The analytical proof of the reflective property of a parabolic mirror

EDVCATIO PHYSICORVM
QVO NON ASCENDAM?
ISSN 1870-9095

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(Received 27 January 2023, accepted 5 April 2023)

Abstract

In this paper, we present an analytical proof of the reflective property of a parabolic mirror that uses the laws of light reflection and an interesting method for solving the resulting non-linear differential equation. We emphasize that within the framework of this approach, students find the sought-for shape of the mirror surface by "discovering".

Keywords: Reflective property, Parabolic mirror, Laws of light reflection, Non-linear differential equation.

Resumen

En este artículo, presentamos una prueba analítica de la propiedad reflectante de un espejo parabólico que utiliza las leyes de la reflexión de la luz y un método interesante para resolver la ecuación diferencial no lineal resultante. Destacamos que en el marco de este enfoque los estudiantes encuentran la forma buscada de la superficie del espejo "descubriendo".

Palabras clave: Propiedad reflectante, Espejo parabólico, Leyes de reflexión de la luz, Ecuación diferencial no lineal.

I. INTRODUCTION

A parabolic mirror is a reflective surface used to collect or project energy such as light, sound, or radio waves. Its shape is part of a circular paraboloid, that is, the surface generated by a parabola revolving around its axis. The parabolic reflector transforms an incoming plane wave travelling along the axis into a spherical wave converging toward the focus. Conversely, a spherical wave generated by a point source placed in the focus is reflected into a plane wave propagating as a collimated beam along the axis.

A parabolic reflector pointing upward can be formed by rotating a reflective liquid, like mercury, around a vertical axis [1]. This makes the liquid-mirror telescope possible. The same technique is used in rotating furnaces to make solid reflectors.

Unlike a spherical mirror, a parabolic mirror is free of spherical aberrations, that is, all rays parallel to the axis of such a mirror are collected after reflection at a single point. This property of the paraboloid was first noticed by ancient Greek scientist Diocles [2].

There is a plenty proofs of the reflective property of a parabolic mirror. Most of these proofs are purely geometric [3]. Some of the approaches are based on testing a trial solution in the form of a quadratic function in the basic differential equation of the theory (see equation (3) of the present paper). In this paper, we present an analytical proof of this property that uses the laws of light reflection and an interesting method for solving the resulting non-linear differential equation. We emphasize that within the framework of this approach, students find the sought-for shape of the mirror surface by "discovering" (the uniqueness of this shape). The issues outlined in this article will be useful for undergraduate students studying the advanced topics of geometrical optics.

II. THE PROBLEM

Let us consider a curved mirror surface that is constructed as follows. We start with a curve, denoted by the $x-y$ plane, that is symmetrical under a reflection through the y axis; i.e. $y(-x)=y(x)$. The y-axis is thus the symmetry-axis of the two-dimensional curve $y(x)$. The three-dimensional curved mirror surface is then obtained by rotating the curve about the y-axis, thereby producing a "surface of revolution" corresponding to the surface of the mirror. The projection of this surface onto the $x-y$ plane yields the original curve $y(x)$. We also assume that our mirror is concave everywhere (the derivative $y'(x)=dy/dx>0$), that is, it is able to transform a parallel beam of light into a converging one.

Due to the symmetry of the three-dimensional surface, it is sufficient to examine the light rays propagating in the $x-y$ plane. Let us consider the ray which is initially propagating in a direction parallel to the y-axis (figure 1). It then strikes the mirror with an angle of incidence θ with respect to the normal to the curve $y(x)$ at the point P, labeled by coordinates (x,y).

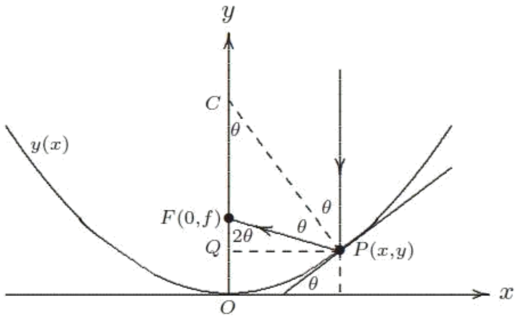

Figure 1. Geometry of the problem.

Using the law of reflection, the angle of reflection of the resulting reflected ray is equal to the angle of incidence θ. The simple geometrical considerations imply that the angle the tangent line makes with the x-axis is also given by θ. Thus,

$$\tan\theta = y'(x). \tag{1}$$

In addition, the angle FCP is also equal to θ (as the two initial light rays are parallel), from which we conclude that the angle QFP is equal to 2θ, as indicated in the above figure. Hence, the distance OF is given by:

$$OF = y + \frac{x}{\tan 2\theta} = y + \frac{x\left(1-\tan^2\theta\right)}{2\tan\theta}. \tag{2}$$

We need to find such a function $y(x)$ that the distance OF (the focal distance f) would be fixed, that is, it would not depend on x and y.

III. THE SOLUTION

Using equation (1) and (2), we have:

$$y + \frac{x\left(1-y'^2\right)}{2y'} = f = const. \tag{3}$$

This is a non-linear differential equation with respect to the function $y(x)$, but it can be easily reduced to a separable variable equation. Solving equation (3) with respect to function $y'(x)$, we get:

$$y' = -\frac{f-y}{x} \pm \sqrt{\left(\frac{f-y}{x}\right)^2 + 1}. \tag{4}$$

Since, we are only interested in the positive solutions of equation (2), in what follows we should consider only the solution corresponding to sign "+" before the square root.

Now, we introduce new function $z(x)$, so that

$$y = xz + f. \tag{5}$$

In this case, equation (4) reduces to

$$xz' = \sqrt{z^2+1}. \tag{6}$$

Equation (4) is a separable differential equation. Separating the variables in it and integrating, we get:

$$\ln\left(z+\sqrt{z^2+1}\right) = \ln(Cx). \tag{7}$$

where C is the integration constant. Solving equation (7), we obtain:

$$(x) = \frac{(Cx)^2-1}{2Cx}. \tag{8}$$

Then, using equation (5), we derive:

$$y(x) = \frac{(Cx)^2-1}{2C} + f. \tag{9}$$

If we require that $y(0) = 0$, then $C = 1/2f$ and, finally

$$y(x) = \frac{x^2}{4f} \tag{10}$$

Thus, we discover that the generating curve is a parabola. Moreover, the focus of the parabola coincides with the focus of the mirror F.

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