RHV, 2024, No 25, 129-148 eISSN 0719-4242
DOI: https://doi.org/10.22370/rhv2024iss25pp129-148
Sección Monográfica / Monographic Section

The Arbitrariness of Symmetry in Mathematical Proofs

La arbitrariedad de la simetría en las demostraciones matemáticas

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Abstract

Symmetry is not an inherent characteristic of mathematical proofs; instead, it is a property that arbitrarily manifests in different modes of presentation. This arbitrariness leads to the conclusion that symmetry cannot be part of the defining or essential properties that characterize proofs. Consequently, contrary to some authors' claims, symmetry does not significantly contribute to the validity, accuracy, or soundness of mathematical proofs. What is more, it does not even play any critical role in heuristic aspects such as explanatory power. The examples developed in this paper constitute compelling evidence supporting these claims.

Keywords: philosophy of mathematics, mathematical practice, explanation in mathematics, mathematical proof.

Resumen

La simetría no es una característica inherente a las demostraciones matemáticas; en cambio, es una propiedad que se manifiesta arbitrariamente en sus diferentes modos de presentación. Esta arbitrariedad lleva a la conclusión de que la simetría no puede ser parte de las propiedades definitorias o esenciales que caracterizan las demostraciones. Por consiguiente, en contra de las afirmaciones de algunos autores, la simetría no contribuye significativamente a la validez,

Received: 03/01/2023. Final version: 08/01/2024

exactitud o solidez de las demostraciones matemáticas. Incluso más, ni siquiera juega ningún papel crítico en aspectos heurísticos como el poder explicativo. Los ejemplos desarrollados en este trabajo constituyen evidencia convincente que respalda estas afirmaciones.

Palabras clave: filosofía de las matemáticas, práctica matemática, explicación matemática, demostración matemática.

1. Introduction

The conception of mathematical proof has undergone a significant evolution from its formal inception with the ancient Greeks to the present day, reflecting broader shifts in epistemological frameworks and mathematical rigor. Initially, the ancient Greeks, particularly through the work of Euclid, established the foundation of deductive reasoning in mathematics, emphasizing axioms, propositions, and the necessity of logical coherence in proofs. Later, the 19th century witnessed a critical transformation in the understanding of proof, driven by the emergence of non-Euclidean geometries and the formalization of mathematical analysis, which led to a more rigorous scrutiny of foundations and the development of formal proof systems. In contemporary times, the advent of computer-assisted proofs and the exploration of proof in mathematical logic and category theory continue to challenge and refine our understanding of what constitutes a mathematical proof.

The exploration of the possibility of a universal and apprehensible notion of mathematical proof has been addressed by Russell, Gödel, Lakatos, and other of the most influential philosophers of mathematics. Yet, the question remains elusive due to the complex interplay of logical rigor, intuitive understanding, and mathematical practice. Attempts to characterize a mathematical proof have historically oscillated between the Hilbertian formalistic rigor and conceptual intuition, reflecting broader philosophical tensions between the objective structure of mathematical reality and the subjective processes of mathematical discovery (as illustrated by Brouwer's philosophical work) and justification, which motivated Frege's program.

Building on the diverse insights yielded by foundational projects, contemporary philosophy of mathematics approaches the concept of mathematical proof as a more multifaceted construct, integrating formal logical structures, intuitive understanding, and social validation within the mathematical community. This pluralistic perspective recognizes that mathematical proofs transcend mere formal artifacts, being intricately woven into the fabric of mathematical practices, conventions, and epistemic values. Such a view renders the pursuit of a singular, well-defined notion of proof both practically and philosophically ambitious, challenging the feasibility of maintaining a uniform definition. Despite this, many philosophers and mathematicians persist, to varying degrees, in the belief that properties generalizable across all mathematical proofs exist. In this context, the present paper contributes to the view that the characteristics of mathematical proofs are not readily generalizable, thereby underscoring the implausibility of achieving a moderately satisfactory characterization.

For instance, Mancosu (1996) underscores the importance of clarity, rigor, and logical coherence as fundamental properties inherent to the nature of mathematical proofs, irrespective of their specific mathematical domain or the era in which they were conceived. He articulates how the evolution of mathematical practice has continuously reaffirmed the necessity for proofs to adhere to stringent criteria of deductive reasoning, thereby highlighting the universality of these properties in ensuring the validity and reliability of mathematical arguments. Although this is a historical perspective and takes into account mathematical practice, it still excludes instances of proof that, in principle, we consider legitimate. Consider, for example, a physical proof of the Pythagorean theorem. These proofs are standard in science museums. Are these proofs less legit because instead of convincing professional mathematicians, they exhibit the statements' truth to the general public?

The compulsion to achieve a characterization of the mathematical proof has led to different attempts to generalize properties that contribute to the sense of proofs in mathematics or at least of certain subclasses such as explanatory proofs. In line with this idea, I will focus on the property of symmetry, which has been used by Lange (2017) as a feature that endows mathematical proofs with explanatory power. That is, symmetry would be a characterizing property of explanatory proofs. The central objective in this paper is to show through a variety of examples that symmetry is a property that can vary arbitrarily not only among different mathematical proofs but also among what we would intuitively consider to be different modes of presentation of the same proof. Symmetry is not an intrinsic property of mathematical proofs. It is rather a property that may or may not be in a particular presentation. Certainly, what is at stake here is the mere idea of mathematical proof as an individual object in which specific characteristics can be identified.

2. Symmetry

Lange (2017) articulates the significance of symmetry within the domain of mathematical explanations, positing that the most compelling explanations are those that can account for observed symmetries in the outcomes by harnessing symmetrical features inherent in the initial conditions of the problem. Among the instances he explores is d'Alembert's theorem – an illustration of explanation through subsumption under a theorem – which posits that for any polynomial equation of n^{th} degree with solely real coefficients, non-real roots invariably occur in conjugate pairs. In his book, *Because Without Cause*, Lange critiques algebraic demonstrations that "merely manipulate symbols" to arrive at this conclusion as lacking explanatory depth, failing to unveil the underlying rationale, which he identifies as the invariance of the axioms of complex arithmetic under the interchange of i with $-i$. He characterizes such algebraic proofs as “magical” and devoid of explanatory power. In a previous analysis (Bueno and Vivanco, 2019), we challenge this perspective by offering an alternative elucidation of the same proof, thereby dispelling its perceived magical quality. Further extending his discussion to a geometrical example, Lange underscores similar considerations. The subsequent sections of this paper are devoted to presenting three distinct proofs, scrutinized for their explanatory capacity, particularly through the lens of symmetry. The discussion culminates in an examination of a proof by induction, leveraging Lange’s criterion to argue for its potential explanatory value — a stance that contrasts with Lange’s earlier position, which characterizes induction proofs as inherently non-explanatory (Lange, 2009). This exploration aims to broaden the understanding of what constitutes an explanatory proof, highlighting the contingent role of symmetry in achieving such explanatory depth.

3. Two Crossed Lines Inscribed in a Trapezoid

Even if we admit that, within the realm of algebraic proofs, the manifestation of symmetry is intrinsically contingent upon the interpretation of the given elements, we still can hold that in geometry, symmetry is a property almost axiomatically self-evident. This distinction is explored through an example analyzed by Marc Lange (Lange, 2017, pp. 245–249), wherein he endeavors to substantiate his argument concerning the explanatory power of mathematical proofs as derived from their symmetrical properties. According to Lange, the geometric proposition in question serves as a case study for a mathematical phenomenon that admits of various proofs, among which one is deemed explanatory by virtue of its capacity to mirror the symmetry from the proof onto the theorem itself. In contrast, other proofs are categorized as non-explanatory, owing to their failure to demonstrate the said symmetry. Through this analysis, Lange aims to illustrate the significant role that symmetry plays in distinguishing explanatory proofs within the mathematical discourse, thereby advancing a nuanced understanding of the criteria that underpin the explanatory nature of mathematical demonstrations. In what follows, I will analyze the proofs provided, making it evident that, due to the arbitrariness and relativity of symmetry, the three proofs should be considered explanatory according to Lange’s criterion.

3.1 Theorem

If ABCD is an isosceles trapezoid; the segment $\overline{AB}$ is parallel to the segment of line $\overline{CD}$ and $\overline{AD} = \overline{BC}$, such that $\overline{AM} = \overline{BK}$, and $\overline{ND} = \overline{LC}$, then $\overline{ML} = \overline{KN}$.

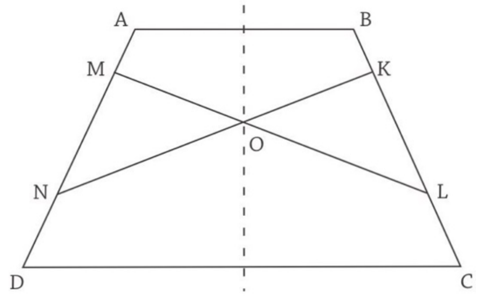

Figure 1. Trapezoid.

The result under consideration exemplifies the quintessential characteristics of early Greek geometry, a mathematical tradition fundamentally anchored in the exploration of lines, points, and their interrelations, typically elucidated through measurements facilitated by the utilization of a ruler and compass. In concluding his examination of this exemplar, Lange furnishes a demonstration that, in his estimation, adequately fulfills the criteria of symmetry he posits as essential for explanatory value. It is within this context that we proceed to a meticulous analysis of those proofs deemed non-explanatory by Lange, aiming to delineate the attributes that distinguish them from their explanatory counterparts in accordance with Lange's theoretical framework.

3.2 The Cartesian Proof

The following demonstration has been characterized by Lange as a *brute- force proof* of 2.2, a designation that merits attention within the context of analytic (Cartesian) geometry, which is seldom invoked for such purposes. The primary utility of this theoretical framework lies in its provision of parameters that facilitate discourse on distances and, by extension, on lengths. Lange's delineation of the brute-force proof is articulated as follows:

> Let *D*'s coordinates be (0, 0), *C*'s be (0, *c*), *A*'s be (*a, s*), and *B*'s be (*b, s*), and then solve algebraically for the two distances *ML* and *KN,* showing that they are equal. [p.245]

In the ensuing discourse, an elaborated rendition of this proof will be presented, with the intention of unveiling the symmetry that underpins the transition from proof to theorem.

Should this endeavor prove successful, it would thereby be demonstrated, in accordance with Lange's criterion, that this instance does not constitute a *non-explanatory proof*. This analysis aims not only to elucidate the potential of finding symmetry in the demonstration but also to engage with Lange's broader theoretical considerations regarding the explanatory capacity of mathematical proofs.

Proof.

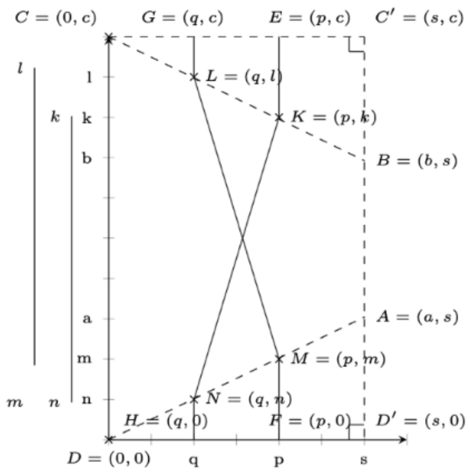

Figure 2. Cartesian proof.

The task at hand necessitates the translation of geometric concepts into a Cartesian framework. Let's use *D* to denote the origin (0*,* 0) and *C* to denote a point of the form (0*, c*), with $c \in \mathbb{R}$, thereby predetermining the plane's region accommodating the trapezoid (given *D* and *C*'s coordinates, the trapezoid's base is aligned along the *y*-axis, initiating from the origin, as depicted in Figure 2). The result displayed in 2.2 has four hypotheses:

1 We have that $\overline{AB}||\overline{CD}$. Given Lange's placement of the trapezoid's base $\overline{CD}$ along the *y*-axis, asserting $\overline{AB}||\overline{CD}$essentially equates to *A* and *B* sharing identical *x*-coordinates (thus, the line through *A* and *B* runs perpendicular to the *x*-axis and parallel to the *y*-axis). Consequently, *A* adopts coordinates (*s,a*) and *B*, (*s,b*), where *s* is any real number, and *a* and *b* are defined by the ensuing hypothesis.

2 We have that $\overline{DA} = \overline{CB}$. The Cartesian plane enables discussions around distances, equating the identity $\overline{DA} = \overline{CB}$ to *d*(*D,A*) = *d*(*C,B*), or $\sqrt{a^2 + b^2} = \sqrt{b^2 + (s-c)^2}$. The symmetry emerges from the trapezoid's inherent identities (e.g., |*a*| = |*c-b*|), a key aspect of the sought-after symmetry between the proof and the result, which will become clearer as the proof progresses.

3 We have that $\overline{AM} = \overline{BK}$. With *M* represented as (*p,m*) such that $|p| \leq |s|$ and $|m| \leq |a|$, and similarly, *K* as (*p,k*) with $|b| \leq |k|$, the equality (symmetry) of segments translates into an equivalence of distances: $\sqrt{(a-m)^2 + s - p^2} = \sqrt{(k-b)^2 + (s-p)^2}$.

4 We have that $\overline{ND} = \overline{LC}$. Analogous to the previous hypothesis, *N* is (*q,n*) where $|q| \leq |s|$ and $|n| \leq |a|$, while *L* is (*q,l*) with $|b| \leq |l|$. This implies *d*(*N,D*) = *d*(*L,C*), or $\sqrt{n^2 + q^2} = \sqrt{(a-m)^2 + (s-p)^2}$.

The objective is to demonstrate that $\overline{ML} = \overline{KN}$. In Cartesian jargon, this is to proof that $d(M,L) = d(N,K)$.

Observing that the trapezoid *ABCD* is inscribed within the right parallelogram *DCD'C'* (where $C' = (s,c)$ and $D' = (s,0)$), whose base is $\overline{DC}$ and height is |s|, and recognizing that *DCD'C'* forms an isosceles shape, it follows that $\triangle ADD'$ and $\triangle BCC'$ are congruent, preserving the corresponding symmetry (verified via the side, angle, side criterion: $\overline{DD'} = \overline{CC'}$; $\measuredangle D'DA = \measuredangle C'CB$; and $\overline{DA} = \overline{CB}$). Additional observations further highlight the symmetry integral to the proof and its results:

i. The equivalence $d(K,E) = d(M,F)$ emerges from the congruence of $\triangle KCE$ and $\triangle MDF$ substantiated through the angle-side-angle criterion of congruence. This congruence is evidenced by the equality of angles $\measuredangle$MDF and $\measuredangle KCE$, which, in turn, is justified by the equivalences $\measuredangle FDM = \measuredangle D'DA$ and $\measuredangle ECK = \measuredangle C'CB$. The transitivity of angle identity ensures that $\measuredangle MDF = \measuredangle KCE$. Further analysis reveals that $\overline{DM} = \overline{CK}$, a conclusion drawn from the isosceles nature of the trapezoid which dictates $\overline{DA} = \overline{CB}$, and by extension, from hypothesis 3, that $\overline{AM} = \overline{BK}$. Thus, the subtraction of $\overline{AM}$ from $\overline{DA}$ and $\overline{BK}$ from $\overline{CB}$ respectively yields $\overline{DM}$ and $\overline{CK}$, confirming their equality. Additionally, the parallelism of $\overline{EK}$ with $\overline{C'B}$ aligns $\measuredangle EKC$ with $\measuredangle C'BC$, just as $\measuredangle FMD$ corresponds with $\measuredangle D'AD$, the congruence of $\triangle ADD'$ with $\triangle BCC'$ reinforcing the angle equivalences. These observations collectively affirm the congruence of $\triangle KCE$ with $\triangle MDF$, thereby establishing the equality of distances $d(K,E)$ and $d(M,F)$ as a direct consequence of the demonstrated symmetrical relations and congruences.
ii. Also, $d(L,G) = d(N,H)$. Analogously, the identity $\overline{LG} = \overline{NH}$ can be demonstrated.

These symmetries pervade the proof and resonate with the final result. The algebraic expressions — potentially deemed 'magical' — yielding the conclusion stem from observations i. and ii. as follows:

For i., $|c - k| = |c - (c - m)| = |m|$, and for ii., $|c - l| = |c - (c - n)| = |n|$. Subsequently, $|c - k + n| = |c - l + m|$, leading to $|l - m| = |k - n|$, and $(l-m)^2 = (k-n)^2$. Thus, $\sqrt{(q-p)^2 + (l-m)^2} = \sqrt{(q-p)^2 + (k-n)^2}$, demonstrating $d(M,L) = d(K,N)$ or, in terms of segments, that $\overline{ML} = \overline{KN}$

■

Unsurprisingly, the symmetry in the Cartesian proof is not very different from the symmetry that Lang points out in the original proof. Symmetry could be a property inherent to the result 2.2 itself, but if this is the case, it seems highly plausible that any proof, regardless of its explanatory power, exhibits such symmetry. In this sense, symmetry would not contribute at all to the distinction between explanatory and non-explanatory mathematical proofs.

3.3 The Euclidean Proof

An additional instance that would elicit critique from Lang for its failure to harness the inherent symmetry present within the explanandum of 2.2, as opposed to the symmetry exploited in the original demonstration, pertains to the Euclidean proof. The proof unfolds as follows:

Proof.

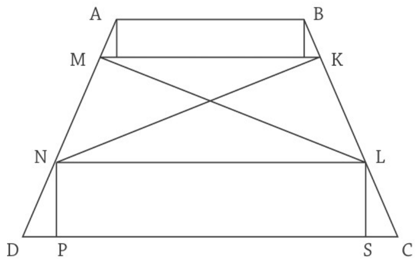

Figure 3. Euclidean proof.

Initiate by constructing a line from point *N* perpendicular to segment $\overline{CD}$, denoting their point of intersection as *P*. Similarly, construct line segment $\overline{LS}$. Analyzing triangles $\triangle DNP$ and $\triangle CLS$, we observe that $\measuredangle PDN = \measuredangle SCL$ due to the trapezoid's isosceles nature, and $d(N,D) = d(L,C)$ as given. Furthermore, the corresponding angles $\measuredangle DPN$ and $\measuredangle CLS$ are both right angles. Consequently, by employing the angle-side-angle criterion, we establish that $\triangle DNP$ is congruent to $\triangle CLS$. It is pertinent to note that these triangles are not merely congruent but also exhibit symmetry relative to the perpendicular bisector intersecting point *O* (figure 3). As a result, side $\overline{NP}$ is equal in length to $\overline{LS}$, given their correspondence in the congruent triangles, and are parallel by virtue of their perpendicular orientation to the same line. This congruence and parallelism of opposite sides validate that quadrilateral *PNLS* forms a parallelogram, thus asserting that segment $\overline{NL}$ is parallel to $\overline{DC}$. Employing a similar rationale with additional auxiliary lines, we deduce that $\overline{AB}$ is parallel to $\overline{MK}$.

This parallelism between $\overline{MK}$ and $\overline{NL}$ in conjunction with their shared parallelism to lines $\overline{AB}$ and $\overline{DC}$ respectively, confirms the trapezoidal configuration of *MKLN*. Given that $\overline{MN} = \overline{AD} - \overline{AM} - \overline{ND}$ and $\overline{KL} = \overline{BC} - \overline{BK} - \overline{LC}$, coupled with the equalities $\overline{AM} = \overline{BK}$, $\overline{AD} = \overline{BC}$, and $\overline{ND} = \overline{LC}$, it follows that $\overline{MN}$ is equal in length to $\overline{KL}$. The congruence of corresponding angles $\measuredangle KLN = \measuredangle LCP$ and $\measuredangle LCS = \measuredangle NDP$, alongside the equality $\measuredangle NDS = \measuredangle MNL$, further corroborates that $\measuredangle KLN = \measuredangle MNL$. This analysis, informed significantly by the symmetrical properties underlying the theorem, leads to the conclusion that triangles $\triangle MNL$ and $\triangle KLN$ are congruent by the side-angle-side criterion. Consequently, their corresponding sides $\overline{ML}$ and $\overline{KN}$ are equidistant.

As can be observed, this proof explores the properties of a geometric figure, specifically a trapezoid, by drawing on the principles of congruence and symmetry. It begins by constructing perpendicular lines from certain points to the sides of the trapezoid, creating new triangles within the figure. By analyzing these triangles and their properties -- such as equal angles and sides -- the proof demonstrates that they are congruent to each other. The key step involves showing that certain segments and angles within the trapezoid and the constructed triangles are equal and parallel, leading to the formation of a parallelogram within the larger trapezoid. This parallelogram's properties help establish that two opposite sides of a newly formed smaller trapezoid are parallel and equal in length. Ultimately, the proof relies on a series of geometric constructions and the properties of congruent triangles (symmetries included) to show that certain lines within the trapezoid are parallel and equal, affirming the symmetry and specific equalities within the geometric figure. This methodical approach, grounded in basic geometric principles, elegantly demonstrates the relationships between various parts of the trapezoid and its internal structures. Nevertheless, according to Lange, the proof does not fit the bill:

> The construction is artificial because the proof using it seems obliged to go to elaborate lengths–all because it fails to exploit the figure's striking feature: its symmetry with respect to the line between the midpoints of the bases. [p. 246]

Indeed, the proof in question effectively leverages the symmetry under discussion, a characteristic equally exploited by the Cartesian proof. It is precisely through the invocation of this symmetry, alongside additional symmetrical considerations, that numerous triangle congruences are deducible. Ultimately, the symmetry central to this discourse emerges inherently from the trapezoid's isosceles nature. Consequently, the formulation of a proof for theorem 2.2 indispensably presupposes this hypothesis, underscoring the intrinsic reliance on the trapezoid's defining isosceles property to establish the proof's foundational arguments.

Upon examination, we can see no substantive divergence between the "original," the Euclidean and the Cartesian methodologies of proof. The underpinnings required to construct these proofs are fundamentally identical, merely articulated within disparate frameworks, with symmetry being salient in all instances. Contrary to Lange's insinuations, there exists no contrivance within the Euclidean approach. The essence of the result – concerning lines, points, and their interrelations – necessitates that any proof must engage with these elements to elucidate the observed phenomenon. Given that the phenomenon inherently embodies a symmetrical aspect, the presence of symmetry within any proof is inevitable, manifesting to varying degrees. The presentation's granularity enhances the visibility of the result's defining characteristics, including the integral symmetrical features. As we will see below, examples of this kind can be extended to different areas of mathematics.

4. The Power Set Cardinality

Beyond the examples previously delineated, it is instructive to explore mathematical proofs beyond the realms of algebra and geometry, where the symmetry previously discussed manifests as a characteristic feature of the proof, contributing to its explanatory power. This discussion extends to include various demonstrations of the theorem articulating the formula for calculating the cardinality of the power set. Of particular interest within this context is a proof employing mathematical induction. Following Lange (2017) criterion, if in a mathematical proof, the observed symmetry can be traced from the proof to the theorem itself, it should be explanatory. This principle also covers areas of mathematics such as set theory and different types of mathematical proof, including inductions. Nonetheless, it is noteworthy that numerous philosophers, including Lange (2009), have classified induction proofs as quintessential examples of non-explanatory proofs, presenting an intriguing paradox within the philosophical discourse on the nature of mathematical explanation.

We hereby elucidate a mathematical theorem concerning the cardinality of the power set, positing that the cardinality of the power set of a given set *X*, possessing cardinality *n*, is precisely double that of the power set of another set *Y*, whose cardinality is $n-1$. This observation may be interpreted as reflecting a symmetry, articulated mathematically as follows:

$$|\mathrm{P}(X)| = |\mathrm{P}(Y)| + |\mathrm{P}(Y)|$$

In the context of assessing Lange's theoretical framework regarding the explanatory nature of mathematical proofs, we shall examine various demonstrations of this theorem to discern how they manifest the aforementioned symmetry (or how different presentations of the same proof might reveal such symmetry). This exploration aims to contribute to the ongoing discourse on the criteria that imbue a proof with its explanatory power, particularly with respect to symmetry as a distinctive element between proofs with different explanatory value.

Theorem 4.1 *Consider a set X possessing a cardinality of n. The cardinality of the power set of X (denoted as* $|P(X)|$*, and which aggregates the cardinalities of all subsets of X) is demonstrably twice the cardinality of the power set of any set comprising* $n-1$ *elements. This relationship is mathematically articulated as follows:*

$$|P(X)| = \sum_{k=0}^{n} \binom{n}{k} = \sum_{k=0}^{n-1} \binom{n-1}{k} + \sum_{k=0}^{n-1} \binom{n-1}{k}$$

This equation elegantly captures a fundamental symmetry in the combinatorial structure of sets and their power sets, offering a profound insight into the growth patterns of such cardinalities as one progresses from a set of cardinality $n-1$ to a set of cardinality *n*. This duality may be interpreted as the intrinsic symmetry present within the combinatorial landscape, providing a clear illustration of the doubling effect observed in the transition between these cardinalities.

To substantiate the aforementioned theorem, our discourse shall endeavor to demonstrate the following mathematical identity:

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

This identity is relevant for our purposes, as its confirmation underpins the result delineated in the cardinality theorem 2.6. Specifically, if the above equation holds true, it logically follows that $2^n = 2 \cdot 2^{n-1} = 2^{n-1} + 2^{n-1}$, thereby .validating the theorem's assertion regarding the relationship between the cardinalities of power sets for sets of cardinality n and $n-1$. This proof, therefore, serves as the foundational pillar upon which the theorem's validity rests, illustrating the exponential growth pattern encapsulated within the symmetrical structure of power sets.

Should Lange's demarcation between explanatory and non-explanatory proofs hold veracity, one might anticipate that for a given theorem, its demonstrations can be categorized distinctly as explanatory or otherwise. This expectation arises despite Lange's omission to furnish a comprehensive framework for the evaluation of all conceivable proofs.

This discourse endeavors to elucidate this premise through the exposition of two proofs pertaining to Theorem 2.6. The initial proof is posited as intuitively explanatory, particularly within the confines of Lange's definition. Conversely, the subsequent proof employs induction, a method often relegated to the non-explanatory domain, as delineated by (Lange, 2009). The objective here is to substantiate that, under the auspices of Lange's criteria for explanatory merit, the inductive proof of Theorem 2.6 not only qualifies as explanatory but does so with parity to the proof deemed more intuitively so.

Proof I (Intuitively Explanatory). Consider the array known as Pascal's triangle, composed of binomial coefficients, presented thus:

$n = 0$							1						
$n = 1$						1		1					
$n = 2$					1		2		1				
$n = 3$				1		3		3		1			
$n = 4$			1		4		6		4		1		
$n = 5$		1		5		10		10		5		1	
$n = 6$	1		6		15		20		15		6		1

The first documented manifestation of a binomial coefficient triangle is traced to the tenth century, articulated within the exegeses of the Chandas Shastra, an archaic Indian treatise on Sanskrit prosody, authored by Pingala circa 200 B.C.[1] This work laid the foundational understanding of binomial coefficients that would later be expanded upon by Blaise Pascal. Pascal's contributions were not merely iterative but transformative, introducing a plethora of heretofore unrecorded applications for the numbers constituting the triangle. His seminal work, the *Traité du triangle arithmétique* (1653), represents perhaps the first mathematical discourse explicitly dedicated to the arithmetic triangle, delineating its utility and theoretical underpinnings in a comprehensive manner.

The structure of Pascal's triangle elucidates the enumeration of subsets of a given cardinality for a set *X* of cardinality *n*.

1 For instance, for $|X| = 0$, the initial row communicates the existence of a singular subset of cardinality 0, namely $\emptyset$. Extending this logic:

2 For $X = \{a\}$ with $|X| = 1$, the triangle's second row signifies *X* possesses one subset of cardinality 0 ($\emptyset$), and one of cardinality 1 ($\{a\}$). Hence, the power set of *X* encompasses $|P(X)| = 1 + 1 = 2$ subsets.

3 For $X = \{a, b\}$ with $|X| = 2$, it's conveyed that *X* comprises one subset of cardinality 0, two of cardinality 1 ($\{a\}$ and $\{b\}$), and one of cardinality 2. Thus, $|P(X)| = 1 + 2 + 1 = 4$.

4 Continuing to $X = \{a, b, c\}$, where $X = |3|$, it's denoted that *X* includes one subset of cardinality 0, three of cardinality 1, three of cardinality 2 ($\{a, b\}, \{a, c\}$, and $\{b, c\}$), and one of cardinality 3 ($\{a, b, c\}$). Consequently, $|P(X)| = 1 + 3 + 3 + 1 = 8$.

5 …

In the framework of polynomial multiplication, the figures in each row of the triangle correspond to the coefficients arising from the expansion of the binomial theorem, where '$\binom{n}{k}$' denotes the count of subsets comprising k elements from a set of cardinality *n*. This interpretation renders the triangle as follows:

[1] (Edwards, 2002)

$$
\begin{array}{lccccccccccc}
n=0 & & & & & & \binom{0}{0} & & & & & \\
n=1 & & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\
n=2 & & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
n=3 & & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
n=4 & & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\
n=5 & \binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5}
\end{array}
$$

Newton's binomial theorem posits the general expression as:

$$\binom{n}{0}a^n b^0 + \binom{n}{1}a^{n-1}b^1 + \cdots + \binom{n}{n-1}a^1 b^{n-1} + \binom{n}{n}a^0 b^n = (a+b)^n$$

Given a set *X* of cardinality *n*, the aggregate cardinality of its subsets, or the cardinality of its power set, can be articulated as:

$$\binom{n}{0}\cdot 1 + \binom{n}{1}\cdot 1 + \cdots + \binom{n}{n-1}\cdot 1 + \binom{n}{n}\cdot 1 = |P(X)|$$

Employing the binomial theorem, justified by polynomial multiplication principles and Pascal's triangle, allows for representing each coefficient 1 as $1 \cdot 1$. This yields:

$$\binom{n}{0}(1^n \cdot 1^0) + \binom{n}{1}(1^{n-1}\cdot 1^1) + \cdots + \binom{n}{n-1}(1^1 \cdot 1^{n-1}) + \binom{n}{n}(1^0 \cdot 1^n) = (1+1)^n$$

Conclusively,

$$\sum_{k=0}^{n}\binom{n}{k} = 2^n$$

■

The structure of this proof is fundamentally underpinned by what can be interpreted as intrinsic symmetry, a characteristic that extends beyond mere aesthetic appeal to touch upon foundational principles of mathematical reasoning. Notwithstanding, a pertinent query arises concerning the breadth of this explanation's generality and its sufficiency to constitute a proof in the most rigorous sense. Setting aside this legitimate concern for the moment, it is imperative to highlight an observable symmetry that permeates the transition from proof to theorem, thereby embodying the very essence of mathematical harmony. This symmetry is not merely coincidental but reflects a deeper congruence between the structure of mathematical arguments and the universal truths they aim to unveil. In aligning with Lange's *desideratum*, the proof not only adheres to the established criteria for explanatory power but also leverages this symmetry to elucidate the theorem's underlying principles, thereby reinforcing the proof's validity and enhancing its explanatory value.

The apparent efficacy of Lange's theoretical framework in this context is, upon closer examination, not serendipitous but rather a consequence of deliberate methodological choices. Specifically, the selection of a test case for analysis was guided by the criterion of showcasing a particular symmetry, integral to both the theorem under consideration and its proof. The highlighted symmetry is not fortuitous, but it is not an essential part of the result either. It is the product of an arbitrary process, orchestrated and designed to accentuate the consonance between mathematical theory and its practical demonstration.

5. Proofs by Induction

In philosophy of mathematics, several authors have criticized proofs by induction for their supposed lack of explanatory power. Regarding this point, in earlier versions of his work, Marc Lange (2014) has articulated specific arguments against the explanatory value of inductive proofs. Lange argues that proofs by induction often fail to provide understanding because they do not reveal the underlying reasons for the mathematical phenomenon in question. He contends that while inductive proofs can show that a proposition holds for all natural numbers, they often do so by relying on the structure of the natural numbers themselves, rather than illuminating the deeper mathematical truths underlying the proposition. Thus, these proofs can verify the truth of a statement without offering insight into why it is true across all cases.

There are several criticisms of inductive proofs. One of them is that they lack a mechanistic explanation, which means that it is unclear how one step leads to another, and how this elucidates the mathematical structure or phenomenon (Lange, 2013). Another perspective is that inductive proofs tend to focus on establishing the generality of a proposition rather than providing specific instances that could offer insight into the workings of the theorem (Baker, 2005). This generality is seen as coming at the expense of explanatory depth. Lastly, some critics argue that the explanatory power of inductive proofs is diminished when they heavily rely on the axiom of induction. They consider this reliance a formal trick rather than a genuine explanation of the phenomenon (Fehige, 2019).

Notwithstanding the aforementioned critiques, the crux of the matter resides in the realization that proofs by induction, conventionally categorized as non-explanatory, can indeed be structured to unveil characteristics interpretable as symmetry. Should this assertion hold, we encounter a paradigm wherein proofs, ostensibly non-explanatory by Lange's criteria, nonetheless fulfill his stipulation of manifesting a symmetry traceable from the proof to the theorem itself. This revelation contributes substantively to the burgeoning corpus of evidence advocating for symmetry as a deliberately chosen attribute within the exposition of mathematical proofs. Such an insight not only challenges prevailing notions regarding the nature of explanatory proofs but also underscores the versatility and depth of symmetry, which, rather than a conceptual lens through which mathematical truths can be discerned and articulated more clearly and coherence, is an inherent property of mathematical results, whose natural essence is the occurrence of patterns.

Proof. (Induction 1)

The objective of this proof is to establish, via mathematical induction, the identity:

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Commencing with a set X of cardinality 1, we select $n = 1$ as the base case for induction (notably, the base case $n = 0$ is equally viable):

$$\sum_{k=0}^{1} \binom{1}{k} = \binom{1}{0} + \binom{1}{1} = \frac{1!}{0! \cdot 1!} + \frac{1!}{1! \cdot 0!} = 1 + 1 = 2^1$$

Assuming the induction hypothesis:

$$\sum_{k=0}^{n} \binom{n}{k} = \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!} = 2^n$$

our task is to demonstrate that:

$$\sum_{k=0}^{n+1} \binom{n+1}{k} = \sum_{k=0}^{n+1} \frac{n+1!}{k!\,((n+1)-k)!} = 2^{n+1} = 2 \cdot 2^n = 2^n + 2^n$$

A salient symmetry, observable within the progression from the proof to the theorem, underlies our induction step. Specifically, each term from $\sum_{k=0}^{n}\binom{n}{k}$ appears twice in $\sum_{k=0}^{n+1}\binom{n+1}{k}$, a reflection of the symmetry inherent in Pascal's triangle. This observation is integral to the induction step, as revealed by the following identity:

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}, \qquad k = 0, \cdots, n.$$

To elucidate this, let us consider the cases $n = 0$ and $n = 1$, followed by a general case:

Case $n = 0$:

$$\sum_{k=0}^{n}\binom{n}{k} = \binom{0}{0} = 1; \sum_{k=0}^{n+1}\binom{n+1}{k} = \binom{1}{0} + \binom{1}{1} = 1 + 1 = \binom{0}{0} + \binom{0}{0}$$

Case $n = 1$:

$$\sum_{k=0}^{n}\binom{n}{k} = \binom{1}{0} + \binom{1}{1} = 1 + 1; \sum_{k=0}^{n+1}\binom{n+1}{k} = \binom{2}{0} + \binom{2}{1} + \binom{2}{2} = 1 + 2 + 1 =$$
$$1 + [1 + 1] + 1 = \left[\binom{1}{0} + \binom{1}{0}\right] + \left[\binom{1}{1} + \binom{1}{1}\right]$$

For the general case, the identity can be proven as follows:

$$\binom{n}{k-1} + \binom{n}{k} = \frac{n!}{(k-1)!(n-(k-1))!} + \frac{n!}{k!(n-k)!} = \left(\frac{k}{k}\right)\frac{n!}{(k-1)!(n-(k-1))!} + \frac{n!}{k!(n-k)!} =$$
$$\frac{n!k}{k!(n-k+1)!} + \frac{n!}{k!(n-k)!} = \frac{n!}{k!(n-k)!}\left(\frac{k}{n-k+1} + 1\right) = \frac{n!}{k!(n-k)!}\left(\frac{k+n-k+1}{n-k+1}\right) =$$
$$\frac{n!}{k!(n-k)!}\left(\frac{n+1}{n-k+1}\right) = \frac{(n+1)!}{k!(n-k+1)!} = \frac{(n+1)!}{k!((n+1)-k)!} = \binom{n+1}{k}.$$

Given $\sum_{k=0}^{n}\binom{n}{k} = 2^n$ as per the induction hypothesis, and demonstrating that each term of this series is duplicated in $\sum_{k=0}^{n+1}\binom{n+1}{k}$, it follows that:

$$\sum_{k=0}^{n+1}\binom{n+1}{k} = 2 \cdot 2^n = 2^{n+1}$$

■

The symmetry principle elucidated in the preceding inductive proof (namely the observation that each term from $\sum_{k=0}^{n}\binom{n}{k}$ manifests twice in $\sum_{k=0}^{n+1}\binom{n+1}{k}$) finds a visual corollary within the geometric structure of Pascal's triangle, as shown below:

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & \underbrace{1+1}_{2} & & 1 & & & \\
 & & 1 & & \underbrace{1+2}_{3} & & \underbrace{2+1}_{3} & & 1 & & \\
 & 1 & & \underbrace{1+3}_{4} & & \underbrace{3+3}_{6} & & \underbrace{3+1}_{4} & & 1 & \\
1 & & \underbrace{1+4}_{5} & & \underbrace{4+6}_{10} & & \underbrace{6+4}_{10} & & \underbrace{4+1}_{5} & & 1
\end{array}$$

In each sequence delineated by n within Pascal's triangle, where individual numerals epitomize the combinatory counts congruent to their respective positions, a distinctive symmetry is observed: each value manifesting in row n is replicated twice in the ensuing row $n+1$. This paper proposes to make salient this symmetry principle through an innovative inductive proof, underscoring its persistence from the proof to the mathematical result.

Given a finite set X, its power set, denoted $P(X)$, is defined as the union of all subsets A_i contained within X, expressed mathematically as $P(X) = \bigcup_{A_i \subset X} A_i$, with the cardinality of $P(X)$ represented by $|P(X)| = \sum_{i\in\Lambda}|A_i|$. The total number of elements within the power set of a set X of cardinality n ($n \in \mathbb{N}$) equates to the aggregate of the cardinalities of all constituent subsets of X. This encapsulates the summation of the cardinalities spanning all subsets of X with cardinalities ranging from 0 to n (acknowledging that $|A| \leq |X|$ for any $A \subseteq X$). Consequently, the equation $|P(X)| = \sum_{i\in\Lambda}|A_i| = \sum_{k=0}^{n}\binom{n}{k}$ emerges.

The endeavor to demonstrate that $|P(X)| = 2^n$ serves not only as a proof of this specific assertion but also substantiates the broader theorem 2.6, thereby reinforcing the underlying symmetry that characterizes both the methodology and the outcomes of this particular way to present the result.

Proof. (Induction 2)

Initiating with the case $n = 1$ as the foundation for our inductive reasoning, we consider a set X constituted as a singleton, specifically $X = \{a\}$. Then, $P(X) = \{\emptyset, \{a\}\}$. Therefore, $P(X) = 2 = 2^0 + 2^0 = 2^1$. (We can see the enclosed symmetry represented by "$2^0 + 2^0$.")

Consequently, it follows that the cardinality of $P(X)$ is $2 = 2^1$, effectively establishing our base case.

Progressing to the inductive step, let us postulate that for a set *X* with cardinality *n*, where $n > 1$, the power set $P(X)$ satisfies the equation $|P(X)| = 2^{n-1} + 2^{n-1} = 2^n$. This assumption constitutes our induction hypothesis.

Consider now a set *X* of cardinality *n+1*. Selecting an arbitrary element *a* from *X*, we observe that the set $X\backslash\{a\}$, which denotes the removal of $\{a\}$ from *X* has cardinality *n*. By our induction hypothesis, the power set $P(X\backslash\{a\})$ is ascertained to have a cardinality of 2^n

Analyzing the composition of any subset *N* within the power set $P(X)$, we discern two distinct scenarios:

(a) *N* remains unaltered subsequent to the exclusion of {*a*} from *X*, denoting that $a \notin N$.

(b) *N* is derived from a precursor set $N' = N \cup \{a\}$, signifying the inclusion of *a* within *N*.

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Accounting for all subsets of *X* through scenarios (a) and (b) facilitates the enumeration of all subsets of $X\backslash\{a\}$. Therefore, the cardinality of $P(X)$ is determined as $|P(X\backslash\{a\})| + |P(X\backslash\{a\})| = 2^n + 2^n = 2^{n+1}$. This derivation, encapsulating the operationalization of $P(X)$ as explicated in conditions (a) and (b), exemplifies the symmetry in the transition from proof to theorem, thereby reaffirming the inductive premise.

It is evident that the two preceding inductions do not constitute two distinct proofs but rather two distinct presentations of the same proof. Although a criterion for distinguishing mathematical proofs is yet to be established, this conclusion can be readily accepted given that both induction 1 and induction 2 necessitate the assumption of precisely the same mathematical principles. Is one of these proofs more symmetrical than the other? No, in both proofs (or versions of the proof) the symmetrical structure of the sets is the same. It is a symmetry of cardinalities, which can be made explicit through notation in different ways. The symmetry of the two proofs is contingent on how the proof of 2.6 is presented, and therefore, it is subject to the intention of the 'narrator.'

6. Final remarks

The tradition in the philosophy of mathematics has, from various perspectives, sought to characterize what constitutes a mathematical proof. Formalist attempts, from ancient Greece through the foundational schools and culminating in the current focus on mathematical practice, have referenced diverse properties in an effort to delineate the class of arguments that should be legitimized as genuine mathematical proofs. The futile search for such properties has led to more modest objectives, such as seeking these characterizations only for certain types of mathematical proofs, for example, explanatory proofs. In this regard, various properties have been proposed as answers to the question of what makes a proof explanatory. This article specifically discusses the property of symmetry as a suggested characteristic element of explanatoriness. Previous sections have presented results in algebra (D'Alembert's theorem), geometry (the relationships between segments of a trapezoid) and set theory (the cardinality of the power set). Beyond these different areas, various types of proof have been explored (some of which are traditionally considered non-explanatory). These proofs incorporate algebraic, analytic, geometric, constructive, and inductive methods. The combination of these elements and the variety of contexts in which they are applied demonstrates that the potential to extract symmetry from proofs is not dependent on a specific area or methodology.

To conclude, it's important to note that the evidence presented in this work isn't intended to fit into explanatory paradigms. Rather, it serves as a collection of illustrative examples that highlight prominent symmetries. The proofs included in this work were carefully crafted to encapsulate and reflect the symmetries evident in the results, incorporating analogous symmetrical structures at key stages of the proof process. The emergence of symmetry is not only a matter of presentation, but also of evaluative subjectivity. Just as we can contemplate the geometric symmetry of a triangle, this discourse extends the concept of symmetry to include algebraic terms and the sequential steps that underpin a mathematical proof, highlighting the flexibility available when we identify symmetry as a defining characteristic of a mathematical proof.

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