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The development of students' mathematical reasoning in Calculus courses

El desarrollo del razonamiento matemático de los estudiantes en los cursos de Cálculo

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Abstract ∞ The aim is to understand how the students' involvement in solving and discussing exploratory tasks, combined with the teacher's actions, can contribute to the development of their mathematical reasoning (MR) in a Differential and Integral Calculus course. The study was qualitative, with an interpretive approach, and the participants were undergraduate engineering students. The data consists of (a) protocols containing written records of students' discussions; (b) audio recordings of these discussions; and (c) video of the plenary discussion facilitated by the teacher. We discuss the MR processes that students mobilize, in particular conjecturing, generalizing, and justifying. We point out which of the teacher's actions can contribute to the development of MR, in a continuous and growing movement, essentially related to the deepening of discussions based on elements presented by the students themselves, and the opportunities that are created in this process.

Keywords ∞ Mathematics Education; Teaching Differential and Integral Calculus; Mathematical Reasoning; Exploratory tasks

Resumen ∞ El objetivo es comprender cómo la participación de los estudiantes en la resolución y discusión de tareas exploratorias, vinculada a las acciones del profesor, puede contribuir al desarrollo de su razonamiento matemático (RM) en el curso de Cálculo Diferencial e Integral. El estudio fue una investigación cualitativa, y los participantes fueron estudiantes de ingeniería de educación superior. Los datos se componen de (a) protocolos que contienen registros escritos de las discusiones de estudiantes; (b) audios de estas discusiones; y (c) vídeo de la discusión plenaria mediada por el profesor. Discutimos los procesos de RM que movilizan los estudiantes, especialmente conjeturar, generalizar y justificar. Señalamos cuál de las acciones del docente puede contribuir al desarrollo del RM, en un movimiento continuo y creciente, relacionado esencialmente con la profundización de las discusiones a partir de elementos presentados por los propios estudiantes, y las oportunidades que se crean en ese proceso.

Palabras clave ∞ Educación Matemática; Enseñanza del Cálculo Diferencial e Integral; Razonamiento matemático; Tareas exploratorias

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1. INTRODUCTION

Differential and Integral Calculus (DIC) is a crucial component of the fundamental core of Exact Science programs in Brazil, particularly in Engineering, as it enhances the necessary reasoning processes for formulating and solving problems in various areas. It also aids in comprehending and validating phenomena through experimentation and effective written, oral, and graphic communication (Brasil, 2019).

However, the fundamental science importance is marred by high failure rates among engineering undergraduate students over the years. In addition to gaps in students' prior mathematical knowledge (Ghedamsi & Lecorre, 2021), students' failure in the DIC subject and subsequent program dropouts can be attributed to the didactic-pedagogical structure of engineering programs. The prevalence of a traditional teaching methodology that prioritizes lectures and teacher-centered instruction (Hieb et al., 2015) contributes to this issue (Thompson & Harel, 2021). Despite the progress made in theoretical frameworks for teaching and learning mathematics that have aided research in Higher Education, they fail to manifest in DIC classrooms (Lithner, 2008; Rasmussen et al., 2014).

The current movement to revamp engineering teaching models, as exemplified by Brazil's National Curriculum Guidelines for Undergraduate Courses (Brasil, 2019), emphasizes the development of skills necessary for formulating and designing innovative solutions, utilizing suitable techniques, communication, teamwork, and an investigative mindset that must be cultivated throughout the educational process. The development of technical and general skills is linked directly to guidelines stemming from research in Mathematics Education. Results indicate that teaching approaches that promote the development of students' mathematical reasoning (MR) through collaborative work, productive mathematical discussions (Stein et al., 2008; Rodrigues et al., 2018), and solving exploratory tasks (Ponte, 2005) show promising potential. Therefore, it is essential to have a learning environment with such features as outlined in the DIC (Trevisan & Mendes, 2018) to effectively teach Engineering in Brazil. When students engage in problem-solving tasks, it can trigger various cognitive processes of Mathematical Reasoning (MR) (Jeannotte & Kieran, 2017), where they form conjectures and proffer justifications, validations, or refutations (Lannin et al., 2011).

The present article aims to understand how the involvement in solving and discussing exploratory tasks, linked to the teacher's actions, can contribute to the development of MR in students of DIC courses. Based on the research aim and hypothesis, we have the following questions i. What MR processes are mobilized by the students of the DIC course when solving tasks of an exploratory nature? ii. How do the teacher's actions, working with the development of exploratory tasks in DIC courses, contribute to the students' MR development?

To accomplish this, we analyze the research on MR and its processes and examine the teacher actions that can facilitate it, such as introducing exploratory tasks. We present a theoretical framework outlining the function and mathematical content of the proposed tasks. Next, we introduce the study context and describe the methodological procedures employed. We subsequently analyze data collected from three tasks proposed in DIC courses. Finally, we discuss the results, consider the proposed theoretical framework, answer the research questions, and highlight implications for teaching and research in DIC.

2. Theoretical framework and background

The development of students' MR is undoubtedly, a challenge for teachers of all levels of education (Niss, 2013). Despite the different theoretical approaches followed to this topic, there is a consensus among researchers that MR — the dynamic process of conjecturing, generalizing, investigating why, and developing and evaluating arguments — is a fundamental skill in mathematical learning.

Jeannotte and Kieran (2017) discuss the process aspect of MR related to the search for similarities and differences (generalizing, conjecturing, identifying a pattern, comparing, and classifying), and others related to validation (justifying, proof, and formal proving); in addition to these processes, and supporting them, exemplifying. Despite the relevance of all processes in the development of students' MR, in this work, we pay special attention to the processes of conjecturing, generalizing, and justifying, since they stand out as essential processes.

According to Jeannotte and Kieran (2017), conjecturing is a "MR process that, by the searching for similarities and differences, infers a narrative about some regularity that has likely or probable epistemic value and that has the potential for mathematical theorization" (p. 10). Students can make valid or invalid conjectures, based on valid or sometimes invalid reasoning, and the latter, while not ideal, can serve as a starting point for understanding mathematical ideas. Conjectures can be written in different ways, or even exist only in the students' minds. Students develop conjectures about concepts and skills, whether they state them or not, by observing commonalities between different facts. Students develop generalizations, that lead them to manipulate and clarify the meaning of concepts, symbols, and representations (Mata-Pereira & Ponte, 2018).

More specifically, a generalization "involves identifying commonalities across cases or extending the inference beyond the domain in which it originated" and "identifying the application of the generalization by recognizing the relevant domain" (Lannin et al., 2011, p.12), which is an important specificity for formulating conjectures. An important aspect of producing a generalization is the identification of a pattern. Jeannotte and Kieran (2017) define identifying a pattern as "An MR process that, by the searching for similarities and differences, infers a narrative about a recursive relationship between mathematical objects or relations" (p. 10).

Justifying, in turn, has to do with explaining a previously made assumption by presenting reasons (arguments) to change the epistemic value first from "probable to more probable" and then to true or false. Araman et al. (2019) state that justification has to do with "identifying relationships that allow us to understand why a statement can be true or false" (p. 468). Thus, understanding justification as closely related to "investigating why" leads the students' elaboration of arguments "to convince themselves and others of why a particular statement is true" (Lannin et al., 2011, p. 35), using various mathematical concepts.

Supported by procedures, properties, and definitions, justifications, and generalizations are seen as central aspects of reasoning, that mathematically validate certain statements. It is up to the teacher to propose situations that promote justifications, emphasizing the "why," and redirecting students to the context of a given situation. In this sense, several studies carried out in different countries show that "only at an advanced level do students appreciate the need for convincing reasoning from an explicit set of assumptions" (Galbraith, 1995, p. 412). Thus, the development of students' MR implies interventions that lead them to make sense of justifications, supported by mathematical knowledge. To mobilize these processes, the way the teacher organizes the lesson is relevant, and the choice of tasks and the way they are presented are of singular importance. Studies show that task-solving episodes, with exploratory or investigative tasks, lead students to develop MR (Ponte et al., 2012).

The work with exploratory tasks does not replace others present in a traditional DIC classroom, such as the teacher's exposition of the concepts or the solution of routine tasks. Therefore, it is necessary to consider teacher's actions. Some studies characterize teacher's actions, especially those that are most effective in promoting students' reasoning, among which we highlight Wood (1998), Ponte et al. (2013), and Ellis et al. (2019).

Wood (1998) believes that mathematical reasoning develops in classrooms where there are frequent interaction situations that require teacher actions that enable exchanges between teacher and students. The author points out that in classrooms where exchanges between teachers and students occur, it is possible to observe three patterns of interaction. The first interaction pattern — report — occurs when the student tells how he solved the problem. In the second interaction pattern — questioning — the student continues to tell how he solved the problem, but the teacher asks him to explain why he did it that way and to clarify for the teacher and the other students why he did it that way. In the third interaction pattern — arguing — students tell how they solved the problem, clarifying their meanings and giving reasons.

In turn, Ponte et al. (2013) propose a model for analyzing the teacher's actions in moments of discussion with small groups of students, or in a plenary session, with the whole class, organized in four categories. The first is to invite, which would be the student's first contact with the topic to be addressed. The second is to guide/support, in which the teacher, through questions, guides the students to continue participating in the solution of a task that has already started. The third is to inform/suggest, the moment when the teacher validates the students' answers, introducing new information and providing new arguments. The last is to challenge, when the teacher "puts the student in the situation of moving on to new territory, whether in terms of representations, interpreting of statements, making connections, or reasoning, arguing or evaluating" (Ponte et al., 2013, p. 59).

Another similar model (Ellis et al., 2019) also organizes the teacher's actions into categories, each of which details possible actions. In this model, we can identify some actions and interactions in pedagogical movements that the teacher can play in the learning process at the moment of collective discussions in the class. In the low potential movements, we notice that the teacher's actions are focused on encouraging the students to present their justifications and conclusions about the task, while in the high potential movements, the actions are focused on elucidating the reasoning developed.

Based on these models, Araman et al. (2019) organized an analysis framework that describes the teacher actions that support mathematical reasoning, shown in Table 1. Through the articulation of these models, the proposal describes the different actions that underline each of the main categories of teacher actions, and which will serve as support for the analysis in the context of one of the tasks presented in this article, in higher education engineering.

Table 1. Teacher's Actions Analysis Table (Araman et al., 2019, p. 476)

Categories	Actions
Invite	Requesting answers to specific questions. Requesting reports on how they did it.
Guide/ Support	Providing clues to students. Encouraging explanation. Leading the students' thinking. Focusing the students' thinking on important facts. Encouraging students to re-state their answers. Encouraging students to re-elaborate their answers.
Inform/ Suggest	Validating correct answers provided by students. Correcting incorrect answers provided by students. Re-elaborating responses provided by students. Providing information and explanations. Encouraging and providing multiple resolution strategies.
Challenge	Requesting students to provide reasons (justifications). Proposing challenges. Encouraging assessments. Encouraging reflection. Pressing for accuracy. Pressing for generalizations.

In preparing the tasks to be used in this teaching model, in order to promote covariational reasoning and a reinterpretation of the concept of function, it is considered that the following skills should be developed: constituting the quantities involved in the situation; reasoning about the process of measuring these quantities; imagining measures of quantities that vary continuously; and coordinating two quantities that vary together.

It should be emphasized that proposing these reflections about variables and functions does not dispense with the algebraic approach and proofs and theorems being taught on DIC. On the contrary, it is intended to make them more meaningful and closer to applications in the context of Engineering.

3. METHODOLOGICAL PROCEDURES

3.1. Research Characterization and Context

The research follows a qualitative-interpretive approach (Bogdan & Biklen, 1994; Crotty, 1998), with design-based research (DBR) assumptions (Cobb et al., 2003). In particular, it adopts the teaching experience methodology as a special type of DBR, organized in task-solving episodes (Trevisan & Mendes, 2018), considering the development of the planning, application, and investigation processes about these episodes in DIC classrooms, and their relationship with the development of students' MR.

The data presented here were collected from Engineering students' DIC courses in classes taught by the first author. The assignment cycles were conducted between 2017 and 2019 in DIC, part of the first-semester grid (32 classes of 150 minutes). The curriculum includes the study of functions, limits, derivatives, and integrals of real functions, of a real variable. This paper focuses on three tasks (each of a class of 150 minutes), where about 50 students were organized into groups of three or four. Group work was followed by a plenary session to share solutions and systematize concepts, following the assumptions of Stein et al's. (2008) assumptions for orchestrating productive mathematical discussions. The tasks (Figures 1, 2, and 3) were designed to get students to mobilize their ability to analyze the change variables, in a coordinated way by articulating different representations (Thompson & Carlson, 2017). In task 3, we chose not to explicitly state that the rate of spread of the rumor begins to decrease when almost everyone has heard of the rumor, leaving it up to students to consider (or not) this assumption.

Figure 1. Task 1. Bottle Task, adapted from Thompson & Carlson (2017)

Water is poured into a spherical bottle at a constant rate. Use this information and the shape of the bottle to answer the following questions. Sketch a graph showing the level of water in the bottle over time. Explain the reasoning that led you to this graph.

Figure 2. Task 2. Mixture Task

A tank contains 5000 liters of fresh water. A mixture containing 750 grams of salt diluted in 25 liters of water is pumped into the tank every minute. Investigate how the concentration of the mixture in the tank behaves for "very large" time values.

Figure 3. Task 3. Rumor Task, adapted from Connally et al. (2009)

When a rumor starts in a small town, initially, the number of people who have heard it starts slowly, and as more and more people know about it and comment on it, it spreads rapidly, even as the number of people who know about it reaches the borders of the region. Plot a graph showing how many people know about the rumor over time.

3.2. Data collection and analysis procedures

The data collected consisted of (a) protocols containing written records of the discussions in small groups of students working on exploratory tasks; (b) audio recordings of the discussions in these small groups; and (c) video recordings of plenary discussions of student solutions facilitated by the teacher.

The analysis of the data is carried out inductively (Erickson, 1986), using different techniques — considering that these data are produced by different sources. The analysis of the collected data, although supported by the theoretical references, is surrounded by the understanding and comprehension of the three researchers.

After the data collection by the first author and the teacher of the class, a systematic, more organized, and rigorous analysis was carried out, based on the stages present in the Powell et al. (2004) model, first listening to the audio in its entirety, then identifying significant moments and transcribing them, and then analyzing them. An initial categorization of the teacher's thought processes and teacher's actions was carried out individually by each of the three researchers and, in a second moment, they all came together to make a comparative analysis, until they reached a consensus, whose analysis is presented in the next section of this article.

In order to identify a greater variety of reasoning processes mobilized by the students, data of types (a) and (b) were considered to identify a greater variety of reasoning processes mobilized by students, in the case of Tasks 1 and 2, data of types (a) and (b) were considered. In this paper, only one group of three students was analyzed. The criteria for selecting the group was to select those in which students were more involved in "presenting, justifying, arguing and negotiating of meanings" (Rodrigues et al., 2018, p. 399). In turn, to understand how the teacher's actions can contribute to the development of MR (Araman et al., 2019), we highlight the data in (c), considering an excerpt from the plenary in which we believe there was expressive participation of students and a greater variety of these actions. The results are organized, sequentially and separately, for each of the proposed tasks, considering the objectives announced in the introduction of the article.

4. Data presentation and analysis

4.1. Task 1

The student members of the group (S1, S2, S3) made a first sketch (Figure 4, left), which we understand as a first conjecture that they tried to validate in the following discussion:

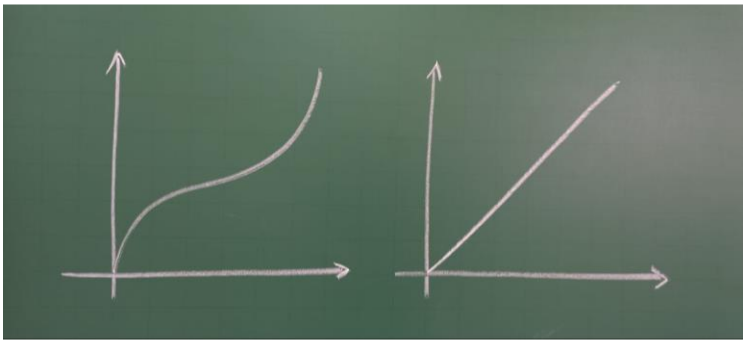
> [1.1] S1: The graph will look like this, won't it?
>
> [1.2] S2: Yes, because first it goes up fast, then it slows down and then comes back faster, because of this here [points to the center of the bottle].
>
> [1.3] S3: So, but then if you're going to measure the height of the bottle by the diameter, it's a straight line [The student might say "diameter" to refer to a vertical axis with which to measure the height of the empty bottle].

[1.4] S2: No, because the radius here is smaller than here [comparing the lower part with the middle part of the bottle] so the amount of water you pour here, it will appear here immediately. When it starts to fill from that point on, it will take time to rise and here it will rise faster [pointing to the diameters of the water surface in the bottle as it would be filled].

[1.5] S3: Yes, but I'm talking about the diameter.

[1.6] S1: I think it would look kind of like this [Figure 4, right]. Would it be a straight line?

Figure 4. Sketches proposed by the group of students in Task 1



In order to validate the first conjecture, student S2 elaborated justifications based on the shape of the bottle, imagining what the diameter of the water surface would be as it was filled [1.2]. He argued that the height would first increase rapidly, then decrease, and then return more rapidly. Student S3, in turn, presented a new conjecture: that the graph relating the water level to the diameter of the water surface would be a straight line [1.3]. Student S2 again elaborated on his justification for the graph format, now mentioning the radius in his argument [1.4]. Student S3 seemed to be confused [1.5], not realizing that S2's argument implicitly considered time as the independent variable, not the radius of the bottle. A new representation (a line — Figure 4, right) emerged, suggested by S1 based on S3's conjecture [1.6]. The discussion continued:

[1.7] S1: So I think the graph would look like this [Figure 4, left], because when you start pouring water here, it's going to go up fast, it's going to slow down a little bit, look at the size here [points to the middle of the bottle], then when it starts to taper here, it's going to go up faster. This graph here [Figure 5, right] would be the water level. The volume will be constant, it will always fill the same amount [Figure 4, right].

In this excerpt, Student S1 reformulated the arguments to validate the initial conjecture based on the "way" in which the water level varied. He also pointed to the other graph, acknowledging that it represented the volume of water in the bottle; however, in his justification, he incorrectly used the word constant, since the rate of change of the volume was assumed to be constant in the task statement.

Although the group managed to validate the initial conjecture, the elaboration of arguments justifying the change of the concavity, or even the fact that the initial part of the graph has a concavity facing downwards and then upwards, was not part of the discussion. These aspects were accepted almost as a matter of course by S2 and S3 from the sketch prepared by S1, although it was an incomplete justification that did not address the concept of concavity.

4.2. Task 2

The students (S1, S2, S3) of the same group started the solution by trying to interpret the information contained in the problem statement:

[2.1] S3: If after one minute 25 liters of water and 750 grams of salt are added, let's say time 1 is 1 minute.

[2.2] S1: So I'll have 5025 liters.

[2.3] S3: I think we should draw a table first.

[2.4] S1: Okay. At t=0 there are 5000 [liters of water]. At time 1, there are 5025 liters of water and 750 grams of salt. At time 2, there are 5050 liters of water and 1500 grams of salt. Now we can get the percentage of concentration. Do you get it? Because then we start to get the difference.

[2.5] S2: Ah, yes, because it would increase infinitely.

[2.6] S3: Do it: 750 divided by 5025. It's 0.14, because concentration is the amount of matter divided by the total volume. Ah! I'll do it here because we're going to follow a pattern. Now 1500 divided by 5050. It's 0.30. It doubled!

[2.7] S1: But I don't think it's going to stay that way.

[2.8] S2: Yes, I don't think it will.

In this first excerpt, students made some conjectures based on the context of the task, for example in [2.1] and in [2.2]. Students began to think about how to organize the data to find a generalization. In this way, they started to try to draw a table [2.3]. From what S1 said [2.4], after reaching some conclusions, they can perceive the function as increasing and supposedly without stabilization, so they started trying to find ways to work with the numbers now. In [2.6] student S3 came up with a new conjecture, trying to generalize that the concentration would double in every unit of time, but soon after other students refuted this conjecture. The discussion continued:

[2.9] S2: We have to find the function to get to a number far ahead.

[2.10] S1: Do it three more times...

[2.11] S3: After three minutes the concentration is 0.44. I think I see a pattern, but let's wait.

[2.12] S2: It increased again by 0.15 from time 2 to time 3. From here to there [from 1 to 2 minutes] and from here to there [from 2 to 3 minutes] it's 0.15. I love it when that works.

[2.13] S3: I think now we should come up with a function. Because then we can draw the limit to infinity.

[2.14] S1: Will there come a time when the concentration stays constant? Because you're always adding more water.

[2.15] S2: Yeah, because if they were just adding salt, it would eventually saturate, but they're adding it diluted in water, so no matter how much it increases, it's not going to grow forever.

In this excerpt, students began to investigate and develop mathematical arguments to explain why the previous generalization (about double) was incorrect. Next, they conjectured that the rate of change of concentration would remain constant (about 0.15) [2.12]. So they began to think about using mathematical relations to prove this new conjecture, such as the concept of limit [2.13]. So the group tried to develop an algebraic expression that generalizes the relation between mixture concentration and time (t):

[2.16] S3: The concentration will be the mass divided by the volume. The mass will be 750t. And the volume is 5000 plus 25t.

[2.17] S1: There you go, now we can apply it and see if it's right.

[2.18] S3: Do it with 4.

[2.19] S2: It worked!

In the excerpt, student S3, in a process of generalization, proposed a mathematical formula that makes sense to everyone. Having found a formula, in [2.17] the student looked for a way to validate it, and for this, he chose certain time values [t=4]. The discussion continued.

[2.20] S3: Now let's try to get to the limit. I don't remember the rule, do you?

[2.21] S1: I think if it goes to infinity multiplied by a very large number, it goes to zero. I don't remember how to do that either.

[2.22] S3: Because look, what is going to happen? This number 750t is going to get really big, and this part 5000 plus 25t is also going to get really big.

[2.23] S3: Look, if it's 1 million, it's 29.99.

[2.24] S2: So it's always going to go up, but up to what number? Because like, it wasn't over 30, but up to what number does it go?

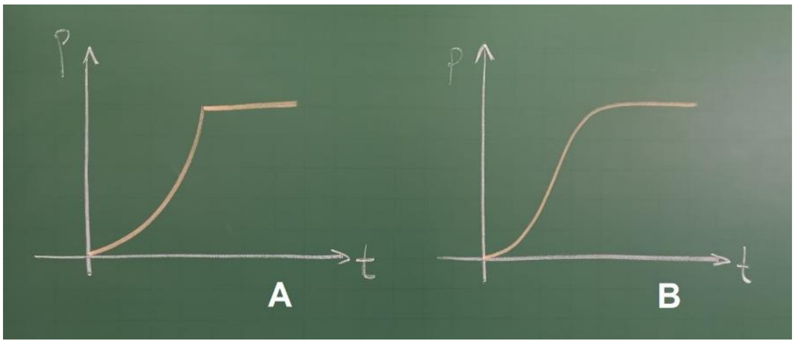
[2.25] S3: I know why; it's because 750 divided by 25 is 30. If that time is too long, that 5,000 becomes negligible, so what will it be? Practically 750 divided by 25, so it's never going to be more than 30 anyway.

Finally, in this last part of the discussion, the students tried to understand and validate their conjectures about the limit [2.20-2.23]. They still did not understand why the concentration remained stable at 30. Soon after, student S3 was able to unravel the reason and explain it to his peers [2.25], thus justifying their choices and validating the generalization they had formulated.

4.3. Task 3

After two different graphs have been reproduced on the blackboard by the group members (Figure 5, graph A by S1 and B by S2 and S3), the teacher (T) leads the discussion to clarify what each group thought, to assess which group(s) followed the task statement and to make comparisons, identifying similarities and differences.

Figure 5. Graphs produced by the groups for Task 3



[3.1] S2: [Graph B] The last part, instead of continuing to grow, it stabilizes, because there will come a time when there are no more people to hear the rumor.

[3.2] T: And how does it stabilize? Is that shown in the drawing?

[3.3] S2: The number of people will stabilize and it starts to stabilize, this in this drawing here is like this [points to a part of the graph B].

[3.4] S1: [Graph A] Our difference is that it doesn't stabilize in a way where it stabilizes all at once. It forms a 90-degree angle as it stabilizes. The same curve grows in time, but it stabilizes at once because it reaches the limit of people and becomes different from them.

[3.5] S3: Sir, just to explain, I don't know if this is right [Graph B]. We thought that the number of people would grow fast over time. It would be the first part [before the turning point], and then it would grow slower. As time goes by, the number of people in the city will run out. So it will stabilize, and over time the same number of people will know, right?

[3.5] T: Did this idea of yours cause this "turnaround" here [inflection point in graph B]?

[3.6] S2: Yes, because it wasn't abrupt.

[3.7] S1: [Graph A]. The maximum growth is reached, therefore the maximum number of people. This is the moment when it stops growing and stabilizes. In the second [graph B], when it reaches the maximum growth, it starts to decrease, then it will reach it.

[3.8] T: Technically speaking, growth declines, what does that mean?

[3.9] S2: The growth decreased.

[3.10] T: What does it mean to say that the growth has decreased?

[3.11] S3: It is growing slower.

Based on the conjecture presented by S2 concerning graph B (in [3.1]), the teacher carried out actions of the guidance/support category, in [3.2]. The teacher encouraged the students to give justifications for the constructions presented, which led students S1, S2, and S3 to better elaborate their justifications. For example, after the teacher's question about the stability described by S2, he justified by pointing to a point in his representation and showing that he recognized a horizontal asymptote in the graph of the function. In turn, S1 justified that the way this asymptote is formed in the two representations is different; S3 reflected by pointing out that this difference is due to the way each of the groups interpreted the task (as a slower or faster growth rate).

He also focused the student's thinking on important facts by trying to differentiate some subtle aspects in graphs A and B, an action that also falls into the category of guidance/support. In particular, he drew attention to what he called a "turnaround" in [3.5], referring to the inflection point. This action led S1 to elaborate his justifications for the inflection point, both for graphs A and B, in [3.7]. The challenging action, in turn, is present in the subsequent discussions, in [3.8] and [3.10], when the teacher asked the students to explain what the expression growth decline means. He intended to push the students for a more precise explanation, which was provided by S3 in [3.11].

The arguments students used to explain their construction, shown in the excerpt above, represent important aspects of the development of covariation reasoning skills. In particular, the context of the task led them to recognize the need to change the concavity of the graph to represent the behavior of a variable (the number of new people who hear about the rumor) that changes the mode as it grows. This allows us to intuitively explore this concept by relating it to the idea of rate of variation: if the rate of variation is increasing, then the graph is concave upward; and if the rate of variation is decreasing, then the graph is concave downward. This allowed students to intuitively explore a concept that would be formalized later in the DIC course, as seen in the following dialog.

> [3.11] T: The problem does not explain whether this slowing down occurs. So if I interpret that there is a moment when this rumor spread starts to slow down, would that slow down tell me which is the correct graph?
>
> [3.12] S1: Graph B.
>
> [3.13] T: Well, if I don't assume that this slowing down has occurred... that it's spreading and spreading, there comes a time when people run out. What graph would that be?
>
> [3.14] S3: The first one [graph A].
>
> [3.15] T: Suppose at some point the pace of rumor spread slows down, leading to a change in the trend that has been going on in the graph. How can we describe this change? What happens on the graph?
>
> [3.16] S3: It has two curves [referring to one concave part up and another concave part down].

In this excerpt, we first highlight a combination of actions from three categories. In order to move students forward in their understanding, the teacher performed informing/suggesting actions, as in [3.11], when he clarified that the task does not explain what the rate of spread of the rumor is, and in [3.15], when he pointed out that at some point the rate of spread of the rumor starts to slow down, leading to a shift in the trend. The teacher also acted as guidance/support by encouraging students to identify which graph would best represent the situation if the rate did or did not slow down, in [3.11] and [3.13]. Finally, the action of asking students to establish a relationship between the slowing behavior of the rumor propagation rate and the corresponding graphical representation [3.15]. This question was answered by S3 in [3.16].

5. Discussion and final considerations

The aim of this work was to understand how the performance of task-solving episodes, linked to the teacher's actions, can contribute to the development of students' MR in DIC courses in three episodes. To achieve this, data analyses were performed in two aspects: in the case of the first and second episodes, the MR processes mobilized by the students while solving exploratory tasks were analyzed. In the third episode, it was analyzed with a moment of discussion with the teacher, highlighting the role of the teacher's actions during the plenary discussion and how these actions supported mathematical reasoning.

Regarding the MR processes, the group that solved Task 1 made a first conjecture, mobilizing their mathematical knowledge related to the rate and direction of growth. This conjecture was illustrated by drawing a graph, which students tried to validate in the following discussion, identifying relationships that allowed understanding why a statement was true or false (Araman, Serrazina & Ponte, 2019). This was done in the subsequent discussion, where students tried to validate the conjecture they had developed based on new relationships between their mathematical knowledge (related to the shape, height, and diameter of the bottle).

The discussion led to the elaboration of a new conjecture, illustrated by a new graph (a line), which was not immediately accepted by the group, leading to new discussions and the establishment of new relationships. The group decided to validate the first conjecture and the first graph by justifying that, although supported by logical arguments based on ideas already understood (Lannin et al., 2011), it misused the expression constant growth and did not address the concept of concavity.

In the case of Task 2, there is evidence of a cyclical movement (Jeannotte & Kieran, 2017) to reason about mathematical relationships and develop statements (conjectures), and to try to recognize and explain the validity (or not) of these statements (justification) (Lannin et al., 2011). At times, it has allowed them to extend regularities observed in specific cases (generalization) - such as the change in the amount of water and the amount of salt as a function of time and, more generally, concentration as a function of time - thus demonstrating conceptual understanding (Mata-Pereira & Ponte, 2018).

Regarding the moment of the teacher's discussion with the students, in the case of Task 3, a continuous and growing movement of the teacher's actions is identified during the plenary, essentially related to the deepening of the discussions based on the elements presented by the students themselves and the opportunities to mobilize different mathematical reasoning processes. Thus, according to Araman, Serrazina, and Ponte (2019), the discussion of the task initially considered guide/support and challenging actions. Guide/support actions are recurrent in the first excerpts and are intended to engage students in the discussion, explain how they performed the task, and focus on important aspects of the task. These actions are very likely to support the progress of the students' thinking since the purpose of the questions and their development contribute to the conjecturing of mathematical ideas. The students, through these actions of the teacher, made explicit and reformulated their initial conjectures.

Actions in the challenge category are more present when the teacher is trying to deepen the justification of the facts presented by the students. For Araman, Serrazina, and Ponte (2019), these actions have the greatest potential for developing more sophisticated MR processes, such as generalizing and justifying. We also recognize that, in the inform/suggest category, the actions were designed to provide information and concepts that were new to the students. For example, when asked by the teacher, they justified their responses by extending their intuitive understanding of mathematical elements present in their representations, such as horizontal asymptote, inflection point, and growth rates.

The three proposed tasks allowed students to explore quantitative relationships between variables, reflect on these quantities and the relationship between them (Thompson & Carlson, 2017), and use different mathematical representations. They were encouraged to coordinate two quantities that vary together, to recognize how quantities are related, the direction of increase and decrease, the existence of rates of variation and possible changes in that rate, different representations of the same function, as well as to apply DIC algebraic tools based on a situation intuitively explored a priori and, in the case of Task 2, to develop justifications as a basis for generalizations. In the case of Task 1 and Task 3, the constructed narratives showed a movement towards a coordinated analysis of the changes that occur in two interdependent variables, without the groups being able to explicitly relate these changes to the concavity of the function graph.

Working with exploratory tasks in a DIC course allowed students to explore concepts associated with the study of functions, such as domain and image, direction of growth, rate of increase/decrease, concavity, and concavity variation. Students were not expected to reach a formal definition of these concepts, but to mobilize reasoning processes that contributed to a more robust understanding of the interdependent relationship that exists between two variables and how changes in this relationship can be observed in engineering problems.

Constant work with exploratory tasks can improve students' ability to mobilize more MR processes, leading them to understand that mathematical justifications are elaborated from logical arguments based on previously understood ideas, without resorting to arguments based on authority, perception, common sense, or specific examples (Lannin et al., 2011). Thus, as implications of this study in the context of engineering education, we highlight some aspects that emerged from this practice: more active and interested students, with initiative to solve tasks proposed in class, compared to students in a traditional DIC classroom in Brazil (Trevisan, 2022).

The results show that task-solving episodes combine the basic skills provided by the new National Curriculum Guidelines for Engineering Courses in Brazil

(Brasil, 2019) with subjects of the basic cycle of an engineering program (as DIC), exploring the same content from approaches that mobilize MR processes in discussions of real situations in a process of proposing, constructing, validating, and generalizing. It should be noted that lessons that follow this type of methodology are hardly found in traditional teaching environments. The background of this text is an environment in which students are constantly asked to reason mathematically, to elaborate conjectures, to find a way to justify them, and to build generalizations, which are fundamental for the development of students' MR in the context of Mathematics for Engineering. The data showed that in the moments of discussion among students and between students and teacher (through their different actions), students provided mathematical arguments supported by the processes of mathematical reasoning.

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El desarrollo del razonamiento matemático de los estudiantes en los cursos de Cálculo

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El objetivo de este artículo es comprender cómo la participación de los estudiantes en la resolución y discusión de tareas exploratorias, vinculadas a la actuación del profesor, puede contribuir al desarrollo de su razonamiento matemático (RM) en un curso de Cálculo Diferencial e Integral (CDI). El estudio fue diseñado como una investigación cualitativa, con un enfoque interpretativo, y los participantes fueron estudiantes de un programa de educación superior de ingeniería en una Universidad Federal en el Estado de Paraná, Brasil. Los datos recolectados consisten en (a) protocolos con registros escritos de discusiones en pequeños grupos de estudiantes sobre tres tareas exploratorias; (b) audios de estas discusiones; y (c) video de la discusión plenaria mediada por el profesor a partir de las resoluciones de los estudiantes. Discutimos los procesos de RM que movilizan los alumnos, especialmente conjeturar, generalizar y justificar. En el caso de la primera tarea, el grupo hizo conjeturas sobre la velocidad y la dirección del crecimiento e intentó validar sus conjeturas basándose en nuevas relaciones entre sus conocimientos matemáticos (relacionados con la forma, la altura y el diámetro de la botella). En el caso de la segunda tarea, hay evidencias de un movimiento cíclico para razonar sobre relaciones matemáticas y desarrollar afirmaciones (conjeturas), tratando de reconocer y explicar la validez (o no) de estas afirmaciones (justificación). A veces ha sido posible extender regularidades observadas en casos particulares (generalización), demostrando comprensión conceptual. Con respecto a la tercera tarea, señalamos cuáles de las acciones del profesor pueden contribuir al desarrollo de la RM, en un movimiento continuo y creciente, esencialmente relacionado con la profundización de las discusiones a partir de elementos presentados por los propios alumnos, y las oportunidades que se crean en este proceso. La actuación del profesor en la gestión de la discusión está en consonancia con las competencias que se deben formar en los futuros ingenieros, ya que favorece el intercambio de ideas, con repreguntas de explicaciones y justificaciones, aunque sean parciales o incorrectas. El trabajo constante con tareas exploratorias puede mejorar la capacidad de los estudiantes para movilizar más procesos de RM, llevándoles a comprender que las justificaciones matemáticas se elaboran a partir de argumentos lógicos basados en ideas previamente comprendidas, sin recurrir a argumentos basados en la autoridad, la percepción, el sentido común o ejemplos concretos. Los resultados también mostraron que un curso DIC dinámico con estudiantes involucrados en tareas exploratorias es consistente con el desarrollo de la RM y las competencias previstas por las nuevas Directrices Curriculares Nacionales para los Cursos de Ingeniería en Brasil.