachers (both preservice and practising) to improve their teaching or supporting teachers to critically analyse and reflect upon their capabilities of making connections.

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Conexiones matemáticas en las estrategias de solución de problemas de álgebra de profesores de matemáticas de secundaria en formación

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Investigaciones previas proporcionan apoyo empírico respecto a la validez de la Teoría Extendida de Conexiones Matemáticas o ETMC (ver Rodríguez-Nieto, Moll et al., 2022) y su potencial para investigar la capacidad de los individuos para establecer conexiones en la enseñanza o el aprendizaje de las matemáticas. Sin embargo, estas investigaciones no han examinado intencionalmente las posibilidades y limitaciones del ETMC para capturar conexiones matemáticas en la resolución de problemas matemáticos. En este artículo, utilizamos el ETMC para investigar los tipos de conexiones que este marco captura y no captura en una muestra de 22 soluciones de 22 futuros profesores de matemáticas de secundaria (PSMT) a cuatro problemas de álgebra.

El número total de respuestas a la pregunta a lo largo de los cuatro problemas fue de 70, con 18 ausencias: *Piensa y explica tantas soluciones posibles al problema como puedas. Nombra las soluciones como Solución A, Solución B, Solución C, etc.*. En total, las respuestas de estos 22 participantes aportaron 128 soluciones a los cuatro problemas, un promedio de 1.8 soluciones por participante y problema. Para averiguar cómo abordaron los participantes los problemas, se registraron las 128 soluciones. Se identificaron las estrategias que aparecían en las soluciones de los participantes y se clasificaron en once categorías (por ejemplo, ecuaciones, resolución simbólica; patrones; numérica, sistemática). A continuación, se examinaron las estrategias de los participantes categorizadas en estas categorías en función de las conexiones matemáticas definidas en el ETMC.

Los resultados han mostrado que el ETMC reveló cuatro tipos de conexiones matemáticas en cuatro problemas: “representaciones diferentes”, “procedimiento”, “parte-todo” y “significado”. Los otros tipos de conexiones definidas en ETMC, como “reversibilidad” o “característica”, no se encontraron en nuestros datos, quizás debido a los problemas específicos que se utilizaron. Algunas conexiones matemáticas no se pusieron de manifiesto al examinar las soluciones a través de ETMC (“significado”, “implicación o si/entonces” y modelado), lo que muestra áreas en las que ETMC podría tener una capacidad limitada para ayudar a los investigadores a identificar conexiones matemáticas en diferentes contextos.

Examinar las estrategias de solución de los PSMT a través del ETMC permitió comprender los aspectos de la resolución de problemas de álgebra relacionados con las conexiones. Se observó si las soluciones de los profesores ofrecían producciones matemáticamente ricas para descubrir las estrategias utilizadas. si las soluciones

incluían errores e imprecisiones matemáticas, si los profesores establecían conexiones o no y si el marco ETMC capturaba o no conexiones disponibles. No nos interesaba utilizar el marco para evaluar la capacidad de los PSMT para establecer conexiones y, de hecho, esto sería inapropiado dado que solo teníamos sus respuestas en un instrumento escrito. Además, reconocemos que hay aspectos de las conexiones matemáticas que no se destacan a través de la elección de este diseño de estudio o este marco en particular. Estas ideas son importantes a la hora de considerar cómo este marco, o cualquier instrumento similar, puede usarse como una herramienta para trabajar con los profesores (tanto en formación como en ejercicio) para mejorar su enseñanza o para apoyar a los profesores a analizar críticamente y reflexionar sobre sus capacidades para establecer conexiones.