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Identifying Mathematical Reasoning Levels in Initial Teacher Training: A GeoGebra-Based Study about conic sections and emphasis on the case of the parabola

Identificando Niveles de Razonamiento Matemático en la Formación Inicial de Profesores: Un Estudio Basado en GeoGebra sobre secciones cónicas y énfasis en el caso de la parábola

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Abstract: This work aims to identify different levels of construction of mathematical reasoning in students of initial formation, in the resolution of a didactic situation about the study of the parabola with the support of GeoGebra. The theoretical support was the Theory of Didactic Situations, considering the levels of mathematical reasoning related to its dialectical movement, together with Didactic Engineering, as a research methodology. The research was carried out undergraduate students at a Brazilian public university. We noticed the strong inclination of students to recognize the parabola only through the perspective of quadratic functions, to the detriment of Analytical Geometry, prioritizing functions as the first alternative solution. The results suggest the need to rethink the parabola approach, aiming at comprehensive teaching, considering it from

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an analytical, geometric, and algebraic point of view, as well as approaches using technology.

Keywords: *Parabola, Mathematical reasoning, Theory of Didactic Situations, Didactic Engineering, GeoGebra.*

Resumen: Este trabajo tiene como objetivo identificar diferentes niveles de construcción del razonamiento matemático en estudiantes de formación inicial, en la resolución de una situación didáctica sobre el estudio de la parábola con el apoyo de GeoGebra. El soporte teórico fue la Teoría de las Situaciones Didácticas, considerando los niveles de razonamiento matemático relacionados con su movimiento dialéctico, junto con la Ingeniería Didáctica, como metodología de investigación. La investigación se llevó a cabo estudiantes de pregrado en una universidad pública brasileña. Notamos la fuerte inclinación de los estudiantes a reconocer la parábola solo desde la perspectiva de las funciones cuadráticas, en detrimento de la Geometría Analítica, priorizando las funciones como primera alternativa de solución. Los resultados sugieren la necesidad de repensar el enfoque de la parábola, con el objetivo de una enseñanza integral, considerándolo desde el punto de vista analítico, geométrico y algebraico, así como los enfoques que utilizan tecnología.

Palabras clave: *Parábola, Razonamiento Matemático, Teoría de Situaciones Didácticas, Ingeniería Didáctica, GeoGebra.*

INTRODUCTION

The parabola is a topic of Mathematics that has great relevance in the development of areas of knowledge such as Architecture, Physics and Engineering. However, its study in the final years of Basic Education in the Brazilian context has occurred in a purely algebraic, fragmented, and little contextualized way, which has caused difficulties in subsequent stages of studies, such as Higher Education (Siqueira, 2016; Vargas & Leivas, 2019).

Despite the frequency with which we come across mathematics in our daily lives, in some cases it is difficult to present real applications of the subjects studied at school to students, or even to demonstrate situations that involve geometric visualizations without a technological support (Maioli *et al.*, 2012). Based on this, we carried out an experiment to address this issue, considering different exploration possibilities using technology, specifically the GeoGebra software.

In this work we present the study of the parabola exploring and articulating its algebraic, geometric, and analytical views with GeoGebra support, from the discussion of its particularities and the link between topics of mathematics that approach it. In this sense, we aimed to identify different levels of construction of mathematical reasoning in students of initial formation, during the resolution of a didactic situation about the parabola with the support of GeoGebra. This article originated from an investigation initiated and completed in a master's course in Brazil.

For this, Didactic Engineering (DE) was adopted as an investigation methodology (Artigue, 2020). As a guiding teaching theory, we used the Theory of Didactic Situations (TDS) (Brousseau, 2008), given the compatibility between DE and TDS and the fact that both are theories of French-speaking origin.

The experiment outlined in this work brings the development of a teaching session developed with eight students of the Licentiate in Mathematics, between the 6th and 9th semesters of a Brazilian public university, in face-to-face mode. Data were collected in the form of photographs, audio and video recordings, construction records in GeoGebra and students' written productions.

THEORY OF DIDACTIC SITUATIONS

The Theory of Didactic Situations (TDS) brings a theoretical model that aims to understand the dialectical relationship established between the teacher, the student and knowledge, as well as the environment (*milieu*) in which the conjuncture of a specific didactic situation develops. Thus, the TDS aims to encourage the student to behave like a researcher, where, from a set of dialectics, he can develop and be able to formulate hypotheses and concepts, while the professor provides favorable situations so that this student, when acting, transforms the information into knowledge for himself.

The conception, organization, and planning of a didactic situation by itself requires steps in which the student is alone facing the problem and seeks to solve it without the direct intervention of the teacher. This situation is called by the author as *adidactic situation* (Brousseau, 2008), in which the student, when interacting with the proposed problem-situation, manages to solve it, without any help or direct response given by the teacher. It is worth emphasizing that the didactic situations are designed so that they coexist with the didactic situations, characterizing and obeying a didactic process predetermined by objectives, methods, resources, and concepts.

The TDS organizes the student's learning process from dialectics, which are action, formulation, validation, and institutionalization, the first three being considered adidactic situations. Institutionalization, on the other hand, is shown to be an integral part of the transformation of knowledge – simple familiarity, but not intimacy with the object of study – into knowing – intellectual, which admits concepts and judgments about it –, through the devolution process, which occurs throughout the didactic situation (Margolinas, 2015).

However, we emphasize that for the development of this work we are interested in the path of mathematical reasoning in the development of the TDS dialectics, discussed in the following section.

LEVELS OF REASONING IN THE COURSE OF TDS

According to Brousseau (2008), in the scope of teaching, we must consider the heuristic component intrinsic to intuition. The author suggests that the demonstration regarding mathematical algorithms can be performed by intuitions that will play a small role in these algorithms. Thus, he reinforces that "these intuitions can be rationalized locally, when the implementation of an already constituted theory will provide the intended demonstration or part of it" (p. 102). In this way, the choice of theories or structures, which are guided by heuristics, can, a posteriori, evoke an intuition to justify the approach followed.

In parallel, Brousseau and Gibel (2005) discuss certain notions regarding the nature of mathematical reasoning, considering the intuition articulated to such notions for the construction of such reasoning in the development of TDS. To build a model of mathematical reasoning from the notion of fundamental situation, it is necessary to understand that reasoning concerns a domain that is not restricted to formal, logical, or mathematical structures, despite being constituted by an ordered set of statements linked, combined, or opposed to each other, respecting certain restrictions that can be made explicit in the solution of a problem.

On several occasions, the teacher directs his interpretation regarding the students' assertions, seeking to adapt them in a convenient and induced way to the subject addressed in the class, rather than according to the student's initial intentions. Thus, inadequate models, created by the student, are often interpreted by the teacher as an inability to reason (Brousseau, 1997; 2008; Brousseau & Gibel, 2005). However, we must consider that students sometimes use different representations or knowledge than what we intend to teach them, which may be the result of children's logic, natural thinking.

According to Brousseau (1997), reasoning can be characterized by the role it plays in a situation, that is, by its function in that situation. Thus, such a function can be to decide about something, to inform, to convince or to explain. From this perspective, the function of reasoning varies according to the type of situation in which it occurs, having a direct relationship with the dialectical movement within the TDS, that is, whether it is a situation of action, formulation, or validation. Thus, Brousseau and Gibel (2005) seek to distinguish the levels of mathematical reasoning, considered more or less degenerate, and that adapt to different types of situations in TDS, as summarized below:

- *1st level reasoning (L1)*: characterized by a type of reasoning that is not formulated as such, however it can be attributed to the subject based on his actions, constructed as a model of that action. It is considered as an implicit model relative to the action situation.
- *2nd level reasoning (L2)*: it is considered as an incomplete reasoning from the formal point of view, but with gaps that can be, implicitly, filled by the student's action in a situation where a complete formulation would not be justified. This type of reasoning appears in situations where communication is necessary, relating to the formulation phase.
- *3rd level reasoning (L3)*: defined as a formal and concluded reasoning, based on a set of correctly related inferences, which make a clear mention of the elements of the situation or knowledge considered as shared by the class, even if it is not yet postulated that such reasoning is absolutely correct. Reasoning at this level is characteristic of situations of validation.

We seek to relate the levels of reasoning presented to the actions that are expected to be carried out by the student, in the course of the TDS dialectics, in a learning situation, as shown in Figure 1:

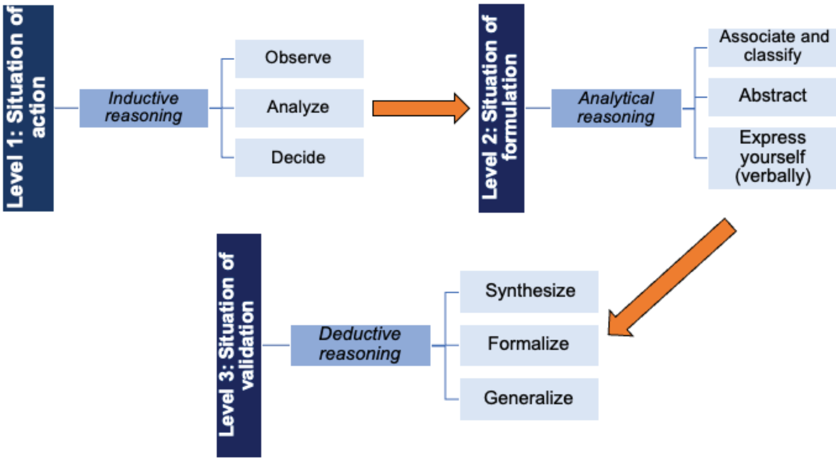


Figure 1. *Levels and types of reasoning within the TDS and actions expected by the student*

The problem presented to the student demands solutions or proofs whose validation can be given independently of the didactic circumstances in which the problem was introduced. The standard solution, that is, a solution that could be produced by the teacher and that is expected of the student, has the form of a sequence of inferences (and calculations), which is correctly connected, that is, according to rules of logic. Thus, we can consider that each stage of reasoning is incorporated into logical and mathematical justifications considered standard, in which their validity and relevance seem to be autonomous. In Brousseau and Gibel's (2005) proposal, the interpretation of students' solutions must consider a larger and more complex system, if it is the teacher's intention to challenge them, instigate them or even explain why such forms of reasoning, correct or not, were produced. In this way, it is recommended that the teacher consider the student's prior knowledge to build his reasoning in an objective situation.

The relationship between levels of mathematical reasoning in TDS was validated in recent research such as Gibel (2015; 2018; 2020) and De Sousa *et al.* (2022; 2023).

METHODOLOGY: DIDACTIC ENGINEERING

According to Artigue (2020), Didactic Engineering (DE) is characterized by an experimental scheme based on didactic achievements within the classroom, that is, in the design, realization, observation and analysis of teaching sessions. In addition, DE can also be considered an experimental research methodology, due to the record in which it is located and its validation mode: the comparison between *a priori* and *a posteriori* analysis.

The planning and execution of a DE can be structured in four stages, which are: i) *Preliminary analysis*, ii) *Conception and a priori analysis of didactic situations*, iii) *Experimentation* and iv) *A posteriori analysis and validation*. In the next sections we describe the four DE stages developed in this research.

Furthermore, in this work we have a *microengineering*, because we seek to observe and improve a DE directed to the teaching of parabolas, focused on the development of the teacher in initial formation, within the scope of the classroom. In addition, the empirical record of the investigation carried out based on this DE provides data for *internal validation*, based on the confrontation between *a priori* analysis, which brings with it the subsidy of a theoretical framework, and *a posteriori* analysis, through a bias that is anchored in the practical dimension.

This research was approved by the Ethics and Research Committee in Brazil, under the Consolidated Opinion Number: 5,267,816.

The first two phases of the DE consist of a theoretical overview and the last two phases focus on the implementation of the experiment and its analysis, respectively.

The data collection instruments were photographs, written records, audio recordings of the meetings and recordings of the computer screen with manipulation in the GeoGebra environment.

Data analysis was carried out based on the theoretical contribution of TDS and the theoretical assumptions of DE as a research methodology.

STAGE 1: PRELIMINARY ANALYSIS

In this stage of the DE, we carried out an epistemological and didactic study on the parabola, investigating how its approach occurs in Basic and its effects on Higher Education. We seek to understand how the transition between these two stages of teaching occurs and the gaps that permeate the learning of this topic.

The parabolas are part of the student's daily life: the trajectory of kicking a ball, launching a projectile, satellite dishes and car headlights, as well as constructions in the field of Architecture and Engineering. Its analytical definition, according to Lima (2014, p. 115) says that "let d be a line and F be a point outside it. In the plane determined by d and F, the set of points equidistant from d and F is called a parabola with focus F and directrix d". This definition can be represented by Figure 2:

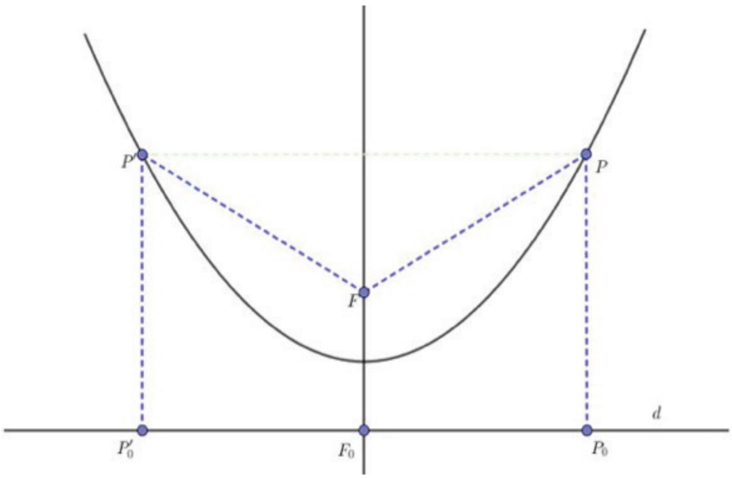


Figure 2. Analytical definition of the parabola.

According to Lima (2014), point P belongs to the parabola with focus F and directrix , since the distance from point to is the same distance between point P and P_0. Then, $d\ (P, F) = d(P, P_0)$, with the segment PP_0 perpendicular to the directrix and the perpendicular FF_0 drawn from the focus on the directrix is configured on an axis of symmetry.

Another definition, from a geometric point of view, considers the parabola as a conic section, forming a curve obtained through the intersection of a cone and a plane that does not pass through its vertex and is parallel to its generatrix (Lima, 2014).

In the context of 2nd degree polynomial functions, within the Brazilian Basic Education curriculum, the definition of parabola is presented in textbooks as the graph of the quadratic function $f(x) = ax^2 + bx + c$, with $a \neq 0$, in which its opening (concavity) can face upwards or downwards (Leonardo, 2016). With regard to this topic of mathematics, the parabola is commonly presented considering only the sign of the coefficient and the discriminant (Δ).

However, textbooks rarely mention why there are these different positions for the parabola in this graphic model, or even the difference between a parabola and a catenary. The catenary has a certain visual similarity with the parabola but given the complexity of its understanding and the components of its equation, this is treated only in higher education (Barbosa, 2013).

Seen as a function of the hyperbolic cosine, the catenary can be defined by a curve generated from a flexible cable, of constant density, hung between two extremes, under the action of its own weight (gravity), where its minimum point is $(0, a)$ with $a > 0$, with equation equal to:

$$y = a.\cosh\left(\frac{x}{a}\right)$$

Possibly this apparent confusion can occur geometrically since the algebraic expressions of the two objects are different. However, we emphasize that this would be an important fact to be pointed out, as it is not just a mere mistake to confuse these two curves, but something that can compromise entire architectural structures. We delineate such similarity in GeoGebra, in a more evident way, as shown in Figure 3:

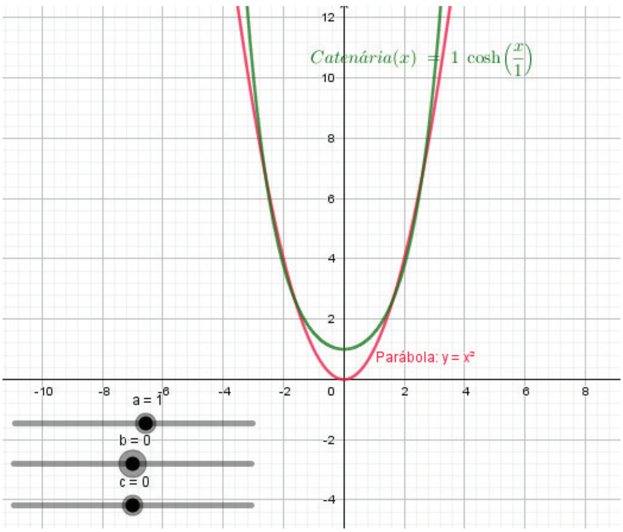


Figure 3. Comparison between parabola and catenary curves.

Note in Figure 3 that, if we look at both curves together, with the same values for parameters a and b and parameter $c = 0$ (since this is only related to the parabola), we can see that the curves, despite having points in common, are different.

These different ways of presenting the parabola routinely appear in school manuals in a fragmented way. Generally, the relationship between these semiotic representations is not mentioned, or when it is mentioned, it is very abbreviated (Bermúdez & Mesa, 2018). Other research points out that the teachers' methodology brings with it some gaps, with traditional classes and little use of technological resources or practical applications (Feltes & Puhl, 2016; Bohrer & Tinti, 2021), which reverberates in the student's difficulty in dealing with this theme when entering higher education in disciplines such as Analytical Geometry and Differential and Integral Calculus.

We clarify that the mathematical object is the parabola as conic, considered from the perspective of Analytical Geometry. Based on the above, we reinforce the importance of teacher development in the epistemic scope and the search for means for a clear presentation of content, with possibilities for practices, reflecting on student learning. In addition, it is worth mentioning the relevance of addressing this topic in initial training, which according to research, has rarely occurred in Mathematics degrees in Brazil (Siqueira, 2016).

STAGE 2: *A PRIORI* ANALYSIS

In this section, we structure a didactic teaching situation that makes it possible to understand the parabola through an algebraic, analytical, and geometric prism, based on a construction in pencil and paper environments and its transposition to GeoGebra. The software, by allowing the manipulation of its elements by the student, provides an environment in which they can demonstrate their mathematical reasoning.

From the prepared situation, we delimit the possible didactic variables (local), as more specific hypotheses focused on the scope of the classroom. Such hypotheses refer to the attitudinal prediction of the student – and in this case we are referring to the undergraduate student and teacher in initial training – in the face of the proposed situation that, at the end of the course, were essential for the validation of Engineering. As local variables, we consider:

(i) Possible difficulties in the development of didactic situations with GeoGebra (translation from paper to software).

(ii) The student's prior knowledge on the subject is not sufficient for understanding and solving the didactic situation.

(iii) The student does not present a coherent path of reasoning levels in the development of TDS.

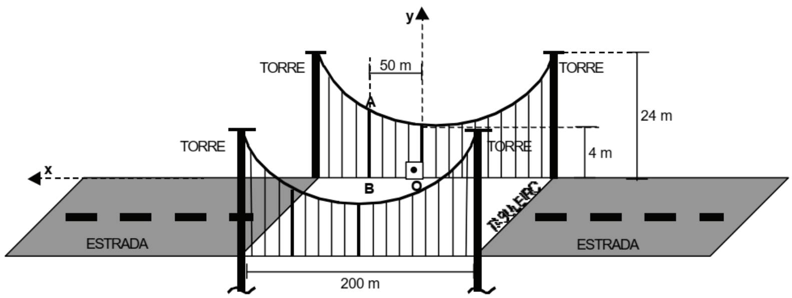
In this step, the didactic situation contextualizes the parabola based on the operating dynamics of a suspension bridge, with the objective of calculating the length of the segment $[BA]$, that connects the cable in a parabolic shape to the base of the bridge.

It is known that, with regard to the Physics application, a suspension bridge is described by the equation of a catenary. However, we emphasize here an error or impropriety in the statement of the exercise (Lima, 2001), when "forcing" the mathematical modeling of the problem, treating the parabola and catenary curves as equal. This fact, still, may not be noticed or mentioned by the participants during the didactic situation, either due to lack of knowledge of the subject, gaps in the initial training or for considering only the elements present in the question, without reflecting on them.

So, it is expected that the student resorts to his previous knowledge about quadratic functions, which can be extended to a discussion about the equation of the parabola with vertex outside the origin to solve the proposed question. The didactic situation is illustrated in Table 1:

Table 1. Proposed didactic situation.

The cables of the suspension bridge, shown in the figure, take the form of second-degree parabola arches. The support towers are 24 m high and there is a gap between them of 200 m. The lowest point of each cable is 4 m from the roadbed. Considering the horizontal plane of the bridge deck containing the $0x$ axis and the axis of symmetry of the parabola to be the $0y$ axis, perpendicular to x, determine the length of the support element BA, which vertically connects the parabolic cable to the bridge deck, located at 50 m from $0y$ axis. Schematize your resolution using GeoGebra.



Source: Adapted from Giovanni and Bonjorno (2005, p. 131).

In the *situation of action*, we expect the student to identify the elements provided in the question, associated with the image of the suspension bridge, in order to elaborate their strategies. One possibility is that the student, involved in *1st level reasoning* mechanisms, observes, and analyses the presented scheme, recognizing elements such as the origin of the coordinate system, the vertex of the parabola at the point $V(0,4)$, or even that the point $(100,24)$ belongs to the described curve.

In the *situation of formulation*, the student is expected to draw up strategies for the solution using knowledge about the quadratic function or the parabola equation itself. However, it is likely that they will opt for the first option, as it is an approach used with greater recurrence. That is, the conjectures and sketches possibly elaborated in this stage, based on what is expected from *2nd level reasoning*, must relate the concept of a parabola as a graph of the quadratic function to the figure of the parabola presented in the scheme. Thus, a possible development to obtain the solution using the quadratic function would be:

The quadratic function can be written as $y = ax^2 + bx + c$, with $a \neq 0$. As the axis of symmetry of the parabola is the axis of the ordinates itself (Oy axis), we have $b = 0$.

According to the proposed scheme, we have that the point $V(0,4)$ is the vertex of the parabola. Replacing in $y = ax^2 + bx + c$, we have $4 = a.0^2 + b.0 + c$, getting $c = 4$. It is also possible to infer that the parabola passes through the point $P(100,24)$. In this way, replacing all the information in $y = ax^2 + bx + c$, we have:

$$24 = a \cdot 100^2 + 0.100 + 4$$
$$24 - 4 = 10000$$
$$a = \frac{20}{10000} \text{ ou } a = \frac{1}{500}$$

Thus, the equation representing the parabolic cable of the bridge is $y = \left(\frac{1}{500}\right)x^2 + 4$. For $x = 50$, we have $y = \frac{1}{500} \cdot 50^2 + 4 = 9m$.

In the *situation of validation*, involved in a *3rd level reasoning*, the student is expected to present his conjectures and strategies that lead to the solution of the question, using formal mathematical language. It is a moment of argument and debate of ideas, given the fact that for the same mathematical question there is more than one way to the solution.

Based on perception, visualization and possible conjectures expressed, students can use the whiteboard to expose their ideas and GeoGebra as a resource to prove the solution.

In the *situation of institutionalization*, the teacher-researcher must resume the didactic situation, summarizing everything that was pointed out by the students. On this occasion, the aim is to discuss the mathematical concept addressed in the question and to present the use of the equation of the parabola with vertex outside the origin as an alternative solution, in addition to comparing the constructions carried out on the board or pencil/paper and in the GeoGebra environment.

STAGE 3: EXPERIMENTATION

Initially, the didactic contract was reinforced by the teacher-researcher, in which the students were instructed to seek information within the proposed situation, try to solve it in the pencil and paper environment and, after that, transpose their solution to GeoGebra, presenting it and discussing it with others. In order to preserve the students' identities, we refer to them as P1, P2, ..., P8.

In the *situation of action* all students interpreted the statement of the question, observing the information presented from the suspension bridge graph. In an attempt to organize the available elements for a later formulation, some of them were scribbling some sketches.

Almost unanimously, the *inductive reasoning* (L1) presented by the students revolved around the use of the quadratic function to solve the problem. As predicted in the *a priori* analysis, due to the fact that the parabola is recurrently studied in the school/academic path through the perspective of quadratic functions, this model was replicated, intuitively, and perhaps even naturally. However, some of them used different initial strategies during the *situation of formulation*, as the case of P5 and P2, reported in the subsequent paragraphs. For example, participant P5 started his elaboration looking for a solution through Plane Geometry, as shown in Figure 4:

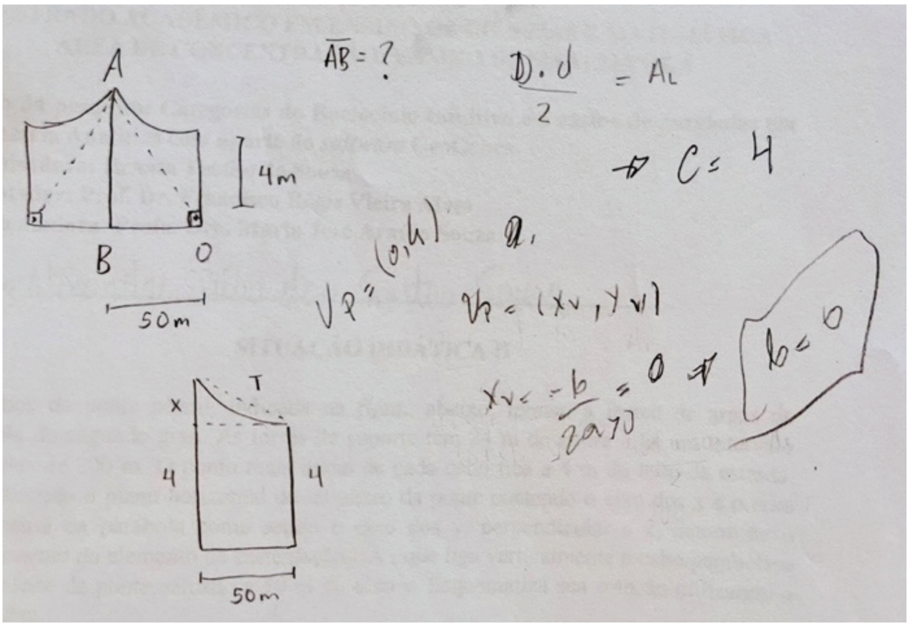


Figure 4. Participant P5's attempted solution.

At first, P5, when conjecturing a solution strategy, thought it possible to calculate the segment $[BA]$ using triangles (Figure 4). However, the analytical reasoning model (L2), elaborated and expressed in this resolution, was not the most appropriate. In the audio recording, when talking with the researcher, P5 stated:

- *I'm trying to create a triangle here and see if I can find the sine and cosine, to see if I can find the measures of the sides of these triangles and it helps me with something (P5)*
- *Are you trying to use plane geometry in the solution? (researcher)*
- *Yes, I'll see if it works. [...] I wanted to remember the diamond formula... (P5)*
- *From the area? (researcher)*
- *Yes (P5).*

After several attempts and when concluding that he would not reach the solution, P5 resorted to the same strategy as participants P4, P6 and P7, adopting the quadratic function as a path, finishing the route, and going through the three levels of reasoning in the pencil and paper environment, shown further forward.

Participant P2, on the other hand, was unable to sketch the solution, neither in the pencil and paper environment, nor in GeoGebra. We observe scribbles of partially elaborated ideas, mechanisms of a *1st level reasoning (L1)*, with conjectures that have not been established enough to become global analytical solutions. If we look at P2's drafts (Figure 5), we can see that he was unable to develop a complete line of reasoning to complete the solution to the problem:

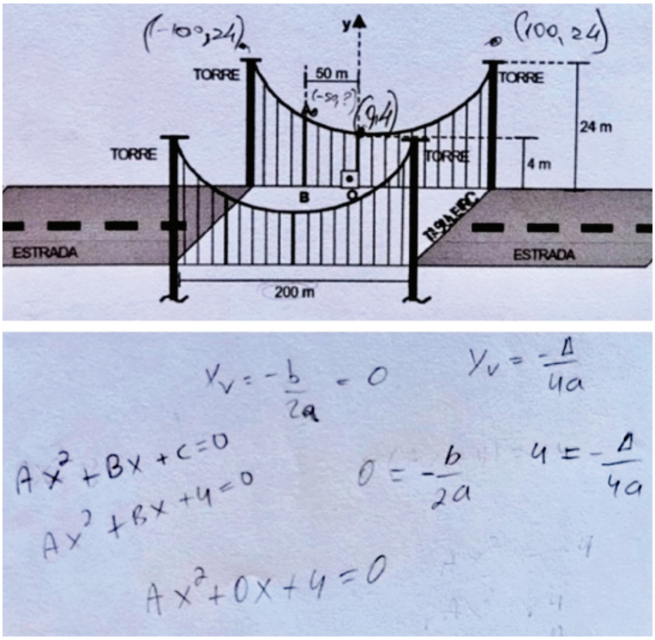


Figure 5. Participant P2's sketch.

In the two parts of Figure 5 we have markings on the drawing, indicating the points and some elements extracted from the graph. "As the amount of new knowledge to be learned increases, it becomes increasingly difficult to keep in mind the increasing number of independent circumstantial connections, and there is considerable risk of confusion" (Brousseau & Gibel, 2005, p. 23). We can consider P2's reasoning manifestations as something broad, non-linear, and dependent on many variables, including well-established prior knowledge, which in this case was not demonstrated.

On the other hand, the formulations of participants P1, P3, P4, P6, P7 and P8 presented a route to the solution concluded from the perspective of quadratic functions, as predicted. However, regarding the use of strategy to arrive at the solution, there are similarities and differences in the path. P1 and P8, in the search for coefficients a, b and c of the function, opted for the use of linear systems (Figure 6):

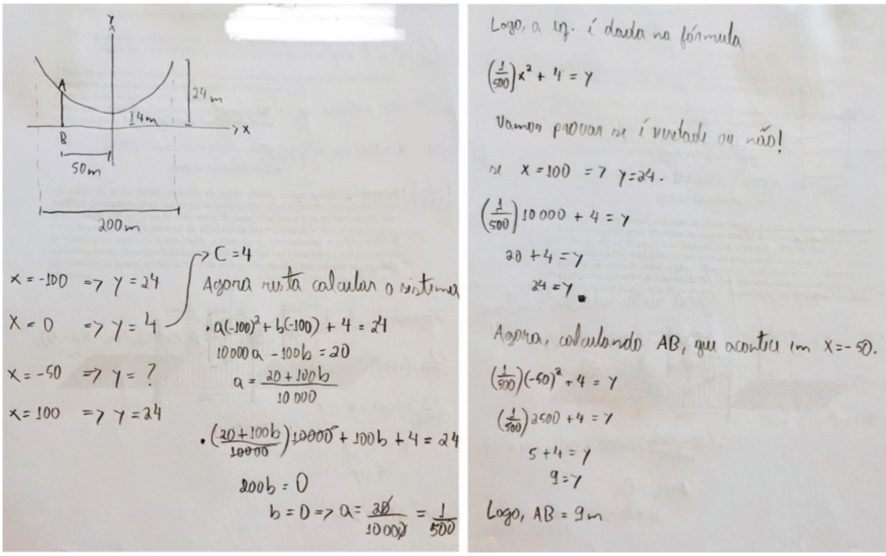


Figure 6. Analytical solution of P1

As the students exchanged ideas over the course of the *situation of formulation*, their written records bear similarities. Note that they used the points given in the problem, replacing them in the structure of a quadratic function $y = ax^2 + bx + c$, concluding that $a = \frac{1}{500}$, $b = 0$ and $c = 4$, and finding $y = \frac{1}{500}x^2 + 4$.

In the case of P4, P6 and P7 solutions, we observe slightly different discussions and elaborations. In the *situation of formulation*, they manifested what we consider to be a *2nd level reasoning*, given the fact that the conjectures elaborated were verbalized from a joint discussion, as shown in the audio recording:

- *I tried to get the data here from the graph and build a 2nd degree function (P7)*
- *We already have the vertex and it's clear that b is zero, because if the x_v is zero and x_v is, $-b/2a$ and we are seeing here that the $a > 0$, so b can only be zero, for the result of division to be zero (P6).*
- *Now just take the points and do the equation, right? (P4).*
- *So... my b was zero, and yours? (P4 asking P6 and P7).*
- *It's really zero, I didn't even need a calculation to see it (P6).*
- *The c I know is 4 because here is the intersection of the graph with the y, so this point here is (0, c) (P7).*

- *I thought the height would be 12, because here, as it is in the middle and the parts seem to increase in the same proportion, I thought it would be 12... but of course I would have to make a geometric shape to prove it (P4).*
- *Didn't you try to equate? (P6).*
- *I'm going to try now... as here c is 4, I'm going to try to find b and a.. as the parabola is facing upwards, I know that the $a > 0$ (P4).*
- *The intersection of the parabola here is 4, so what did I do, I did $ax^2 + 4 = 24$ and when the x is 100, the y is 24, so I put it here, look ..., and gave that my coefficient $a = 1/500$ (P7).*
- *That's it, (participant P7)! Don't you already have the general equation? Now you just take the point here and replace... (P6).*

It should be noted that these students sought to extract as much information as possible based on the problem data, structuring an *analytical reasoning (L2)* and starting with a *deductive reasoning (L3)*. According to the perspective of Brousseau and Gibel (2005), we understand that there is a need for subjects to prove that the reasoning elaborated was intentional, purposeful, and useful, with regard to their mathematical knowledge. Still regarding the formulation situation, one participant used the parabola equation in Analytical Geometry to solve the question, which was not foreseen in the *a priori* analysis. P3 used the presented model (Figure 7):

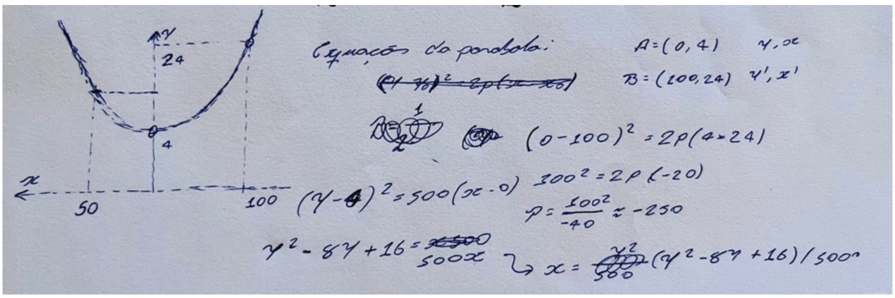


Figure 7. Participant P3's Formulation

The participants' solution proposals were presented and discussed in the *situation of validation* using the whiteboard and GeoGebra. As the discussions took place in small groups, the solutions showed similarities in general, with the exception of P2 and P3. The participants P1 and P7 presented the solution on the board to the others.

About the observations of the levels of reasoning and validation of the didactic situation, we describe what was observed in the construction files in GeoGebra and in the video records of the computer screen.

Participant P1 started his construction by typing the function $f(x) = 1/500x^2 + 4$, previously found in the pencil and paper environment, in the GeoGebra Input field. Based on *2nd level reasoning (L2)*, P1 drew a segment connecting the point $(\text{-}50, y)$, observing the approximate value of y, first finding the value $8{,}99$ (Figure 8):

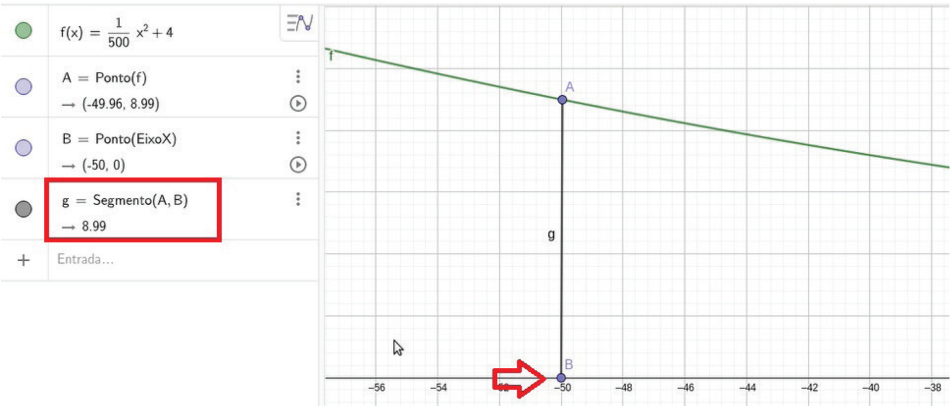


Figure 8. P1's validation.

However, when observing that the value approached $y = 9$, P1 enlarged the graph using the zoom tool and noticed that the created segment was not exactly parallel to the axis Oy, what caused such an approximation. In this way, through inductive reasoning (L1), and understanding that the ends of the segment $[BA]$ were not fixed, manipulated the point A to the position $y = 9$, finding the value of the segment $[BA]$ and validating what had previously been conjectured in written form.

Participants P4 and P6 initially built the function $f(x) = 1/500x^2 + x + 4$ in GeoGebra, but they were unsuccessful. Returned to the pencil and paper environment to calculate the correct value of b and returned to GeoGebra to enter the correct function. Based on *2nd level reasoning (L2)*, they created the points $(-50{,}0)$ and $(-50{,}9)$, given the fact that these values had already been previously calculated. After that, a little different from P1, through *deductive* *reasoning (L3)*, P4 and P6 used the "distance, length or perimeter" tool, finding the value $\overline{AB} = 9$ (Figure 9):

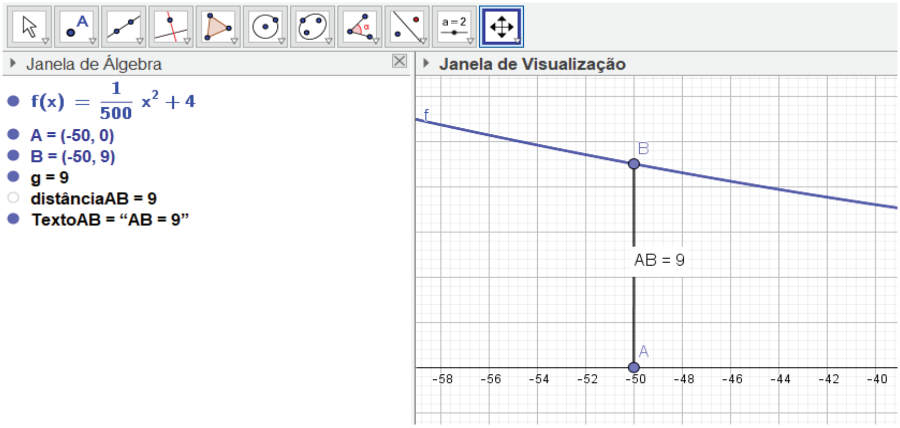


Figure 9. P4's validation.

The beginning of the situation of validation for both P5 and P8 is similar to that of P1. They started by building the function $f(x) = 1/500x^2 + 4$ and a segment with $x = -50$ and y as a point on the parabola that has an image that satisfies the function (Figure 10):

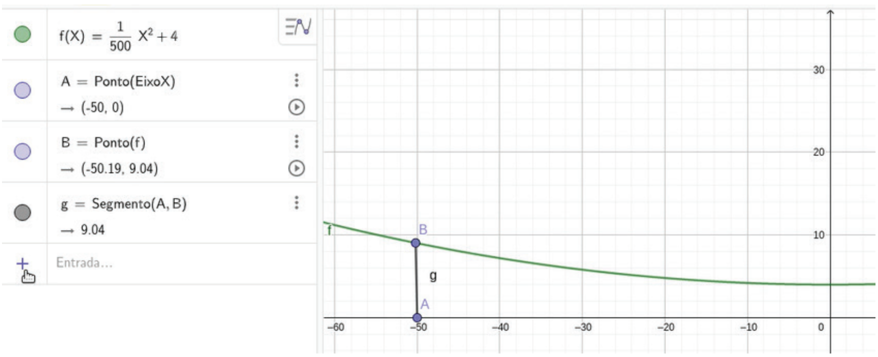


Figure 10. P5's validation (similar to P8).

Note that approximate values appear $(-50{,}19;\,9{,}04)$. Then, using *2nd level reasoning (L2)*, P8 calculated the value of $f(-50)$ and got the value 9. After confirming his conjecture, P8 configured point B, inserting the coordinates $(-50{,}9)$. The difference between the solutions of P5 and P8 is that P8 did not calculate, as well as P5 and the others, considering the positive coordinates, that is, the x in the first quadrant. We interpret that this may have been done intuitively, due to an influence of the visual model presented, which configures a *1st level reasoning (L1)*.

P7 started his construction by inserting the equation $1/500x^2 + 4 = 0$ in the input field but noticing that GeoGebra did not provide the expected graph, inserted the function $f(x) = 1/500x^2 + 4$. After a pause, P7 calculated $f(50)$, in which two symmetrical lines parallel to the Oy axis $(eq1)$, that could represent the bridge towers (Figure 11):

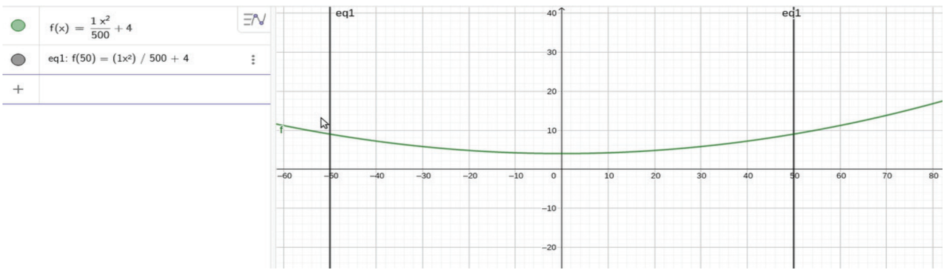


Figure 11. P7's validation.

P7 used the "intersection between two objects" tool, finding the coordinates of the intersection point of the line $eq1$ with the parabola, obtaining $(-50{,}9)$. Tracing the segment from A to B, he obtained $\overline{BA} = 9$.

Unlike his colleagues, P2 started the solution of the didactic situation in GeoGebra, without initially using notes on paper. As relevant information was revealed during construction, he developed the supposed solution, which allowed him to further explore the software, getting to know tools and resources. There were several attempts and errors, but throughout P2 he tried to use the Analytical Geometry view to solve the situation, which caught our attention. One of his (unsuccessful) attempts can be seen in Figure 12:

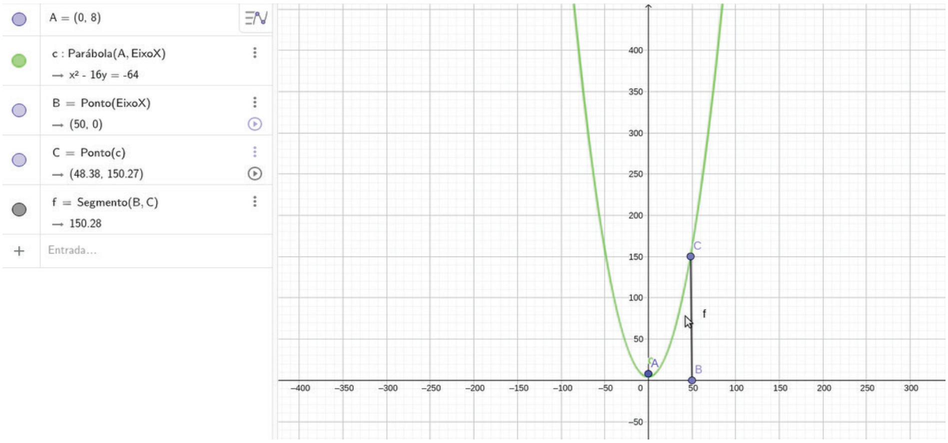


Figure 12. P2's validation.

In Figure 12, P2 has created a parabola with $A(0,8)$ as vertex and tried an approximation of the support element $[BA]$, using the "segment" tool, considering $x = 50$ and y the ordinate of a point belonging to the parabola. As this did not work, he created a line perpendicular to the Oy axis, passing through $x = 50$ and marked its intersection with the parabola. It should be noted that its path to validation, despite the similarity with the others, did not present the expected result. Such a solution was inadequate, as the equation did not satisfy the conditions of the question. In this case, P2 disregarded the value of parameter p.

When listening to the dialogues of colleagues in the moments of formulation, he realized that something was inconsistent and called the researcher to exchange ideas. He didn't understand what his mistake was. He was instigated by the researcher to describe its construction. She asked P2 to test the question data in her build and see what would happen. So, he understood that if the equation was really that, the solution would be right; otherwise, he should look for the correct equation and correct the solution. After a pause, P2 created the points $(-100,24)$, $(100,24)$ and $(0,4)$ and tried to draw a parabola passing through these points, considering $(0,4)$ as its vertex and using the "parabola" tool, but also without success. Another failed attempt occurred when P2 chose the "semicircle" tool and the points $(-100,24)$ and $(100,24)$, because it was never possible for the point $(0,4)$ to belong to the curve.

It is important to understand the reasoning that does not reach the solution of problems and, especially, the reasons why these solutions are not reached.

Participant P2, despite several attempts, did not complete the didactic situation and did not present an appropriate solution. We noticed that he relied only on the points given in the question but did not verify the feasibility of the value of the ***p*** parameter and the focus, having difficulties in concluding the reasoning.

Participant P3 elaborated a partially correct solution, using Analytical Geometry, but presented only a written record. He did not record the construction on video in GeoGebra and .ggb file, not complying with the provisions of the didactic contract. However, he stated that the value of $\overline{BA}$ was 9 and said that "with the equation you can calculate in your head". We understand that P3 elaborated a mental scheme based on inductive reasoning (L1), structured by him through the elements available in the problem. For Alves (2012; 2016), this type of reasoning can be manifested based on *perception* or *insight*, as an inciting form of thinking and directly related to the intuitive field.

In the *situation of institutionalization*, the researcher actually presented Lima's (2014) definition as planned, mentioning the equation of the parabola with vertex outside the origin and axis of symmetry parallel to one of the coordinate axes. The researcher pointed out that, although the curve is physically a catenary, the mathematical modeling of the problem refers to the parabola. Therefore, the proposed solution was presented according to the work from which the situation was extracted and the definitions discussed in the preliminary analysis, pointing out the parabola from the perspective of Analytical Geometry.

After institutionalization, the participants were able to show associations between the parabola equation from the point of view of Analytical Geometry and the quadratic function. At this time, they confirmed the gap of this study in the course of its formation and the fact of not knowing the catenary curve, corroborating in some way what was pointed out in our preliminary analysis and conjectured in the hypotheses and didactic variables mentioned in the a priori analysis.

A POSTERIORI ANALYSIS AND VALIDATION

In the action situation, the participants read and extracted information from the utterance, verbalizing that it was a question that involved quadratic functions, being characteristic of 1st level reasoning, generated from the perception of visual elements. This fact may be due to what Fischbein (1987) calls *status of theory*, which consists of an intuition generated from the representation of a problem situation by means of a scheme or model. During this stage, students sketched representative drawings of the bridge, suggesting it as the graph of a quadratic function, which we understand to be the representation of inductive reasoning (L1), based on their prior knowledge.

In the formulation situation, in fact, we expected the development of the situation using both the quadratic functions and the parabola equation but considering that the use of functions would be more likely. We also foresee the manifestation of *1st level reasoning* from the direct association of the geometric shape of the parabola to a quadratic function, as well as *2nd level reasoning* during the elaboration and discovery of terms to schematize its equation.

The second level reasoning (L2) and its analytical character were remarkable when involving the use of Plane Geometry for the solution (participant P5). In addition, one of the participants (P3) outlined a strategy using the parabola equation. Both facts were not foreseen in our a priori analysis. Another pertinent situation was the case of participant P2, who was unable to solve the issue in any of the environments (pencil and paper/GeoGebra), due to a lack of prior knowledge and lack of connection or alignment between his ideas, as shown in the data records collected. The other participants elaborated their solutions using quadratic functions, which were built from linear systems (P1 and P8) or just the law of formation of this type of function (P4, P6 and P7).

It is important to highlight the observations recorded in audio in the dialogue between P4, P6 and P7, which triggered a sequence of thoughts that, from inductive reasoning (L1) and analytical (L2), generated a path to deductive reasoning (L3). As these participants discussed and extracted as much information as possible from the problem, they tried to solve the necessary parts for the final solution in an objective way, reserving the calculations only for steps considered more complex.

In the *situation of validation*, we wanted students to present their solutions both on paper and in GeoGebra, also using the whiteboard to expose their ideas. At this stage, participants P1 and P7 presented their solutions using the quadratic function and the discussion was fruitful among those present. During the meeting and this presentation, the articulation of the ideas proposed by P1 and P7 permeated the three levels of reasoning proposed in Brousseau and Gibel (2005).

However, we emphasize that P2 was unable to validate his solution, since he disregarded the value of the parameter in the elaboration of the equation, which mischaracterized his solution as correct for the problem. P3, on the other hand, used Analytical Geometry, but only presented a partial solution, claiming the possibility of "solving the calculations in his head", also breaching the established didactic contract.

With regard to the *situation of institutionalization*, progress occurred as expected. The researcher presented Lima's (2014) definition, which deals with the parabola equation with vertex outside the origin. In addition, it provided an alternative solution from the perspective of Analytical Geometry using the framework and a construction in GeoGebra. The presented institutionalization generated a discussion among the participants, in which they pointed out associations between the quadratic function, commenting that this is widely studied throughout Basic Education, and the parabola equation, which is rarely mentioned in parallel. In addition, they reinforced the gap in this discussion during the undergraduate course.

CONCLUSIONS

The main focus of this work was to understand how future Mathematics teachers see the topic of parabolas within Analytical Geometry and in related disciplines during their training. Thus, we carried out a survey that starts from the connection between points such as the relevance of teaching the parabola in Analytical Geometry in initial training, the importance of the articulation between Algebra and Geometry and the use of software or applications for the didactic transposition of this topic.

In the experimentation, the vision of the parabola through an algebraic prism was remarkable, linked to the topic of quadratic functions, recurrently explored in Brazilian Basic Education. We noticed that the students knew the software in an elementary way. However, the didactic situation allowed them to explore some tools, as well as trigger geometric and algebraic perceptions through the manipulation and visualization of elements within the constructions.

With the contribution of TDS, GeoGebra and the entire structure elaborated in the *a priori* analysis, we were able to capture moments in which the manifestations of different levels of reasoning occurred, in addition to providing a fertile discussion environment. In the course of experimentation, we noticed the presence of three levels of reasoning, with levels L1 and L2 being the most recurrent. The TDS was fundamental in structuring the didactic sessions, with regard to interpreting these levels of reasoning associated with the main actions of the students, recognizing them in their dialectics.

We hope that the results presented can be a contribution to the field of Mathematics Education, given the importance of teaching this topic associated with reality and relating its concepts correctly. We also hope that the research activity carried out can be replicated and/or adapted to other contexts, as complementary material to the teaching methodology, considering it as a possibility of working on this subject using technology.

As a future perspective, it is possible to expand this research by exploring other conics, in search of improving ways of working on Analytical Geometry associated with GeoGebra through an intuitive prism, contributing to teacher training.

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