DEPÓSITO LEGAL ZU2020000153
ISSN 0041-8811
E-ISSN 2665-0428

Revista de la Universidad del Zulia

Fundada en 1947
por el Dr. Jesús Enrique Lossada

Ciencias Exactas, Naturales y de la Salud

Año 14 N° 40
Mayo - Agosto 2023
Tercera Época
Maracaibo-Venezuela

REVISTA DE LA UNIVERSIDAD DEL ZULIA. 3ª época. Año 14, N° 40, 2023
Bulatnikova Irina Vyacheslavovna // Method of Mathematical Theory of Moments, 84-102
DOI: https://doi.org/10.46925//rdluz.40.05

Method of Mathematical Theory of Moments

Bulatnikova Irina Vyacheslavovna*

ABSTRACT

Infinite matrices play an important role in many aspects of analysis, algebra, differential equations, and the theory of mechanical vibrations. Jacobi matrices are interesting because they are the simplest representatives of symmetric operators in infinite-dimensional space. they are used in interpolation theory, quantum physics, moment problem. In this paper, based on the elements of Jacobi matrix, it will be determined the type of the operator that occurs when processing the results of measurements of random variables. The first type of operators are matrices, for which the moment problem has a unique solution, and Jacobi matrix generates a specific moment problem. The second type of operators are matrices, for which the moment problem has many solutions, and Jacobi matrix is said to generate an indeterminate moment problem.

KEY WORDS: Infinite matrices, Jacobi matrices, moment problem, type of operator.

*Moscow Aviation Institute (National Research University) MAI. ORCID: https://orcid.org/0000-0002-5514-8198. E-mail abcdeik@mail.ru

Recibido: 18/01/2023 Aceptado: 09/03/2023

DOI: https://doi.org/10.46925//rdluz.40.05

Método de la Teoría Matemática de los Momentos

RESUMEN

Las matrices infinitas juegan un papel importante en muchos aspectos del análisis, el álgebra, las ecuaciones diferenciales y la teoría de las vibraciones mecánicas. Las matrices de Jacobi son interesantes porque son las representantes más simples de los operadores simétricos en el espacio de dimensión infinita; se utilizan en teoría de interpolación, física cuántica, problema de momento. En este trabajo, con base en los elementos de la matriz de Jacobi, determinaremos el tipo de operador que se presenta al procesar los resultados de las mediciones de variables aleatorias. El primer tipo de operadores son las matrices, para las cuales el problema de momento tiene una solución única, y la matriz de Jacobi genera un problema de momento específico. El segundo tipo de operadores son las matrices, para las cuales el problema de momento tiene muchas soluciones, y se dice que la matriz de Jacobi genera un problema de momento indeterminado.

PALABRAS CLAVE: Matrices infinitas, Matrices de Jacobi, problema de momentos, tipo de operador.

Introduction

Infinite matrices play an important role in many aspects of analysis, algebra, differential equations, and theory of mechanical vibrations. They are also connected with one type of algebraic continued fractions widely applied by classics of XIX century, and, first of all, by P.L. Chebyshev. In particular, the so-called systems of orthogonal polynomials were introduced through the mediation of these fractions (Akhiezer, 1961).

Infinite matrices can be found in different branches of mathematics (Dyukarev, 2015). They were initially studied in the theory of summation of divergent sequences and series, in quantum mechanics and theory of solving infinite systems of linear equations with finite number of unknown variables. For example, the transformations specified with the help of infinite matrix **A** = (aij) are considered in the theory of series.

The solutions of two linear equations in infinite matrices are used in Heisenberg-Dirac theory in quantum mechanics: AX-XA=1 and AX-XD=0 (the first equation is called the quantization equation). The operator spectrum theory in Hilbert space is used to find the solutions.

DOI: https://doi.org//10.46925//rdluz.40.05

Algebraic properties of infinite matrices and infinite-dimensional linear or classical groups are investigated in many papers and monographs. This is done from many points of view, among which the theory of associative rings and modules, algebraic K-theory, theory of Lie algebras and algebraic groups, theory of infinite groups, functional analysis (operator rings, spectral analysis), elemental analysis (theory of functions, sequences and series), theory of representations, theory of models, infinite combinatorial analysis and probability theory can be highlighted.

Infinite matrices are summed up as ordinary matrices. But when multiplying infinite matrices, their specific character is revealed. Namely, the multiplication of infinite matrices is not always specified. In analysis, in which complex-valued and real-valued infinite matrices are used, this situation is overcome by applying conditions of convergence of coefficients in lines and columns to the matrices. Matrices with the coefficients from arbitrary ring R with one are considered in algebra, thus, other finitude conditions, finite-lineness and finite-columnness type are applied. Besides, multiplication can be specified but, at the same time, be non-associative. Third, the invertibility of infinite matrices has its specific character – for example, there are infinite matrices with infinite number of reciprocals.

Jacobi matrices can occur in different mathematical problems (continued fraction theory, differential equations). Mathematical models of elemental processes, the physical nature of which is known, are written down as formulas and dependencies known for these processes. As a rule, static problems are expressed as algebraic expressions, dynamic – as differential or finite-difference equations. At the same time, any differential equation has an infinite number of solutions in partial derivatives.

The solution method often consists in transition to non-stationary problems, which are approximated by the systems of finite-difference equations. In practice, the solutions satisfying the additional conditions are of the most interest. As a rule, the problems describing physical or chemical processes in the frameworks of differential equations in partial derivatives comprise the boundary conditions.

Jacobi matrices are of interest since they are the simplest representatives of symmetrical operators in infinite-dimensional space L^2. They are used in interpolation theory, quantum physics, moment problem.

The term "moment problem" was for the first time found in the paper of 1894-1895 by

DOI: https://doi.org/10.46925//rdluz.40.05

T. Stieltjes. However, one important problem, related to moment problem, had already been set and, in particular case, solved by P.L. Chebyshev as early as in 1873. P.L. Chebyshev referred to his problem several times during the whole last period of his life. However, in his problem P.L. Chebyshev saw, first of all, the way to obtain some limiting theorem of probability theory. The comprehensive study of the problem and its different generalizations are the merit of A.A. Makarov.

Based on Jacobi matrix elements it is possible to recognize the operator type occurring while processing the measuring results of random variables, i.e. it is possible to understand the classification of operators generated by Jacobi matrices in two-dimensional space. In particular, it is necessary to be able to recognize the operator type by elements of Jacobi matrices. The types, to be found further, can be of two types.

The first type comprises such matrices, for which the corresponding moment problem has a unique solution, therefore, it is said that Jacobi matrix, in this case, generates a certain moment problem.

The second type comprises such cases when the moment problem has many solutions. In this case, the moment problem is called indeterminate.

The problem is urgent and sparks interest with specialists in different branches of mathematics. Finding the operator class with the help of infinite Jacobi matrices can serve as a new solution for some aspects.

1. Definitions and terms

Let C be a set of complex numbers. Let us designate Hilbert space of infinite sequences

of vectors u=$(u_0, u_1, \dots)$ through L^2, where $u_i \in C$ and $v = (v_0, v_1, \dots)$, where $v_i \in C$, i=0,1,2,… .

As it is known, u$\in L^2$, if series $\sum_{k=0}^{\infty} |\mathbf{u}_k|^2 < +\infty$ and scalar product of elements u, v$\in L^2$

are found by the equality: (u, v) = $\sum_{k=0}^{\infty} \mathbf{u}_k \mathbf{v}_k$. Thus, L^2 is completely separable Hilbert space.

DOI: https://doi.org/10.46925//rdluz.40.05

Vector u∈L^2 is called finite, if it has a finite number of non-zero components.

We will call the finite matrix

$$J=\begin{pmatrix} a_0 & b_0 & 0 & 0\dots \\ b_0 & a_1 & b_1 & 0\dots \\ 0 & b_1 & a_2 & b_2.. \\ .. & .. & \dots & \dots \end{pmatrix}$$

Jacobi matrix, in which a_κ are real and b_κ are positive, k=0,1,2,… .

Matrix *J* by the operations

$$(lu)_j = b_j u_{j+1} + a_j u_j + b_{j-1} u_{j-1} \,,$$

j=0,1,2,… , (1)

where $u_{-1}=0, \quad u_0, u_1, \ldots \in C$

defines on the variety of all finite vectors, i.e. vectors with a finite number of non-zero coordinates of L^2 space, the nonclosed symmetric operator whose closure we designate through *L*.

The linear operator *L* is called symmetric in Hilbert space *H*, if its definition region D_L is dense in *H* and if the following equality is relevant for any *u*, *v* (finite vectors):

$(Lu, v)=(u, Lv)$

It is known that *L* is a minimum closed symmetric operator generated by the expression (1) and boundary condition u_{-1} = 0 in space L^2, generally speaking, it is not self-conjugated and defect indices n_+ and n_- of this operator equal the pairs (0, 0) or (1, 1), i.e. $n_+ = n_- =1$ or $n_+ = n_- =0$.

Let A be a symmetric operator. Let us consider the homogeneous equation Ax=λx, where N_A is linear space of solutions of this equation. It can be proved that if A is a symmetric operator and λ is an arbitrary complex number from the upper half-plane (Imλ>0), then dimensionality N_A does not change, i.e. it is the same for any λ. This number (dimensionality N_A) is called the upper defective number. The lower defective number is, in this case, when λ is taken from the lower half-plane (Im λ<0).

DOI: https://doi.org/10.46925//rdluz.40.05

The main theorems, their application for classifying operators and facts given below are taken from the book (Akhiezer, 1961), (Suetin, 1979), (Refaat El Attar, 2006), (Koralov, Sinai, 2013), papers (Kostyuchenko, Mirzoev, 1998), (Kostyuchenko, Mirzoev, 2001), (Korepanov, 1999) and other literature sources.

2. Main theorems and their application for classifying operators

The main theorems and facts given below are taken from the book (Akhiezer, 1961; Koralov, 2013; Suetin, 1979), papers (Koralov, 2013; Kostyuchenko, Mirzoev, 1998; Korepanov, 1999) and other literature sources.

The finite-difference equation

$$b_j u_{j+1} + a_j u_j + b_{j-1} u_{j-1} = \lambda u_j$$

$$b_j u_{j+1} + b_{j-1} u_{j-1} = (\lambda - a_j) u_j, \quad j = 1,2,... \qquad (1)$$

has two linearly independent solutions: $P_k(\lambda)$ and $Q_k(\lambda)$, corresponding to the initial

conditions $P_0(\lambda)=1$, $P_1(\lambda)=\frac{\lambda-a_0}{b}$, $Q_0(\lambda)=0$, $Q_1(\lambda)=\frac{1}{b}$

Solution $P_k(\lambda)$ of this equation we seek in the form: $P_k(\lambda) = c_K \chi_K(\lambda)$, k = 0,1,..., where c_K does not depend on λ and is defined for the conditions to be fulfilled

$$b_K c_{K+1} = -b_{K-1} c_{K-1}, \quad k=0,1,... \qquad (2)$$

and $c_0 = \frac{1}{b}$, $c_1 = 1$. From these conditions we obtain by the mathematical induction method:

DOI: https://doi.org/10.46925//rdluz.40.05

$$k=1 \quad b_1c_2=-b_0c_0 \qquad c_0=\frac{1}{b_0} \qquad b_1c_2=-1$$

$$k=2 \quad b_2c_3=-b_1c_1 \qquad b_2c_3=-b_1$$

$$c_3=-\frac{b_1}{b_2}$$

$$k=3 \quad b_3c_4=-b_2c_2, \qquad c_2=-\frac{1}{b_1}, \qquad c_4=\frac{(-1)^2 b_2}{b_1b_3}$$

$$k=4 \quad b_4c_5=-b_3c_3, \qquad c_3=-\frac{b_1}{b_2}, \qquad c_5=\frac{(-1)^2 b_1b_3}{b_2b_4}$$

$$k=5 \quad b_5c_6=-b_4c_4, \qquad c_4=\frac{(-1)^2 b_2}{b_1b_3}, \qquad c_6=\frac{(-1)^3 b_2b_4}{b_1b_3b_5}$$

$$\ldots \qquad \ldots. \qquad \ldots$$

$$k=2j \qquad c_{2j}=(-1)^j\frac{b_2b_4\ldots b_{2j-2}}{b_1b_3b_5\ldots b_{2j-1}}$$

$$k=2j+1 \quad c_{2j+1}=(-1)^j\frac{b_1b_3\ldots b_{2j-1}}{b_2b_4\ldots b_{2j}}$$

So, C_k can be written down as follows:

$$c_k=\begin{cases}(-1)^j b_{2j-1}^{-1}b_{2j-2}\ldots b_2b_1^{-1}, \text{если} & k=2_j \\ (-1)^j b_{2j}^{-1}b_{2j-1}\ldots b_2^{-1}b_1, \text{ if} & k=2j+1\end{cases} \quad (3)$$

Let us substitute u_j for $P_j(\lambda)$ in the formula (1)

$$b_jc_{j+1}\chi_{j+1}(\lambda)+b_{j-1}c_{j-1}\chi_{j-1}(\lambda)=(\lambda-a_j)c_j\chi_j(\lambda), \quad j=1,2,\ldots$$

DOI: https://doi.org/10.46925//rdluz.40.05

Taking into account the equality (2), we have:

$$\chi_{j+1}(\lambda)-\chi_{j-1}(\lambda)=\frac{(\lambda-a_j)c_j\chi_j(\lambda)}{b_jc_{j+1}}=R_j(\lambda)\chi_j(\lambda),$$

where

$$R_j(\lambda)=\frac{(\lambda-a_j)c_j}{b_jc_{j+1}} \tag{4}$$

Let us demonstrate that the following formula is correct

$$\frac{1}{b_kc_{k+1}}=(-1)^k c_k \tag{5}$$

First, we will prove the formula (5) for $k = 2j$

$$\frac{1}{b_{2j}c_{2j+1}}=\left(\frac{(-1)^j b_1b_3\ldots b_{2j-1}}{b_2b_4\ldots b_{2j}}\right)^{-1}\cdot\frac{1}{b_{2j}}=\frac{b_2b_4\ldots b_{2j}}{(-1)^j b_1b_3\ldots b_{2j-1}b_{2j}}=$$

$$=(-1)^j\frac{b_2b_4b_6\ldots b_{2j-2}}{b_1b_3\ldots b_{2j-1}}=c_{2j}$$

Then

$$\frac{1}{b_{2j}c_{2j+1}}=c_{2j}$$

Let us consider the case for $k = 2j - 1$

$$\frac{1}{b_{2j-1}c_{2j}}=(-1)^j\frac{b_1b_3b_5\ldots b_{2j-1}}{b_2b_4\ldots b_{2j-2}b_{2j-1}}=$$

$$=(-1)^j\frac{b_1b_3b_5\ldots b_{2j-3}}{b_2b_4\ldots b_{2j-2}b_{2j-2}}=-c_{2j-1}$$

DOI: https://doi.org/10.46925//rdluz.40.05

Then

$$\frac{1}{b_{2j}c_{2j}}=-c_{2j-1}$$

We assured ourselves that the formula (5) is correct. Taking into account the formula (4), we have:

$$R_j(\lambda)=(\lambda-a_j)c_j\cdot(-1)^j c_j=(-1)^j c_j^2(\lambda-a_j)$$

Now let us consider the equality:

$$\chi_{j+1}(\lambda)-\chi_{j-1}(\lambda)=R_j(\lambda)\chi_j(\lambda)$$

$$\chi_0(\lambda)=b_0,\ \chi_1(\lambda)=b_0^{-1}(\lambda-a_0)$$

We select

Let us apply the mathematical induction method, we will take *j* as odd

$$\begin{array}{ll}
j=1 & \chi_2(\lambda)-\chi_0(\lambda)=R_1(\lambda)\chi_1(\lambda) \\
j=3 & \chi_4(\lambda)-\chi_2(\lambda)=R_3(\lambda)\chi_3(\lambda) \\
j=5 & \chi_6(\lambda)-\chi_4(\lambda)=R_{51}(\lambda)\chi_5(\lambda) \\
\quad\dots & \quad\dots \\
j=2k-1 & \chi_{2k}(\lambda)-\chi_{2k-2}(\lambda)=R_{2k-1}(\lambda)\chi_{2k-1}(\lambda)
\end{array}$$

Let us sum up all the obtained equalities. We have:

$$-\chi_0(\lambda)+\chi_{2k}(\lambda)=R_1(\lambda)\chi_1(\lambda)+R_3\chi_3(\lambda)+\ldots+R_{2k-1}(\lambda)\chi_{2k-1}(\lambda)$$

Let us express $\chi_{2k}(\lambda)$

$$\chi_{2k}(\lambda)=\chi_0(\lambda)+\sum_{j=1}^{k}R_{2j-1}(\lambda)\chi_{2j-1}(\lambda)$$

We do the same for all even *j*. We have:

DOI: https://doi.org//10.46925//rdluz.40.05

$$\begin{cases} \chi_{2k}(\lambda)=\chi_0(\lambda)+\sum_{j=1}^{k} R_{2j-1}(\lambda)\chi_{2j-1}(\lambda) \\ \chi_{2k+1}(\lambda)=\chi_1(\lambda)+\sum_{k=1}^{k} R_{2j}(\lambda)\chi_{2j-1}(\lambda) \end{cases}$$

The following theorems are true.

Theorem 1.1 (see (Koralov, 2013), theorem 2.1)

Let elements of matrix *j* are such that

а) $\sum_{k=1}^{+\infty} |c_k|^2 < +\infty$ и б) $\sum_{k=1}^{+\infty} |c_k|^2 |a_k|$ <and b

where sequence c_k is defined by the following equality:

$$c_k = \begin{cases} (-1)^j b_{2j-1}^{-1} b_{2j-2} \dots b_2 b_1^{-1}, \text{ if} & k = 2_j \\ (-1)^j b_{2j}^{-1} b_{2j-1} \dots b_2^{-1} b_1, \text{ if} & k = 2j+1 \end{cases}$$

Then there is quite determinate case for matrix *j*.

Proof:

Let M be the space of limited sequences of matrices with dimensionality *p x p* with norm $\|u\|_1 = \sup_j |u_j|$, where $u=(u_0,u_1,\dots)$, and *i* is a natural number.

Let us define the expression of $F_i : M \to M$ by the following equalities:

DOI: https://doi.org/10.46925//rdluz.40.05

$$(F_i u)_0 = (F_i u)_1 = \ldots = (F_i u)_{i+2} = 0$$

$$(F_i u)_{i+2k+1} = \sum_{s=1}^{k} R_{i+2s}(\lambda) u_{i+2s} \qquad (k=1,2,\ldots)$$

$$(F_i u)_{i+2k} = \sum_{s=2}^{k} R_{i+2s-1}(\lambda) u_{i+2s-1} \qquad (k=2,3,\ldots),$$

and element $u^{(0)}$ by the following equalities:

$$u_j^{(0)} = \chi_j(\lambda) \qquad (j=0,1,\ldots,i+1)$$

$$u_{i+2k}^{(0)} = \chi_i(\lambda) + R_{i+1}(\lambda)\chi_{i+1}$$

$$u_{i+2k+1}^{(0)} = \chi_{i+1} \qquad (k=1,2,\ldots)$$

Thus, $u^{(0)} \in M$. Besides, from the definition of operator F_1 and identities:

$$\begin{cases} \chi_{2j}(\lambda) = \chi_0(\lambda) + \sum_{k=1}^{j} R_{2k-1}(\lambda)\chi_{2k-1}(\lambda) \\ \chi_{2j+1}(\lambda) = \chi_1(\lambda) + \sum_{k=1}^{j} R_{2k}(\lambda)\chi_{2k-1}(\lambda) \end{cases}$$

it follows that sequence $(\chi_0(\lambda), \chi_1(\lambda), \ldots)$ is a fixed point of the expression of $u \to F_i u + u^{(0)}$. On the other hand, number i can be selected so for the expression of F_i to be contractive. Actually, if, for example, $|\lambda| \le 1$, then

$$|(F_i u)_{i+2k} - (F_i v)_{i+2k}| \le \|u-v\|_1 \left(\sum_{s=2}^{k} |c_{i+2s-1}|^2 + \sum_{s=2}^{k} |c_{i+2s-1}|^2 |a_{i+2s-1}|\right) \quad (\text{k}=2,\ 3,\ldots)$$

$$|(F_i u)_{i+2k+1} - (F_i v)_{i+2k+1}| \le \|u-v\|_1 \left(\sum_{s=1}^{k} |c_{i+2s}|^2 + \sum_{s=1}^{k} |c_{i+2s}|^2 |a_{i+2s}|\right) \quad (\text{k}=1,2,\ldots)$$

These inequalities follow from the properties of absolute value, definition of the expression of F_i and equality

$$R_j(\lambda) = (-1)^j c_j^2 (\lambda - a_j)$$

Let us fix $0<p<1$. From the demonstrated inequalities and theorem conditions it follows that number i can be taken so large that with all $\lambda \in \{\lambda : |\lambda| \le 1\}$ the following inequality is

DOI: https://doi.org/10.46925//rdluz.40.05

fulfilled:

$$|F_i u - F_i v|_1 \le \rho \|u - v\|_1$$

which was to be proved. So, with fixed $\lambda \in \{\lambda : |\lambda| \le 1\}$ sequence $\{\chi_0(\lambda), \chi_1(\lambda), \ldots\}$ belongs to space M. Further, from this fact, equalities

$$P_k(\lambda) = c_k \chi_k(\lambda), \qquad k = 0, 1, \ldots$$

and condition a) of the theorem it is seen that

$$\sum_{k=0}^{+\infty} |P_k(\lambda)|^2 < +\infty$$

with any $\lambda \in \{\lambda : |\lambda| \le 1\}$.

The conclusion is derived from this theorem.

Conclusion 1.1 (see (Koralov, 2013), conclusion 2.1)

Let the elements of matrix *J* to be such that

1) $|b_{k-1}| \cdot |b_{k+1}| \le |b_k|^2, \qquad (k=1,2,\ldots)$

2) $\sum_{k=1}^{+\infty} \frac{1}{|b_k|} < +\infty$

3) $\sum_{k=1}^{+\infty} \frac{|a_k|}{|b_k|} < +\infty$

Then there is quite indeterminate case for matrix *J*.

Proof.

First, let us demonstrate that $|c_{2j}|^2 \le \frac{1}{|b_{2j}||b_0|}$, $|c_{2j+1}|^2 \le \frac{|b_1|}{|b_{2j+1}|}$, where j = 1, 2, ... (6)

Actually, from the formula

$$c_k = \begin{cases} (-1)^j b_{2j-1}^{-1} b_{2j-2} \ldots b_2 b_1^{-1}, \text{ if} & k = 2_j \\ (-1)^j b_{2j}^{-1} b_{2j-1} \ldots b_2^{-1} b_1, \text{ if} & k = 2j+1 \end{cases}$$

DOI: https://doi.org/10.46925//rdluz.40.05

and condition 1) it follows that, for example,

$$|c_{2j}|^2 \leq \left(|b_{2j-1}^{-1}| \cdot |b_{2j-3}^{-1}| \cdot \ldots \cdot |b_1^{-1}|\right)^2 \cdot \left(|b_{2j-2}| \cdot |b_{2j-4}| \cdot \ldots \cdot |b_2|\right)^2 \leq \frac{1}{|b_{2j}| \cdot |b_o|}$$

Applying the inequalities (6) we find that condition 1) of theorem 1.1 is fulfilled, since

$$\sum_{k=2}^{+\infty} |c_k|^2 \leq \frac{1}{|b_0|} \sum_{j=1}^{+\infty} \frac{1}{|b_{2j}|} + |b_1| \sum_{j=1}^{+\infty} \frac{1}{|b_{2j+1}|} < +\infty,$$

condition b) of theorem 1.1 is fulfilled based on condition 2), since

$$\sum_{k=2}^{+\infty} |c_k|^2 |a_k| \leq \frac{1}{|b_0|} \sum_{j=1}^{+\infty} \frac{|a_{2j}|}{|b_{2j}|} + |b_1| \sum_{j=1}^{+\infty} \frac{|a_{2j+1}|}{|b_{2j+1}|} < +\infty$$

based on condition 3).

Consequently, if the elements of Jacobi matrix satisfy conditions 1) – 3), the assertion of theorem 1.1 is true.

Let us take some nonreal point λ and construct the sequence of circles for it $K_n(\lambda)$. Since $K_{n+1}(\lambda) \subset K_n(\lambda)$ there is either a limit circle or a limit point $K_\infty(\lambda)$.

Theorem 1.3 (see (Akhiezer, 1961), theorem 1.3.1)

a) With any nonreal λ there is at least one solution $\{u_j\}_0^\infty$ of the equation

$$\lambda u_j = b_j u_{j+1} + a_j u_j + b_{j-1} u_{j-1}, \qquad j = 0,1,\ldots,$$

for which

$$\sum_0^\infty |u_j|^2 < \infty,$$

i.e. the solution, which belongs to ℓ^2.

b) Any solution of this equation belongs to ℓ^2, only if $K_\infty(\lambda)$ is a circle.

Theorem 1.2 (see (Koralov, 2013), theorem 1.3.2)

If $K_\infty(\lambda)$ is a circle for some nonreal λ, then $K_\infty(\lambda)$ will be a circle for any nonreal λ. Moreover, if the series

DOI: https://doi.org/10.46925//rdluz.40.05

$$\sum_{0}^{\infty} |P_k(\lambda)|^2$$

converges in some nonreal point of λ, then it converges uniformly in each finite part of complex λ-plane.

J matrix is called a matrix of C type, if there is a case of limit circle for it, and of D type, if there is a case of limit point for it. Defect indices for matrices of D type will be $n_+ = n_- = 0$, and for matrices of C type – $n_+ = n_- = 1$.

3. Main results of using the method of classifying operators by elements of infinite Jacobi matrix

Example 1.

Jacobi matrix with elements $a_n = a \cdot n^{\alpha}$, $b_n = b \cdot n^{\beta}$ is given. What relations between a, b, α, β must be for theorem 1.1 or conclusion 1.1 to be fulfilled?

Solution.

Let us consider for conclusion 1.1 assuming that n=k.

1. Let us check the fulfillment of the first condition: $b_{k-1} b_{k+1} \leq b_k^2$

$$b(k-1)^{\beta} \cdot b(k+1)^{\beta} \leq b^2 k^{2\beta}$$

$$b^2 (k-1)^{\beta} (k+1)^{\beta} \leq b^2 k^{2\beta}$$

$$\cancel{b^2} (k^2-1)^{\beta} \leq \cancel{b^2} k^{2\beta}$$

$$(k^2-1)^{\beta} \leq k^{2\beta}$$

$$k^2 - 1 \leq k^2$$

This is true for any positive b, β.

2. Let us check the fulfillment of the second condition, i.e. the series

$$\sum_{k=1}^{+\infty} \frac{1}{bk^{\beta}} = \frac{1}{b} \sum_{k=1}^{+\infty} \frac{1}{k^{\beta}}$$ converges.

DOI: https://doi.org/10.46925//rdluz.40.05

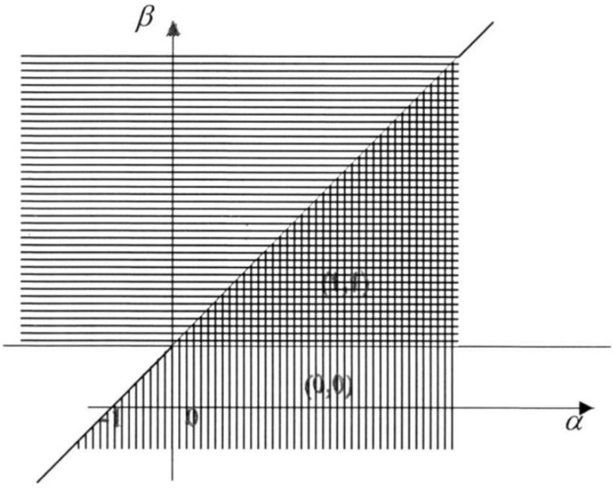
This can be proved with the help of integral Cauchy criterion on the series convergence.

$$\int_1^{+\infty} \frac{dk}{k^\beta} = \frac{k^{-\beta+1}}{-\beta+1}, \qquad -\beta+1<0 \Rightarrow \beta>1$$

This series converges at $\beta > 1$.

3. $$\sum_{k=1}^{+\infty} \frac{|a \cdot k^\alpha|}{b \cdot k^\beta} = \frac{a}{b}\sum_{k=1}^{+\infty} \frac{k^\alpha}{k^\beta} = \frac{a}{b}\sum_{k=1}^{+\infty} \frac{1}{k^{\beta-\alpha}}$$
$$\beta-\alpha>1 \Rightarrow \alpha<\beta-1$$

So, Jacobi matrix with elements $a_n = a{\cdot}n^\alpha$ and $b_n = b{\cdot}n^\beta$, where a, b – any positive numbers, $\alpha < \beta - 1$ and $\beta > 1$ fulfills condition 1.1, i.e. there is an indeterminate case and defect indices will be (1, 1).



Carleman theorem

If series $\sum_{k=0}^{+\infty} \frac{1}{b_k} = +\infty$, then there is a determinate case for such matrix, i.e. defect indices will be (0, 0)

(regardless of α_k).

If in our example $\beta \leq 1$, then it follows from Cauchy integral criterion that se $\sum_{k=0}^{+\infty} \frac{1}{b_k}$ diverges. Thus, applying Carleman theorem, we have that in the band $0 < \beta \leq 1$ and α any of our Jacobi matrices has the defect index (0, 0).

DOI: https://doi.org//10.46925//rdluz.40.05

Example 2.

Jacobi matrix with elements $a_k = a \cdot k^k$ and $b_k = b^k$ is given, where a, b, α – given numbers. What relations between a, b, α must be for theorem 1.1 or conclusion 1.1 to be fulfilled?

Solution.

Let us check if the conditions of conclusion 1.1 are fulfilled.

1. $b^{k+1} \cdot b^{k-1} \le b^{2k} \Leftrightarrow b^{2k} \le b^{2k}$.

   This is fulfilled for any b.

2. Let us consider series $\sum_{k=1}^{+\infty} \frac{1}{b^k}$.

$$\lim_{k \to +\infty} \sqrt[k]{\frac{1}{b^k}} = \frac{1}{b}$$

Using Cauchy criterion on series convergence, we found out that this series converges at $b > 1$.

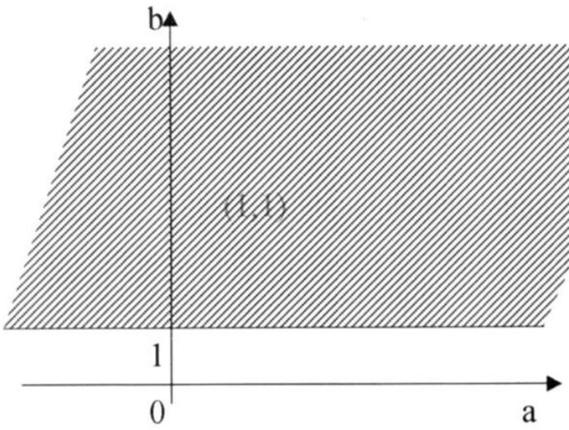
3. Let us consider series $\sum_{k=1}^{+\infty} \frac{a \cdot k^\alpha}{b^k}$. Let us apply D'Alembert criterion once again

$$\lim_{k \to +\infty} \frac{a \cdot (k+1)^\alpha}{b^{k+1}} \cdot \frac{b^k}{a \cdot k^\alpha} = \lim_{k \to +\infty} \frac{1}{b} \cdot \left(\frac{k+1}{k}\right)^\alpha = \frac{1}{b}$$

We found out that series $\sum_{k=1}^{+\infty} \frac{a \cdot k^\alpha}{b^k}$ converges at any a, α and $b > 1$.

We came to the conclusion that Jacobi matrix with elements $a_k = a \cdot k^k$ and $b_k = b$, where a, α – any numbers, $b > 1$, fulfills conclusion 1.1, i.e. there is an indeterminate case and defect indices will be (1, 1).

DOI: https://doi.org/10.46925//rdluz.40.05



Example 3.

Jacobi matrix with elements $a_k = a^k$ and $b_k = b{\cdot}k^\beta$ is given. What relations between a, b, β must be for theorem 1.1 and conclusion 1.1 to be fulfilled?

Solution.

Let us check if the conditions of the problem for conclusion 1.1 are fulfilled.

1. Let us write down the first condition of the conclusion: $b_{k+1}b_{k-1} \le b_k^2$. Let us substitute $b_k = b{\cdot}k^\beta$. We have

$$b(k+1)^\beta \cdot b(k-1)^\beta \le b^2 \cdot k^{2\beta}$$

$$(k^2-1)^\beta \le k^{2\beta}$$

$$k^2-1 \le k^2$$

This is fulfilled for any b and β.

$$\sum_{k=1}^{+\infty}\frac{1}{bk^\beta}=\frac{1}{b}\sum_{k=1}^{+\infty}\frac{1}{k^\beta}$$

2. Let us apply Cauchy integral criterion to prove the second condition of the conclusion

$$\frac{1}{b}\int_1^{+\infty}\frac{1}{k^\beta}dk=\frac{1}{b}\left(\left.\frac{k^{-\beta+1}}{-\beta+1}\right|_1^{+\infty}\right)$$

From this we have that $-\beta+1<0$, i.e. $\beta>1$.

3. Let us check the fulfillment of the third condition of conclusion 1.1, substituting

$$\sum_{k=1}^{+\infty}\frac{a^k}{bk^\beta}$$

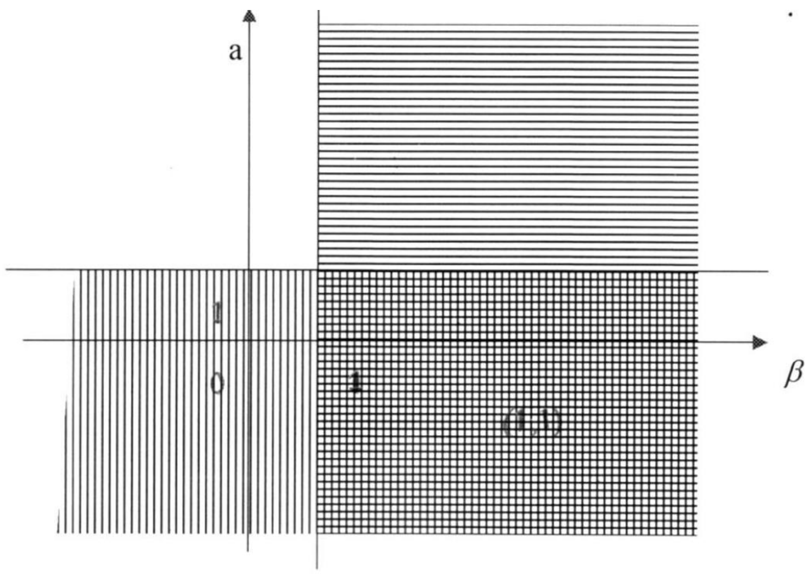
DOI: https://doi.org/10.46925//rdluz.40.05

instead of $a_k = a^k$ and $b_k = b{\cdot}k^{\beta}$. We obtain series . Let us apply D'Alembert criterion on the series convergence

$$\lim_{k\to+\infty}\frac{a^{k+1}}{\cancel{b}\cdot(k+1)^{\beta}}\cdot\frac{\cancel{b}k^{\beta}}{a^{k}}=\lim_{k\to+\infty}\frac{\cancel{a^{k}}\cdot a}{\cancel{a^{k}}}\cdot\left(\frac{k}{k+1}\right)^{\beta}=a$$

For the given series to converge, it is necessary that $a < 1$.

We came to the conclusion that Jacobi matrix with elements $a_k = a^k$ and $b_k = b{\cdot}k^{\beta}$ fulfills conclusion 1.1 under the condition that $a < 1$, $\beta > 1$ and b – any number. Then there is an indeterminate case, the defect indices will be the same (1, 1).



Conclusion

Finding the operator class with the help of infinite Jacobi matrices can serve as a new solution for some aspects, which are rather urgent and which spark interest with specialists in different branches of mathematics.

As a result, based on Jacobi matrixes it is possible to recognize the operator type generated in space L^2 by these matrices. It was found out that they can be of two types.

The first type is D (limit point case) – the moment problem has a unique solution and it is called determinate. The second type is C (limit circle case) – it occurs when the moment problem has many solutions. In this case, the moment problem is called indeterminate. It was also revealed that the defect indices for C type are $n_- = n_+ = 1$ and for D type – $n_- = n_+ = 0$. In the abovementioned examples it is demonstrated how it is possible to use theorem 1.1, Carleman theorem and conclusion 1.1 to define the operator classification.

Mathematical models, in which systems of linear equations are used, are widely applied in different fields, including the processing of results of random variables. For example, in Markovian processes used in queue theory.

The moment method in mathematical statistics is one of the most general methods of finding statistical estimates for unknown parameters of distributing probabilities based on the observation results.

References

Akhiezer, N.I. (1961). Classical moment problem and some analysis aspects connected with it. — State edition of physical and mathematical literature, Moscow.

Dyukarev, Yu. M. (2015). Block Jacobi matrices and matrix problem of Hamburger moments. — Scientific news, series: *Mathematics. Physics*, №5(202), issue 3.

Koralov, L. B.; Ya.G., (2013). Sinai. Probability theory and random processes (translated from English by E.V. Perekhodtseva. — Moscow: Moscow Center of Continuous Mathematical Education, 407 p.

Korepanov, I. G. (1999). Fundamental mathematical structures of integrated models. — Theory of mathematical physics, 01/1999; 118(3):405-412. DOI:10.4213/tmf713.

Kostyuchenko, A. G.; Mirzoev, K. A. (1998). Three-term recurrences with matrix coefficients. Quite indeterminate case. — *Mathematical Notes*, volume 63, issue 5, pp. 709-716.

Kostyuchenko, A. G.; Mirzoev, K. A. (2001). Features of quite indeterminateness of Jacobi matrices with matrix elements. — Functional analysis and its applications, volume 35, issue 4, pp. 32-37.

Refaat El Attar, (2006). Special Functions and Orthogonal Polynomials. — Lulu Press, Morrisville NC 27560.

Suetin, P. K. (1979). Classical orthogonal polynomials. — Nauka, 2nd edition, 1979.