RHV, 2023, No 22, 85-103 eISSN 0719-4242
DOI: https://doi.org/10.22370/rhv2023iss22pp85-103
Sección Monográfica / Monographic Section

Semantic Closure and Classicality

Cierre semántico y clasicalidad

Edson Bezerra
Sociedad Argentina de Análisis Filosófico, Argentina
edson.vinber92@gmail.com

Abstract

The semantic paradoxes show that semantic theories that internalize their semantic concepts, such as truth and validity, cannot validate all classical logic. That is, it is necessary to weaken some connective of the object language, taken as the guilty of the paradoxes, or give up some property of the consequence relation of the logical theory. Both strategies may distance us from classical logic, the logic commonly used in our current mathematical theories. So, a desirable solution to semantic paradoxes cannot distance us from classical logic. This paper analyzes two interesting proposals that aim to maintain classical logic as most as possible. The first strategy, the Barrio & Pailos & Szmuc-approach (2017) (BPS-approach), proposes the paraconsistent logic MSC that contains in its object language a connective capable of recovering classical inference whenever the sentences at issue are consistent. So, they show that it is possible to build a semantic theory over this logic that is immune to the semantic paradoxes. The second approach is based on the hierarchy ST_ω of non-transitive systems ST_n, proposed by Pailos (2020a). This hierarchy recovers classical metainferences as many as possible in higher levels of the hierarchy. We argue in favor of the second approach by arguing that the first strategy must adopt weak self-referential procedures to avoid the paradoxes.

Keywords: semantic paradoxes, recovery operators, paraconsistent logics, substructural logics, classical logic.

Resumen

Las paradojas semánticas muestran que las teorías semánticas que internalizan sus propios conceptos semánticos, como la verdad y la validez, no pueden validar toda la lógica clásica. Es decir, es necesario debilitar algún conectivo del lenguaje objeto, tomado como culpable

Received: 11/07/2023. Final version: 28/10/2023
eISSN 0719-4242 – © 2023 Instituto de Filosofía, Universidad de Valparaíso

de las paradojas, o renunciar a alguna propiedad de la relación de consecuencia de la teoría lógica. Ambas estrategias pueden alejarnos de la lógica clásica, que es la lógica comúnmente utilizada en nuestras teorías matemáticas actuales. Por tanto, una solución deseable a las paradojas semánticas no debería alejarnos de la lógica clásica. En este trabajo analizamos dos propuestas interesantes que pretenden mantener la lógica clásica al máximo posible. La primera estrategia, Barrio & Pailos & Szmuc-approach (2017) (BPS-approach), propone la lógica paraconsistente MSC que contiene en su lenguaje objeto un conectivo capaz de recuperar la inferencia clásica siempre que las oraciones en cuestión sean consistentes. Así, muestran que es posible construir una teoría semántica sobre esta lógica que sea inmune a las paradojas semánticas. El segundo enfoque se basa en la jerarquia ST_ω de sistemas no transitivos ST_n, propuesta por Pailos (2020a). Esta jerarquía recupera tantas metainferencias clásicas como sea posible en los niveles superiores de la jerarquía. Argumentamos a favor del segundo enfoque, argumentando que la primera estrategia tiene que adoptar procedimientos autorreferenciales débiles para evitar las paradojas.

Palabras clave: paradojas semánticas, operadores de recuperación, lógicas paraconsistentes; lógicas subestructurales; lógica clásica.

1. Introduction

The semantic paradoxes impose a limit in the project of formulating a semantic theory T capable of expressing its own semantic concepts. First, by Tarski's undefinability theorem (Tarski, 1956; 1933), it is impossible to introduce the *truth predicate* that states "[*A*] is true if and only if *A*", where [*A*] is the name of *A*, in the object language of semantic theories T based on classical logic that are able to express their own syntax, due to the inconsistency caused by *Truth paradox*. Second, as Beall & Murzi (2013), these theories are not capable to express their on validity predicate that states "there is a valid derivation of [*B*] from [*A*]" due to the inconsistency caused by *Curry paradox*. These impossibilities results are due to the expressive capability of T of formulating self-referential statements.

As it is widely known, many solutions to these paradoxes propose to weaken the base logic *L* of T by weakening the deductive behaviour of the logical connectives (Kripke, 1976; Goodship, 1996; Priest, 2006; Field, 2008; Barrio *et al.*, 2017; Pailos, 2020b) or by weakening the deductive properties of the deductive relation of *L* (Ripley, 2013; Zardini, 2013; Meadows, 2014; Weber, 2014; French, 2016; Murzi and Rossi, 2021). Each solution blocks these paradoxes in a specific way.

On the other hand, in blocking these inconsistency results, many of these solutions pay a high cost. By dropping one or more properties of classical logic, these non-classical solutions fail to validate many inferences that are usually performed in the current scientific practice. This preocupation with the recapture of classical inferences is not new in the literature. Indeed, the forefather of paraconsistency Da Costa (1974) introduced his hierarchy of paraconsistent systems that are able to recover classical inferences with the aid of classicality connectives. Although his works on paraconsistency did not have the preocupation with semantic paradoxes, it shows that the preocupation with classical recapture is not a novelty. Priest (2006) introduces the *minimally inconsistent logic* mLP in order to recover classical inferences when there is no paradoxical sentence involved.

The classical recapture was vastly explored in recente years. Da Costa's original idea deeply influenced the *Logics of Formal Inconsistency* (Carnielli *et al.* 2007; Carnielli *et al.*, 2016). The Logics of Formal Inconsistency (henceforth, LFIs) are a family of paraconsistent systems that recovery classical inferences once some consistency assumptions are made. These logics have classicality connectives that makes possible the recovery of classical inferences. This interesting idea was widely discussed and applied to distinct contexts. Antunes (2020) and Tajer (2020) show that these logics have sufficient expressive power to accommodate Priest's recapture proposal as well as Beall's *shrieking method* (Beall, 2013).

However, this recovery strategy does not work well in the context of semantic paradoxes. As Barrio *et al.* (2017) proves, many theories of truth that have the expressive strenght of LFIs are trivial. So it is also necessary to weaken the self-referential procedures of these semantic theories with the aid of weak biconditionals. But, as Rosenblatt (2021) argues, this movement is *ad hoc*, because there is no clear justification of why it is necessary to weaken the self-referential procedures. Moreover, as Picollo (2020) shows, these classicality connectives fail in recovering the validity of instances of the induction axioms in some non-classical theories of truth. Then, as these criticisms shows, the introduction of recovery operators face significant difficulties in the context of semantic theories.

In this paper, we argue that the semantic theory STT_{ω}, introduced by Pailos (2020a) and explored in Barrio *et al.* (2021), offers an adequate approach to a semantic theory of truth, and can be non-trivially extended with a validity predicate. STT_{ω} is comprised by an infinite hierarchy of non-transitive systems that allows to recover the classical metainferences we lose at each step of the hierarchy. This paper is organized as follows. In the Section 2, we present the paradoxes of truth and validity. We also discuss the problems concerning the deductive weakeness of the non-classical solutions to these paradoxes. In Section the 3, we present Barrio *et al.* (2017)'s proposal, which proposes to introduce a consistency connective in paraconsistent theories of truth in order to recover classical inferences whenever the sentences involved are not paradoxical. We also present some criticisms that this approach faces. In the Section 4, we present the hierarchy of strict tolerant logics, introduced by Pailos (2020a), that can be a basis to a non-trivial theory of truth. In the Subsection 4.1, we discuss the advantages of this approach, some criticisms that this proposal faces, and we respond these criticisms. In the Section 5, we close the discussion with a few remarks.

2. Paradoxes of truth and validity

Tarski's undefinability theorem shows that no theory T based on classical logic and that has expressive power to talk about its own syntax can have a truth predicate Tr that satisfies the following schema:

(Schema-*Tr*) $Tr([A]) \leftrightarrow A.$

Where $[A]$ is the name of A. The reason for such impossibility is that these theories have sufficient deductive power to prove *Diagonalization lemma*, which allow them self-referential statments and, among them, the sentence that state its own falsity:

(Liar sentence) $\lambda \leftrightarrow \sim\mathrm{Tr}([\lambda])$.

Let $|=_{\mathrm{T}}$ be the consequence relation of T. The (Liar sentence) generate the inconsistency known as *truth paradox*, whose derivation is given as follows:

1.	$\models_{\mathrm{T}} \lambda \leftrightarrow \sim\mathrm{Tr}([\lambda])$	(Liar sentence)
2.	$\models_{\mathrm{T}} \lambda \leftrightarrow \mathrm{Tr}([\lambda])$	(Schema-Tr)
3.	$\models_{\mathrm{T}} \mathrm{Tr}([\lambda]) \leftrightarrow \sim\mathrm{Tr}([\lambda])$	Transitivity of ↔1, 2
4.	$\models_{\mathrm{T}} \mathrm{Tr}([\lambda]) \wedge \sim Tr([\lambda])$	Elimination of ↔, 3
5.	$\models_{\mathrm{T}} (\mathrm{Tr}([\lambda]) \wedge \sim\mathrm{Tr}([\lambda])) \rightarrow \bot$	Explosion
6.	$\models_{\mathrm{T}} \bot$	Modus Ponens 4, 5

It is also possible to formulate the sentence that says an absurdity follows from its own truth:

(Curry sentence) $c \leftrightarrow (\mathrm{Tr}([c]) \rightarrow \bot)$

The (Curry sentence) generate the *Curry paradox*, whose derivation is given as follows:

1.	$\models_{\mathrm{T}} c \leftrightarrow (\mathrm{Tr}([c]) \rightarrow \bot)$	(Curry sentence)
2.	$\models_{\mathrm{T}} c \leftrightarrow \mathrm{Tr}([c])$	(Schema-Tr)
3.	$\models_{\mathrm{T}} \mathrm{Tr}([c]) \leftrightarrow (\mathrm{Tr}([c]) \rightarrow \bot)$	Transitivity of ↔ 1, 2
4.	$\models_{\mathrm{T}} \mathrm{Tr}([c]) \rightarrow (\mathrm{Tr}([c]) \rightarrow \bot)$	Elimination of ↔ 3
5.	$\models_{\mathrm{T}} \mathrm{Tr}([c]) \rightarrow \bot$	Contraction 4
6.	$\models_{\mathrm{T}} (\mathrm{Tr}([c]) \rightarrow \bot) \rightarrow \mathrm{c}$	Elimination of ↔ 1
7.	$\models_{\mathrm{T}} \mathrm{c}$	Modus Ponens 5, 6
8.	$\models_{\mathrm{T}} c \rightarrow \mathrm{Tr}([c])$	Elimination of ↔ 2
9.	$\models_{\mathrm{T}} \mathrm{Tr}([c])$	Modus Ponens 7, 8
10.	$\models_{\mathrm{T}} \bot$	Modus Ponens 5, 9

So, no theory based on classical logic that has expressive power to talk about its own syntax can have such a truth predicate *Tr*. As we know, Tarski original solution appeal to a transfinite hierarchy of metalanguages where the truth predicate relative to the language *L* is only definable in its metalanguage.

On the other hand, the basic idea of non-classical solutions to both paradoxes is to maintain the truth predicate in the object language *L* of T by changing the underlying logic L. So, as Beall & Murzi (2013) observe, the solutions that serve to block the *truth paradox* by weakening the negation, dropping Explosion (in symbols $A, \sim A \models B$) or Excluded Middle ($\models A \vee \sim A$), are not sufficient. It is also necessary to weaken the connective $\rightarrow$. So, most *paraconsistent* logics – i.e. logics that drop Explosion- and *paracomplete* logics – i.e., logics that drop Excluded Middle- are not adequate to give a uniform solution to both paradoxes. Among the paraconsistent solutions to truth paradoxes, we find Goodship (1996), Priest (2006), Barrio *et al.* (2017) and Pailos (2020). Among the paracomplete ones, we find Kripke (1976) and Field, 2008.[1]

The situation become even worse when one wants to internalize other logical concepts into the object language of the logic. For example, consider the binary validity predicate Val, that internalizes the following relation between sentences: there is a valid derivation of *B* from *A*. This predicate is supposed to satisfy the following rules:

(Validity Proof) Given a valid derivation of *B* from *A*, infer Val([*A*],[*B*]).

(Validity Detachment) From *A* and Val([*A*],[*B*]) infer *B*.

The rule (Validity Proof), henceforth (VP), says that the valid inferences of T are also valid with respect of the predicate Val. The rule (Validity Detachment) says that truth is preserved under Modus Ponens. According to Beall & Murzi, this predicate, when introduced in the object language of T, has catastrophic effects. The main reason is that its derivation uses mainly the structural rules of the consequence relation $\models_T$ of T, such as:

(Reflexivity) $\Gamma, A \models_T A$;

(Contraction) If $\Gamma, A, A \models_T B$, then $\Gamma, A \models_T B$.

(Cut) If $\Gamma \models_T B$ and $\Delta, B \models_T C$, then $\Gamma, \Delta \models_T C$.

Given that the T has expressive power to generate self-referential statements, it is possible to formulate the following statement:

(Validity Curry) $\quad c \leftrightarrow \mathrm{Val}([c],[\bot])$

[1] This list is not meant to be exhaustive. We refer the reader to Barrio (2014) for a good presentation of the non-classical solutions to truth paradoxes.

The sentence (Validity Curry) generates the *validity paradox*, a.k.a. *v-Curry*, whose derivation is given as follows:

1. $\models_T c \leftrightarrow \mathrm{Val}([c],[\bot])$	(Validity Curry)
2. $c \models_T c$	(Reflexivity)
3. $\models_T c \rightarrow \mathrm{Val}([c],[\bot])$	Elimination of $\leftrightarrow$,1
4. $c \models_T \mathrm{Val}([c],[\bot])$	Modus Ponens 2, 3
5. $c,c \models_T \bot$	(VD) 2, 4
6. $c \models_T \bot$	(Contraction) 5
7. $\models_T \mathrm{Val}([c],[\bot])$	(VP) 6
8. $\models_T \mathrm{Val}([c],[\bot]) \rightarrow c$	Elimination of $\leftrightarrow$,1
9. $\models_T c$	Modus Ponens 7, 8
10. $\models_T \bot$	(VD) 7,9

Although (Transitivity) was not explicitly used in the derivation, it is hidden in a formulation of the rule (VD), which can be formulated in the following two ways:

(VD_1) A, $\mathrm{Val}([A],[B]) \models_{\mathbf{T}} B$.

(VD_2) $\models_T A$ and $\models_T \mathrm{Val}([A],[B])$ imply $\models_T B$.

The rule (VD_2) is used in the derivation of v-Curry. As we will show now, this rule can be proven from (VD_1) and (Transitivity).

1. $\models_T A$	Hypothesis
2. $\models_T \mathrm{Val}([A],[B])$	Hypothesis
3. A, $\mathrm{Val}([A],[B]) \models_T B$	(VD_1)
4. $\mathrm{Val}([A],[B]) \models_T B$	(Transitivity) 1, 3
5. $\models_T B$	(Transitivity) 2, 4

The distinction between (VD_1) and (VD_2) is important, because the non-classical solutions that dispense the property (Transitivity) cannot introduce the rule (VD_2) given it pressupposes (Transitivity).

As the derivation of v-Curry shows, restrictions on the connectives may not be sufficient, given that the above derivation mainly uses the rules of the predicate Val as well as the properties of $\models_{\mathbf{T}}$. Some non-classical solutions proceed by dispensing one or more properties of $\models_T$. These solutions are called *substructural solutions*. For example, French (2016), Murzi

& Rossi (2017) propose theories of validity and truth that dispense (Reflexivity); Zardini (2013) and Weber (2014) propose theories that dispense (Contraction); and Ripley (2013) and Barrio *et al.* (2016) propose theories that dispense (Transitivity).

Although v-Curry mainly uses the properties of the consequence relation of T, it is possible to present theories of validity and truth that keep all these properties of $\models_T$ intact. For example, Pailos (2020b) proposes a paraconsistent theory of validity and truth that respects the three properties of $\models_T$. But, in order to avoid v-Curry, he adopts a weaker self-referential procedure in his theory.

As widely discussed in the literature about semantic paradoxes, many non-classical solutions fail in delivering a theory that is strong enough to validate some inferences that are usually performed in our inferential practice. For example, if a semantic theory fails in validating the rule of Modus Ponens, one is entitled to say that this theory fail in delivering a good theory of reasoning about these semantic concepts. The same could be said with respect to a theory that does not have any valid formula. So, the closeness to the classical reasoning is still a good way for adopting a non-classical solution to paradoxes. In what follows, we will discuss two possible approaches to classicality. One of them is proposed by Barrio, Pailos and Szmuc (2017), the BPS approach, and the other is the hierarchy of non-transitive systems proposed by Pailos (2020a) and Barrio *et al.* (2021). Although both approaches have their merits and own internal problems, we will argue in favour of the latter approach.

Athough the validity paradoxes play an important role in the debate, we will focus on the theories of truth. The main reason is that our discussion is more focused on the classicality of the theories, and that solutions below are also able to block the validities paradoxes when properly extended.

3. Classicality and paradoxes

As we discussed in the latter Section, many non-classical solutions sacrifice their deductive power in order to introduce semantic notions in their object language. So, it is desirable that some method of classical recapture should be available whenever problematic sentences such as (Liar sentence) are not involved. In the literature about classical recapture, we find many strategies such as Priest's logic mLP (Priest, 2006) Beall's shrieking method (Beall, 2013). Both strategies show that it is possible to recover classical inferences whenever we are facing neither sentences that are both true and false nor sentences that are neither true nor false. On the other hand, as Antunes (2020) and Tajer (2020) argue, the strategy of introducing *recovery operators* in the object language is able to reproduce these strategies in a more elegant way.

The strategy of introducing recovery operators in the object language of formal theories is not new in the literature. Indeed, Da Costa (1974) introduced these operators that separate the *well-behaved* sentences from the sentences that are not. This idea was widely explored by the founders of the *Logics of Formal Inconsistency* (LFIs) (Carnielli *et al.*, 2007; Carnielli *et al.*, 2016). LFIs are paraconsistent logics that, although they do not validate the inference A, ~A |= B, they have a consistency connective *o* such that:

(i) *oA*, *A* |=/ *B* (|=/ means that the relation |= does not hold)

(ii) *oA*, *~A* |=/ *B*

(iii) *oA*, *A*, *~A* |= *B*

This last condition is called *gentle principle of explosion*. Intuitively, it says that if *A* is consistent and both *A* and *~A* are true, then anything follows. LFIs comprise a wide family of paraconsistent logics that have *o* as primitive or that is capable to define *o* by means of other primitive connectives of the language. Although we will focus on these logics, its is important to mention the existence of the *Logics of Formal Undeterminedness* (LFUs) (Marcos, 2005). LFUs are logics that, although they do not validate |= *A* ∨ *~A*, they have a determinedness connective ♠ such that:

♠*A* |= *A* *~A*

BPS propose a semantic theory of truth based on a LFI. The logical system they propose is called MSC (*Matrix Logic for Semantic Closure*). This logic has the language L_{MSC} = {Var, ~, *o*, ∧, ∨, $\rightarrow_{MSC}$, $\leftrightarrow_{MSC}$}. Its set of formulas $For_{\mathbf{MSC}}$ is generated as usual. The semantic structure is the matrix M_{MSC} = ({1, ½,0}, ~, o, ∧, ∨ , $\rightarrow_{MSC}$, $\leftrightarrow_{MSC}$, {1, ½}), where {1, ½,0} is the set of truth values, {1, ½} is the set of designated values, ~, o, ∧, ∨, $\rightarrow_{MSC}$, $\leftrightarrow_{MSC}$ are operations that have the following truth-tables:

	¬
1	0
½	½
0	1

1	1
½	0
0	1

∧	1	½	0
1	1	½	0
½	½	½	0
0	0	0	0

∨	1	½	0
1	1	1	1
½	1	½	½
0	1	½	0

→MSC	1	½	0
1	1	1	0
½	1	1	½
0	1	1	1

↔MSC	1	½	0
1	1	1	0
½	1	1	½
0	0	½	1

Let v: $For_{MSC} \rightarrow \{1, ½,0\}$ be a valuation and SEM_{MSC} be the set of all valuations v. We say that v is a *model* of $A \in For_{MSC}$ if $v(A) \in \{1, ½,0\}$. A is a *tautology* if every valuation v is a model of A. A is a *semantic consequence* of a set of formulas Γ ($\Gamma \models_{MSC} A$) if and only if: if v is a model of every γ of Γ, then v is also a model of A.

Given the truth tables of the negation and the consistency connective, it is easy to see that MSC is a LFI. As the following proposition shows, one can easily show that $\rightarrow_{MSC}$ does not validate Modus Ponens.

Proposition A: $A, A \rightarrow_{MSC} B \not\models B$.

Proof. Let $v \in SEM_{MSC}$ be a valuation such that $v(A) = ½$ and $v(B) = 0$. By the semantic definition of $\rightarrow_{MSC}$, $v(A \rightarrow_{MSC} B) = ½$. Then, $A, A \rightarrow_{MSC} B \not\models B$. This concludes the proof. Q.E.D.

As said before, MSC is a LFI. With the aid of the connective *o* it is possible to recover classical inferences. This is possible due to *Derivability Adjustment Theorem* (DAT). According to this Theorem, classical inferences hold if the formulas involved are consistent. One of the first formulation of this result in the literature is given by Da Costa (1974) for his systems Cω. There are many versions of this result in the literature. For example, Carnielli *et al.* (2020) provide a wide investigation of these connectives in the field of LFIs and LFUs, and Ciuni & Carrara (2020) investigate these connectives in many-valued logics.[2] Let CPL be the classical propositional logic. The following Theorem is a version of DAT for MSC. Its proof follows from Ciuni & Carrara's results:

Theorem B. For every subset Γ of For_{MSC} and for every A of For_{MSC},

$$\Gamma \models_{CPL} A \text{ if and only if } \{op_1, ..., op_n\}\ \Gamma \models_{MSC} A,$$

where $\{p_1, ..., p_n\}$ ($n \in N$) is the set of propositional variables that occur in $\Gamma \cup \{A\}$.

Thus, Theorem B allows to recover classical inferences whenever the sentences at issue are not the problematic ones, such as (Liar sentence) and (Curry sentence). Now, consider a semantic theory of truth T_{MSC} whose logical basis is MSC. T_{MSC} is supposed to be strong enough to provide names $[A]$ for its sentences A. In this theory, the truth predicate is semantically characterized as:

$$v(A) = v(Tr([A]))$$

Unfortunately, the strong expressiveness of T_{MSC} is a problem. As Tajer (2020) observes, recovery operators and self-reference do not match well. The main reason is the formulation of the following sentence:

(Stronger liar) $\qquad A \leftrightarrow (oA \wedge \sim Tr([A]))$

[2] We also refer the reader to Corbalán (2012)'s master dissertation for a deep study of these connectives.

As BPS observe, the presence of (Stronger liar) makes many semantic theories trivial. In order to avoid triviality, they adopt a weak self-referential principle for T_{MSC}, which stated as follows:

Weak Self-Referential Principle (WSRP). Let T be a theory that has a name forming device [.]. If for every formula $\psi(x)$, with x as the only free variable in $\psi(x)$, there is a sentence φ such that the formula $\varphi \leftrightarrow \psi([\varphi])$ is true in T, then we say that T adopts WSRP.

In the case of T_{MSC}, the biconditional of WSRP is $\leftrightarrow_{MSC}$. By the semantic conditions of this connective, if we assign ½ to the left-hand side formula, the whole biconditional will receive ½, disregarding the truth-value of the right-hand side formula. So, the non triviality of T_{MSC} is guaranteed when T_{MSC} is extended with WSRP, where the biconditional in question is $\leftrightarrow_{MSC}$. As BPS observe, the adoption of any stronger self-referential principle leads T_{MSC} to triviality. For example, if we allow a self-referential principle be stated in terms of '=' instead of the biconditional $\leftrightarrow_{MSC}$, T_{MSC} becomes trivial.

However, BPS' approach face some objections. Rosenblatt (2021) argues that BPS' proposal is *ad hoc*. As he argues, it is possible to extend the language of T_{MSC} with primitive recursive functions that make it possible to prove a stronger self-referential principle. But, as we said before, this makes T_{MSC} trivial. If non-triviality is the only reason to introduce WSRP instead of any stronger self-referential principle, BPS's proposal is philosophically unsatisfactory.

Another criticism comes from Picollo (2020), who argues that theories of truth based on LFIs fail in validating important arithmetical axioms, such as the induction axiom. In her work, she focuses on the truth theory based on the logic LP^{o}, that has the same truth-tables of conjunction, negation and consistency connective as MSC. Its implication connective also fails to validate Modus Ponens. So, the connective *o* fails in its basic aim: recovering the validity of classical inferences. As she argues, our modifications in the logical theories may affect the way we do mathematics.

Since LFIs fail in capturing classical inferences in the contexts of stronger theories of truth, the ones that extend theories T in the arithmetical language, one might say that BPS approach has a narrow scope. In what follows, we argue that the theory of truth based on the metainferential logic ST_{ω} does better at recovering classical inferences. As we will see, the logical base ST has many merits, by sharing the same set of tautologies and inferences as classical logic, differing on the level of metainferences (*i.e.,* inferences between inferences). But, as Pailos (2020a), Barrio *et al.* (2020) and Barrio *et al.* (2021) shows, the metainferences that ST fails to validate are recovered in the logic ST_1, thus creating the hierarchy ST_{ω} where the logics in the higher steps recover the metainferences of the logics of the lower levels. We also respond some objections made against the hierarchy ST_{ω}.

4. The theory of truth STT_ω

In this Section we introduce the theory of truth STT_ω. First, we will introduce some basic definitions. We also respond some possible criticism against STT_ω. We will present now the *Strict Tolerant Logic* (ST) (Cobreros *et al.*, 2012; Ripley, 2012, 2013). The logic ST is characterized by the matrix $M_{ST} = (\{1, ½,0\}, \sim, \wedge, \vee, \{1\}, \{1, ½\})$, where $\{1, ½,0\}$ is the set of truth-values, $\sim, \wedge, \vee$ are operations that share the same truth tables as the truth tables of the logic MSC, $\{1\}$ is the set of strict truth, and $\{1, ½\}$ is the set of tolerant truth.

Let v: $For_{ST} \rightarrow \{1, ½, 0\}$ be a valuation and SEM_{ST} be the set of all valuations v. We say that v is a *model* of $A \in For_{ST}$ if $v(A) \in \{1, ½, 0\}$. A is a *tautology* if every valuation v is a model of A. A is a *semantic consequence* of a set of formulas Γ ($\Gamma \models_{ST} A$) if and only if:

$$\text{if } v(\gamma) = 1, \text{ for every } \gamma \in \Gamma, \text{ then } v(A) \in \{1, ½\}$$

Given the definition of $\models_{ST}$, one can show that it is non-transitive. Suppose that $\Gamma = \Delta = \emptyset$, $v(A) = 1$, $v(B) = ½$ and $v(C) = 0$ for every v of SEM_{ST}. By definition of $\models_{ST}$, $A \models_{ST} B$ and $B \models_{ST} C$, but $A \not\models_{ST} C$. For being cut-free, this logic can be a non-trivial basis for semantic theories of truth.

From the truth tables of $\sim, \wedge, \vee$, one may define the semantic structure $M_{TS} = (\{1, ½, 0\}, \sim, \wedge, \vee, \{1, ½\}, \{1\})$ for the *Tolerant Strict Logic* (TS) (French, 2016). This logic will be important for our objectives in this Section. Let v: $For_{TS} \rightarrow \{1, ½, 0\}$ be a valuation and SEM_{TS} be the set of all valuations v. We say that v is a *model* of $A \in For_{TS}$ if $v(A) = 1$. A is a *tautology* if every valuation v is a model of A. A is a *semantic consequence* of a set of formulas Γ ($\Gamma \models_{TS} A$) if and only if:

$$\text{if } v(\gamma) \in \{1, ½\}, \text{ for every } \gamma \in \Gamma, \text{ then } v(A) = 1.$$

The logic TS is non-reflexive. Let v: $For_{TS} \rightarrow \{1, ½,0\}$ be a valuation of SEM_{TS} such that $v(A) = ½$. By definition of $\models_{TS}$, it is clear that $A \not\models_{TS} A$. This logic can also be used as a basis for a semantic theory of truth giventhat it blocks the rule of reflexivity.

In (Ripley, 2013), Ripley propose a theory of truth, which we call here STT, that is obtained by extending the language of ST with the predicate Tr as well as names for the sentences of ST. Given that ST is cut-free, STT is immune to the paradoxes of truth and validity. An interesting advantage of ST over the others non-classical solutions is its closeness to classical logic. As Ripley shows, ST has the same tautologies and valid inferences as classical logic. However, Ripley's proposal faces some criticisms. The main criticism his theory faces is that it is not closed under its own validities. For example, although ST validates explosion, it validates neither the metarule of explosion nor the metarule of modus ponens:

(Meta-explosion) $\models_{ST} A$ and $\models_{ST} \sim A$ imply $\models_{ST} B$.

(Meta-modus ponens) $\models_{ST} A$ and $\models_{ST} A \rightarrow B$ imply $\models_{ST} B$.

Because of such a failure, some truth-theorists claim that it is not possible to reason using this logic because it is not closed under its own valid inferences. But, there are proposals in the literature that shows that ST can be extended in such a way that classical metainferences (inferences between inferences) are always recovered. In the last few years, the Buenos Aires Logic group published a series of papers where they propose an infinite hierarchy of metainferential logics based on ST (Pailos, 2020a; Barrio *et al.*, 2018; Barrio *et al.*, 2019; Barrio *et al.*, 2021). The intuition behind this interesting idea is the following: suppose that the metainference

(1) From $\Gamma_1 \models A_1 \ldots \Gamma_n \models A_n$ we infer $\Gamma \models A$

is valid in classical logic, but not in ST. This metainference will be recovered in the next step of the hierarchy, which is ST_1. The same reasoning applies to ST_1: the classical metametainferences (inferences between metainferences) that are not valid in ST_1 are recovered in ST_2, and so on. This creates an infinite hierarchy of metainferences, and the system that reunites the logics ST_n (for $n \in \omega$) is called ST_ω. The resulting theory of truth is called STT_ω. Now we will present some basic definitions that are taken from Pailos (2020a).

Definition 4.1. An inference on L_L is a pair (Γ, Δ), where $\Gamma, \Delta \subseteq For_L$ (written $\Gamma \models \Delta$). $INF^0(L_L)$ is the set of all inferences on L_L.[3]

Note that Definition 4.1 talks about inferences between inferences between sets of formulas whereas we have defined inferences between sets of formulas and a formula. Fortunately, we can restate our former definitions as follows:

$\Gamma \models \Delta$ (in L) iff: if every γ of Γ is satisfied according to L, then so is some δ of Δ.

Definition 4.2. A metainference of (a finite) level n (for $1 \leq n \leq \omega$) is an ordered pair (Γ, Δ), where $\Gamma,\Delta \subseteq INF^{n-1}(L_L)$ (written $\Gamma \models_n \Delta$). $INF^0(L_L)$ is the set of all metainferences of level $n\text{-}1$ on L_L.

Before we continue, let us introduce a notion from matrix semantics and a terminology. As the reader can note, the matrix of MSC has only one set of designated values, whereas the matrices for ST and TS have two sets. As Chemla *et al.* (2017) calls, these sets are called *standards*. A standard is a subset of the set of values of the matrix of L that determines what means for a valuation v of SEM_L to confirm or to refute an inference. So, the matrices for ST and TS have two different standards, one for the premises and one for the conclusions. Second, the valuations that respect the truth tables of negation, conjunction and disjunction above presented are called *Strong Kleene valuations* (for short, SK-valuations).

Definition 4.3. A metainference $\Gamma \models_n \Delta$ is valid in a logic L if and only if, for every v of SEM_L, if v satisfies every γ of Γ according to L, then v satisfies some δ of Δ according to L.

[3] Here we will omit the subscript on the relation '$\models$' whenever the context is clear.

Definition 4.4. Let L_1 and L_2 be two logics characterized by SK-valuations. L is a *mixed system* if its consequence relation $|=_L$ is defined as follows: $\Gamma \models_n \Delta$ if and only if it is not the case that v satisfies all γ of Γ according to L_1 and not satisfies any δ of Δ according to L_2.

Given the basis ST of the hierarchy ST_ω, we will define the others ST_n that are mixed, in the sense of Definition 4.4. When $n = 1$, we have the mixed logic TSST, whose definition of consequence relation is stated as follows:

Definition 4.5. $\Gamma \models \Delta$ is valid in a logic TSST if and only if, for every v of SEM_{TSST}, if v satisfies every γ of Γ according to TS, then v satisfies some δ of Δ according to ST.

As Definition 4.5 highlights, the logic TSST is defined in such a way that it preserves the original intuition of ST because its consequence relation is defined "from the strict to tolerant." As we can see, in a valid metainference of TSST, the premises receive the value 1 and the conclusion or 1 or ½. As we said before, ST_1 (= TSST) recovers the valid inferences of ST. The following example shows how this recovery happens. Consider again (Meta-explosion). This metainference is invalid in ST, because both $\models_{ST} A$ and $\models_{ST} {\sim}A$ are satisfied by the value ½. So, TSST recovers such a metainference as follows: according to Definition 4.5, both $\models_{ST}$ A and $\models_{ST} {\sim}A$ should be satisfied by every v of SEM_{TS}. Since no v of SEM_{TS} gives 1 to A and ${\sim}A$ simultaneously, (Meta-explosion) becomes valid in TSST.

On the other hand, the MetaMeta-explosion ($Meta_2$-explosion) is not valid in TSST. That is, the following metametainference:

$$\{\Gamma \models_1 A, \Delta, \Gamma \models_1 {\sim}A, \Delta\} \models_1 \{ \Gamma \models_1 B, \Delta \},$$

Where Γ and Δ are inferences of ST, is not valid in TSST. That is, TSST is not closed under its metametainferences. So, analogously, ST_2 recovers the classical metametainferencs that TSST fails to validate. The same applies to ST_2 with respect to the metametametainferences. So, Pailos (2020a) introduces an infinite hierarchy of metainferences where the logics that are on the higher levels recover the metainferences from the logics on the lower levels. In what follows, L_1/L_2 denotes a mixed system. Consider the following definitions:

Definition 4.6. For any metainferential consequence relation L_1/L_2, $\#L_1/L_2\# = L_2/L_1$.

Definition 4.7. The first step of the hierarchy is the following:

(i) ST_1 = TSST;

(ii) For every $2 \leq n < \omega$, the consequence relation of ST_n is defined as follows:

Let $\Gamma_1, \ldots, \Gamma_k$ and Δ be metainferences of level *n-1*. $\Gamma_1, \ldots, \Gamma_k \models_n \Delta$ is valid if and only if every valuation v that satisfies $\Gamma_1, \ldots, \Gamma_k$ according to $\#ST_{n-1}\#$, also satisfies Δ according to ST_{n-1}.

Now we present some collapse results between ST_n and classical logic.

Fact 4.8. For every level n ($1 \leq n < \omega$), a metainference $\Gamma_1, \ldots, \Gamma_k \models_n \Delta$ is valid in classical logic if and only if it is valid in ST_n.

However, as Pailos (2020a) observes, ST_n is not closed under its metainferences of level n + 1. For example, there will be instances of Cut, Meta-explosion, Meta-modus ponens in the level $n + 1$ that are not valid in ST_n. This is expressed by the following fact:

Fact 4.9. For every level n ($1 \leq n < \omega$), ST_n invalidates infinitely many classical metainferences of level $n + 1$. (And its metainferences of level $n + 1$ are properly included in the n + 1 classical metavalidities.)

In order to overcome this limitation, Pailos (2020a) defines ST_ω as the union of the logics ST_n.

Definition 4.10. A metainference of level n $\Gamma \models_n \Delta$ is satisfied in ST_ω if and only if it is satisfied in some ST_n.

The logic ST_ω makes it possible to recover every classical metainference. Every classical metainference that is lost in some ST_k is recovered in its successor ST_{k+1}. Then so it is recovered in ST_ω.

Now, it is important to make some observation regarding the above construction. As Ferguson & Ramírez-Cámara (2022) observe, the objects of the inferences of the logics ST_n are not formulas, but metainferences.[4] Second, each logic of this hierarchy is treated as an instance. Given that there is no language that take these metainferences as primitive objects, the metainferences are instances as well as each logic in this hierarchy. This is by no means a limitation of Barrio *et al.*'s construction. Indeed, it can be seen as an advantage, because this allows generalizations that can be applied to other logics.

In what concerns the truth predicate, Pailos extends the language L_{ST} with Tr, thus obtaining the theory STT_ω. As he defines, this theory is supposed to contain a name forming operator that provides names [A] for every sentence *A*. In the language of first-order logic, every sentence *A* has a *distinguished name* that is represented by an individual constant, that we can denote by c_A to make our presentation simpler. So, the individual constant c_A stands for [*A*] in the first order language of the theory STT_ω. Thus, in what concerns the first-order models, the domain *D* of the interpretation contains those distiguished names that are represented by these individual constants.

Definition 4.11. A model v for STT_ω is a model for ST_ω that satisfies the following restriction: $v(A) = v(\mathrm{Tr}([A]))$.

From this definition, it is clear that (Schema-Tr) is valid in STT_ω. In what concerns the semantic paradoxes, we have the following situation: since that each ST_k fails to validate the metarule of Cut in the level $k + 1$, the inconsistency caused by (Liar sentence) and (Curry

[4] Ferguson & Ramírez-Cámara (2022) propose a different approach to ST-hierarchy, by treating metainferences as objects in this hierarchy. That is, they define an extended language, call it L^+, that contains terms for each metainference of ST_ω.

sentence) is avoided in STT_ω. Pailos (2020a) proves that is non-trivial. That is, it has a model that satisfies all the sentences of its sentences. As proved in Barrio & Bezerra (2023) show, STT_ω can be non-trivially extended with a validity predicate.

4.1. STT_ω, classicality and objections

The semantic theory STT_ω is a good candidate for the philosophical project of providing a semantically closed language, *i.e.* a language that can talk about its own semantic concepts. Moreover, as we showed in Section 4, the logic ST_ω recovers classical metainferences in its hierarchy because there is always a ST_n that recovers the classical metainferences at issue. But it raises the following question: is ST_ω classical logic? The answer is *no*. The reason is the following: even if ST_ω always recovers classical metainferences at some point ST_k in the hierarchy, it fails in validating metainferences of level $k + 1$. On the other hand, the metainferential hierarchy based in classical logic is classical at every metainferential level. Here we have the following passage taken from Barrio *et al.* (2020) (hierarchy of classical and paraconsistent logic):

> Remarkably, such a criterion allows to provide an answer to our initial question: what identifies CL as such? It is, in fact, the inferences it validates at every inferential level that are crucial to its identity and which allow to tell it apart not only from closely related systems like ST, but also from systems that are even more similar to it—like TS/ST or any of the recursively defined systems appearing in our hierarchy, for that matter. What identifies CL are all its valid inferences of every inferential level, and each of those systems differ with it at some point. This clearly explains why ST is not CL, and neither is TS/ST or any of the systems of our hierarchy. Subsequently, this criterion makes it also easy to tell the difference between the systems in the hierarchy themselves. (Barrio *et al.*, 2020, p. 114)

Although ST_ω is not classical logic, it recovers classical metainferences as most as possible. In comparison to other non-classical solutions to semantic paradoxes, STT_ω is a remarkably strong system. For this reason, we claim that STT_ω is an adequate solution to semantic paradoxes, for being classical as far as possible.

However, cut-free solutions based on STT_ω faces objections in the literature. For example, Scambler (2020), Golan (2022) and Porter (2023) argue that ST_ω is not a logic because it is not closed under its own metainferences. As we showed above, ST_n is not closed under the metainferences of level $n + 1$. For this reason, they claim that it is not possible to reason with ST_ω because it does not allow to connect inferences in order to make another inferences. As argued pointed by Barrio & Bezerra (2023), this failure of closure is not specific to ST_ω, but to all the logics characterized by SK-valuations. While ST fails to be closed under its metainferences, TS has no valid inferences and no tautologies, the logic LP (Asenjo, 1969; Priest, 1979) and K_3 (Kleene, 1938) have mismatches between inferences and tautologies:

LP has the same tautologies as classical logic, but fails to validate modus ponens; K_3 has no tautologies while it validates many classical inferences. That is, all non-classical solutions based on SK-valuation will present some kind of mismatch between its inferences, validities and metainferences.

Second, as we pointed before, the hierarchy ST_ω shows the way how we perform inferences using this logic. As Barrio *et al*. (2021) explains, the whole hierarchy formalizes reasoning according to the standard *from strict to tolerant*. That is, the inferences formalized by this logic should be understood from the perspective of *mixed logics*. So, ST_ω indeed connect metainferences when we read the premises, that are also metainferences, as *strictly valid* and the conclusion, another metainference, as *tolerantly valid*. It is according to this perspective that the inferences formalized by ST_ω should be understood.

Ferguson & Ramírez-Cámara (2020) also raise some criticisms concerning the construction of the hierarchy ST_ω. According to them, the hierarchy ST_ω has a limited expressivity that, by its turn, does not allow to represent metainferences involving metainferences of distinct levels. So, they propose a language that considers metainferences as primitive objects as well as a new semantics for the hierarchy that is based on the conditional → of LP, that is definable as $\sim A \vee B$.

In order to respond Ferguson & Ramírez-Cámara's criticisms, we note the following: even if Barrio and his collaborators' proposal do not take metainferences as primitive objects in the language, Ferguson & Ramírez-Cámara can be properly embedded in the hierarchy ST_ω, as defined above. The reason is the following: Barrio *et al.* (2020) and Roffe & Pailos (2021) offer a procedure of translating metainferences into formulas. Intuitively, this procedure works as follows: an inference of the form

$$\Gamma \models_0 \Delta$$

is converted into a formula of the form:

$$(\gamma_1 \wedge \ldots \wedge \gamma_n) \rightarrow (\delta_1 \vee \ldots \vee \delta_n)$$

where $\gamma_1 \ldots \gamma_n$ belong to Γ and $\delta_1 \ldots \delta_n$ belong to Δ. This procedure can be applied to metainferences of higher levels. In the case of ST, its conditional has the same truth-table as in LP. For this reason, we claim that Ferguson & Ramírez-Cámara's proposal can be reproduced in the original ST_ω as presented above. So, the construction of the hierarchy ST_ω enjoys expressivity enough to reproduce metainfereces of distinct levels.

5. Conclusion

We argued that the theory STT_ω is an adequate non-classical theory of truth since it keeps classical reasoning intact as most as possible, and it is able to recover the classical metainferences with the aid of the logics in the hierarchy. Since the metarule of Cut of level $n + 1$ is not valid in ST_n, one cannot expect to reproduce stronger versions of liar and Curry paradoxes in ST_ω. As showed by Barrio & Bezerra (2023), STT_ω can be extended with a validity predicate that satisfies both (VP) and (VD) without trivializing the theory. So, the cut-free approach to semantic paradoxes is capable to block the paradoxical results mutilating classical logic the least possible.

Acknowledgments

The author acknowledges the financial support of Agencia Nacional de Promoción de la Investigación, el Desarrollo Tecnológico y la Innovación (Agencia I+D+I).

References

Antunes, H. (2020). Enthymematic classical recapture. *Logic Journal of the IGPL*, *28*(5), 817-831.

Asenjo, F. G. (1966). A calculus of antinomies. *Notre Dame Journal of Formal Logic*, *7*(1), 103-105.

Barrio, E. (2014). *La lógica de la verdad*. Eudeba.

Barrio, E., & Bezerra, E. (2023). The logics of a universal language. *Manuscript*, 1-26.

Barrio, E., Rosenblatt, L., & Tajer, D. (2016). Capturing naive validity in the cut-free approach. *Synthese*, *199*(Suppl 3), 707-723.

Barrio, E., Pailos, F., & Calderón, J. T. (2021). Anti-exceptionalism, truth and the BA-plan. *Synthese*, *199*, 12561–12586. https://doi.org/10.1007/s11229-021-03343-w.

Barrio, E., Pailos, F., & Szmuc, D. (2020). A hierarchy of classical and paraconsistent logics. *Journal of Philosophical Logic*, *49*(1), 93-120.

Barrio, E., Pailos, F. M., & Szmuc, D. E. (2017). A paraconsistent route to semantic closure. *Logic Journal of the IGPL*, *25*(4), 387-407.

Beall, J. (2013). LP+, K3+, FDE+, and their 'classical collapse'. *The Review of Symbolic Logic*, *6*(4), 742-754.

Beall, J., & Murzi, J. (2013). Two flavors of curry's paradox. *The Journal of Philosophy*, *110*(3), 143-165.

Carnielli, W., Coniglio, M.E., & Marcos, J. (2007). Logics of Formal Inconsistency. In D. Gabbay & F. Guenthner (Eds.), *Handbook of Philosophical Logic*, Vol. 14 (pp. 1-93). Springer. https://doi.org/10.1007/978-1-4020-6324-4_1

Carnielli, W., Coniglio, M. E., & Rodrigues, A. (2020). Recovery operators, paraconsistency and duality. *Logic Journal of the IGPL*, *28*(5), 624-656.

Carnielli, W. A., & Coniglio, M. E. (2016). *Paraconsistent logic: Consistency, contradiction and negation*. Logic, Epistemology, and the Unity of Science, Vol. 40. Springer.

Chemla, E., Égré, P., & Spector, B. (2017). Characterizing logical consequence in many-valued logic. *Journal of Logic and Computation*, *27*(7), 2193-2226.

Ciuni, R., & Carrara, M. (2020). Normality operators and classical recapture in many-valued logic. *Logic Journal of the IGPL*, *28*(5), 657-683.

Cobreros, P., Egré, P., Ripley, D., & van Rooij, R. (2012). Tolerant, classical, strict. *Journal of Philosophical Logic*, *41*(2), 347-385.

Corbalán, M. I. (2012). *Conectivos de restauração local*. Dissertação de mestrado. Universidade Estadual de Campinas.

Da Costa, N. C. (1974). On the theory of inconsistent formal systems. *Notre dame journal of formal logic*, *15*(4), 497-510.

Ferguson, T. M., & Ramírez-Cámara, E. (2022). Deep ST. *Journal of Philosophical Logic*, *51*, 1261-1293. https://doi.org/10.1007/s10992-021-09630-8

Field, H. (2008). *Saving truth from paradox*. Oxford University Press.

French, R. (2016). Structural reflexivity and the paradoxes of self-reference. *Ergo, an Open Access Journal of Philosophy*, 3(5). https://doi.org/10.3998/ergo.12405314.0003.005

Golan, R. (2023). On the Metainferential Solution to the Semantic Paradoxes. ournal of Philosophical Logic, *52*, 797-820. https://doi.org/10.1007/s10992-022-09688-y

Goodship, L. (1996). On dialethism. *Australasian Journal of Philosophy*, *74*(1), 153-161.

Kleene, S. C. (1938). On notation for ordinal numbers. *The Journal of Symbolic Logic*, *3*(4), 150-155.

Kripke, S. (1976). Outline of a theory of truth. *The journal of philosophy*, *72*(19), 690-716.

Marcos, J. (2005). Nearly every normal modal logic is paranormal. *Logique et Analyse*, *48*(189/192), 279-300.

Meadows, T. (2014). Fixed points for consequence relations. *Logique et Analyse*, *57*(227), 333-357. https://www.jstor.org/stable/44085292

Murzi, J., & Rossi, L. (2021). Naïve validity. *Synthese*, 199(Suppl 3), 819-841. https://doi.org/10.1007/s11229-017-1541-6

Pailos, F. M. (2020a). A fully classical truth theory characterized by substructural means. *The Review of Symbolic Logic*, *13*(2), 249-268.

Pailos, F. M. (2020b). Validity, dialetheism and self-reference. *Synthese*, *197*(2), 773-792.

Picollo, L. (2020). Truth in a logic of formal inconsistency: How classical can it get? *Logic Journal of the IGPL*, *28*(5), 771-806.

Porter, B. (2023). *Three Essays on Substructural Approaches to Semantic Paradoxes*. PhD thesis, City University of New York.

Priest, G. (1979). The logic of paradox. *Journal of Philosophical logic*, *8*(1), 219-241.

Priest, G. (2006). *In Contradiction*. Oxford University Press.

Ripley, D. (2012). Conservatively extending classical logic with transparent truth. *The Review of Symbolic Logic*, *5*(2), 354-378.

Ripley, D. (2013). Paradoxes and failures of cut. *Australasian Journal of Philosophy*, *91*(1), 139-164.

Roffe, A. J., & Pailos, F. (2021). Translating metainferences into formulae: satisfaction operators and sequent calculi. *The Australasian Journal of Logic*, *18*(7), 724-743.

Rosenblatt, L. (2021). Expressing consistency consistently. *Thought: A Journal of Philosophy*, *10*(1), 33-41.

Scambler, C. (2020). Classical logic and the strict tolerant hierarchy. *Journal of Philosophical Logic*, *49*(2), 351-370.

Tajer, D. (2020). and methods of classical recapture. *Logic Journal of the IGPL*, *28*(5), 807-816.

Tarski, A. (1956). *Logic, semantics, metamathematics: papers from 1923 to 1938*. Oxford Clarendom Press.

Weber, Z. (2014). Naive validity. *The Philosophical Quarterly*, *64*(254), 99-114.

Zardini, E. (2014). Naive truth and naive logical properties. *The Review of Symbolic Logic*, *7*(2), 351-384.