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An evaluation of various probability density functions for predicting diameter distributions in pure and mixed-species stands in Türkiye

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Abstract

Aim of study: To assess the capabilities of some infrequently used probability density functions (PDFs) in modeling stand diameter distributions and compare their performance to that of typical PDFs.

Area of study: The research was conducted in pure and mixed stands located in the OF Planning Unit of the Trabzon Forest Regional Directorate in Northern Türkiye.

Materials and methods: A set of 17,324 DBH measurements, originating from 608 sample plots located in stands of even-aged and pure and mixed stands, were obtained to represent various stand conditions such as site quality, age, and stand density in OF planning unit forests. In order to ensure a minimum of 30-40 trees in each sample plot, the plot sizes ranged from 0.04 to 0.08 hectares, depending on stand density. The parameters of PDFs include Weibull with 3P and 2P, Rice, Rayleigh, Normal, Nagakami, Lognormal with 2P and 3P, Lévy with 1p and 2P, Laplace, Kumaraswamy, Johnson's SB, and Gamma were estimated using the maximum likelihood estimation (MLE) prediction procedure. Additionally, the goodness of fit test was combined with the Kolmogorov-Smirnov test (statistically at a 95% confidence interval).

Main results: The Rayleigh distribution was the model that best explained the diameter distributions of pure and mixed forests in the OF Planning Unit (as Fit Index (FI) = 0.6743 and acceptance rate 96.4% based on the result of one sample Kolmogorov-Smirnov test).

Research highlights: Less commonly used PDFs such as Rice, Nakagami, and Kumaraswamy-4P demonstrated superior predictive performance compared to some traditional distributions widely used in forestry, including Weibull-2P and -3P, Johnson's SB, Normal, Gamma-3P, and Lognormal-3P.

Additional key words: Rayleigh distribution; maximum likelihood estimation; stand structures

Abbreviations used: AIC (Akaike information criterion); BIC (Bayesian information criterion); DBH (diameter at breast height); EI (error index); FI (fit index); MAE (mean absolute error); MLE (maximum likelihood estimation); PDFs (probability density functions); PMF (probability mass function); RMSE (root mean squared error).

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Introduction

Diameter distribution models play a crucial role in forestry, providing valuable information through tables that depict the number and distribution of trees across specific diameter classes (Bettinger et al., 2022). Furthermore, stand structures and the variety of wood products that can be obtained from stands can be estimated thanks to the diameter distributions (Gadow & Hui, 1999). Diameter distributions, which differ from the models that provide general projections of various stand characteristics, provide more specific information about the distribution of tree sizes in the stand. In terms of estimation level, diameter distribution models are approach that falls between stand and single tree models (Vanclay, 1994; Gadow & Hui, 1999).

Diameter distributions in forestry have long been a subject of extensive research and analysis. Over the years, numerous studies have been conducted to understand and estimate the patterns of diameter distributions in forest stands. Various methods and approaches have been employed to tackle this complex task, reflecting the continuous quest for accurate and reliable estimations. In the initial research conducted on this topic, Gram in 1883 observed that the diameter distributions of beech stands exhibit suitability for normal distribution. Similarly, De Liocourt in 1898, found that the diameter distributions of old stands are well-suited for exponential distribution (Bailey & Dell, 1973). In the 1930s, there was a notable surge of interest in modelling diameter distributions using various mathematical series. In the 1960s, there was a significant shift in the modeling of diameter distributions in stand, with the introduction and utilization of statistical density functions in forestry (Packard, 2000). According to Packard (2000), the first diameter distribution study that utilized a PDF was conducted by Clutter & Bennett in 1965. Indeed, the modeling of diameter distributions in forestry has been the subject of numerous studies over the years, with researchers utilizing various PDFs to capture the patterns and characteristics of diameter distributions within forest stands (Packard, 2000).

To represent the diameter distributions of stands in forestry, researchers commonly utilize various PDFs. PDFs are statistical functions that describe the probability of different diameter values occurring within a given distribution. With the help of PDFs and their corresponding cumulative distribution functions (CDFs), it is possible to obtain various characteristics of stands, including basal area, volume distributions, and the number of trees (Waldy et al., 2022). Models play a crucial role in forest management planning and decision-making processes. They provide valuable insights and predictions related to forest stand dynamics, growth, and yield, as well as assist in planning forest management operations such as thinning and rotation. Indeed, diameter distribution models continue to play a fundamental role in modern forest planning systems (Liu et al., 2014). Even today's modern forest planning systems are thought to be based on diameter distribution models, according to various researchers. Furthermore, diameter distribution models, when combined with complementary forecasting models, provide a straightforward approach to obtaining data on biomass, carbon stock, or wood energy by diameter class for forest management objectives (Özçelik et al., 2016).

Over the last few decades, as statistical density functions, different functions such as Normal, Lognormal, Gamma, Beta, Johnson's SB, and Weibull distributions have been widely used in forestry. The simulating diameter distributions for specific periods and stand conditions involve estimating the future values of the parameters associated with the statistical density functions that represent these distributions. Each statistical density function used to model diameter distributions in forestry has its own advantages and disadvantages. The selection of an appropriate function depends on various factors, including stand age, stand structure (even-aged, uneven-aged, or irregular), species composition (pure or mixes stands), and the characteristics of the data set being analyzed (Liu et al., 2014). Pogoda et al. (2019), Sakıcı (2021), Sakıcı & Dal (2021), among others, found that Johnson's SB distribution is well-suited for modeling the diameter distributions of various stand sizes. However, its complex structure and practical difficulties limit its widespread application. On the other hand, the Weibull function is simple, adaptable, and convenient. It offers considerable flexibility in the number of parameters used, making it the most commonly applied theoretical distribution in practice (Siipilehto & Mehtätalo, 2013; Korkmaz et al., 2022).

According to Gadow & Hui (1999), it is recommended to include diameter class models based on PDFs within the framework of diameter class models. Indeed, incorporating diameter distribution models based on different statistical density functions, such as the Weibull, Johnson, and Beta functions, within diameter class models can provide more detailed estimates and information about the stand structure.

Besides these traditional and well-known PDFs with Gamma, Beta, Weibull, and Johnson's SB functions, the science of statistics has developed other distribution functions such as Laplace, Rayleigh, Nakagami, Lévy, Rice, and Kumaraswamy (Michalowicz et al., 2013). The Rice distribution has important connections to other well-known several distributions, including the Chi-Square, Normal, Log-Normal, and Rayleigh distributions, and is valid for real positive numbers (Jiang et al., 2018). The Rayleigh distribution is a special example of the 2-parameter Weibull distribution and is named after the English Lord Rayleigh (Aslam et al., 2015). The Nakagami distribution is a relatively new PDF that first appeared in 1960 and is one of the most widely used for modeling right-skewed, positive datasets (Akgül & Şenoğlu, 2023). The Lévy distribution is a probability distribution that is characterized as both continuous (for non-negative random variables) and stable (for random variables; x+y) (Knopova & Schilling, 2013; Yousof et al., 2022). The Laplace distribution, one of the oldest known probability distributions, is unimodal (just one peak), symmetrical, and has a sharper peak than the Normal distribution (Liu & Kozubowski, 2015). Kumaraswamy's distribution is a probability distribution that shares many of the same characteristics with Beta distribution but offers certain advantages in terms of tractability (it's a broader PDF distribution for double-bounded random processes) and its applicability to a wide range of natural events (El-

Table 1. Summary diameter at breast height (DBH, cm) statistics of sample trees per tree species

Tree species	N	Min.	Max.	Mean	SD
Scotch pine (*Pinus sylvestris* L.)	723	8	68	23.8	10.0
Caucasian fir (*Abies nordmanniana* Link.)	370	8	86	26.3	15.1
Oriental spruce (*Picea orientalis* (L.) Link.)	4737	8	127	22.9	12.5
Oriental beech (*Fagus orientalis* Lipsky)	2731	8	140	27.2	17.8
Oak (*Quercus* ssp.)	46	8	51	21.6	9.4
Hornbeam (*Carpinus* ssp.)	584	8	79	16.2	8.6
Alder (*Alnus* ssp.)	4144	8	61	18.4	8.1
Poplar (*Populus* ssp.)	20	8	35	17.0	7.2
Chestnut (*Castanea sativa* Mill.)	3357	8	71	18.6	8.3
Maple (*Acer* ssp.)	37	8	47	18.4	9.6
Elm (*Ulmus* ssp.)	8	8	25	13.1	6.1
Hazelnut (*Corylus* ssp.)	194	8	21	10.1	2.2
Rhododendron ssp.	148	8	14	9.0	1.2
Walnut (*Juglans* ssp.)	7	15	28	21.9	4.7
Acacia (*Acacia* ssp.)	3	12	28	19.0	8.2
Locust tree (*Robinia pseudoacacia* L.)	4	8	20	11.3	5.9
Other broadleaf	212	8	41	14.3	6.9

Sagheer, 2019). Diameter distribution modeling studies continue to be the subject of numerous national (such as Sivrikaya & Karakaş, 2020; Sakıcı, 2021; Sakıcı & Dal, 2021; Seki, 2022) and international (such as Pogoda et al., 2019; Schmidt et al., 2020; Ciceu et al., 2021; Guo et al., 2022; Yang et al., 2022) researches, whıch use well-known traditional PDFs many of them. However, evaluation of the application and success of some other PDFs in forestry has been limited. Therefore, our study aims to assess if various PDFs, which are not frequently used in forestry, are compatible with the diameter distributions of the stands in the OF Planning Unit (Trabzon province), which is located in the northeastern part of Türkiye.

Material and methods

Material

The data used in this study were collected from even-aged, pure, and mixed stands located in OF forests in northwestern Türkiye (40º 36′ 26″ - 40º 59′ 13″ N, 40º 12′ 21″ - 40º 36′ 00″ E) (Fig. 1). The natural tree species distributed in the OF Planning Unit are Oriental spruce (*Picea orientalis* (L.) Link.), Scotch pine (*Pinus sylvestris* (L.)), Caucasian fir (*Abies nordmanniana* subsp. *nordmanniana Spach.*), Oriental beech (*Fagus orientalis* Lipsky), hornbeam (*Carpinus betulus* L.), alder (*Alnus glutinosa* subsp. *barbata* (C.A. Mey.) Yalt.) and other non-primary stand types (Table 1). The study area is characterized by an altitude ranging from 50 to 2100 m a.s.l., slopes varying from 5% to 85%. Geomorphologically, the study regions are described as high mountainous areas with moderate to steep slopes.

The average annual temperature in the study area ranges from 4 ºC to 17.6 ºC. The climatic regime is typical characteristic of the Black Sea region, with temperate winters and cool summers. The average annual rainfall ranges from 1000 mm to 2020 mm, with a uniform precipitation pattern throughout the year. The abundant rainfall in the region contributes to the presence of dense forest areas. Also, the significant rainfall in the study area of the OF Planning Unit contributes to the development of dense forest areas. Adequate rainfall provides the necessary moisture for plant growth, allowing trees to thrive and form dense stands. The challenging terrain and geographical location often result in limited human activities such as afforestation or extensive management practices. The forests in the study area, characterized by high slopes and limited human intervention, have a unique capacity for self-renewal and natural development.

The even-aged OF Planning Unit comprises approximately 14650 hectares of productive forests, which include both pure stands and mixed stands. This sizeable area of forested land provides significant opportunities for various forest management activities and the sustainable utilization of forest resources.

In the study, a systematic sampling procedure was employed to sample the entire forest area at regular intervals of 400 × 400 m. This systematic approach ensures a representative coverage of the forest and allows for the

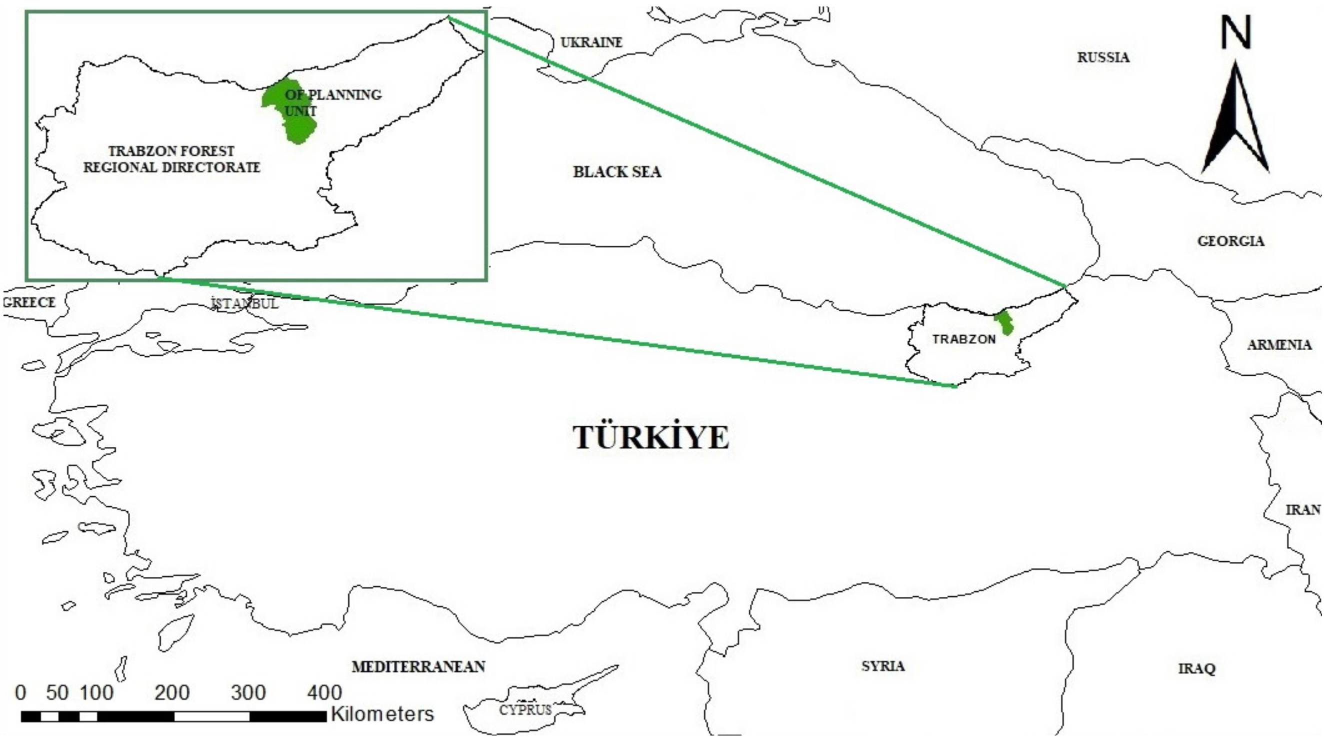

Figure 1. Study area.

collection of data across different areas of the OF Planning Unit. In the summer of 2008 we selected 608 sample plots with 17,324 DBH (diameters at breast height) measurement data to represent various stand conditions such as site quality, age, and stand density. To attain a minimum of 30-40 trees in these sample plots, the plot sizes ranged from 0.04 to 0.08 ha, depending on stand density; DBHs (1.3 m) of all live trees in the sample plots were measured. The minimum, mean, maximum, and standard deviation (SD) of measurement data based on tree species are indicated in Table 1.

Methods

PDFs actually hold a significant place in statistics and are widely utilized in forestry for various purposes, including modeling the number of trees at specific diameter classes or levels and developing diameter distribution models. These functions give the ratio of the number of trees in a certain diameter class to the total number of trees in the stand and thus make estimations between 0 and 1. Normal, Lognormal, Gamma, Beta, Johnson's SB, and Weibull distributions are the main PDFs that can be considered as examples of these functions. These functions express the ratio of the number of trees in a specific diameter class to the total number of trees in the stand, representing the probability or likelihood of finding a tree within that diameter range.

In forestry, various estimation techniques are employed for modeling diameter distributions, including maximum likelihood estimation (MLE), percentile estimation, and method of moment estimation (Diamantopoulou et al., 2015). Among these techniques, MLE has been widely utilized for estimating distribution parameters due to its asymptotic efficiency (Cao, 2022). According to Lu & Zhang (2010), MLE outperforms other parameter estimation techniques by providing the least variance for sample data, given that the assumptions of normality and homoscedasticity are met.

To estimate the parameters of PDFs using the MLE technique, numerical analysis methods involving iterative processes are commonly employed. The MLE technique involves maximizing the maximum likelihood function in order to apply estimation process and obtain parameter estimates (Harter & Moore, 1965). Because it is commonly used in forestry and involves stable processes in parameter estimation (Michalowicz et al., 2013; Sedighi et al., 2021), the choice of MLE technique was reasonable for our study, and the success of 14 different PDFs (Table 2) in modeling diameter distribution was evaluated. In the context of our study, where we are modeling diameter distributions using various PDFs, the MLE procedure can be represented by the following formula:

$$L(\theta|x) = F(x_1, x_2, \ldots, x_n|\theta) = \prod_{i=1}^{n} f(x_i|\theta) \qquad (1)$$

In this formula, a parameter's probability mass function (PMF) is represented by the symbol f(x, θ). The probability values for an integer x are provided by this function when the parameter is θ. We define the joint function to get the function of the unknown parameter vector, θ, where x is the

Table 2. Various probability density functions (PDFs) for modelling diameter distributions

No.	Distribution	Density function	Parameters
1	Gamma (3P)	$f(x)=\frac{(x-\gamma)^{\alpha-1}}{\beta^{\alpha}\Gamma(\alpha)}\exp(-(x-\gamma)/\beta)$	α: continuous shape parameter ($\alpha > 0$) β: continuous scale parameter ($\beta > 0$) γ: continuous location parameter Γ: Gamma function $\gamma \leq x < +\infty$
2	Johnson's SB (4P)	$f(x)=\frac{\delta}{\lambda\sqrt{2\pi}\,z(1-z)}\exp\left(\frac{-1}{2}\left(\gamma+\delta ln\left(\frac{z}{1-z}\right)\right)^2\right)$	γ, δ:continuous shape parameters ($\delta > 0$) λ: continuous scale parameter (λ>0) ξ: continuous location parameter $\xi \leq x \leq \xi + \lambda$
3	Kumaraswamy (4P)	$f(x)=\frac{\propto_1 \propto_2 z^{\alpha_1-1}(1-z^{\alpha_1})^{\propto_2-1}}{(b-a)}$	α_1, α_2: continuous shape parameters (α_1, $\alpha_2 > 0$) a, b: continuous boundary parameters ($a < b$) $a \leq x \leq b$
4	Laplace (2P)	$f(x)=\frac{\lambda}{2}\exp(-\lambda\lvert x-\mu\rvert)$	λ: continuous inverse scale parameter (λ>0) μ: continuous location parameter $-\infty < x < +\infty$
5	Lévy (1P)	$f(x)=\sqrt{\frac{\sigma}{2\pi}}\ \frac{\exp(-0.5\,\sigma/x)}{(x-\gamma)^{3/2}}$	σ: continuous scale parameter ($\sigma > 0$)
6	Lévy (2P)	$f(x)=\sqrt{\frac{\sigma}{2\pi}}\ \frac{\exp(-0.5\,\sigma/(x-\gamma))}{(x-\gamma)^{3/2}}$	γ: continuous location parameter (γ≡0 yields the one-parameter Lévy distribution) $\gamma < x < +\infty$
7	Lognormal (2P)	$f(x)=\frac{\exp\left(\frac{-1}{2}\left(\frac{lnx-\mu}{\sigma}\right)^2\right)}{x\sigma\sqrt{2\pi}}$	σ and μ: continuous parameters ($\sigma > 0$)
8	Lognormal (3P)	$f(x)=\frac{\exp\left[\frac{-1}{2}\left(\frac{\ln(x-\gamma)-\mu}{\sigma}\right)^2\right]}{(x-\gamma)\sigma\sqrt{2\pi}}$	γ: continuous location parameter (γ≡0 yields the two-parameter Lognormal distribution) $\gamma < x < +\infty$
9	Nakagami (2P)	$f(x)=\frac{2m^m}{\Gamma(m)\Omega^m}x^{2m-1}\exp\left(\frac{-m}{\Omega}x^2\right)$	m: continuous parameter ($m \geq 0.5$) Ω: continuous parameter ($\Omega > 0$) $0 \leq x < +\infty$
10	Normal (2P)	$f(x)=\frac{\exp\left(\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)}{\sigma\sqrt{2\pi}}$	σ: continuous scale parameter ($\sigma > 0$) μ: continuous location parameter $-\infty < x < +\infty$
11	Rayleigh (1P)	$f(x)=\frac{x}{\sigma^2}\exp\left(\frac{-1}{2}\left(\frac{x}{\sigma}\right)^2\right)$	σ: continuous scale parameter ($\sigma > 0$)
12	Rice (2P)	$f(x)=\frac{x}{\sigma^2}\exp\left(\frac{-(x^2+v^2)}{2\sigma^2}\right)I_0\left(\frac{xv}{\sigma^2}\right)$	v, σ: continuous parameters ($v \geq 0$; $\sigma > 0$) I_0: modified Bessel function of the first kind of zero $0 \leq x < +\infty$
13	Weibull (2P)	$f(x)=\frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1}\exp\left[-\left(\frac{x}{\beta}\right)^{\alpha}\right]$	α: continuous shape parameter ($\alpha > 0$) β: continuous scale parameter ($\beta > 0$)
14	Weibull (3P)	$f(x)=\frac{\alpha}{\beta}\left(\frac{x-\gamma}{\beta}\right)^{\alpha-1}\exp\left(-\left(\frac{x-\gamma}{\beta}\right)^{\alpha}\right)$	γ: continuous location parameter (γ≡0 yields the two-parameter Weibull distribution) $\gamma \leq x < +\infty$

* x is the tree diameter here.

collection of sample data. The sum of the individual PMFs is the combined PMF of the n independent and identically distributed (iid) observations from this experiment. The "likelihood function" is known as a function of the parameter θ, for a fixed sample $X = (X_1, X_2, \ldots, X_n)$, and it is denoted by L $(\theta\,|\,x)$.

Table 3. Goodness-of-fit criteria for numerical comparisons for PDFs

EI (error index by Reynolds et al., 1988)	$EI=\sum_{i=1}^{n}\left\|N_i-\widehat{N}_i\right\|$
MAE (mean absolute error)	$MAE=\frac{\sum_{i=1}^{n}\left\|N_i-\widehat{N}_i\right\|}{n}$
RMSE (root mean squared error)	$RMSE=\sqrt{\frac{\sum_{i=1}^{n}\left(N_i-\widehat{N}_i\right)^2}{n-p}}$
RMSE%	$RMSE\%=\left(\left[\sqrt{\sum_{i=1}^{n}\left(N_i-\widehat{N}_i\right)^2 (n-p)}\right]\overline{N}_i\right).100$
FI (fit index)	$R^2=1-\frac{\sum_{i=1}^{n}\left(N_i-\widehat{N}_i\right)^2}{\sum_{i=1}^{n}\left(N_i-\overline{N}_i\right)^2}$
AIC (Akaike information criterion)	$AIC=-2\log(L)+2p$
BIC (Bayesian information criterion)	$BIC=-2\log(L)+plog(n)$

Due to its simplicity, the logarithm of the likelihood is chosen in practice. Because of this, we chose the logarithmic form:

$$\Lambda(\theta)=\log L(\theta|x)=\sum_{i=1}^{n}\log f\left(x_i|\theta\right) \quad (2)$$

The discrete distribution with the highest likelihood is what we were aiming for. So, determining the parameter values that maximize the log-likelihood function was necessary. This is called the maximum likelihood estimate *θ*.

$$\hat{\theta}=\theta\ with\,L(\theta)=max_{\theta}L(\theta) \quad (3)$$

The analytical approach is used when we can find a closed-form solution for the equation $\Lambda'(\theta)$; otherwise, we apply optimization strategies to solve the problem numerically (Mathwave, 2014).

By using all the diameter values measured in the sample areas, these diameter values were grouped into 4-cm diameter classes, what is generally preferred in Turkish forestry, and the observed values for the number of trees in each diameter class were obtained. Converting the number of trees in diameter classes to hectares is a common practice in forestry studies to standardize the calculations and facilitate comparisons between different sample areas. Since our study has different sample area sizes (from 400, 600, or 800 m^2) converting the number of trees to hectares ensures that the calculations are performed on a consistent basis.

Then, the estimation of the parameters of the PDFs (formulas given in Table 2), was carried out by using the package program with the Easy-Fit software library (version 5.3, Mathwave Technologies). Easy-Fit provides a user-friendly interface and a variety of statistical distributions that can be fitted to data.

The number of trees in diameter classes of these sample areas was estimated by using PDFs whose parameters were estimated using this Easy-Fit software based on the observation values classified in diameter classes in each sample area.

The error criteria indexes with EI (error index by Reynolds et al. (1988), MAE (mean absolute error), RMSE (root mean squared error), RMSE%, FI (fit index), AIC (Akaike information criterion), and BIC (Bayesian information criterion) were calculated (formulas given in Table 3) and used to compare the fitting ability of these PDFs. The fact that EI, MAE, RMSE, RMSE%, AIC, and BIC values are as small as possible and the FI value is as close to 1 as possible indicates that the PDF predicts values close to the observation value and is quite successful.

In the formulas listed above: N_i = calculated number of trees; $\hat{N}_i$ = estimated number of trees; $\overline{N}_i$ = mean number of trees; n = a number of data; L = maximum value of the log-likelihood function; p = a number of parameters within the model.

Besides these fitting criteria for these PDFs, the differences between observed and predicted diameter distributions by these PDFs were tested at the 5% significance level ($p<0.05$) with a Kolmogorov-Smirnov (K-S) one-sample test using the Mathwave EasyFit 5.3 package program (Poudel & Cao, 2013):

$$D_n=max.\left|F\left(d_i\right)-\widehat{F}\left(d_i\right)\right| \quad (4)$$

Results

The goodness-of-fit statistics of EI, MAE, RMSE, RMSE%, FI, AIC, and BIC for the studied PDFs that

Table 4. Comparison of the predictive performance for PDFs modeling diameter distribution

No.	PDF	EI	MAE	RMSE	RMSE%	FI	AIC	BIC
1	Gamma (3P)	283205.46	899.2207	29.9870	40.8483	0.5228	31313.5044	32733.5701
2	Johnson's-SB (4P)	174343.47	2647.1164	51.4501	70.0854	0.5781	33528.6198	34948.6855
3	Kumaraswamy (4P)	227929.27	971.1588	31.1634	42.4508	0.5420	31471.3990	32891.4647
4	Laplace (2P)	625561.99	5050.8286	71.0692	96.8105	0.1950	34854.1250	36274.1907
5	Lévy (1P)	850136.90	4449.0134	66.7009	90.8601	0.2909	34593.8377	36013.9034
6	Lévy (2P)	448542.75	1081.7557	32.8901	44.8029	0.2900	31692.6661	33112.7318
7	Lognormal (2P)	287202.29	3595.0041	59.9584	81.6753	0.4270	34156.5638	35576.6295
8	Lognormal (3P)	214726.75	940.1438	30.6618	41.7675	0.5580	31404.8096	32824.8753
9	Nakagami (2P)	270387.02	3208.3502	56.6423	77.1582	0.4887	33923.1150	35343.1807
10	Normal 2P	291134.55	3319.2551	57.6130	78.4805	0.4710	33992.8358	35412.9015
11	Rayleigh (1P)	145301.94	733.5491	27.0841	36.8940	0.6743	30895.7273	32315.7930
12	Rice (2P)	185850.69	2564.7601	50.6435	68.9866	0.5912	33463.7768	34883.8425
13	Weibull (2P)	216529.06	2784.1889	52.7654	71.8771	0.5563	33632.1966	35052.2623
14	Weibull (3P)	172062.66	761.7998	27.6007	37.5977	0.6059	30973.2560	32393.3217

model the diameter distributions are given in Table 4. In these values, EI ranged from 145301.94 to 850136.90, MAE from 733.5491 to 5050.8286, RMSE from 27.0841 to 71.0692, RMSE% from 36.894 to 96.8105, AIC from 30895.7273 to 34854.125, BIC from 32315.793 to 36274.1907, and FI from 0.1950 to 0.6743. According to the goodness-of-fit statistics (Table 4), Laplace's PDF had the worst predicting performance, with higher EI, MAE, RMSE, RMSE%, AIC, and BIC, as well as the lowest FI of all PDFs. From the various distribution functions tested, the Rayleigh function gave the best predictive fitting results with an EI value of 145301.94, MAE of 733.5491, RMSE of 27.0841, RMSE% of 36.894, AIC of 30895.7273, BIC of 32315.793, and FI of 0.6743. These results suggest that the Rayleigh distribution function outperformed the other distribution functions.

The relationships between the observed and predicted number of trees in various diameter classes for the different tested distributions (PDFs) tested are shown in Figs. 2 and 3. The models with these PDFs that model the number of trees in different diameter classes inclined to a 45º angle with axes, as shown in these graphs. From these graphs, with more correlated relationships between predicted and observed values around the 1:1-line, the PDF of Rayleigh resulted in better predictions than those of other functions.

Table 5 provides the number and percentages of hypothesis acceptance by the one-sample K-S test for the studied PDFs, on which the hypothesis was based, as there was no difference between observational and estimated densities at the 5% significance level ($p<0.05$). When these analysis results were evaluated, it was seen that the Rayleigh distribution had the highest acceptance rate (96.4%), followed by the Nagakami (95.9%) and 2-parameter Weibull (95.6%) functions, respectively.

The relationships between the observed and predicted number of trees for the four PDFs (a) Normal PDF, (b) Weibull-3P PDF, (c) Johnson's SB PDF, and (d) Rayleigh PDF, that delivered the best result for any sample plot are shown in Fig. 4. Upon evaluating the relationships of various PDFs in this graph, it becomes evident that these functions were highly effective in representing the diameter distributions in the sample plot. Apart from that, Fig. 5 shows the graphical variation of Bias and RMSE' values for the most successful Rayleigh-1P (commonly used) and Weibull-3P (traditional and second most successful) functions based on diameter classes. When assessing the deviations between observation and predictions across multiple sample areas with diverse structures, the Rayleigh distribution emerged as successful in modeling the diameter distribution. As demonstrated by this example scatter plot, the Rayleigh distribution stood out when modeling the diameter distribution of OF forests.

Discussion

The study aimed to assess the compatibility of various diameter distribution functions with the stands in the OF Planning Unit using the MLE technique, 14 different diameter distribution functions, including both well-known and newly tested functions, were evaluated for parameter estimation of pure and mixed stands in OF Planning Unit in northern Türkiye.

Rayleigh (1P) distribution demonstrated superior predictive results than commonly used distribution functions

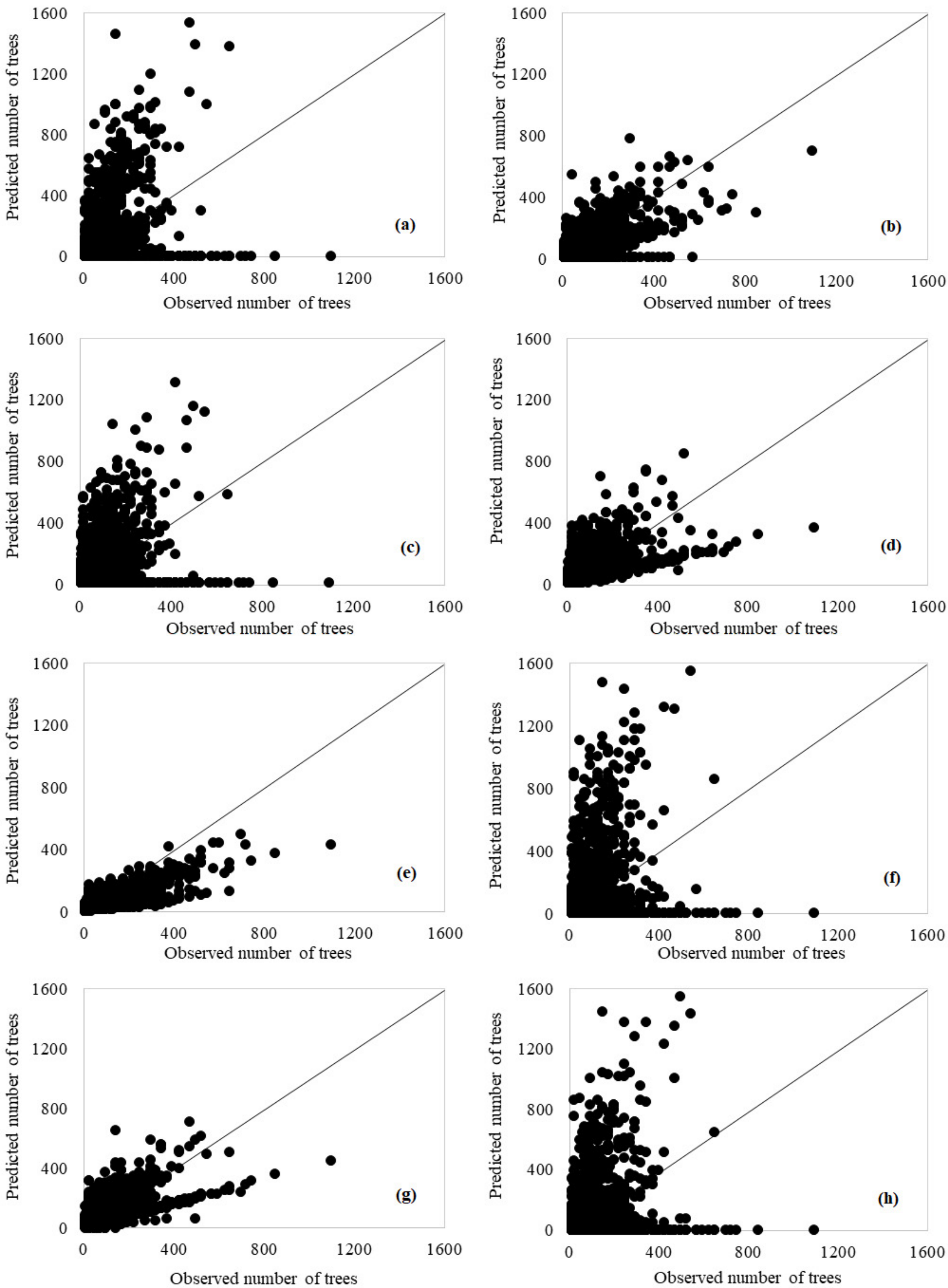

Figure 2. The relationships between observed (x-axis) and predicted number of trees (y-axis) (N/ha) according to the various PDFs: a) Gamma (3P) PDF, b) Johnson's SB (4P) PDF, c) Kumaraswamy (4P) PDF, d) Laplace (2P) PDF, e) Lévy (1P) PDF, f) Lévy (2P) PDF, g) Lognormal (2P) PDF, h) Lognormal (3P) PDF.

Table 5. Results of the one-sample Kolmogorov-Smirnov test of the difference between observed and predicted distributions (p<0.05)

No.	PDFs	No. of sample plots		Acceptance rate (%)
		Hypothesis[1] accepted	Hypothesis rejected	
1	Gamma (3P)	412	196	67.8
2	Johnson's-SB (4P)	576	32	94.7
3	Kumaraswamy (4P)	419	189	68.9
4	Laplace (2P)	488	120	80.3
5	Lévy (1P)	31	577	5.1
6	Lévy (2P)	318	290	52.3
7	Lognormal (2P)	552	56	90.8
8	Lognormal (3P)	401	207	66.0
9	Nakagami (2P)	583	25	95.9
10	Normal 2P	553	55	91.0
11	Rayleigh (1P)	586	22	96.4
12	Rice (2P)	568	40	93.4
13	Weibull (2P)	581	27	95.6
14	Weibull (3P)	433	175	71.2

[1] Hypothesis: there is no difference between observation and estimated frequencies at the 5% significance level.

such as Weibull-3P and -2P, Johnson's SB, Normal, Gamma-3P, and Lognormal-3P in terms of several error values, such as EI, MAE, RMSE, RMSE%, AIC, and BIC, among the 14 studied distribution functions, with a FI of 0.67. According to these results, the Rayleigh function shows its strong performance in accurately representing the observed data and so, it was determined as the most successful function in modeling the diameter distributions of the stands of OF forests with complex multi-layered structures of different structures formed by the combination of various tree species. Furthermore, the study's findings revealed that other distribution functions (i.e. Rice, Nakagami, Kumaraswamy), which are used in other sciences but not so much in forestry, can also produce good outcomes. The Rayleigh function provided a compatible estimate of the observed and expected tree numbers inside 586 of the 608 sample plots, according to the K-S test results (Table 5). The Rayleigh-1P function has the acceptance rate of 96.4%; however, this value for RMSE% was 36.894%. This acceptance rate of 96.4% indicates that the Rayleigh function was able to provide satisfactory estimates for the majority of the sample plots, capturing the observed tree numbers within an acceptable range. Alternatively, RMSE% values near or above 36%, also obtained by Siipilehto & Mehtätalo (2013), Diamantopoulou et al. (2015), and Schmidt et al. (2020) suggest that there may be inherent difficulties in accurately modeling diameter distributions in forestry. The acceptance rate of Rayleigh-1P was 96.4% using the K-S test, even if the RMSE% was around 36%, what indicates that the observed and expected tree numbers inside the sample plots were in good agreement. This may be because, in particular, the positive and negative deviations in the diameter classes somewhat balanced one another out, passing the test.

The graphical checking of Bias and RMSE by diameter class for the Rayleigh-1P and Weibull-3P functions are also given in Fig. 5. As seen in the graph, although RMSE is close to each other in both functions, bias was always higher in Weibull-3P than Rayleigh-1P function (for each diameter class). In addition, in the Weibull-3P and Rayleigh-1P functions, the Bias varied between -147.56 to 155.677 and -9.715 to 24.219, respectively, while the RMSE varied between 4.288 to 225.589 and 2.906 to 73.705, respectively.

As opposed to that, the RMSE for the Weibull-3P function was found to be quite high and variable, especially in thin-diameter (less than 20 cm) trees. The observation that the RMSE for the Weibull-3P function is high and variable, particularly for thin-diameter trees, suggests that this function may not accurately capture the distribution of trees in those diameter classes The variability here may also increase with the inadequacy of the number of trees in the relevant diameter classes.

On the other hand, as expected, the frequently utilized distribution functions (Weibull, Johnson's SB, Normal, Log-normal, and Gamma-3P) performed well in predicting the diameter distribution of the forests in question. However, the newly tested Lévy and Laplace distribution functions proved to be ineffective. The Lévy function

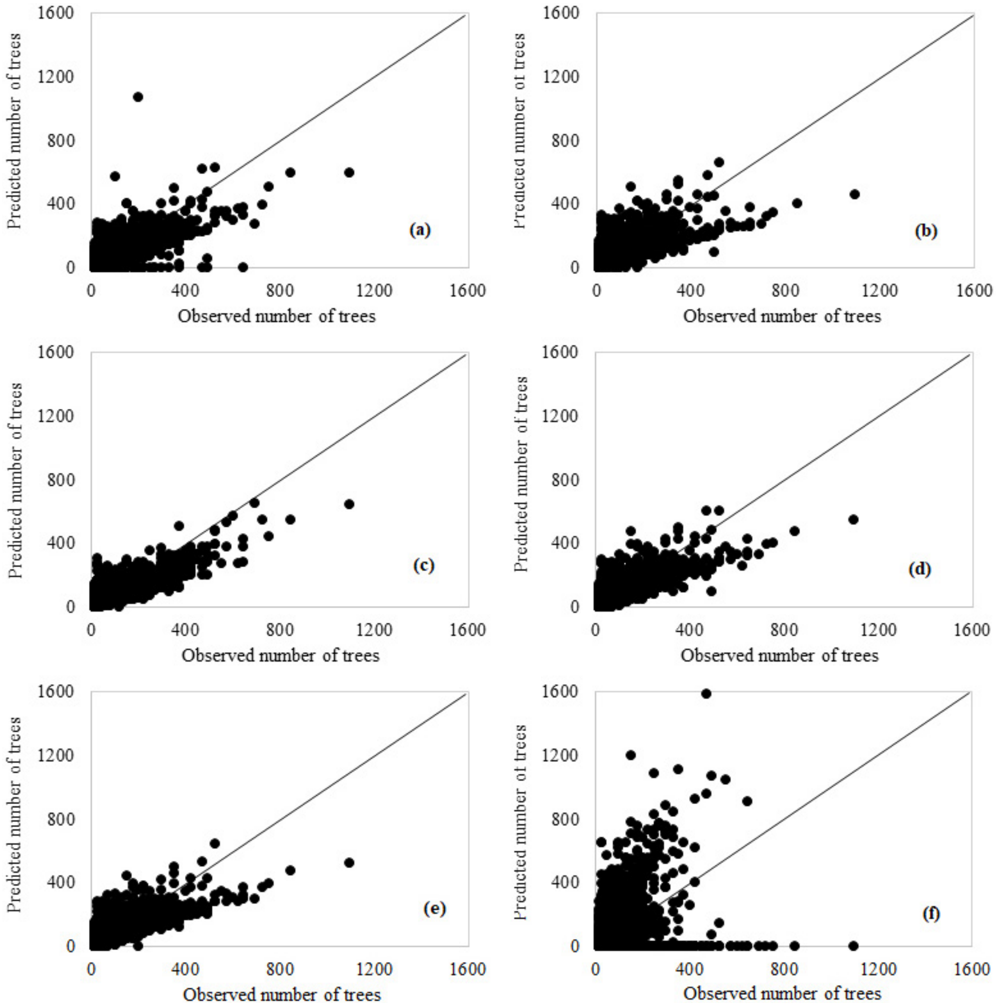

Figure 3. The relationships between observed (x-axis) and predicted number of trees (y-axis) (N/ha) according to the various PDFs: a) Nakagami PDF (2P), b) Normal PDF, c) Rayleigh (1P) PDF, d) Rice PDF (1P), e)Weibull PDF (2P), f) Weibull PDF (3P).

performed the worst of all, with consistent findings in only 5.1% of the 608 sample plots.

In some international academic research done up to now, some PDFs, equations, or models that are not used much in forestry also have been used in addition to the customary PDFs (Weibull, Johnson's SB, Gamma, etc.). For instance, Podlaski (2008) concluded in his research in fir-beech mixed forests that the Birnbaum-Saunders distribution is more accurate compared to the Gamma and Weibull distributions. Binoti et al. (2012) indicated that the Log-logistic (3P), Burr (3P and 4P), Hyperbolic (2P and 3P), Weibull (3P), Fatigue Life (3P), and Nakagami functions provide more satisfactory values in the diameter distribution of young Teak stands than the commonly used Weibull distribution. Duan et al. (2013) concluded that the Richards distribution (0.80% non-rejection rate) provides more satisfactory values than Weibull (3P) (72.33% non-rejection rate) in the diameter distribution of Chinese fir stands. Ogana et al. (2018) found that Logistic-Dagum (LLD-2), Burr XII-2, Dagum-2, Log-Logistic (LL-2), and Kumaraswamy-2 functions were the most successful distributions after Johnson's SB function in diameter distribution of *Eucalyptus* stands.

Recent studies have examined the effectiveness of two and three-parameter Weibull distribution functions using various methodologies (Pogoda et al., 2019; Sun et al., 2019; Schmidt et al., 2020; Schutz & Rosset, 2020; Ciceu et al., 2021). However, no research evaluating alternative distribution functions has been found.

Several studies on diameter distribution modeling in Türkiye have been conducted. Carus (1996) using the Gamma function and observed 65.9% variation in diameter distribution based on site and age in Oriental beech stands in the Western Black Sea region of Türkiye. Ercanlı & Yavuz (2010) concluded that Johnson's SB PDF is suitable for modeling Oriental spruce stands, while the Weibull (3P) PDF is suitable for Scotch pine in Oriental spruce-Scotch pine mixed stands. Sönmez et al. (2010) determined that Johnson SB PDF was the most successful in modeling diameter distributions of Oriental spruce in the Artvin region of Türkiye. Kahriman & Yavuz (2011) found Johnson's SB (4P) function to be successful in

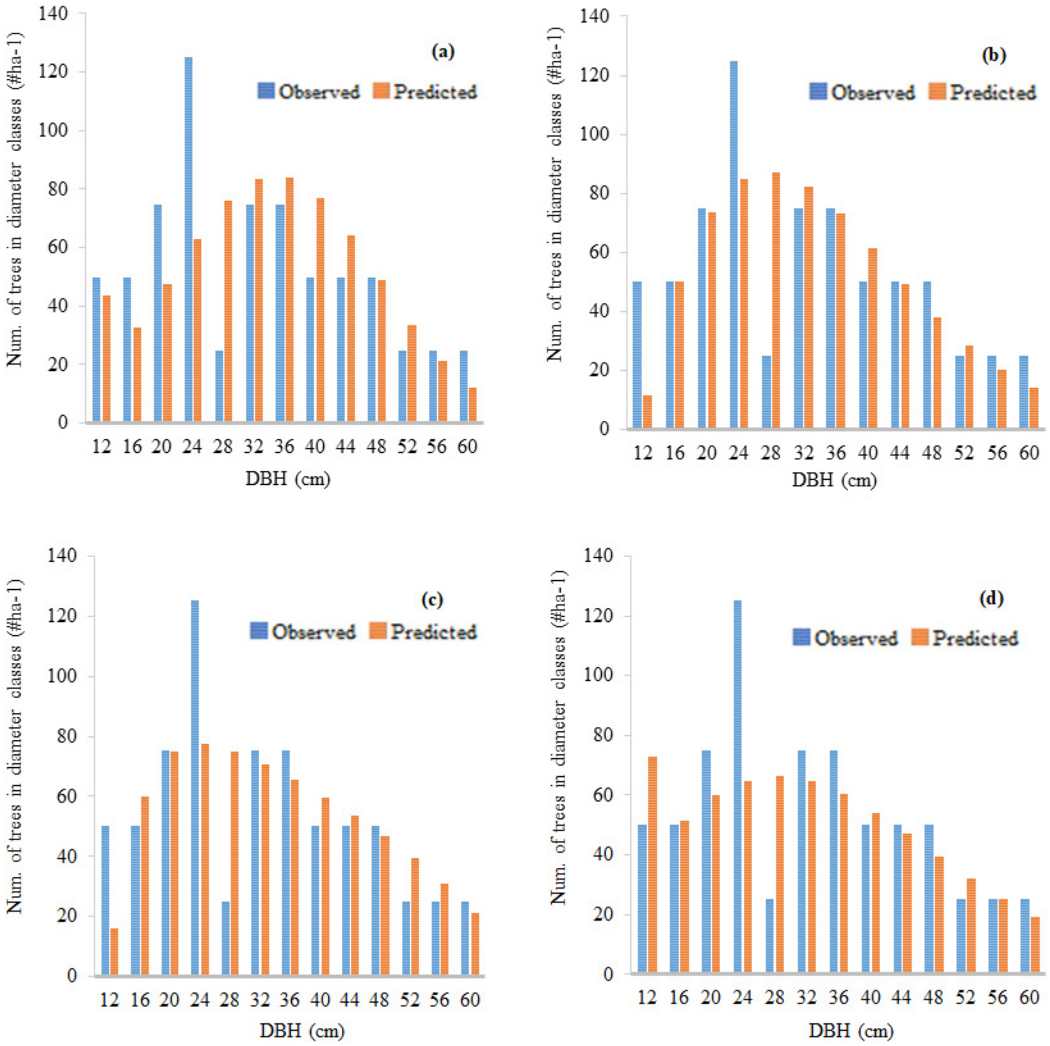

Figure 4. Relationships between the observed and predicted number of trees by (a) Normal PDF, (b) Weibull (3P) PDF, (c) Johnson's SB PDF, and (d) Rayleigh PDF for any sample plot.

modeling the diameter distribution of Scotch pine-Beech mixed stands. Sakıcı & Gülsunar (2012) discovered that the Weibull (2P and 3P) function yielded successful results in modeling the diameter distribution of Bornmullerian fir in mixed coniferous stands. Diamantopoulou et al. (2015) concluded that the MLE procedure, coupled with Levenberg-Marquardt Artificial Neural Network (ANN) modeling method, provided the most reliable estimates for modeling the diameter distribution of Crimean Juniper stands using the Weibull (2P) distribution parameters. Sönmez et al. (2015) modeled diameter distributions in even-aged and pure spruce stands using the Beta, Weibull (3P), and Johnson's SB functions across different site and age classes. Ercanlı et al. (2016) found the Weibull (3P) function to be successful in modeling the diameter distribution of Vezirköprü-Sarıçiçek forests. Özçelik et al. (2016) employed parameter recovery and a novel approach based on the unrestricted (i.e., without restrictions) MLE technique for Johnson's SB theoretical function in Brutian pine diameter distribution modeling. Özdemir (2016) found the Weibull (2P) function successful in modeling the diameter distribution of Douglas fir. Bolat & Ercanlı (2017) reported a 97.7% success rate for the Weibull (3P) function in modeling the diameter distribution of Bursa-Kestel forests. Sivrikaya & Karakaş (2020), utilized the percentile technique, and Weibull (3P) function to model the diameter distributions of Stone pine stands in Kahramanmaraş. Sakıcı & Dal (2021) determined that Johnson SB function was the most successful PDF in modeling the diameter distributions of Scotch pine stands and alternative stand characteristics did not significantly affect the choice of the most successful PDFs. Seki (2022) estimated parameter values of the Weibull function for diameter distributions of Oriental beech stands using the MLE method and correlated them with the stand characteristics. He found that the regression model using the arithmetic mean diameter as the independent variable provided superior estimates of the scale parameter, while the model using the maximum stand diameter as the independent variable was superior in estimating the shape parameter. These studies demonstrate that while common PDFs have been frequently used, some less commonly

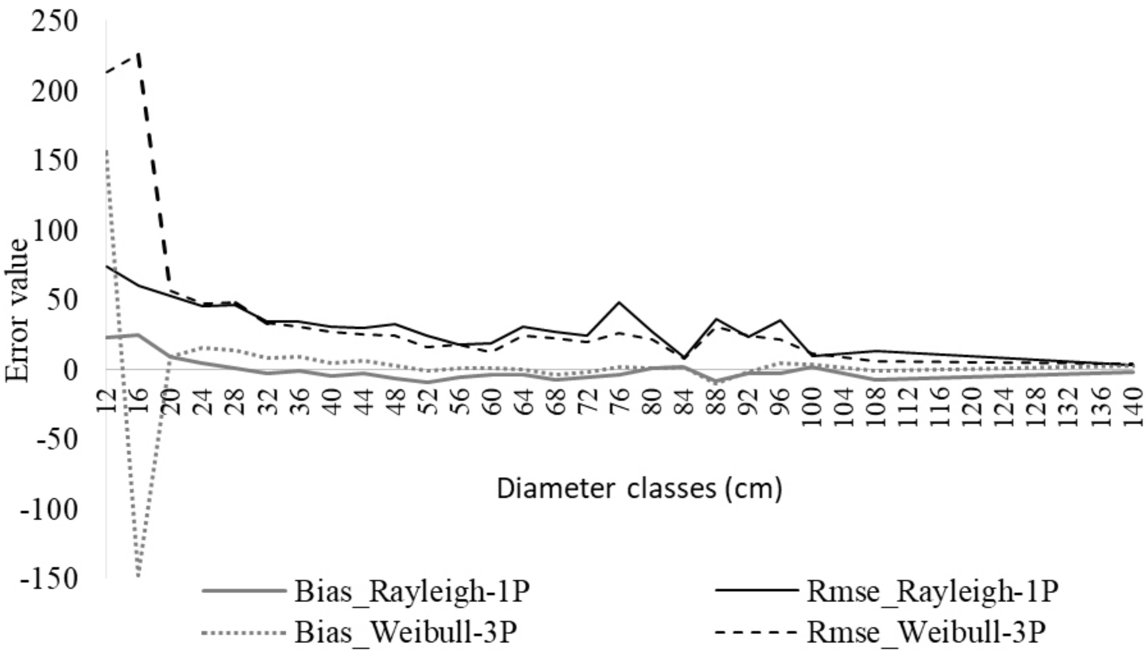

Figure 5. A graphic checking of Bias and RMSE by diameter class for the Rayleigh PDF and Weibull (3P) PDF.

used PDFs have also been tested in this study to model forest diameter distribution.

Conclusion

In this study, the compatibility of various PDFs with the diameter distributions of forests in the OF Planning Unit was examined. Based on the study's results; the Rayleigh function provided the most accurate predictions and had an advantage over the other PDFs, despite being a one-parameter function. The superiority of the Rayleigh function was also confirmed by the K-S tests. While some functions such as Weibull (3P), Gamma (3P), Lognormal (3P), and Kumaraswamy (4P) yielded similarly successful results, they were unsuccessful in modeling tree numbers. Additionally, functions like Rice and Weibull (2P) produced outcomes close to those of Rayleigh but did not surpass its performance (Table 4 and Figs. 2-3). Interestingly, various PDFs used in other scientific disciplines but less popular in forestry, demonstrated success in modeling the diameter distributions of forests.

The Rayleigh distribution, which is widely used in various fields such as engineering, medicine, lifetime analysis, wind speed, energy, physics, and communication, has emerged as the most successful function in accurately modeling diameter distributions in the pure and mixed forests of the OF Planning Unit. Thanks to its practical structure, Rayleigh PDF, which finds applications in diverse areas, has proven to be the most suitable function for modeling the diameter distributions of pure and mixed forests in the research area.

Based on the study's results, it was concluded that by utilizing parameterized PDFs and updated yield tables, it becomes feasible to estimate various stand sharacteristics, including tree number, basal area, volume, biomass, and carbon storage across diameter classes in OF Planning Unit forests. The preference for alternative PDFs, as tested in this study, holds promise for future investigations.

Authors' contributions

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Project administration: Not applicable.
Resources: A. Sahin, I. Ercanli
Software: I. Ercanli, A. Sahin
Supervision: A. Sahin, I. Ercanli
Validation: A. Sahin, I. Ercanli
Visualization: Not applicable.
Writing – original draft: A. Sahin, I. Ercanli
Writing – review & editing: A. Sahin, I. Ercanli

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