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Some variants of Lagrange's mean value theorem

Algunas variaciones del Teorema de valor medio de Lagrange

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Abstract
In this note we prove some variants of Lagrange's mean value theorem. The main tools to prove these results are some elementary auxiliary functions.
Keywords . Flett's theorem, Myers' theorem, Sahoo-Riedel's theorem, Çakmak-Tiryaki's theorem.

Resumen
En esta nota demostramos algunas variaciones del Teorema de valor medio de Lagrange. Las herramientas principales para provar estos resultados son algunas funciones auxiliares elementares.
Palabras clave. Teorema de Flett, Teorema de Myers, Teorema de Sahoo-Riedel, Teorema de Çakmak-Tiryaki.

1. Introduction. We know that mean value theorems are important tools in real analysis. The first one that we learn is the famous Lagrange's mean value theorem ([3, Theorem 2.3] or [7, Theorem 4.12] e.g.) and it asserts that a function f continuous on $[a, b]$ and differentiable on (a, b) ensures the existence of $\eta \in (a, b)$ such that

$$f(b) - f(a) = f'(\eta)(b - a).$$

If $f(a) = f(b)$, then the Lagrange's mean value theorem reduces to Rolle's theorem (see [3, Lemma 2.2] or [7, Theorem 4.11] e.g.), which is another important result in real analysis.

Many authors generalized the Lagrange's mean value theorem. The first variant of Lagrange's mean value theorem was given by T.M. Flett [2] in 1958.

Theorem 1.1 (Flett's Theorem). *Let $f : [a, b] \to \mathbb{R}$ be a differentiable function on $[a, b]$ and $f'(a) = f'(b)$. Then there exists $\eta \in (a, b)$ such that*

$$f(\eta) - f(a) = f'(\eta)(\eta - a). \tag{1.1}$$

In 1977, R.E. Myers [6] proved a slight variant of Flett's theorem.

Theorem 1.2 (Myers' Theorem). *Let $f : [a, b] \to \mathbb{R}$ be a differentiable function on $[a, b]$ and $f'(a) = f'(b)$. Then there exists $\eta \in (a, b)$ such that*

$$f(b) - f(\eta) = f'(\eta)(b - \eta). \tag{1.2}$$

In this note we prove some variants of Lagrange's mean value theorem (Theorems 2.2 and 2.4 in Section 2 and Theorems 3.3 and 3.4 in Section 3). To do this we use some simple auxiliary functions.

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2. Some variants of Lagrange's Theorem. P.K. Sahoo and T. Riedel (see [8, Theorem 5.2]) gave a generalization of Flett's theorem where they removed the boundary condition on the derivative of f, that is, $f'(a) = f'(b)$.

Theorem 2.1 (Sahoo-Riedel's Theorem). *Let $f : [a, b] \to \mathbb{R}$ be a differentiable function on $[a, b]$. Then there exists $\eta \in (a, b)$ such that*

$$f(\eta) - f(a) = f'(\eta)(\eta - a) - \frac{1}{2}\frac{f'(b) - f'(a)}{b - a}(\eta - a)^2. \tag{2.1}$$

Now we prove our first result, which is, a variant of Sahoo-Riedel's Theorem.

Theorem 2.2. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable function on $[a, b]$. Then there exists $\eta \in (a, b)$ such that*

$$f(\eta) - f(a) = f'(\eta)(\eta - a) - \frac{(n-1)}{n}\frac{f'(b) - f'(a)}{(b - a)^{n-1}}(\eta - a)^n, \quad n \in \mathbb{N}. \tag{2.2}$$

Proof: Let $n \in \mathbb{N}$ and consider the auxiliary function $\psi : [a, b] \to \mathbb{R}$ given by $\psi(x) = f(x)+\lambda(x-a)^n$, where $\lambda \in \mathbb{R}$. We choose λ in such a way to satisfy the condition $\psi'(a) = \psi'(b)$.

We can see that ψ is differentiable on $[a, b]$ and $\psi'(x) = f'(x) + n\lambda(x - a)^{n-1}$. Then

$$\begin{aligned} \psi'(a) = \psi'(b) &\Leftrightarrow f'(a) = f'(b) + n\lambda(b - a)^{n-1} \\ &\Leftrightarrow \lambda = -\frac{1}{n}\frac{f'(b) - f'(a)}{(b - a)^{n-1}}. \end{aligned}$$

Thus, we have the simple auxiliary function $\psi(x) = f(x) - \frac{1}{n}\frac{f'(b)-f'(a)}{(b-a)^{n-1}}(x - a)^n$ which satisfy the conditions of Flett's theorem (Theorem 1.1). Then there exists $\eta \in (a, b)$ such that

$$\psi(\eta) - \psi(a) = \psi'(\eta)(\eta - a),$$

which implies,

$$\begin{aligned} \Big\{f(\eta) - \frac{1}{n}\frac{f'(b) - f'(a)}{(b - a)^{n-1}}(\eta - a)^n\Big\} - f(a) &= \Big\{f'(\eta) - \frac{f'(b) - f'(a)}{(b - a)^{n-1}}(\eta - a)^{n-1}\Big\}(\eta - a) \\ f(\eta) - f(a) - \frac{1}{n}\frac{f'(b) - f'(a)}{(b - a)^{n-1}}(\eta - a)^n &= f'(\eta)(\eta - a) - \frac{f'(b) - f'(a)}{(b - a)^{n-1}}(\eta - a)^n \\ f(\eta) - f(a) &= f'(\eta)(\eta - a) - \big(1 - \frac{1}{n}\big)\frac{f'(b) - f'(a)}{(b - a)^{n-1}}(\eta - a)^n \\ f(\eta) - f(a) &= f'(\eta)(\eta - a) - \big(\frac{n-1}{n}\big)\frac{f'(b) - f'(a)}{(b - a)^{n-1}}(\eta - a)^n. \end{aligned}$$

■

Remark 1. *Note that:*

If $n = 1$ in (2.2), we get the Flett's theorem (Theorem 1.1).

If $n = 2$ in (2.2), we get the Sahoo-Riedel's theorem (Theorem 2.1). *In this case the auxiliary function takes the form*

$$\psi(x) = f(x) - \frac{1}{2}\frac{f'(b) - f'(a)}{(b - a)}(x - a)^2 \ \ ([8, \text{Theorem } 5.2]).$$

If $n = 3$ in (2.2), we get a slight variant of (2.1)

$$f(\eta) - f(a) = f'(\eta)(\eta - a) - \frac{2}{3}\frac{f'(b) - f'(a)}{(b - a)^2}(\eta - a)^3 \tag{2.3}$$

In this case, we have the auxiliary function

$$\psi(x) = f(x) - \frac{1}{3}\frac{f'(b) - f'(a)}{(b - a)^2}(x - a)^3.$$

D.Çakmak and A.Tiryaki (see [1, Theorem 2.1]) proved a slight modification of Sahoo-Riedel theorem (Theorem 2.1) and its reduces to Myers' theorem (Theorem 1.2) when $f'(a) = f'(b)$.

Theorem 2.3 (Cakmak-Tiryaki's Theorem). *Let* $f : [a, b] \to \mathbb{R}$ *be a differentiable function on* $[a, b]$*. Then there exists* $\eta \in (a, b)$ *such that*

$$f(b) - f(\eta) = f'(\eta)(b-\eta) + \frac{1}{2}\frac{f'(b) - f'(a)}{b-a}(b-\eta)^2. \tag{2.4}$$

Now we prove our second result, which is, a variant of Çakmak-Tiryaki's theorem.

Theorem 2.4. *If* $f : [a, b] \to \mathbb{R}$ *is a differentiable function on* $[a, b]$*, then there exists* $\eta \in (a, b)$ *such that*

$$f(b) - f(\eta) = f'(\eta)(b-\eta) + \frac{n-1}{n}\frac{f'(b) - f'(a)}{(b-a)^{n-1}}(b-\eta)^n, \quad n \in \mathbb{N}. \tag{2.5}$$

Proof: Let $n \in \mathbb{N}$ and consider the function $\phi : [a, b] \to \mathbb{R}$ given by $\phi(x) = f(x) + \lambda(x-b)^n$, where $\lambda \in \mathbb{R}$. We choose λ in such a way to satisfy the condition $\phi'(a) = \phi'(b)$.

The function ϕ is differentiable on $[a, b]$ and $\phi'(x) = f'(x) + n\lambda(x-b)^{n-1}$. Then

$$\begin{aligned}\phi'(a) = \phi'(b) &\Leftrightarrow f'(a) + n\lambda(a-b)^{n-1} = f'(b)\\ &\Leftrightarrow \lambda = \frac{1}{n}\frac{f'(b) - f'(a)}{(a-b)^{n-1}}.\end{aligned}$$

Thus, we have the simple auxiliary function $\phi(x) = f(x) + \frac{1}{n}\frac{f'(b)-f'(a)}{(a-b)^{n-1}}(x-b)^n$ which satisfy the conditions of Myers' theorem (Theorem 1.2). Then there exists $\eta \in (a, b)$ such that

$$\phi(b) - \phi(\eta) = \phi'(\eta)(b-\eta),$$

which implies,

$$f(b) - \left\{f(\eta) + \frac{1}{n}\frac{f'(b) - f'(a)}{(a-b)^{n-1}}(\eta-b)^n\right\} = \left\{f'(\eta) + \frac{f'(b) - f'(a)}{(a-b)^{n-1}}(\eta-b)^{n-1}\right\}(b-\eta)$$

$$f(b) - f(\eta) - \frac{1}{n}\frac{f'(b) - f'(a)}{(a-b)^{n-1}}(\eta-b)^n = f'(\eta)(\eta-b) + \frac{f'(b) - f'(a)}{(a-b)^{n-1}}(\eta-b)^{n-1}(b-\eta). \tag{2.6}$$

(**i**) If n is even, from (2.6) we have

$$\begin{aligned}f(b) - f(\eta) + \frac{1}{n}\frac{f'(b) - f'(a)}{(b-a)^{n-1}}(b-\eta)^n &= f'(\eta)(\eta-b) + \frac{f'(b) - f'(a)}{(b-a)^{n-1}}(b-\eta)^n\\ f(b) - f(\eta) &= f'(\eta)(b-\eta) + \frac{n-1}{n}\frac{f'(b) - f'(a)}{(b-a)^{n-1}}(b-\eta)^n.\end{aligned}$$

(**ii**) If n is odd, from (2.6) we have

$$\begin{aligned}f(b) - f(\eta) + \frac{1}{n}\frac{f'(b) - f'(a)}{(b-a)^{n-1}}(b-\eta)^n &= f'(\eta)(\eta-b) + \frac{f'(b) - f'(a)}{(b-a)^{n-1}}(b-\eta)^n\\ f(b) - f(\eta) &= f'(\eta)(b-\eta) + \frac{n-1}{n}\frac{f'(b) - f'(a)}{(b-a)^{n-1}}(b-\eta)^n.\end{aligned}$$

Now, from (**i**) and (**ii**) we get (2.5). ■

Remark 2. *Note that:*

If $n = 1$ *in* (2.5)*, we get* Theorem 1.2.

If $n = 2$ *in* (2.5)*, we get Çakmak-Tiryaki's theorem* (Theorem 2.3)*. In this case the auxiliary function takes the form*

$$\phi(x) = f(x) + \frac{1}{2}\frac{f'(b) - f'(a)}{(a-b)}(x-b)^2.$$

If $n = 3$ in (2.5), we get a slight variant of (2.4)

$$f(b) - f(\eta) = f'(\eta)(b-\eta) + \frac{2}{3}\frac{f'(b)-f'(a)}{(b-a)^2}(b-\eta)^3. \tag{2.7}$$

In this case, we use the auxiliary function

$$\phi(x) = f(x) + \frac{1}{3}\frac{f'(b)-f'(a)}{(b-a)^2}(x-b)^3.$$

3. Generalizations of Sahoo-Riedel and Çakmak-Tiryaki Theorems. In 2012, A.N. Mohapatra [5] generalized Sahoo-Riedel's theorem (Theorem 2.1) and Çakmak-Tiryaki (Theorem 2.3) using two functions f and g.

Theorem 3.1 ([5, Theorem 2.5]). *If $f, g : [a,b] \to \mathbb{R}$ are differentiable functions on $[a,b]$, then there exists $\eta \in (a,b)$ such that*

$$\begin{aligned}&\big[g(b)-g(a)\big]g'(b)\big[f(\eta)-f(a)-f'(\eta)(\eta-a)\big]\\ &\quad= \big[f'(b)-f'(a)\big]\,\big[g(\eta)-g(a)\big]\Big[\tfrac{1}{2}\big(g(\eta)-g(a)\big)-g'(\eta)(\eta-a)\Big]\end{aligned} \tag{3.1}$$

Proof: Consider the function $\mathcal{M} : [a,b] \to \mathbb{R}$ given by

$$\mathcal{M}(x) = \big[g(b)-g(a)\big]g'(b)f(x) - \tfrac{1}{2}\big[f'(b)-f'(a)\big]\big[g(x)-g(a)\big]^2.$$

The function $\mathcal{M}$ is differentiable on $[a,b]$ and

$$\mathcal{M}'(x) = \big[g(b)-g(a)\big]g'(b)f'(x) - \big[f'(b)-f'(a)\big]\big[g(x)-g(a)\big]g'(x).$$

Also, $\mathcal{M}'(a) = \big[g(b)-g(a)\big]g'(b)f'(a) = \mathcal{M}'(b)$. Then, by Flett's theorem (Theorem 1.1), there exists $\eta \in (a,b)$ such that $\mathcal{M}(\eta) - \mathcal{M}(a) = \mathcal{M}'(\eta)(\eta-a)$, i.e.,

$$\begin{aligned}&\big[g(b)-g(a)\big]g'(b)f(\eta) - \tfrac{1}{2}\big[f'(b)-f'(a)\big]\big[g(\eta)-g(a)\big]^2 - \big[g(b)-g(a)\big]g'(b)f(a)\\ &\quad= \Big[\big[g(b)-g(a)\big]g'(b)f'(\eta) - \big[f'(b)-f'(a)\big]\big[g(\eta)-g(a)\big]g'(\eta)\Big](\eta-a)\\ &\Leftrightarrow \big[g(b)-g(a)\big]g'(b)\big[f(\eta)-f(a)-f'(\eta)(\eta-a)\big]\\ &\quad= \big[f'(b)-f'(a)\big]\big[g(\eta)-g(a)\big]\Big[\tfrac{1}{2}\big(g(\eta)-g(a)\big)-g'(\eta)(\eta-a)\Big].\end{aligned}$$

■

Remark 3. *If $g(x) = x$ in (3.1), we get* Sahoo-Riedel's theorem

$$\begin{aligned}(b-a)\big[f(\eta)-f(a)-f'(\eta)(\eta-a)\big] &= \big[f'(b)-f'(a)\big]\big[\eta-a\big]\Big[\tfrac{1}{2}(\eta-a)-(\eta-a)\Big]\\ (b-a)\big[f(\eta)-f(a)-f'(\eta)(\eta-a)\big] &= -\tfrac{1}{2}\big[f'(b)-f'(a)\big](\eta-a)^2\\ f(\eta)-f(a) &= f'(\eta)(\eta-a) - \frac{1}{2}\frac{f'(b)-f'(a)}{b-a}(\eta-a)^2.\end{aligned}$$

Theorem 3.2 ([5, Teorema 2.6]). *If $f, g : [a,b] \to \mathbb{R}$ are differentiable functions on $[a,b]$, there exists $\eta \in (a,b)$ such that*

$$\begin{aligned}&\big[g(b)-g(a)\big]g'(a)\big[f(b)-f(\eta)-f'(\eta)(b-\eta)\big]\\ &\quad= \big[f'(b)-f'(a)\big]\,\big[g(\eta)-g(b)\big]\Big[\tfrac{1}{2}\big(g(b)-g(\eta)\big)-g'(\eta)(b-\eta)\Big]\end{aligned} \tag{3.2}$$

Proof: Consider the function $\mathcal{M}_1 : [a,b] \to \mathbb{R}$ given by

$$\mathcal{M}_1(x) = \big[g(b)-g(a)\big]g'(a)f(x) - \tfrac{1}{2}\big[f'(b)-f'(a)\big]\big[g(x)-g(b)\big]^2.$$

The function $\mathcal{M}_1$ is differentiable on $[a,b]$ and

$$\mathcal{M}_1'(x) = \big[g(b)-g(a)\big]g'(a)f'(x) - \big[f'(b)-f'(a)\big]\big[g(x)-g(b)\big]g'(x).$$

Also, $\mathcal{M}_1'(a) = \big[g(b)-g(a)\big]f'(b)g'(a) = \mathcal{M}_1'(b)$. Then, by Myers' theorem (Theorem 1.2), there exists $\eta \in (a,b)$ such that $\mathcal{M}_1(b) - \mathcal{M}_1(\eta) = \mathcal{M}_1'(\eta)(b-\eta)$, i.e.,

$$\begin{aligned}
&\big[g(b)-g(a)\big]g'(a)f(b) - \Big[\big[g(b)-g(a)\big]g'(a)f(\eta) - \tfrac{1}{2}\big[f'(b)-f'(a)\big]\big[g(\eta)-g(b)\big]^2\Big]\\
&= \Big[\big[g(b)-g(a)\big]g'(a)f'(\eta) - \big[f'(b)-f'(a)\big]\big[g(\eta)-g(b)\big]g'(\eta)\Big](b-\eta)\\
&\Leftrightarrow \big[g(b)-g(a)\big]g'(a)\big[f(b)-f(\eta)\big] + \tfrac{1}{2}\big[f'(b)-f'(a)\big]\big[g(\eta)-g(b)\big]^2\\
&= \big[g(b)-g(a)\big]g'(a)f'(\eta)(b-\eta) - \big[f'(b)-f'(a)\big]\big[g(\eta)-g(b)\big]g'(\eta)(b-\eta)\\
&\Leftrightarrow \big[g(b)-g(a)\big]g'(a)\Big[f(b)-f(\eta)-f'(\eta)(b-\eta)\Big]\\
&= \big[f'(b)-f'(a)\big]\big[g(\eta)-g(b)\big]\Big[\tfrac{1}{2}\big(g(b)-g(\eta)\big) - g'(\eta)(b-\eta)\Big].
\end{aligned}$$

■

Remark 4. *If* $g(x) = x$ *in* (3.2)*, we get* Çakmak-Tiryaki's theorem

$$\begin{aligned}
(b-a)\big[f(b)-f(\eta)-f'(\eta)(b-\eta)\big] &= \big[f'(b)-f'(a)\big](\eta-b)\big[\tfrac{1}{2}(b-\eta)-(b-\eta)\big]\\
(b-a)\big[f(b)-f(\eta)-f'(\eta)(b-\eta)\big] &= \tfrac{1}{2}\big[f'(b)-f'(a)\big](b-\eta)^2\\
f(b)-f(\eta) &= f'(\eta)(b-\eta) + \tfrac{1}{2}\frac{f'(b)-f'(a)}{b-a}(b-\eta)^2.
\end{aligned}$$

Now we prove our last results, which are, some variants of Theorem 3.1 and 3.2 ([5, Theorem 2.5 and Theorem 2.6]).

Theorem 3.3. *If* $f,g:[a,b]\to\mathbb{R}$ *are differentiable functions on* $[a,b]$*, then there exists* $\eta\in(a,b)$ *such that*

$$\begin{aligned}
&\big[g(b)-g(a)\big]^{n-1}g'(b)\big[f(\eta)-f(a)-f'(\eta)(\eta-a)\big]\\
&= \big[f'(b)-f'(a)\big]\big[g(\eta)-g(a)\big]^{n-1}\Big[\tfrac{1}{n}\big(g(\eta)-g(a)\big)-g'(\eta)(\eta-a)\Big], n\geqslant 2.
\end{aligned}\tag{3.3}$$

Proof: Let $n\geqslant 2$ and consider the function $\mathcal{G}:[a,b]\to\mathbb{R}$ given by

$$\mathcal{G}(x) = \big[g(b)-g(a)\big]^{n-1}g'(b)f(x) - \tfrac{1}{n}\big[f'(b)-f'(a)\big]\big[g(x)-g(a)\big]^n.$$

The function $\mathcal{G}$ is differentiable on $[a,b]$ and

$$\mathcal{G}'(x) = \big[g(b)-g(a)\big]^{n-1}g'(b)f'(x) - \big[f'(b)-f'(a)\big]\big[g(x)-g(a)\big]^{n-1}g'(x).$$

Also,

$$\begin{aligned}
\mathcal{G}'(a) &= \big[g(b)-g(a)\big]^{n-1}g'(b)f'(a)\\
\mathcal{G}'(b) &= \big[g(b)-g(a)\big]^{n-1}g'(b)f'(b) - \big[f'(b)-f'(a)\big]\big[g(b)-g(a)\big]^{n-1}g'(b)\\
&= \big[g(b)-g(a)\big]^{n-1}g'(b)\big[f'(b)-(f'(b)-f'(a))\big]\\
&= \big[g(b)-g(a)\big]^{n-1}g'(b)f'(a).
\end{aligned}$$

Thus, $\mathcal{G}'(a) = \mathcal{G}'(b)$. Then, by Flett's theorem, there exists $\eta\in(a,b)$ such that $\mathcal{G}(\eta)-\mathcal{G}(a) = \mathcal{G}'(\eta)(\eta-a)$, i.e.,

$$\begin{aligned}
&\big[g(b)-g(a)\big]^{n-1}g'(b)f(\eta) - \tfrac{1}{n}\big[f'(b)-f'(a)\big]\big[g(\eta)-g(a)\big]^n - \big[g(b)-g(a)\big]^{n-1}g'(b)f(a)\\
&= \Big[\big[g(b)-g(a)\big]^{n-1}g'(b)f'(\eta) - \big[f'(b)-f'(a)\big]\big[g(\eta)-g(a)\big]^{n-1}g'(\eta)\Big](\eta-a)\\
&\Leftrightarrow \big[g(b)-g(a)\big]^{n-1}g'(b)\big[f(\eta)-f(a)-f'(\eta)(\eta-a)\big]\\
&= \big[f'(b)-f'(a)\big]\big[g(\eta)-g(a)\big]^{n-1}\Big[\tfrac{1}{n}\big(g(\eta)-g(a)\big)-g'(\eta)(\eta-a)\Big].
\end{aligned}$$

■

Remark 5. *If* $g(x) = x$ *in* (3.3)*, we get a variant of* Sahoo-Riedel's theorem (Theorem 2.2)

$$(b-a)^{n-1}\big[f(\eta)-f(a)-f'(\eta)(\eta-a)\big] = \big[f'(b)-f'(a)\big](\eta-a)^{n-1}\Big[\tfrac{1}{n}(\eta-a)-(\eta-a)\Big]$$
$$f(\eta)-f(a)-f'(\eta)(\eta-a) = -\frac{n-1}{n}\frac{f'(b)-f'(a)}{(b-a)^{n-1}}(\eta-a)^n$$
$$f(\eta)-f(a) = f'(\eta)(\eta-a) - \frac{n-1}{n}\frac{f'(b)-f'(a)}{(b-a)^{n-1}}(\eta-a)^n.$$

Theorem 3.4. *If* $f, g : [a,b] \to \mathbb{R}$ *are differentiable functions on* $[a,b]$*, then there exists* $\eta \in (a,b)$ *such that*

$$\begin{aligned}&\big[g(a)-g(b)\big]^{n-1}g'(a)\big[f(b)-f(\eta)-f'(\eta)(b-\eta)\big]\\ &= \big[f'(b)-f'(a)\big]\big[g(\eta)-g(b)\big]^{n-1}\Big[\tfrac{1}{n}\big(g(\eta)-g(a)\big)+g'(\eta)(b-\eta)\Big].\end{aligned} \tag{3.4}$$

Proof: Let $n \geqslant 2$ and consider the function $\mathcal{G}_1 : [a,b] \to \mathbb{R}$ given by

$$\mathcal{G}_1(x) = \big[g(a)-g(b)\big]^{n-1}g'(a)f(x) + \tfrac{1}{n}\big[f'(b)-f'(a)\big]\big[g(x)-g(b)\big]^n.$$

The function $\mathcal{G}_1$ is differentiable on $[a,b]$ and

$$\mathcal{G}_1'(x) = \big[g(a)-g(b)\big]^{n-1}g'(a)f'(x) + \big[f'(b)-f'(a)\big]\big[g(x)-g(b)\big]^{n-1}g'(x).$$

Also,

$$\begin{aligned}\mathcal{G}_1'(a) &= \big[g(a)-g(b)\big]^{n-1}g'(a)f'(a) + \big[f'(b)-f'(a)\big]\big[g(a)-g(b)\big]^{n-1}g'(a)\\ &= \big[g(a)-g(b)\big]^{n-1}g'(a)\big[f'(a)+f'(b)-f'(a)\big]\\ &= \big[g(a)-g(b)\big]^{n-1}g'(a)f'(b)\\ \mathcal{G}_1'(b) &= \big[g(a)-g(b)\big]^{n-1}g'(a)f'(b).\end{aligned}$$

Thus, $\mathcal{G}_1'(a) = \mathcal{G}_1'(b)$. Then, by Myers'theorem, there exists $\eta \in (a,b)$ such that $\mathcal{G}_1(b)-\mathcal{G}_1(\eta) = \mathcal{G}_1'(\eta)(b-\eta)$, i.e.,

$$\begin{aligned}&\big[g(a)-g(b)\big]^{n-1}g'(a)f(b)-\big[g(a)-g(b)\big]^{n-1}g'(a)f(\eta)-\tfrac{1}{n}\big[f'(b)-f'(a)\big]\big[g(\eta)-g(b)\big]^n\\ &= \Big[\big[g(a)-g(b)\big]^{n-1}g'(a)f'(\eta)+\big[f'(b)-f'(a)\big]\big[g(\eta)-g(b)\big]^{n-1}g'(\eta)\Big](b-\eta)\\ &\Leftrightarrow \big[g(a)-g(b)\big]^{n-1}g'(a)\Big[f(b)-f(\eta)-f'(\eta)(b-\eta)\Big]\\ &= \big[f'(b)-f'(a)\big]\big[g(\eta)-g(b)\big]^{n-1}\Big[\tfrac{1}{n}(g(\eta)-g(b))+g'(\eta)(b-\eta)\Big].\end{aligned}$$

■

Remark 6. *Se* $g(x) = x$ *in* (3.4)*, we get a variant of* Çakmak-Tiryaki's theorem (Teorema (2.4))

$$(a-b)^{n-1}\big[f(b)-f(\eta)-f'(\eta)(b-\eta)\big] = (f'(b)-f'(a))(\eta-b)^{n-1}\big[\tfrac{1}{n}(\eta-b)+(b-\eta)\big]$$
$$(a-b)^{n-1}\big[f(b)-f(\eta)-f'(\eta)(b-\eta)\big] = \tfrac{n-1}{n}(f'(b)-f'(a))(\eta-b)^{n-1}(b-\eta)$$
$$f(b)-f(\eta)-f'(\eta)(b-\eta) = \tfrac{n-1}{n}\frac{f'(b)-f'(a)}{(a-b)^{n-1}}(\eta-b)^{n-1}(b-\eta). \tag{3.5}$$

(**i**) *If* n *is even, from* (3.5) *we get*

$$f(b)-f(\eta) = f'(\eta)(b-\eta) + \frac{n-1}{n}\frac{f'(b)-f'(a)}{(b-a)^{n-1}}(b-\eta)^n.$$

(**ii**) *If* n *is odd, from* (3.5) *we get*

$$f(b)-f(\eta) = f'(\eta)(b-\eta) + \frac{n-1}{n}\frac{f'(b)-f'(a)}{(b-a)^{n-1}}(b-\eta)^n.$$

From (**i**) *and* (**ii**) *we have* (2.5) *of* Theorem (2.4).

4. Conclusion. The use of elementary auxiliary functions to prove type mean value theorems is an effective didactic resource. This has already been seen in the proof of Lagrange's mean value theorem, see references [4], [6], [8] and [9], for example.

In this note using some elementary auxiliary functions and differential Calculus some variants of Lagrange's mean value theorem were proved (Theorems 2.2, 2.4, 3.3 and 3.4).

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