

Different Languages, Different Mathematics Learning

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Abstract

Culture shapes pupils' mathematical learning, their performances and life trajectories of participation (César, 2013a, 2013b). It also contributes to the senses they attribute to mathematical learning (Bakhtin, 1929/1981). Using collaborative work and inter-empowerment mechanisms facilitates knowledge appropriation (César, 2009). This is particularly important for pupils participating in minority cultures, socially undervalued and whose L1 is not the instruction language. Bi-univocal culture mediation (César, 2017b) is important regarding empowerment. We used an instrument to evaluate pupils' abilities and competencies (IACC), conceived by the *Interaction and Knowledge (IK)* team (Machado, 2014), and other mathematical tasks. The goal we address is to trace the differences between their approaches to problems, mathematical reasoning and solving strategies used by pupils whose L1 is ideographic (Creole, Cape Verde) or phonetic (Portuguese). We developed an intrinsic case study (Stake, 1995). The main participants are the pupils from almost 600 classes (all over Portugal and Cape Verde) who participated in the IK. The analysis of some examples illustrates that L1 shapes pupils' approaches to problems, mathematical reasoning and solving strategies. This evidence plays an important role in their access to school achievement and in teachers' understanding about how they can promote pupils' mathematical learning.

Keywords

Mathematics learning, language (L1), solving strategies, inter-empowerment mechanisms, bi-univocal cultural mediation

To cite this article: César, M., & Machado, R. (2024). Different Languages, Different Mathematics Learning. *Journal of Research in Mathematics Education*, 13(2), pp. 180-198. <http://dx.doi.org/10.17583/redimat.14421>

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Diferentes Lenguas, Diferentes Aprendizajes Matemáticos

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Resumen

La cultura configura el aprendizaje matemático, los rendimientos escolares y trayectorias de vida y de participación (César, 2013a, 2013b). También contribuye a los sentidos atribuidos a los aprendizajes matemáticos (Bakhtin, 1929/1981). Utilizar trabajo colaborativo y mecanismos de *inter-empowerment* facilita la apropiación de conocimientos (César, 2009), esencial para quien participa en culturas minoritarias, socialmente devaluadas, cuya L1 no es la lengua de instrucción. La mediación cultural biunívoca (César, 2017b) es importante para el *empowerment*. Usamos un instrumento para evaluar habilidades y competencias (IACC), concebido por el equipo del *Interacción y Conocimiento* (Machado, 2014) y tareas matemáticas utilizadas en clase. El objetivo es rastrear las diferencias entre el enfoque, el razonamiento matemático y las estrategias de resolución utilizados por los alumnos cuya L1 es ideográfica (criollo, Cabo Verde) o fonética (portuguesa). Desarrollamos un estudio de caso intrínseco (Stake, 1995). Los principales participantes son los alumnos de casi 600 clases (Portugal y Cabo Verde) de lo *Interacción y Conocimiento*. El análisis de algunos ejemplos ilustra como la L1 configura el enfoque, el razonamiento matemático y las estrategias de resolución. Esta evidencia juega un papel importante en el rendimiento escolar y en la comprensión de los profesores sobre como pueden promover el aprendizaje matemático de los alumnos.

Palabras clave

Aprendizaje matemático, lengua (L1), estrategias de resolución, mecanismos de *inter-empowerment*, mediación cultural biunívoca

Cómo citar este artículo: César, M., y Machado, R. (2024). Diferentes lenguas, diferentes aprendizajes matemáticos. *Journal of Research in Mathematics*, 13(2), pp. 180-198. <http://dx.doi.org/10.17583/redimat.14421>

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Europe has become more and more multicultural in the last decades. People from countries at war, poor countries or countries under totalitarian regimes try to come to Europe searching for better life conditions. Portugal is no exception. Nowadays around 10% of those living in Portugal are foreign. In 2022 national statistics showed that 25% of the babies born in Portugal had a foreign mother. There are also second and third generation communities whose families came from Cape Verde. Although Portuguese, at home they speak Creole which is their mother language (L1), the one in which they count and think. The Cape Verde community is one of the biggest in Portuguese public schools. They often experience underachievement, namely in Mathematics. Thus, understanding their approaches to problem solving, mathematical reasoning and solving strategies was essential. Teachers' awareness about the way L1 shapes mathematical performances is very important to find ways to overcome underachievement and to develop a learning community. In the schools that used bi-univocal cultural mediation (César, 2017b) the improvements in academic achievement were higher and pupils tended to define life trajectories of participation that more often included long-term school paths.

Theoretical Background

Mathematical experiences shape pupils' life trajectories of participation (César, 2013a, 2017b), particularly in the school context and in learning situations (César, 2009; Machado, 2014). Culture also plays a major role in mathematical performances and in school achievement (César, 2013b, 2014). In Portugal there are many early school dropouts. Mathematics is often rejected, and underachievement is high (Machado, 2014). Thus, mathematics teachers need to be aware of cultural differences to develop practices that facilitate pupils' access to school achievement. Those participating in vulnerable cultures usually trace life trajectories of participation that do not include long-term studies. For them, respecting their cultures, distributing power, and promoting empowerment and a high positive self-esteem, namely through the implicit messages, are important features to improve equity and social justice. Using collaborative work and inter-empowerment mechanisms facilitates pupils' knowledge appropriation and the development of abilities and competencies (César, 2013a, 2017a). Collaborative work, particularly in dyads and in small groups, favours the internalization of inter-empowerment mechanisms, allowing pupils to transform them into intra-empowerment mechanisms they can use later in an autonomous way, in other contexts, scenarios and situations (César, 2009; César, Machado, & Ventura, 2014). Through social interactions with their peers and with their teachers, pupils give a meaning to school knowledge. They also learn how to express their voices (Bakhtin, 1929/1981), facilitating their access to a legitimate participation (Lave & Wenger, 1991). For those who are underachieving, who usually act as peripheral participants (Lave & Wenger, 1991), teachers' practices and regulatory dynamics of participation between school and families play a fundamental role (César, 2013b). Collaborative work also helps pupils to express their cultural identities, including the different I-positions they assume (e.g., as students, as son or daughter, as friend, or as a member of a particular cultural group). It also facilitates changing their Me-positions, both regarding how

they see themselves, as students and citizens, and how they think the others see them (César, 2013b, 2017b; Hermans, 2001, 2003).

In a different context, other authors illustrate the differences between formal and visual argumentations (Cervantes-Barraza & Cabañas-Sánchez, 2018), particularly in collective discussions. The same authors (2022) also illuminate the importance of argumentation and refutation in mathematical knowledge appropriation. The differences between formal and visual argumentations, as well as the importance of refutation, were also explored by us in previous papers (César, 2009, 2013a, 2014, 2017a). Back then we were analysing dyads and groups from action-research projects, when they were working together and in the whole-class discussion. We were exploring the importance of a learning community in pupils' mathematical performances, an issue that García-Carrión and Díez-Palomar (2015) also address. Dialogical talk favours mathematical learning (César, 2009, 2014) and it is an essential feature of a learning community, as stated by Díez-Palomar, Chan, Clarke and Padros (2021). It facilitates the existence of thinking spaces, a concept coined by Perret-Clermont (2004) that underlines the importance of spaces in which pupils feel safe enough to express their reasonings, argumentations, doubts, and questions.

When we had access to the whole *corpus empiricus* of the IK project we realized the importance of L1 in pupils' mathematical performances and in the development of learning communities. Portuguese schools have a diverse population whose cultural roots shape pupils' approaches to problem solving, mathematical reasoning and solving strategies (César & Machado, 2012; Machado, 2014; Meyer, Prediger, César, & Norén, 2016). Pupils participating in Cape Verde culture are one of the groups experiencing school underachievement, particularly in Mathematics. This culture does not highly value school as a learning opportunity. They believe that experts from their community, older people with more life experience, friends and families are the ones who will be the most adequate partners to learn with. They usually speak Creole at home and with friends. Thus, often they do not master Portuguese, which is the language of instruction (César, 2013b, 2017b). But even when they do, they count, dream, and think in Creole, which is an ideographic language. For many years Creole only had an oral tradition and no written form. In Creole nouns and adjectives have no gender, no singular and plural. Verbs do not change according to the subjects and words have different meanings if they occupy a different position in the sentence. Having Creole as one's mother language (L1) develops spatial abilities, like the mathematical spatial reasoning, as well as geometric and graphic representation solving strategies. It also favours global approaches to problem solving. Portuguese is a phonetic language with a solid written and oral tradition. It has sounds, syllables, words, and a complex syntax. Nouns and adjectives vary according to gender, there is singular and plural, and verbs have many differences in each tense and subject. Thus, it develops analytical reasoning, and favours arithmetic and algebraic solving strategies and a step-by-step approach to problem solving. The characteristics of these languages shape pupils' mathematical performances and their access to school achievement (César, 2009, 2013a, 2017b; Machado, 2014; Meyer et al., 2016). If they are not allowed to use their own approach, reasoning and solving strategies, they begin rejecting Mathematics because they feel they are not appreciated and respected.

César (2013b) illuminated that in schools facing cultural conflicts and misunderstandings, in which we observe disruptive ways of acting and reacting, and school underachievement, it is important to develop regulatory dynamics of participation between school and families. But to be fruitful, this process must be based on a bi-univocal cultural mediation, a construct she coined (César, 2017b). Many authors describe cultural mediation processes that fail, despite the good intentions behind them. They are usually based on top-down decisions and the cultural mediation only goes in one sense: from the dominant culture to the vulnerable culture. This is what César (2017b) calls univocal cultural mediation. Bi-univocal cultural mediation is based on collaborative work between the two (or more) cultures involved in the process. It distributes power among cultural agents. Decision-making and the coordination of the process are shared between representatives of the two (or more) cultures. Members of different cultures act as legitimate participants of that school community, shaping the school culture. Both cultures feel engaged in the cultural mediation process. Thus, this is a transformative process that has impacts even on people's identities, particularly in their dialogical self (César, 2013a, 2013b, 2017b). Using bi-univocal cultural mediation is very important for all school agents, but particularly for pupils participating in vulnerable cultures that are socially undervalued. For them it constitutes an opportunity to trace life trajectories of participation based on longer school paths or professional lives in which they use their abilities and competencies in a job they love. Thus, it allows them to experience more equity and social justice.

Method

We assumed an interpretative approach (Denzin, 2002) and developed an intrinsic case study (Stake, 1995). The *Interaction and Knowledge* (IK) project had three research designs: (1) *quasi-experimental* studies; (2) action-research projects; and (3) intrinsic case studies (Stake, 1995). We focus on an intrinsic case study which analyses the IACC (Machado, 2014), and on a dyad interaction from the action-research projects.

The main participants were the pupils who participated in the IK (5th to 12th grades, around 600 classes from all over Portugal and Cape Verde), 69 mathematics teacher/researchers, and four psychologists. Other participants were pupils' families, educational agents from these schools, and internal (IK team, but from another school) and external observers. This allowed for the triangulation of the sources. Having many researchers also allowed to triangulate the researchers and their interpretations. Pupils whose excerpts were used also discussed the interpretations. They allowed for their use, or not, suggested further excerpts, or a different or more complete interpretation. This is asked in every paper that is written. Thus, we use a more demanding notion of informed consent. Data was collected through the IACC (*Instrumento de Avaliação de Capacidades e Competências*), which evaluates pupils' abilities and competencies and was elaborated by the IK team (see Machado, 2014). The IACC was used in the first week as it is essential to know pupils' abilities and competencies since the beginning of the school year. Data also included questionnaires, tasks inspired in projective techniques, interviews, informal conversations, students' protocols, and reports from external observers. Thus, we also triangulated the collecting instruments. In research using qualitative data treatment and analysis, triangulations are quality criteria.

The IK lasted 12 years (1994/95-2005/06) and had a 10-years follow-up that allowed us to better understand its impacts in pupils' life trajectories of participation (see César, 2009, 2013b, 2017b). A complete analysis of the data was only possible when the follow-up ended. Thus, it was after 2016 that meta-analysis took place, and some new constructs were coined (e.g., bi-univocal cultural mediation). It was also in the meta-analysis that the influence of L1 in the approach to problems, mathematical reasoning, and solving strategies emerged. The *empirical corpus* is very rich. Thus, we are still doing this job. Data treatment and analysis was based on a narrative content analysis (Clandinin & Connelly, 1998), from which inductive categories and patterns emerged. This narrative analysis allowed pupils' life trajectories of participation to be traced (see César, 2009, 2013a, 2013b, 2017).

A previous *quasi-experimental* study (César, 1994) showed that dyad work was better than individual work to promote pupils' mathematical performances. But this was only possible under some working instructions. César's work as supervisor of trainee teachers and many informal conversations with teachers also made her realize how important it was to know pupils' abilities and competencies since the first week of classes. Thus, she and some Mathematics teachers decided to elaborate a developmental instrument (IACC) that could be used and interpreted by teachers. They first used each task in several classes, then chose the five that corresponded better to their goals, and then tested several orders of presentation of the tasks to avoid rejection and get the best results possible. Thus, it took several years to elaborate the IACC, and it was a collaborative and dialogical work between a psychologist and many Mathematics teachers (see Machado, 2014).

We focus on the analysis of the answers that pupils gave in the IACC and in another mathematical task. The IACC is composed by five tasks: Task A evaluates critical sense and the interpretation of written mathematical information; Task B, mathematical intuition, persistence, and creativity; Task C, logical reasoning (concrete or abstract); Task D, the preference for a geometrical or analytical reasoning, and confirms the logical reasoning evaluated in Task C; and Task E, the preference for a global or step-by-step approach and the use of mathematical knowledge in daily life (e.g., selling, buying, profit) (see Machado, 2014). We focus on Tasks B and D. The tasks used in classes were selected, adapted, or elaborated by teacher/researchers to promote dyad or group work followed by rich whole-class discussions.

The goal addressed in this paper is: to trace the differences between the problem-solving approaches, the mathematical reasoning, and the solving strategies of pupils whose mother language (L1) is ideographic (Creole) or phonetic (Portuguese or other European languages). The research questions we address are: (1) Which different approaches, mathematical reasoning and solving strategies do pupils use in each task? and (2) Are they shaped by pupils' L1?

Results

The IACC – First Week of Classes

Pupils knew that the IACC was not for evaluation regarding school marks. It was used to find out their ways of reasoning, their abilities, and competencies. They could not use calculators, but any way to explain how they had solved those problems was accepted: drawings, sums, schemes, sentences, anything that helped them. They also knew that pupils using similar reasoning and solving strategies would not form a dyad. Copying was not a great idea because that would not put them together and would not allow the teacher to know what each one of them needed to develop (see César, 2009; César et al., 2014; Machado, 2014).

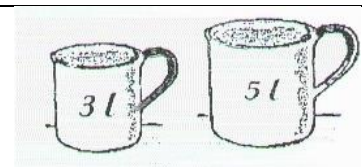
Task B

Task B (see Figure 1) mainly evaluates mathematical intuition, creativity, and persistence in the task. According to a full analysis of the answers in the IACC (Machado, 2014), mathematical intuition is not shaped by the type of reasoning or solving strategies used, as the percentage of pupils showing it and having Creole or Portuguese as L1 is similar. But answers based on persistence in the task instead of mathematical intuition are more common in those who use an analytical reasoning and an arithmetic or algebraic solving strategy, whose L1 is usually Portuguese. Creativity is more often used by pupils whose L1 is Creole, those who prefer geometrical reasoning and graphic representation solving strategies.

Figure 1

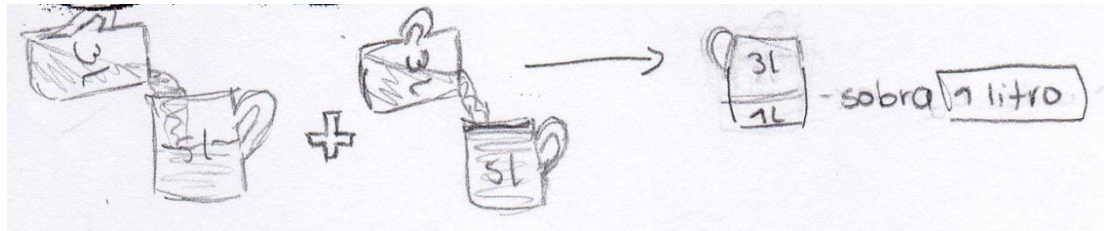
Task B

Task B
Mrs. Isaura went to Mr. Timóteo's shop to buy one litre of milk. Mr. Timóteo only has two measures: a 3 litre and a 5 litre cup. What does Mr. Timóteo have to do to sell Mrs. Isaura just one litre of milk?



Pupils whose L1 is Creole usually prefer graphic representation solving strategies. Thus, those who mobilize mathematical intuition often prefer to use drawings in their answers, as illustrated in Figure 2. This pupil used drawings and then he wrote “1 litre remains” (*sobra 1 litro*).

Figure 2

Graphic Representation Solving Strategy – L1: Creole

Pupils whose L1 is Portuguese, or another phonetic language, usually prefer arithmetic or algebraic solving strategies, which are connected to an analytical reasoning. Those who mobilize mathematical intuition often prefer to write a short text to explain their reasoning, as in the next quote: “He fills the 3l cup and puts it in the 5l [cup]; then he refills the 3l again and puts it in the 5l, 1l remaining in the 3l and 5l in the 5l.” (short mathematical composition solving strategy, L1: Portuguese).

Others use computations and then they write, to clarify their reasoning, as in the quote below:

It is $3 + 3 = 6$ and $6 - 5 = 1$.

So, he puts milk up to the top of the 3l cup. He does that 2 times and always puts that milk in the 5l cup. At the end, he will have 1l in the 3l cup and 5l in the 5l cup.

When they mobilize mathematical intuition, two moves are enough to solve the task, as they fill the 3l cup twice, put its milk in the 5l cup and at the end 1l remains in the 3l cup. Those who cannot mobilize mathematical intuition but show great persistence in the task, need to do many more moves. They are also able to solve the task and to measure 1l of milk, they just need more time. They also waste a lot of milk each time they empty the 3l cup and this would not be what the seller would want. Thus, in these answers there are more moves and less critical sense. This type of answer is much more common among pupils whose L1 is Portuguese, as the next quote illustrates:

We pick the 5l cup and put milk in it. Then we put milk from this cup into the 3l cup and in the 5l cup remain 2l. We empty the 3l cup. We put the remaining 2l in the 5l cup in the 3l cup. We fill the 5l cup again and put the milk (1l) to fill the 3l cup completely. Thus, 4l remain in the 5l cup. We empty the 3l cup again and put milk from the 5l cup, which had 4l remaining, into the 3l cup. The milk that remains in the 5l cup is 1l and we can sell it to the client.

Empirical evidence allowed teachers to understand how important it was to respect pupils’ ways of reasoning and their solving strategies, particularly when starting a new content. Pupils felt valued as in the whole-class discussion their colleagues were also writing down their strategies. As each student with a different solving strategy went to the board and explained it, later they were able to learn new solving strategies and to improve their performances and achievement in Mathematics.

It is important to underline that for many Cape Verde pupils who were underachieving in previous grades, this was the first time they went to the board to explain a solving strategy that was successful. This experience had a huge impact on them (see César, 2009, 2013b; 2017a). It was also particularly important for pupils who needed specialized educational support (César, 2014; César et al., 2014). Thus, the impact of a first task that was not for evaluation, with mathematical tasks that are not common and in which they could use any mathematical reasoning and solving strategy, was strong. This different first week of classes is often mentioned in a very positive way in interviews, questionnaires, and informal conversations both during the project and in the follow-up.

Task D

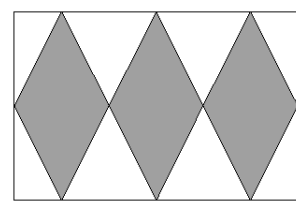
Task D (see Figure 3) illuminates the preference for geometrical or analytical reasoning, and confirms pupils' access to concrete or abstract reasoning, which is mainly evaluated through Task C (see Machado, 2014).

Figure 3

Task D

Task D

The tile in this figure measures 15cm by 10cm. What is the area of the coloured part?



Some pupils give an incomplete answer, as they calculate the area of the rectangle instead of the one of the coloured rhombuses. This is a more common answer in younger pupils who only have access to concrete reasoning. The percentage of pupils whose L1 is Creole or Portuguese is similar. The first example (Figure 4) illustrates a geometrical reasoning and a graphic representation based on drawing, common in pupils whose L1 is Creole. The drawing shows this pupil realizes that the coloured part was half of the rectangle. This means that his geometrical reasoning was developed. In the whole-class discussion he explained that he did not know the formula of the area of the rectangle because in previous grades he always failed in Mathematics. But when he realized that, in this class, he could use geometrical reasoning and graphic representation solving strategies, he was soon able to learn many other solving strategies, including algebraic and arithmetic ones.

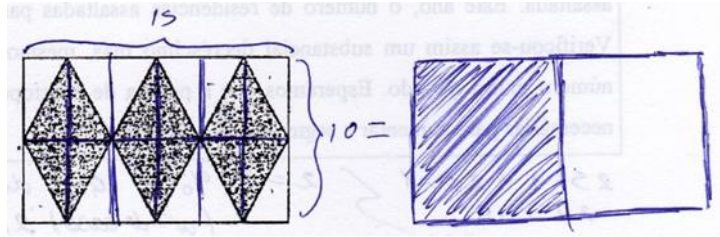
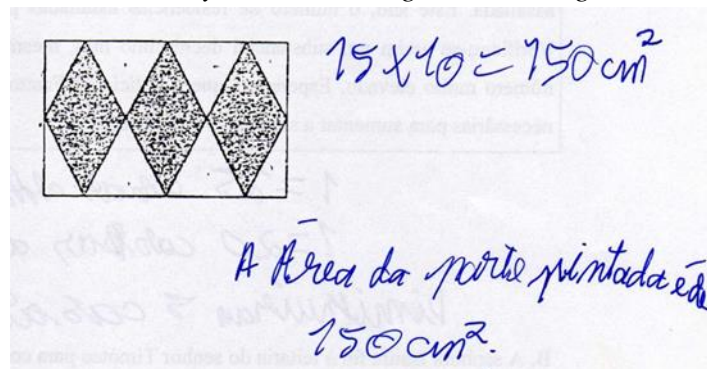
Figure 4*Incomplete Answer Based on Geometrical Reasoning – L1: Creole*

Figure 5 illustrates the same kind of incomplete answer but produced by pupils whose L1 is Portuguese. They usually prefer an analytical reasoning and arithmetic or algebraic solving strategies. In this answer, the pupil did not realize that the coloured part was half of the rectangle. Thus, although knowing the formula of the area of the rectangle, her comprehension of the problem was less accurate than the one in Figure 4.

Figure 5*Incomplete Answer Based on Analytical Reasoning – L1: Portuguese*

When the answers are complete ones, the differences between geometrical reasoning and analytical reasoning remain. The percentage of pupils with complete answers and whose L1 is Creole or Portuguese is similar. The first example (Figure 6) illustrates a geometrical reasoning and a graphic representation based on drawing, very common in pupils whose L1 is Creole. This is the most common answer among them. They were able to decompose the rhombus and to prove, by drawing them in another way, that their area was 75cm^2 . It is particularly interesting that they even show why it is 5cm , because it is half of 10cm .

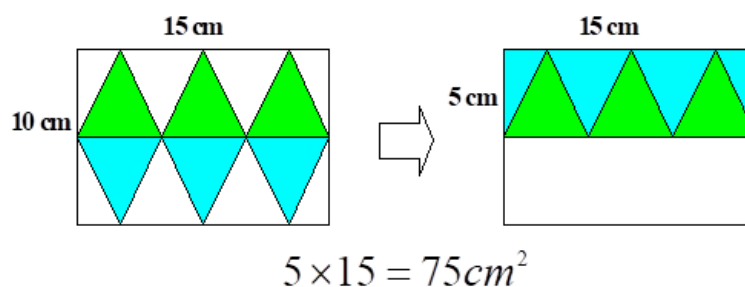
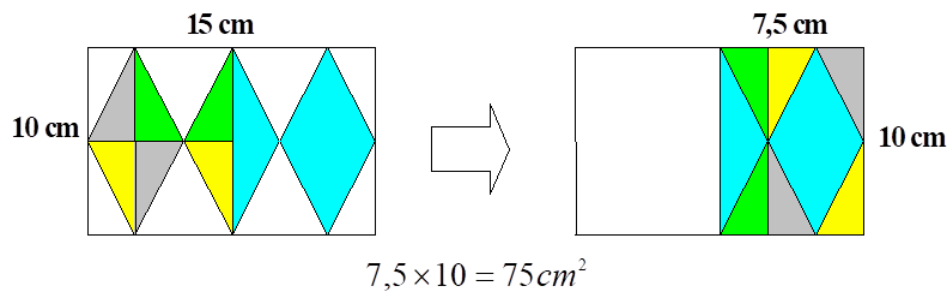
Figure 6*Complete Answer Based on Geometrical Reasoning – L1: Creole*

Figure 7 illuminates a less common answer but still based on a geometrical reasoning. This was a rare answer probably because the decomposition of the rhombus was much more complex. The pupils who draw this answer, many times without using any rule, show a highly developed geometrical reasoning and a very accurate way of drawing their solving strategies.

Figure 7

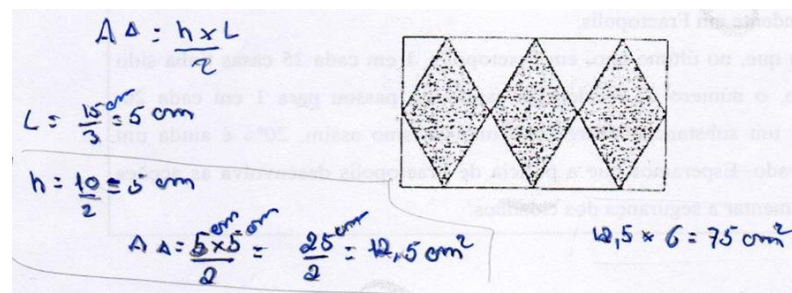
Complete Answer based on geometrical reasoning – L1: Creole



In Figure 8 this pupil used the triangle area formula (applied to half of the rhombus) and then multiplied the result by 6. Others used the rectangle formula and divided the resulting area by 2, or that for the rhombus and multiplied the result by 3. But they all used formulas and an algebraic solving strategy based on analytical reasoning. This is much more common for pupils whose L1 is a phonetic language, like Portuguese.

Figure 8

Complete Answer Based on Analytical Reasoning – L1: Portuguese



The analysis of pupils' answers in the IACC showed differences regarding the approaches to problem solving, mathematical reasoning and solving strategies. These choices are connected to L1. To Cape Verde pupils it was particularly important to see their reasonings and solving strategies accepted and valued by their teacher and, later, by their colleagues.

These results are usually ignored by Mathematics teachers, but they shape pupils' mathematics performances and their school achievement. Diversified mathematical experiences, namely in class, facilitate pupils' decisions regarding their life trajectories of participation (César, 2013a, 2017a), particularly in the school context (César, 2009, 2013b; Machado, 2014). Thus, it is particularly important to accept different reasoning and solving strategies since the first week of classes. This is the best way to promote legitimate participation

and to facilitate rich dyad and group work followed by accurate and diversified whole-class discussions. To do so, having a sound knowledge of pupils' abilities and competencies and the ones they still need to develop is an essential step to adjust mathematical tasks to each class.

Dyad Work in Class

Pupils were curious about the first dyads. After such a different first week and the impact of the whole-class discussion of the IACC, in which each pupil went to the blackboard to explain a successful solving strategy, or part of it, they guessed that Mathematics classes would be different. They also understood the implicit message: each one of you is capable of learning Mathematics.

At the beginning of the school year, deciding about forming dyads was the teacher/researchers' job. They would not explain the criteria, but pupils should try to find them and if their hypotheses were the ones used, they would know. Pupils also knew that dyads and spatial positions in the class would be changed. This allowed pupils to actualize their potentialities and to develop new competencies. Changing the spatial position in the room was a way of allowing each pupil to sit in the front or in the back of the room. Thus, pupils realized their teacher/researcher was trying to be fair.

In the first week, they could choose their position in the room. In this 9th grade class, four very successful girls from upper middle class and literate families were in the front row, as close to the teacher's table as possible. Then, there was a free row. Cape Verde pupils formed a second group. Many had experienced underachievement in Mathematics. They were in the back, far away from these girls and close to the windows and to the door. The third group were Portuguese pupils from the same poor neighborhood of the Cape Verde pupils. They were in the last rows behind the four girls (see César, 2009, 2013a). Thus, even the spatial position of the pupils told us implicitly how distant these cultural communities were.

The class did not have many pupils who achieved in Mathematics. Thus, the teacher/researcher decided to put V. (first letter of his name), whose family was from Cape Verde, and who experienced some disruptive school situations and had Level 1 (the lowest) in Mathematics, with M., one of the four girls who succeeded. He knew they would not be pleased, but V. was one of the pupils who was able to solve all the IACC tasks. In Task B he used a similar graphic solving strategy as shown in Figure 2 which shows mathematical intuition. But his drawings were more complex: he drew Mr. Timóteo, the cups and arrows to express the movements. It was clear that he could draw very well.

In Task D he was one of the rare pupils who used the solving strategy shown in Figure 7. Thus, he showed a well-developed geometric reasoning, and he was able to decompose geometrical figures. His angles were accurate, he drew all the details easily and he was creative. His solving strategies in Tasks A, C and E showed that he had critical sense when reading news including mathematical information, he had access to abstract reasoning and preferred a global approach. His teacher's guess was that if he worked with M. in a dyad, he would learn the contents from previous grades easily and that she could also progress, because he could mobilize competencies she needed to develop.

V., and many other pupils with similar school paths, illuminate how different it is to be successful in school from having quite developed abilities and competencies. His performance in the IACC also showed that under the adequate learning situations he could progress very quickly. But to achieve that his teacher needed to choose very carefully the tasks used in class. Although being a top Mathematics achiever (she had Level 5, the highest), M. was not able to finish all the IACC tasks. Sometimes she was able to use abstract reasoning, others she was not. In Task B she showed no mathematical intuition, but persistence in the task, as she used a solving strategy like the last example. In Task D she used a solving strategy like the one in Figure 5, but complete, i.e., she divided by 2 and gave the answer 75m^2 . She used a step-by-step approach, an analytical reasoning and preferred arithmetic or algebraic solving strategies. She was able to solve Task C (the easiest) and Task E, that related Mathematics to daily life situations (buying and selling goods). But she was not able to solve Task A and showed no critical sense connected to mathematical data in the news.

As expected, when the teacher told them they were a dyad, they were not pleased. But M. was someone who wanted to please her teachers. She accepted her peer and V., who had been appreciated in a Mathematics class for the first time, also did that. He did not want a conflict with a teacher allowing him “to see Mathematics” (V., several interviews) and who appreciated his solving strategies. Thus, they moved to their table and sat as far away from each other as possible while still sharing the same table. But in the second class, there was a task, already analysed in César (2009), that changed their relation. At that point we were not yet doing meta-analysis and discussing the importance of L1 in pupils’ performances.

The teacher was trying to make a smooth introduction to equations. Many pupils do not like this content. Thus, he was trying to use tasks that allowed different reasonings and solving strategies. He wanted to avoid rejection. He expected they could use more intuitive forms to solve problems and then they would formalize the mathematical knowledge. There was an exam at the end of the 9th grade. Thus, he knew that pupils needed to master intuitive and formal solving strategies.

Problem – *A grocer sold half a cheese, then a quarter and finally a sixth. He then checked that 125gr. were left over. How many kilos did the cheese weigh in the first place?*

[V. starts drawing a circumference and then stops to read the problem again.]

1 M. – *What’s that?*

2 V. – *It’s a cheese.*

3 M. – *A cheese?... What for?*

4 V. – *Now I’m going to draw what he sold...*

5 M. – *But I think you do this with sums...*

6 V. – *I don’t know how to do it with sums... so I’m going to see if it works this way...*

7 M. – *Then do yours, I’ll do mine and then we’ll explain.*

[Each one uses his/her own solving strategy. V. finishes first.]

If we connect this excerpt to the reasonings and solving strategies usually preferred by pupils whose L1 is Creole and those whose L1 is Portuguese, we realize how important it is to use tasks that allow for a diversity of reasonings and solving strategies.

As it is common in students whose L1 is Creole, V. uses a geometrical reasoning and a graphic representation solving strategy. Later, he connects it to an arithmetic strategy. M. uses an analytical reasoning and an arithmetic strategy from the start. She also tries to persuade V. to “do it with sums” because she thinks that in Mathematics classes pupils do sums, not drawings. She told us later, she thought “sums were serious work and drawings small jokes little kids did” (M., interview, December, end of the 1st term).

She tries to lead (Talks 5 and 7) because she is the top achieving pupil. But the effects of the whole-class discussion of the IACC (inter-empowerment mechanism, used by the teacher/researcher) were already internalized by V. and transformed into intra-empowerment mechanisms. Thus, he believed he could use his own solving strategies. He even confesses that he does not know how to do those sums. But he does not want to give up. M., who does not believe in V.’s performances, decides each one will work individually. She does not respect the didactic contract that wanted them to work in dyads. But she adds that later they will explain.

What made M. uncomfortable was that V. was very fast and he ended with a big smile on his face. That puzzled her because she never expected him to achieve, much less to do it sooner than her. So, she decided to ask him to explain how he had solved the problem (Talk 8).

8 M. – *How’s yours?*

9 V. – *I drew the cheese, then I divided it into 6 equal parts... so as to be easier... Get it?*

10 M. – *More or less... I understand what you did, but I haven’t yet figured out why you divided the cheese in 6 parts and not in 2... first he sells half...*

11 V. – *I know... but I had to know how to mark a half, a quarter and a sixth... a half and a quarter is easy... the hardest one is the sixth, so I started with that... or else I wouldn’t know how to go about it, after marking the half and the quarter I wouldn’t see where a sixth was...*

12 M. – *I beg your pardon?*

13 V. – *Draw a circle! [M. draws the circle]*

14 V. – *Now, mark half, which is what you sold. [M. does as he tells her to]*

15 V. – *Now mark another quarter, which is the other bit you sold. [She does it]*

16 V. – *Now mark a sixth, which is the third bit you sold. [M. stops, pencil in hand and says]*

17 M. – *Oh! I see! It’s much harder like this... do yours! It must be better.*

[V. draws the cheese again, divides it in six parts, traces half, then a quarter and a sixth. M. follows his steps carefully and says the numbers out loud as he goes along.]

V. had no doubts about his solving strategy, and he begins to explain it. For him, with his highly developed geometrical reasoning and a global approach to the problems, it was clear that he had to begin drawing the $\frac{1}{6}$ (Talk 9). But he looked at M. and realized she was not understanding. She confirms it (Talk 10) but tries to persuade him that she uses the right approach: step-by-step. The trouble is that in this problem this approach works when using an

arithmetic solving strategy, but not if you use a graphic representation. Thus, V. decides to lead (Talk 13 to Talk 16). He gives precise instructions and what is amazing is that M. follows them. But when she was supposed to draw $1/6$ she is not able to do it (Talk 16). So, she recognizes that V.'s solving strategy "must be better".

18 V. – *See what's not traced?*

19 M. – *Yes.*

20 V. – *I think it's half of a sixth... so it's $1/12$. If $1/12$ is 125gr, then the whole cheese is $125\text{gr} \times 12$, which is 1500 gr. [He had done the sum on the calculator] *That's 1,5 kg.**

21 M. – *But I didn't get that!*

22 V. – *How did you do it?*

23 M. – *With sums. I added $1/2 + 1/4 + 1/6$ and ended up with $3/12$. That's what he sold. One cheese is $12/12$. So, I subtracted $3/12$ from $12/12$ and got $9/12$, which are 125gr... but now I don't know how to go on.*

24 V. – *I don't understand your sums because I don't know Maths... but you've got that wrong... because you say he sold $3/12$ and that's a quarter of the cheese...*

25 M. – *Don't be dumb! No, it's not... It's the sum of all that...*

26 V. – *You wish, but that's not what your sum did... See... [He draws another cheese, divides it into 12 parts and marks 3. Then he looks at M.]*

27 M. – *What a mess! I can't understand why... the sums should also give...*

In Talk 20, once again V. shows a highly developed geometrical reasoning. Thus, he sees that what is not traced is half of $1/6$, which corresponds to $1/12$. If he had been obliged to calculate half of $1/6$ with fractions, he would not know how to do it. But when looking at his drawing, that was very accurate, he was able to say that this bit was $1/12$. Then, to do the sums, i.e., when he used an arithmetic solving strategy to finish his graphic solving strategy, he used the calculator.

By then, M. was convinced that he had done it right. But that meant a new issue: she had a different result (Talk 21). Thus, now it was V.'s turn to listen to her solving strategy (Talk 23). V. pays attention, but he goes on using a geometrical reasoning to conclude her answer is wrong (Talk 24). Once again, he declares that he does not know Mathematics, so he cannot do sums. But he is sure that when he uses geometrical reasoning, he knows the answer. The implicit messages of the first week and the whole-class discussion were so impactful on him that in the second week he was already engaged in doing mathematics and using intra-empowerment mechanisms to resist to M.'s implicit criticisms.

M. was not used to failing, much less failing when V. was able to solve the problem. She felt so frustrated that she says: "Don't be dumb!" (Talk 25). Usually the well-behaved girl, she is the one who uses a tone and a name that are not allowed in classes. Reflecting on that, we realized that many of the Cape Verde pupils experienced much higher levels of frustration during their whole school path in Mathematics classes. They were not allowed to use their ways of reasoning and solving strategies and were criticized by teachers and peers when they used them.

These excerpts show how the reasonings and solving strategies that are allowed in Mathematics classes shape pupils performances. Later in this class, the teacher approached

them, and M. understood her mistake. V. was listening attentively, and he never forgot what he needed to do to sum up fractions. He learned so much in one month while he was working in a dyad with M. that he had his first positive mark in the October test. By the end of the school year, he was one of the best pupils in Mathematics and he got the top mark in the final exam. But M. also learned a lot from him. She developed her geometrical reasoning, something very important because the 10th grade had a 1st term that was only based on Geometry and she wanted to study Medicine, so she needed top marks. She developed her mathematical intuition and when she was struggling, she tried to imagine how V. would approach that problem. And when their teacher changed dyads, they just wished they could still go on working together (see César, 2009, 2013a, 2013b, 2017a).

One of the most striking pieces of evidence of V.'s progress was in one external evaluator's report: "It is amazing how V. is able to use graphs and relate them to equations, how he solves problems and explains them to his peer" (...) "I would never think he'd have got [Level] 1 last year. (...) Just loved his ability to connect geometrical and analytical solving strategies" (External Observer 2, report, April, 3rd term).

It was very important to let V. use his own approach, reasoning and solving strategies. At the beginning of the school year, he struggled to learn the contents from previous grades and to believe in his own mathematical abilities and competencies in formal evaluations. Going from being a peripheral participant to a legitimate participant meant effort. But that would not have been possible if pupils did not feel those classes as thinking spaces (Perret-Clermont, 2004).

Final Remarks

The examples follow the general pattern we observed when doing the meta-analysis of all data from the IK project. Both in the IACC and in tasks used in classes pupils whose L1 is Creole, an ideographic language, usually prefer a global approach to problems. They tend to see the problem as a whole and use a geometrical reasoning. They prefer graphic representation solving strategies (e.g., drawings, schemes, pie graphs), which are usually less valued in school practices. Those graphic representations illustrate the way they understand the problem. Thus, if they cannot use them, they feel lost and are not able to learn the other solving strategies their colleagues use. But teachers usually have Portuguese as L1. Thus, they feel more comfortable in a step-by-step approach, mainly use analytical reasonings and prefer algebraic or arithmetic solving strategies. Some only accept solving strategies that are like their own. Thus, they tend to undervalue the mathematical performances of pupils whose L1 is ideographic, like Cape Verde or Roma pupils.

Pupils whose L1 is the same as the instruction language (Portuguese), a phonetic language, prefer a step-by-step approach to problems, analytical reasoning and arithmetic or algebraic solving strategies, the ones teachers use and value the most. This information is often ignored but it should play an important role in teachers' practices to facilitate pupils' access to school achievement. If teachers explore the existence of differentiated approaches to problem solving, mathematical reasoning and solving strategies, they favour a productive whole-class

discussion, after dyad or group work. Respecting each pupil's way of reasoning and solving strategies was one of the regulatory dynamics between school and families used in the IK project. Teacher/researchers were aware of the importance of diversity as they discussed these ways of acting during their pre- and in-service teacher education.

In the IK project the analysis of pupils' answers to the IACC shaped teachers' practices in each class, allowing them to use inter-empowerment mechanisms and to choose, adapt or elaborate mathematical tasks which facilitated pupils' learning and development. Respecting pupils' cultural roots made them feel accepted at school. Thus, they engaged in mathematics activities, and were open to learning new solving strategies and ways of reasoning. Their self-esteem improved. They began expressing their voices and acting like legitimate participants in mathematics classes, even when they had experienced school underachievement in previous school grades (see César, 2009, 2013a, 2013b, 2017a). These were very important steps to promote their school achievement and to allow them to trace life trajectories of participation that included longer school paths.

These results would not have been possible without using the IACC, an instrument that allowed teachers to find out pupils' abilities and competencies since the first week of classes, facilitating the decisions about dyad formation and the tasks that would be used in class. In the schools that used bi-univocal cultural mediation outside classes the improvement of pupils' performances was higher. In these schools, cultural agents learnt how to respect and interact with members participating in other cultures, and decision making was shared among agents from the cultural communities, the families and the school board, teachers, and pupils. This had positive impacts in the school community, and in the social community. These were the schools in which more pupils participating in undervalued cultures traced long-term school paths, something not common in their communities. Thus, bi-univocal cultural mediation is an effective way to improve equity and social justice.

Illuminating how cultural differences, particularly L1, shape mathematical reasoning and solving strategies is an important issue. Teachers need to be aware of these differences. This should be discussed in teachers' pre- and in-service education.

Acknowledgments

The *Interaction and Knowledge* (IK) project was partially supported by the *Instituto de Inovação Educacional*, medida SIQE 2 (project n.º 7/96), in 1996/97 and 1997/98, and by *Centro de Investigação em Educação da Faculdade de Ciências da Universidade de Lisboa* (1996 – 2008). Our gratitude goes to the teachers, pupils and colleagues who participated in the IK. Our special thanks to Sofia Coelho, a psychologist who studied and lived in London, bilingual in English and Portuguese, for her editing of the English language.

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