Solution of Fractional Optimal Control Problems with Specified Final State by using Orthogonal Collocation and Differential Evolution

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Abstract

In the last years, the study of Fractional Optimal Control Problem (FOCP) configures a very interesting challenge due to numerical difficulties inherent in this type of investigation. Traditionally, this problem has been solved by considering three different approaches, namely, Direct and Indirect strategies and Hamilton–Jacobi–Bellman (HJB) equation. The first consists in solving the FOCP through the discretization of state and/or control variables. The resulting nonlinear optimization problem is solved by considering either classical or heuristic methods. On the other hand, the indirect approach consists in obtaining the necessary conditions, i.e., the original FOCP is converted into a two-point boundary value problem. The third strategy considers an extension of the well-known HJB equation for fractional order dynamic systems. In the present contribution, the solution of FOCP with specified final state variable is addressed by using the direct approach. For this purpose, the association involving the Orthogonal Collocation Method (OCM) and the Differential Evolution (DE) algorithm is investigated. In order to evaluate the proposed methodology, a classical mathematical problem and a two degree-of-freedom given by a spring-mass-damper system are considered. As expected, the results indicate that the variation of the fractional order implies different values for the original objective function. Furthermore, depending on the fractional order value, it may not be possible to find a solution that satisfies the boundary conditions for a given application. Finally, it is pointed out that the proposed methodology is considered as a promising strategy to solve FOCPs.

Keywords: Fractional Optimal Control Problem, Orthogonal Collocation, Differential Evolution.

Resumen

En los últimos años, el estudio del Problema de Control Óptimo Fraccionado (FOCP) configura un reto muy interesante debido a las dificultades numéricas inherentes a este tipo de investigación. Tradicionalmente, este problema se ha resuelto considerando tres enfoques diferentes, a saber, estrategias directas e indirectas y la ecuación de Hamilton-Jacobi-Bellman (HJB). La primera consiste en resolver el FOCP mediante la discretización de variables de estado y/o control. El problema de optimización no lineal resultante se resuelve considerando métodos clásicos o heurísticos. Por otro lado, el enfoque indirecto consiste en obtener las condiciones necesarias, es decir, el FOCP original se convierte en un problema de valores en la frontera de dos puntos. La tercera estrategia considera una extensión de la conocida ecuación HJB para sistemas dinámicos de orden fraccionario. En la presente contribución, la solución de FOCP con una variable de estado final especificada se aborda utilizando el enfoque directo. Para ello, se investiga la asociación entre el Método de Colocación Ortogonal (OCM) y el algoritmo de Evolución Diferencial (DE). Para evaluar la metodología propuesta se considera un problema matemático clásico y dos grados de libertad dados por un sistema resorte-masa-amortiguador. Como era de esperar, los resultados indican que la variación del orden fraccionario implica valores diferentes para la función objetivo original. Además, dependiendo del valor del orden fraccionario, puede que no sea posible encontrar una solución que satisfaga las condiciones de contorno para una aplicación determinada. Finalmente, se señala que la metodología propuesta se considera como una estrategia promisoria para solucionar los FOCP.

I. INTRODUCTION

Fractional Optimal Control Problem (FOCP) configures a great challenge in sciences and engineering as due to nonlinearities inherent to algebraic-differential models, existence of equality and inequality constraints, boundary conditions and fractional order [1, 2, 3]. Although the fractional order in FOCP represents an additional difficulty, it also makes possible the physical interpretation of the system through fractional differential concepts, i.e., it is possible to evaluate the influence of the fractional order on the behavior of the system [4]. Thus, the FOCP is a generalization of traditional Optimal Control Problem (OCP) and has been under development for several years [5]. However, it is important to mention that the fractional optimal control theory is a rather new topic [6].

To solve the FOCP, three kind of strategies can be used: Direct, Indirect and the Hamilton-Jacobi-Bellman Equation. The first consists in transforming the original problem into an equivalent one through the discretization of state and/or control variables. The nonlinear optimization problem can be solved considering classical, heuristic or hybrid methods. The indirect approach consists in obtaining the necessary conditions of optimality. In this case, the original FOCP is converted into a two-point boundary value problem. Finally, the third approach describes the necessary and sufficient condition for optimality of a OCP given as a nonlinear partial differential equation. In this case, a more complex problem should be solved considering this last approach.

As examples of these strategies, we can refer to the use of Caputo derivative and Legendre orthonormal polynomials [6]. Biswas and Sen [1] proposed and solved FOCP with specified boundary conditions. For this aim, the authors presented the necessary conditions of optimality. Similar strategy was considered by Toledo-Hernandez et al. [7] for the solution of two FOCP in chemical engineering. In this case, the authors associated the Euler–Lagrange optimality conditions and Caputo derivative to evaluate the influence of the fractional orders on the optimal results. Sweilam and Al-Ajami [8] solved FOCP by using Legendre Spectral-Collocation Method and Caputo sense. Alinezhad and Allahviranloo [9] developed a numerical approach based on fuzzy logic and Caputo sense. Effati et al. [10] proposed a new numerical scheme to solve multi-delay fractional order optimal control problems considering the Grunwald–Letnikov sense. Nemati *et al*. [11] solved FOCP considering the Caputo sense and the Riemann–Liouville integral operator. Li and Zhou [12] evaluated the fractional spectral collocation discretization governed by a space-fractional diffusion equation. Soleiman *et al*. [8] solved Delay Fractional Optimal Control Problem (DFOCP) considering the association involving Caputo sense and Padé approximation. Similarly, Bahaa [13] solved a DFOCP by using an analytical scheme for variable order fractional optimal control. Hassani and Avazzadeh [14] developed a new operational matrix of variable order fractional derivatives to solve FOCP by using the Lagrange multiplier optimization technique. Razminia et al. [15] proposed the extension of the classical HJB to the fractional context. More recently, Lima *et al*. [3] proposed a Multi-objective Optimization Stochastic Fractal Search to solve FOCP in chemical engineering problems.

As mentioned earlier, FOCP with constraints are inherently more complex and difficult to resolve. In the present contribution, the Orthogonal Collocation Method (OCM) is associated with the Differential Evolution (DE) algorithm to solve FOCP with specified final state variable. For this aim, the influence of the fractional order is evaluated by considering two case studies: a classical mathematical problem and a two degree-of-freedom system given by a spring-mass-damper model.

This work is organized as follows. Section II presents some definitions related with fractional integrals and derivatives. Section III revisits general aspects regarding the FOCP. Section IV is dedicated to a brief review of the OCM strategy and its extension to solve the FOCP. Section V briefly describes the DE algorithm. Sections VI and VII discuss the methodology and two case studies, respectively. Finally, the conclusions are outlined in the last section.

II. DEFINITIONS

This section presents some definitions required to approximate the operator $d^{\mu}f(t)/dt^{\mu}$, where μ is a fractional order in fractional differential equations [16, 17, 18, 19, 20, 21, 22].

A. *Riemann–Liouville Fractional Integral*

Being f a generic function and $I^{\mu}f(t)$ the representation of the Riemann–Liouville type fractional integral of order μ ($\mu>0$), the fractional integral valid in the domain $(0,\infty) \to \mathbb{R}$ is defined by:

$$I^{\mu} f(t) = \frac{1}{\Gamma(\mu)} \int_0^t (t-\tau)^{\mu-1} f(\tau) d\tau \tag{1}$$

where Γ denotes the Gamma function.

B. *Riemann–Liouville Fractional Derivative*

Being f a generic function and $D^{\mu}f(t)$ the approximation of Riemann–Liouville type fractional derivative of order μ ($\mu>0$), the derivative valid in the domain $(0,\infty) \to \mathbb{R}$ is defined by:

$$D^{\mu} f(t) = \frac{d^n}{dt^n} \frac{1}{\Gamma(n-\mu)} \int_0^t (t-\tau)^{n-\mu-1} f(\tau) d\tau \tag{2}$$

where $n=[\mu]+1$ and $[\mu]$ is an operator that represents the integer part of μ.

C. *Caputo Fractional Derivative*

The Caputo type fractional derivative ($D^{\mu}f(t)$) of order μ ($\mu>0$) of a generic function f in the domain $(0,\infty) \to \mathbb{R}$ is defined by:

$$D^{\mu} f(t) = \frac{1}{\Gamma(n-\mu)} \int_0^t (t-\tau)^{n-\mu-1} f^{n}(\tau) d\tau \tag{3}$$

where $n=[\mu]+1$ and $[\mu]$ is an operator that represents the integer part of μ.

D. *Shifted Grunwald Fractional Derivative*

The Shifted Grunwald fractional derivative ($D^{\mu}f(t)$) of order μ ($1<\mu<2$) applied to a generic function f in the domain $(0,\infty) \to \mathbb{R}$ is defined by:

$$D^{\mu} f(t) = \frac{1}{h^{\mu}} \sum_{k=0}^{M} \left((-1)^k \frac{\Gamma(\mu+1)}{\Gamma(k+1)\Gamma(\mu-k+1)} \right) f(x-(k-1)h) \tag{4}$$

where M is the number of discretization points and $h = (t_f - t_0)/M$ is the integration step (t_0 and t_f represent the initial and final times, respectively).

III. FRACTIONAL OPTIMAL CONTROL PROBLEM

Consider the following FOCP [1]:

$$\min J(x,u,T) = S(x(t_f),t_f) + \int_0^{t_f} V(t,x(t),u(t))\, dt \qquad (5)$$

where J is the performance index (objective function), x and u are the state and control variables, respectively, t is the time (t_f is the final time), S and V are arbitrary continuous function. For this problem, the goal is to find an optimal control profile u to minimize the integral given in Eq. (5), subjected to the dynamic system:

$$D_t^{\mu} x(t) = f(t,x(t),u(t)) \qquad (6)$$

where f is an arbitrary continuous function with the following boundary conditions:

$$x(a)=x_a \text{ and } x(b)=x_b \qquad (7)$$

where x_a and x_b are fixed real numbers and μ is the fractional order.

If (x,u,t) is a minimizer for Eqs. (5)–(7), then there exists an adjoint state λ for which the triple (x, u, t) satisfies the optimality conditions [1]:

$$D_t^{\mu} x(t) = \frac{\partial H(t,x(t),u(t),\lambda(t))}{\partial \lambda}, \qquad x(0)=x_a \qquad (8)$$

$$D_t^{\mu} \lambda(t) = -\frac{\partial H(t,x(t),u(t),\lambda(t))}{\partial x}, \qquad \lambda(t_f)=\frac{\partial S}{\partial x} \qquad (9)$$

$$\frac{\partial H(t,x(t),u(t),\lambda(t))}{\partial u} = 0 \qquad (10)$$

for all t belonging to $[0, t_f]$, and the Hamiltonian H is defined by:

$$H = V(t,x(t),u(t)) + \lambda(t) f(t,x(t),u(t)) \qquad (11)$$

Under some additional assumptions on the functional V and the right-hand side f, i.e., the convexity of V and the linearity of f in x and u, the optimality conditions given by Eq. (8)–(10) are also sufficient [1].

Finally, it is worth mentioning that the practical application of necessary conditions depends on the kind of fractional derivative considered.

IV. SOLUTION OF THE FRACTIONAL OPTIMAL CONTROL PROBLEM

Obviously, to solve the FOCP it is necessary to integrate the system of fractional differential equations. For this aim, the Orthogonal Collocation Method (OCM) extended for the fractional context is employed. The original OCM strategy consists of two steps [23], as follows: *i*) selecting the Number of Collocation Points - *NCP* (the discretization points), and *ii*) choosing a function to approximate the profile of the dependent variable.

To find the collocation points, an orthogonal function is obtained by considering the following recursive relation [23,24]:

$$\Gamma_i^{(\psi,\eta)}(X) = (X+\psi_i)\Gamma_{i-1}(X) + \eta_i \Gamma_{i-2}(X) \qquad (12)$$

where Γ_{i-1}, Γ_{i-2}, ψ, and η are coefficients defined for each type of polynomial approximation.

The collocation points are the roots of the orthogonal polynomial of degree *NCP* and weight $W(X)$ together with the following Galerkin condition [23]:

$$\int_0^1 W(X)(\psi X + \eta)\Gamma_i^{(\psi,\eta)}(X)\,dX = 0, \quad i = 0, \ldots, NCP-2 \qquad (13)$$

Multiplying Eq. (12) by Γ_{i-2}, the following relation is determined:

$$\eta_i = -\frac{\int_0^1 X W(X)\Gamma_{i-1}(X)\Gamma_{i-2}(X)\,dX}{\int_0^1 W(X)\Gamma_{i-2}^2(X)\,dX} \qquad (14)$$

Multiplying Eq. (12) by Γ_{i-1}, the parameter ψ can be obtained as given by Eq. (15).

$$\psi_i = -\frac{\int_0^1 X W(X)\Gamma_{i-1}^2(X)\,dX}{\int_0^1 W(X)\Gamma_{i-1}^2(X)\,dX} \qquad (15)$$

The values of ψ and η can be estimated as a function of Γ_{i-2}, Γ_{i-1}, $W(X)$ and η_1. Thus, the proposed approximation ($\Gamma_{NCP}^{(\psi,\eta)}(X)$) can be evaluated [23, 24].

After obtaining the collocation points, it is necessary to represent the approximation function. For this goal, the Lagrange Polynomial (LP) methodology is considered. The choice of this kind of function is due to the reduction of the computational cost associated with the numerical approximation of the derivatives, as compared with other existing approximations [23, 24]. Consider the set of data points (X_1, Y_1), (X_2, Y_2), …, (X_{NCP+1}, Y_{NCP+1}) and an interpolation formula passing through these points (an *NCP*-th degree interpolation polynomial), as follows:

$$Y_{NCP}(X) = \sum_{i=1}^{NCP+1} Y_i l_i(X) \qquad (16)$$

where $l_i(X)$ is the Lagrange interpolation polynomial defined as:

$$l_i(X) = \prod_{j=1}^{NCP+1} \frac{X - X_j}{X_i - X_j} \qquad (17)$$

In this approximation function, if the subscript i is equal to j, $l_i(X)$ is equal to 1. Otherwise, $l_i(X)$ is equal to 0. The first derivative for a specific root X_j can be expressed as:

$$\frac{dY_{NCP}(X_j)}{dX} = \sum_{i=1}^{NCP+1} Y_i \frac{dl_i(X_j)}{dX}, \quad j = 1,2,\ldots,NCP+1 \qquad (18)$$

If the original problem has one independent variable, the polynomial approximations defined as:

$$x(t) = \sum_{i=1}^{NCP} l_i \phi_i(t) \qquad (19)$$

where ϕ is a function that depends on time *t*. If this approximation is replaced in the original model, the differential equation is converted into an algebraic equation (residual equation). This residual equation is evaluated by considering each *i*-th root (collocation points). The algebraic system is solved by using a particular methodology, i.e., the Newton Method (NM) for nonlinear equations or LU Decomposition for linear equations. It is important to mention that the original model must satisfy both the boundary conditions and the collocation points.

The presented methodology was proposed to solve systems of differential equations with integer order. However, the OCM can be easily extended to the fractional context, i.e., by considering a system of fractional differential equations and a particular definition (see section II), the term that contains the fractional derivative is replaced and the original fractional differential equation is converted into an algebraic equation that depends on the fractional order.

V. DIFFERENTIAL EVOLUTION ALGORITHM

The DE algorithm is a heuristic optimization strategy based on combinations involving individuals belonging to a population of potential candidates to the solution of the problem [25]. The corresponding classical optimization algorithm can be summarized according to the following steps:

- Initially, a population is randomly generated with *NP* feasible solutions, i.e., the design variable vector satisfies the limits established by the user.
- In general, an individual (X_1) is randomly selected in the population to be replaced. Two other individuals (X_2 and X_3) are randomly selected from the population to perform the vector subtraction.
- The result of the subtraction operation between X_2 and X_3 is weighed by a parameter, namely the perturbation rate (*F*). Then, ($F\times(X_2\text{-}X_3)$) is added to the individual (X_1). Therefore, the new (potential) candidate(X) is given by: $X = X_1+\text{F}\times(X_2\text{-}X_3)$. It is worth mentioning that other schemes to generate potential candidates can also be used [25].
- If the resulting vector (X) has a better value in terms of the objective function, it can replace the previously chosen candidate. This operation happens if a random number generated is less than the crossover probability (*CR*), which is also defined by the user. Otherwise, the previously chosen candidate survives in the next generation. This procedure is repeated until *NP* completes the candidates (formed by new and current individuals).
- To finalize the algorithm, a stopping criterion is defined by the user (generally the maximum number of generations).

VI. METHODOLOGY

The methodology of this work consists first in defining the variables of the FOCP and an approximation for the fractional derivative. In the present contribution the Caputo derivative is considered. It is important to mention that the use of the Caputo derivative is due to the memory effect by means of a convolution between the integer order derivative and a power of time [26]. Then, the DE algorithm is initialized to generate potential candidates (design variables) regarding the solution of the optimization problem. For each candidate, a system of equations is obtained by applying the OCM. This system is solved and the objective function *J* is calculated. The stopping criterion for the evolutionary process in each algorithm is the maximum number of generations. Each case study was run ten times in order to obtain average values presented in tables using the Matlab® software in a computer Desktop Intel Core i7-4770 with 8GB Memory.

VII. RESULTS AND DISCUSSION

A. *Fixed Final State Problem*

This first example consider a time invariant fixed final state FOCP proposed and solved by Biswasand Sen [1]. Mathematically, this is formulated as:

$$J\left(u\right)=\frac{1}{2}\int_0^1\left(x^2+u^2\right)dt \tag{20}$$

$$D_t^{\mu}x(t)=-x(t)+u(t) \tag{21}$$

$$x\left(0\right)=1,\ x\left(1\right)=0 \tag{22}$$

where *J* is the performance index (objective function), *x* and *u* are the state and control variables and *t* is the time. In this application, the state variable is defined in two boundary conditions. To solve this problem, Biswasand Sen [1] developed the general transversility condition associated with the Grünwald–Letnikov definition to represent the fractional derivative.

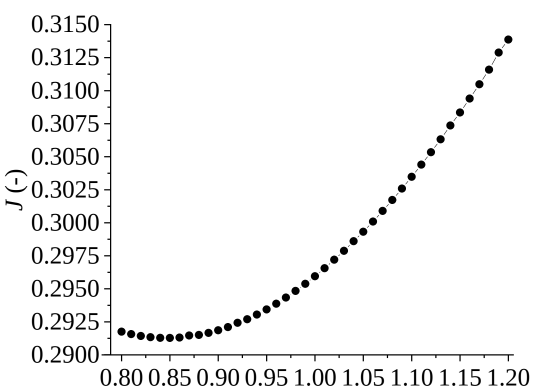
Table I present the results obtained considering the proposed methodology (DE associated with OCM) considering different values for the fractional order (α=[0.8 0.9 1.0 1.1 1.2]) and different values for the number of collocation points (*NCP*=[2 3 4 5]) for the mathematical FOCP. For this purpose, the DE parameters used during the optimization procedure are [25]: 25 individuals, the perturbation rate, and probability crossover are equal to 0.8, respectively, and 500 generations (the total computational cost is equal to 25+25×500 objective function evaluations). In addition, to apply the DE algorithm, the following design space is considered: $0 \leq t_{si} \leq 1$ (i=1, ..., NCP-1) and $-10 \leq u_j \leq 0$ (j=1, ..., NCP). These values were chosen from Biswasand Sen (2011).

TABLE I. Results obtained considering different values for α and *NCP* for the proposed FOCP.

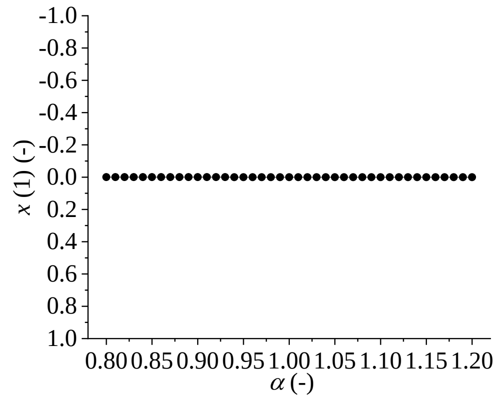
α	*NCP*=2	*NCP*=3	*NCP*=4	*NCP*=5
0.8	0.292284	0.291887	0.291758	0.291758
0.9	0.292814	0.291915	0.291861	0.291861
1.0	0.301424	0.296754	0.295951	0.295951
1.1	0.327434	0.317433	0.303477	0.303477
1.2	0.333043	0.327430	0.313870	0.313870

In this table, in relation to results reported by Biswasand Sen [1] considering α equal to unity (J=0.295858), we can observe that, for *NCP* greater than 3, a good approximation for the solution was obtained. This result demonstrates the good performance of the proposed methodology. In addition, if the value of *NCP* increases, the accuracy increases as well, as expected. In practice, the *NCP* means the dimension of the algebraic system to be solved, i.e., *NCP* equal to 5 leads to a system with 5 algebraic equations, which implies a system with a smaller dimension as compared to traditional techniques to discretize systems of differential equations during the solutions of optimal control problems. In terms of processing time for the solution of the FOCP considering the DE parameters, this processing time was approximately 63 seconds, in average.

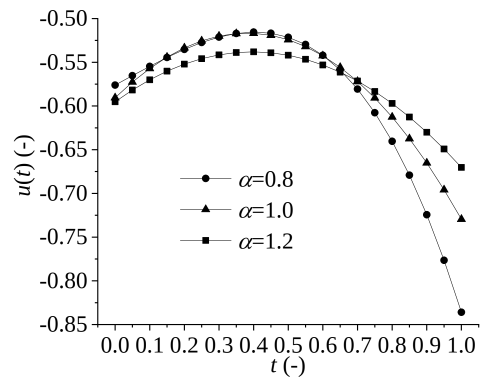
Figure 1(a) presents the objective function considering different values for the fractional order ([0.8 0.81 0.82 ... 1.19 1.2]). In this figure, it is possible to observe J varying with α in a quadratic way, where the profile is decreasing for α < 0.85 and increasing for α > 0.85. Thus, for this mathematical application, we can conclude that, depending on the fractional order the value of the objective function can either increase or decrease. Figure 1(b) shows the value of the state variable at the final time (t_f). For each value of the order fractional, the proposed methodology was able to find the specified final state variable, i.e., an optimal solution was found. Figures 1(c) and 1(d) present the control and state variable profiles considering α equal to [0.8 1.0 1.2], respectively. For all cases analyzed, the control and state variables profiles assume a quadratic behavior, except for α equal to 1.2 for the state variable for which a linear profile is observed, approximately. These results are in agreement with the profiles obtained by Biswas and Sen [1].



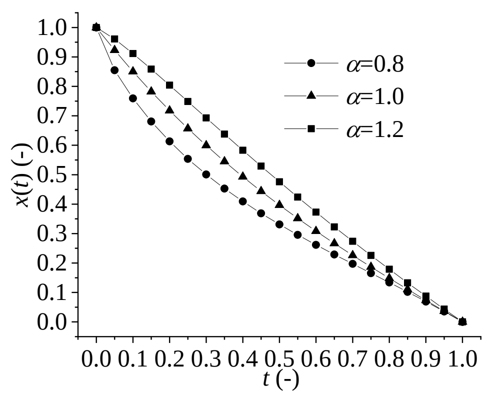
(a) Objective function.



(b) State variable at final time.



(c) Control variable.



(d) State variable.

FIGURE 1. Influence of the fractional order in relation to the objective function, boundary condition at t_f, state variable and control variable.

B. *Two Degree-of-Freedom Spring-Mass-Damper System*

The second application considers a classical optimal control problem from mechanical engineering, as given by the dynamics of a spring-mass-damping system with two degrees of freedom under external forces (see Fig. 2) [27]. The response of the system is represented by the displacements x_1 and x_2 at time t. Both masses m_1 and m_2 are connected with springs with stiffness constants k_1, k_2

and k_3, respectively. The damping factors are given by c_1, c_2 and c_3, respectively. Two external forces u_1 and u_2 are responsible for the external excitation.

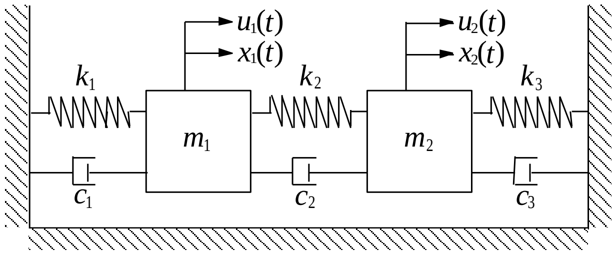


FIGURE 2. Schematic representation of a two degree-of-freedom spring-mass-damper system (Adapted from Veeraklaew and Malisuwan [27]).

The mathematical model that represents the dynamic behavior of the system is described by two second order ordinary differential equations defined in terms of the accelerations of the masses m_1 and m_2 ($\ddot{x}_1$ and $\ddot{x}_2$, respectively). To solve these equations an order reduction is performed, i.e., the following auxiliary variables are defined as: $X_1(t) \equiv x_1(t)$, $X_2(t) \equiv \dot{x}_1(t)$, $X_3(t) \equiv x_2(t)$ and $X_4(t) \equiv \dot{x}_2(t)$. After mathematical manipulation, a first order system of ordinary differential equations is obtained. In matrix form, one has [27, 28]:

$$\dot{X} = AX + Bu \tag{23}$$

where the matrices A and B are defined as:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\dfrac{k_1+k_2}{m_1} & -\dfrac{c_1+c_2}{m_1} & \dfrac{k_2}{m_1} & \dfrac{c_2}{m_1} \\ 0 & 0 & 0 & 1 \\ \dfrac{k_2}{m_2} & \dfrac{c_2}{m_2} & -\dfrac{k_2+k_3}{m_2} & -\dfrac{c_2+c_3}{m_2} \end{bmatrix} \tag{24}$$

$$B = \begin{bmatrix} 0 & 0 \\ \dfrac{1}{m_1} & 0 \\ 0 & 0 \\ 0 & \dfrac{1}{m_2} \end{bmatrix} \tag{25}$$

where the boundary conditions are defined as: $X(0)=[5\ 0\ 10\ 0]^T$ and $X(2)=[0\ 0\ 0\ 0]^T$, respectively.

The objective function (J) for this application is defined as [27]:

$$\min J = \int_0^2 \left(u_1 + u_2\right) dt \tag{26}$$

This problem was originally solved by Veeraklaew and Malisuwan [27] by using the Simpson collocation technique associated with nonlinear programming algorithm. Rutquist and Edvall [28] also solved this problem considering a Matlab toolbox for nonlinear programming through the discretization of state and control variables. It is important to mention that in both cases the authors considered an integer order (α equal to unity). In the present contribution, the influence of the fractional order on physical profiles is evaluated.

As mentioned by Rutquist and Edvall [28], this problem can be solved considering an on-off strategy for both control variables. Thus, for each control variable three control elements can be defined, as follows:

$$u_1 = \begin{cases} 0, & \text{if } 0 \le t \le t_{s1} \\ 9, & \text{if } t_{s1} < t \le t_{s2} \\ 0, & \text{if } t_{s2} < t \le 2 \end{cases} \tag{27}$$

$$u_2 = \begin{cases} 0, & \text{if } 0 \le t \le t_{s3} \\ 9, & \text{if } t_{s3} < t \le t_{s4} \\ 0, & \text{if } t_{s4} < t \le 2 \end{cases} \tag{28}$$

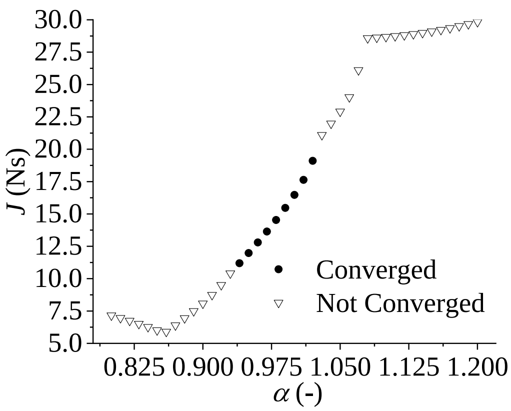
In this case, as the strategy for each control variable is defined, the design variables for this application are evaluated for t_{si} (i=1, 2, 3, 4).

Table II shows the results obtained considering the proposed methodology considering α=[0.8 0.94 1.0 1.02 1.2], NCP=5 and the following model parameters [27,28]: $m_1=m_2=1$ kg, $c_1=c_3=1$ Ns/m, $c_2=2$ Ns/m, $k_1=k_2=k_3=3$ N/m and final time equal to 2 s. The DE parameters used along the optimization procedure are the following [25]: 50 individuals, perturbation rate and probability crossover are equal to 0.8, respectively, and 500 generations (the total computational cost is equal to 50+50×500 objective function evaluations). In addition, the following design space is considered: $0 \le t_{si} \le 2$ s (i=1, 2, 3, 4).

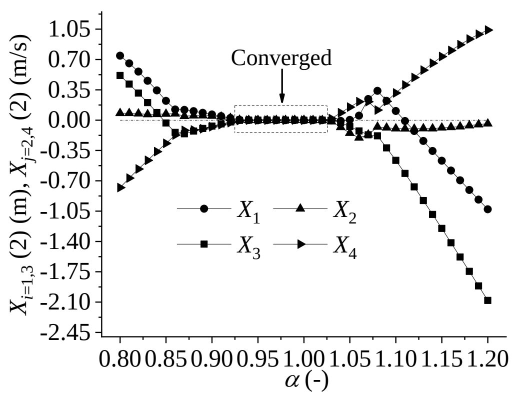
TABLE II. Results for different values of α for the two degree-of-freedom spring-mass-damper system.

	α=0.8	α=0.94	α=1	α=1.02	α=1.2
t_{s1} (s)	1.2000	0.7244	0.6319	0.6866	0.2000
t_{s2} (s)	1.9459	1.5613	1.5280	1.5511	1.9131
t_{s3} (s)	1.2000	1.1331	0.6270	0.3695	0.2000
t_{s4} (s)	1.2010	1.5657	1.5622	1.6092	1.5883
$X_1(2)$ (m)	0.7438	0.0002	-0.0004	0.0000	-1.0284
$X_2(2)$(m/s)	0.0807	-0.0008	0.0001	-0.0002	-0.0402
$X_3(2)$ (m)	0.5153	-0.0003	-0.0010	0.0000	-2.0809
$X_4(2)$ (m/s)	-0.7782	-0.0001	-0.0004	0.0000	1.0395
J (Ns)	7.1552	11.1932	16.4707	19.1052	29.8331

In this table, the value of the objective function considering the fractional order equal to one is similar to the one reported by Veeraklaew and Malisuwan [27] and Rutquist and Edvall [28], i.e., J equal to 16.4853 Ns. As observed in Fig. 3(b), the boundary condition specified at final time is not satisfied for all α values considered, i.e., there is a solution for the problem only for a restricted range of fractional orders and close to unity ($0.94 \le \alpha \le 1.02$). Thus, the influence of the order fractional with respect to the objective function can be analyzed only for the range $0.94 \le \alpha \le 1.02$. In this case, we can observe that the increase on the value of the fractional order implies no increase on the objective function value, as observed in Fig. 3(a) for $0.94 \le \alpha \le 1.02$. From the physical point of view, for the fractional order smaller than 0.94 and greater than 1.02 it was not possible to find control strategies for the dynamics associated with the problem. In addition, for the first control variable, the values of t_{s1} and t_{s2} are identical for α equal to 0.94 and 1.02 and for the second control variable the amplitude (t_{s3}-t_{s4}) decreases with the increase of the fractional order value. In average, for the solution of the FOCP considering the DE parameters the processing time was approximately 95 seconds.



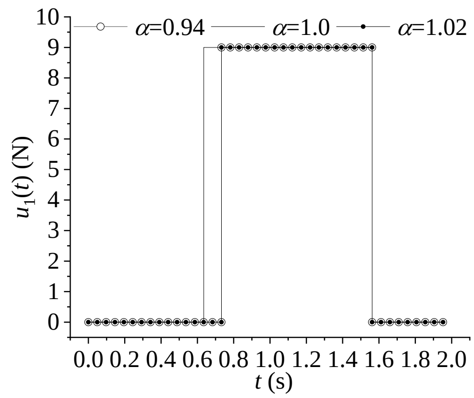
(a) Objective function.



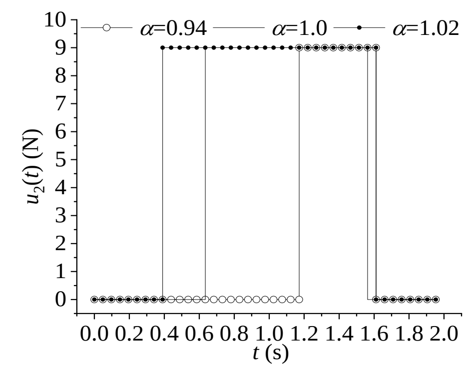
(b) State variables at final time.

FIGURE 3. Influence of the order fractional on the objective function and specified final state variables for the two degree-of-freedom spring-mass-damper system.

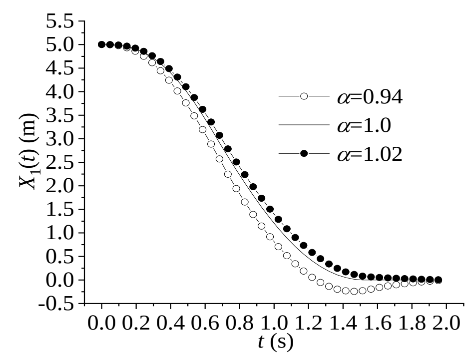
Figure 4 presents the control and state variables considering the fractional order equal to [0.94 1.0 1.02] and *NCP* equal to 5. In Figs. 4(a) and 4(b), for each α value considered, to find the specified state variable at final time, both control variables u_1 and u_2 present similar behavior and are in agreement with the control strategy described by Eqs. (27) and (28). This result was expected since these variables represent the applied forces on the masses during the operation the mechanical system. Figures 4(c) and 4(e) present the displacement and Figs. 4(d) and 4(f) depict the velocity of each mass as a function of time. In these figures we can observe that all specified state variables at final time are fully satisfied. In addition, as the values of the fractional order are close to those that lead to convergence to the optimal solution ([0.94 1.0 1.02]), both profiles are similar.



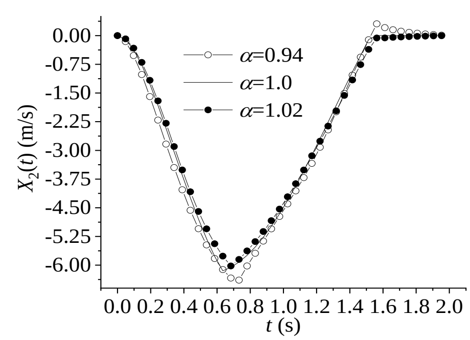
(a) First control variable.



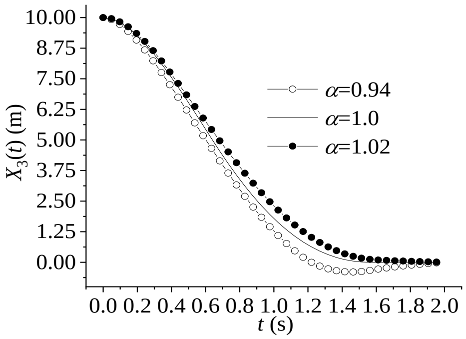
(b) Second control variable.



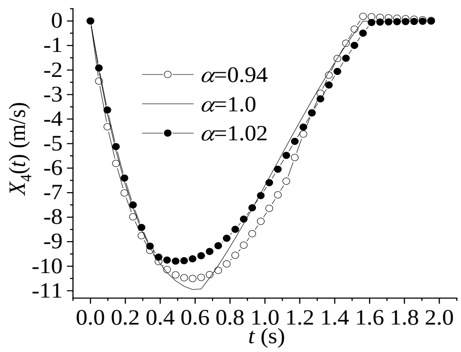
(c) First state variable.



(d) Second state variable.



(e) Third state variable.



(f) Fourth state variable.

FIGURE 4. Control and state variables considering α = [0.94 1.0 1.02] for the two degree-of-freedom spring-mass-damper system.

VIII. CONCLUSION

In this contribution, the Fractional Optimal Control Problem (FOCP) with specified final state variables was studied. For this aim, the proposed methodology consists in the association involving both the Orthogonal Collocation Method (OCM) in the fractional context and the classical Differential Evolution (DE) algorithm. The OCM was used to represent the state and control profiles and the DE was performed to find the events and/or control variable profile. In general, the obtained results considering two classical FOCP (a mathematical FOCP and a spring-mass-damper system) demonstrated that the proposed strategy was able to obtain good approximation for the solution considering different values for the fractional order.

In order to evaluate the influence of the fractional order, each FOCP was solved for different values of this parameter. The obtained results for each application indicate that the variation of the fractional order can lead to different profiles and, consequently, different values for the objective function. However, depending on the values of the fractional order, the specified final state variable is not satisfied, i.e., an optimal solution was not found, as observed for the spring-mass-damper problem. For this case study, for the fractional order smaller than 0.94 and greater than 1.02 it was not possible to find control strategies for the dynamics associated to the physical problem proposed.

Finally, it is worth mentioning that the OCM strategy requires only *NCP* equations for each simulation. Thus, the accuracy of the method associated with the size of the system to be solved appears as the main advantage of the methodology conveyed. Further research work will be focused on the inclusion of reliability and robustness to solve the resulting inverse problems. Different case studies will be evaluated.

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