A NUMERICAL MODEL USING THE FINITE ELEMENT METHOD FOR THE SIMULATION OF ONE PHASE FLOW IN TWO DIMENSIONS

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A Finite Element Model (FEM) is presented to simulate flow of a single phase fluid through a porous media in 2-dimensions. This model is the initial phase of a research project that builds on the development of a general reservoir simulator for three phase flow. The numerical model can be applied to heterogeneous fields with constant thickness but complex irregular geometry. No restrictions are imposed in the number of wells in the field. Governing equations and the boundary conditions are specified, followed by a description of the FEM formulation. Analytical solutions are used to verify the predictions of the model in a circular reservoir. It is also applied in the preliminary simulation of a sector in the Orito oil field (Ecopetrol) located in the Putumayo Department (Southern Colombia).

Se presenta un modelo en elementos finitos para simular el flujo de un fluido monofásico en un medio poroso bidimensional. Este modelo constituye la fase inicial de un proyecto destinado al desarrollo de un simulador general de yacimientos para flujo multifásico. El modelo numérico desarrollado considera que el dominio tiene un espesor constante, puede ser aplicado a campos heterogéneos con geometrías irregulares y complejas, y no hay restricción en el número de pozos que pueda tener el campo. Las siguientes suposiciones fueron hechas para hacer el problema manejable: un fluido ligeramente compresible, isotérmico, ausencia de efectos gravitacionales y un medio poroso isotrópico. Las predicciones del modelo desarrollado son verificadas con soluciones analíticas en el caso de un yacimiento circular y aplicado a la simulación muy preliminar de un sector del campo Orito (Ecopetrol) localizado en el departamento del Putumayo (al Sur de Colombia).

Palabras Claves: *métodos numéricos, simulación, elementos finitos.*

* *A quien debe ser enviada la correspondencia.*

NOMENCLATURE

B	:Volumetric Factor
c_t	:Total Compressibility
h	:Depth
k	:Permeability
N	:Number of elements
n	:Normal unit vector
n_x	:Component in the x direction of the normal unit vector
n_y	:Component in the y direction of the normal unit vector
p	:Pressure
p_i	:Initial Pressure
q	:Source or Sink
Q_n	:Flow of mass that enters or leaves through the boundaries
R	:Radius of the reservoir
t	:Time
x, y	:Cartesian Coordinates
μ	:Fluid viscosity
ρ	:Fluid density
ϕ	:Porosity
Γ	:Boundary of the reservoir
Γ_e	:Boundary of the element
ψ	:Interpolation function
Ω	:Domain of the reservoir
Ω_e	:Domain of the element

INTRODUCTION

Reservoir Simulation is a technological tool that has achieved a high degree of development during the last two decades, becoming now a mature technology. There exist a number of reservoir simulators available in the market, most of them developed using the numerical technique called Finites Differences or their modifications.

Recently some researchers (Nansen, 1993; Khalid, 1993; Abdou, Pham and Al - Aqeel, 1993) have analized new possibilities and trends in the developments of numerical simulators and optimization using the Finite Element Method. In the last ten years, several articles have been published related to the application of the Finite Element Method in reservoir simulation (Forsyth, 1990; Fung, Hiebert and Nghiem, 1992)

Reservoir simulation requires as a first step, the knowledge of basic parameters or input data which must be previously fixed by using techniques or tools different from the Reservoir Simulation. The most common input data are: permeability (k), porosity (φ), fluid viscosities (μ), total compressibility (c_t), thickness (h), initial pressure (p_i), injection rates (q_i), initial production rates (q_p) and boundaries of the reservoir domain (Γ).

The results or output data after a simulation run include: prediction of production rates versus time (as a production curves - after matching process or as a vectorial distribution of velocity on the problem domain) and predictions of pressure variations all over the reservoir (isobaric maps).

The development of a mathematical model that describes the displacement of fluids in a porous media involves differential equations that are usually too complex to be solved by analytical methods. Therefore, numerical methods are an alternative that allows the obtention of approximate solutions to the problem. Finite Difference, Finite Element and Boundary Element method are among the most used numerical methods.

The Finite Element Method (FEM), which is used in this project, has certain advantages in comparison to other methods such as the Finite Difference Method. These advantages are due to a better representation of complex domains and the case of imposing boundary conditions on boundaries of arbitrary shape (Lancaster and Salkauskas 1990; Zienkiewicz 1977).

The goal of the present work was to develop a computer tool to be used in numerical simulation of petroleum reservoirs with complex geological characteristics, such as irregular edges (non regular geometrical borders) and the presence of multiple geological faults through the reservoir. These characteristics are present in most Colombian fields.

Initially, the mathematical equations were presented based on the rules of flow in porous media. Afterwards, the numerical formulation was developed, in which the differential equation obtained is approximated spatially by the Galerkin method and temporary by a finite difference scheme. Then, the model was first verified for a radial flux system, comparing the results with the analytical solution. Finally, the model was applied to

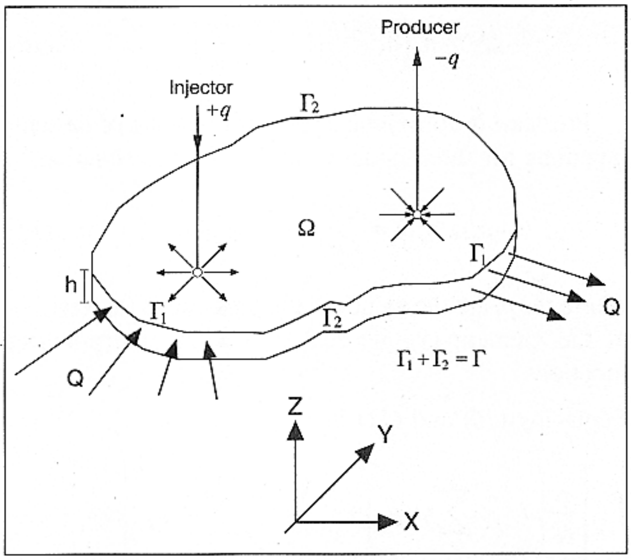

Figure 1. Physical model of the one phase problem in two dimensions and boundary conditions.

the simulation of the flow of oil in a irregular domain, where the velocity vectors and the pressure contours are obtained in different time steps.

MATHEMATICAL FORMULATION

A porous media (Ω) with constant thickness (*h*) saturated with fluid, slightly compressible, which is flowing in a horizontal plane (X, Y plane) through a homogenous media was considered. The domain can contain one or more production wells (-*q*) o injection wells (+*q*) as it is shown in Figure 1.

The equation that governs flow of fluids in porous media (Aziz and Setari, 1979) are obtained using Darcy's law (mass conservation and the state equations):

$$\frac{\partial}{\partial x}\left(\frac{k_x}{\phi\mu c_t}\frac{\partial p}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{k_y}{\phi\mu c_t}\frac{\partial p}{\partial y}\right)= \frac{\partial p}{\partial t}+\frac{1}{\rho\phi c_t}q \qquad \text{in } \Omega \qquad (1)$$

Equation (*1*) is known as the equation of diffusion, and it's subject to the following initial and boundary conditions.

Initial Conditions

It represents the initial state of the reservoir (pressure of the porous media at $t=0$).

$$p(x,y;0) = p_i \qquad x,y \in \Omega \qquad (2)$$

Boundary Conditions

Two types of boundary conditions are considered in the present study.

Boundaries with specified flow

When the flow rate is specified $Q(\Gamma)$ at a given part of the boundary (Γ1 in Figure 1), the normal component of the vector velocity by unit area in the boundary should be equal to the rate of flow. In other words, the directional derivative of the pressure is specified in a normal direction to the boundary of the domain $\partial p/\partial n$.

The rate of flow $Q(\Gamma)$ is calculated by the escalar product between the velocity (Darcy's Law) and the normal vector to the boundary $\vec{n}$:

$$-\frac{k}{\mu}\left(\vec{\nabla}p\right)\cdot\vec{n}=Q(\Gamma) \qquad \text{on} \qquad (3)$$

In the special case, when there is no flow through a given part of the boundary (Γ_2 in Figure 1), the normal component of the vector velocity in the boundary has to be zero.

$$\vec{\nabla}p\cdot\vec{n}=0 \qquad \text{on } \Gamma_2 \qquad (4)$$

Boundaries with specified pressure

The distribution or value of pressure is specified at the boundary by:

$$p(x,y;t) = p \qquad \text{on } \Gamma \qquad (5)$$

FORMULATION BY FINITE ELEMENTS

The method of finite elements has been used for the solution of continuum problems since the fifties. It's explained in detail in the references (Reddy, 1984; Strang and Fix, 1973; Zienkiewicz, 1977; Desai, 1979; Lancaster and Salkauskas, 1990).

Initially, the ruling equation (*1*) is written in an abridged form:

$$-\frac{\partial}{\partial x}\left(a_x\frac{\partial p}{\partial x}\right)-\frac{\partial}{\partial y}\left(a_y\frac{\partial p}{\partial y}\right)+\frac{\partial p}{\partial t}+\tilde{q}=0 \qquad (6)$$

where: $a_x=\frac{k_x}{\phi\mu c_t}$; $a_y=\frac{k_y}{\phi\mu c_t}$ y $\tilde{q}=\frac{q}{\rho\phi c_t}$

After that, the porous media is discretized in *N* sub-regions or triangular elements (Figure 2), which make it easier to simulate irregular geometries.

Spatial Discretization

Applying the method of weighted residuals (Reddy,

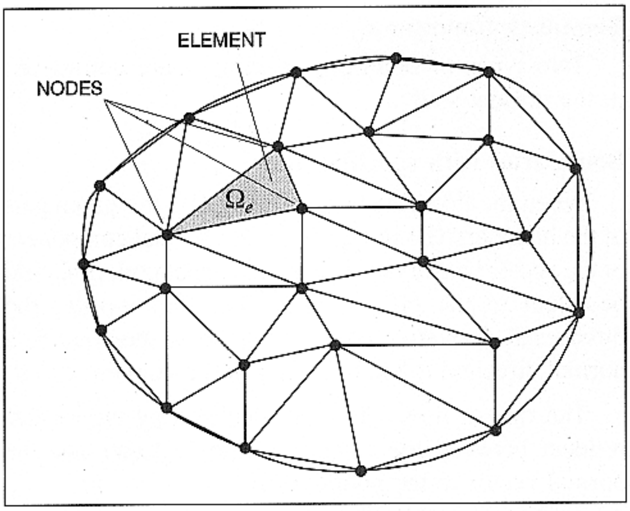

Figure 2. Discretization of domain by finite elements and identification of basic components.

1984) the result of equation (*6*) multiplied by a function of weight v and then integrated over a domain composed of element Ω_e gives:

$$\int_{\Omega_e} v\left[-\frac{\partial}{\partial x}\left(a_x\frac{\partial p}{\partial x}\right)-\frac{\partial}{\partial y}\left(a_y\frac{\partial p}{\partial y}\right)+\frac{\partial p}{\partial t}+\tilde{q}\right]d\Omega=0 \qquad (7)$$

Applying the theorem of Green-Gauss to the first term in equation (*7*),

$$\int_{\Omega_e} v\left[-\frac{\partial}{\partial x}\left(a_x\frac{\partial p}{\partial x}\right)\right]d\Omega=$$
$$\int_{\Omega_e}\left[\frac{\partial v}{\partial x}\left(a_x\frac{\partial p}{\partial x}\right)\right]d\Omega-\int_{\Gamma_e} v\left[\left(a_x\frac{\partial p}{\partial x}\right)n_x\right]d\Gamma \qquad (8)$$

and repeating the same process for the second term and replacing gives,

$$\int_{\Omega_e}\left[\frac{\partial v}{\partial x}\left(a_x\frac{\partial p}{\partial x}\right)+\frac{\partial v}{\partial y}\left(a_y\frac{\partial p}{\partial y}\right)\right]d\Omega$$
$$-\oint_{\Gamma_e} v\left[n_x\left(a_x\frac{\partial p}{\partial x}\right)+n_y\left(a_y\frac{\partial p}{\partial y}\right)\right]d\Gamma$$
$$+\int_{\Omega_e} v\left(\frac{\partial p}{\partial t}\right)d\Omega+\int_{\Omega_e} v\tilde{q}\,d\Omega=0 \qquad (9)$$

The boundary terms in the equation (*9*) corresponds to the flux of mass through the boundaries which is the scalar product of the normal vector and the flow, represented by:

$$Q_n=n_x\left(a_x\frac{\partial p}{\partial x}\right)+n_y\left(a_y\frac{\partial p}{\partial y}\right) \qquad (10)$$

Pressure is aproximated by interpolation of defined functions for three nodal elements, by the equation:

$$p_e(x,y,t)=\sum_{j=1}^{3}\psi_j(x,y)p_j(t) \qquad (11)$$

where the p_j are the values of the pressure in the vertices of the element (x_j, y_j) and ψ_j are the interpolating functions.

Replacing (*10*) and (*11*) in (*9*),

$$\int_{\Omega_e}\left[\frac{\partial v}{\partial x}\left(a_x\frac{\partial}{\partial x}\sum_{j=1}^{3}\psi_j p_j\right)+\frac{\partial v}{\partial y}\left(a_y\frac{\partial}{\partial y}\sum_{j=1}^{3}\psi_j p_j\right)\right]d\Omega$$
$$-\oint_{\Gamma_e} vQ_n\,d\Gamma+\int_{\Omega_e} v\left(\frac{\partial}{\partial t}\sum_{j=1}^{3}\psi_j p_j\right)d\Omega+\int_{\Omega_e} v\tilde{q}\,d\Omega=0 \qquad (12)$$

and applying the method of Galerkin $(v=\psi_i)$ the following equation is obtained:

$$\sum_{j=1}^{3}\left\{\int_{\Omega_e}\left[\frac{\partial\psi_i}{\partial x}\left(a_x\frac{\partial\psi_j}{\partial x}\right)+\frac{\partial\psi_i}{\partial y}\left(a_y\frac{\partial\psi_j}{\partial y}\right)\right]d\Omega\right\}p_j$$
$$-\oint_{\Gamma_e}\psi_i Q_n\,d\Gamma+\sum_{j=1}^{3}\int_{\Omega_e}\psi_i\psi_j\left(\frac{\partial p_j}{\partial t}\right)d\Omega+$$
$$\int_{\Omega_e}\psi_i\tilde{q}\,d\Omega=0 \qquad (i=1,\cdots N) \qquad (13)$$

or $\left[K^{(e)}\right]\{p\}+\left[M^{(e)}\right]\{\dot{p}\}=\left\{F^{(e)}\right\}$ for each element e

where:

$$K_{ij}^{(e)}=\int_{\Omega_e}\left[\frac{\partial\psi_i}{\partial x}\left(a_x\frac{\partial\psi_j}{\partial x}\right)+\frac{\partial\psi_i}{\partial y}\left(a_y\frac{\partial\psi_j}{\partial y}\right)\right]d\Omega$$

$$M_{ij}^{(e)}=\int_{\Omega_e}\psi_i\psi_j\,d\Omega$$

$$F_i^{(e)}=\int_{\Omega_e}\psi_i\tilde{q}\,d\Omega-\oint_{\Gamma_e}\psi_i Q_n\,d\Gamma$$

Matriz $\left[K^{(e)}\right]$ includes the coefficients obtained in the spatial discretization; matriz $\left[M^{(e)}\right]$ contains the coefficients obtained in the spatial discretization that accompany the temporal discretization, and the vector $\left\{F^{(e)}\right\}$ contains the terms related to the sources (producing wells) and/or sinks (injecting wells) plus the flow of mass that enters or leaves the element by the boundaries.

Temporal Discretization

According to Reddy (1984), the derivate of pressure versus time, in equation (*13*), is aproximated using a scheme in finite differences (Ritchmyer and Marton, 1967; Smith, 1987). In other words, a recurrence relationship is used which linearly weights the values of pressure in two time steps (t_n, t_{n+1}):

$$\theta\{\dot{p}\}_{n+1} + (1-\theta)\{\dot{p}\}_n = \frac{\{p\}_{n+1} - \{p\}_n}{\Delta t_{n+1}} \qquad (14)$$

$$\text{for } 0 \leq \theta \leq 1$$

Galerkin's scheme was used for this particular problem with $\theta = 2/3$ which gives unconditional stability.

Combining the different subsystems (*13*) that appear for each element *e*, and taking in consideration the connectivity between adjacent elements, a linear system for the whole domain is obtained.

$$[K]\{p\} + [M]\{\dot{p}\} = \{F\} \qquad (15)$$

where [*K*] is the connectivity matriz.

Applying the time marching process this linear system is solved at each time step, and the time - history of the pressure at each node is obtained.

VERIFICATION AND RESULTS

Circular homogenous reservoir with a production well at the center

The model was first applied to simulate the transient behavior of the pressure in a circular reservoir with a producing well, for which an analytical solution exists.

The geometry of the domain is shown in Figure 3 only a fourth of the domain is discretized, due to the symmetry of the problem. For the simulation, a total of 52 elements and 36 nodes were used. Initially, the reservoir had a constant pressure of 20,67 MPa (3.000 psi) a well is open at t=0 and the variation of pressure as a function of time is calculated. The fluid and porous media properties are: k=0,1 md, ϕ=0,23, μ=0,72 cp, c_t=1,5 Pa (1,5x10^{-5} psi), h=45,7 m (150 ft), R=30,5 m (100 ft), B=1,475 RB/STB, q=3,15 m^{3}d^{-1} (20 BPD), p_i=20,68 MPa.

Figure 4 shows the behavior of pressure as a function of time for three observations points. It's seen that the pressure dropout is more severe for the observation

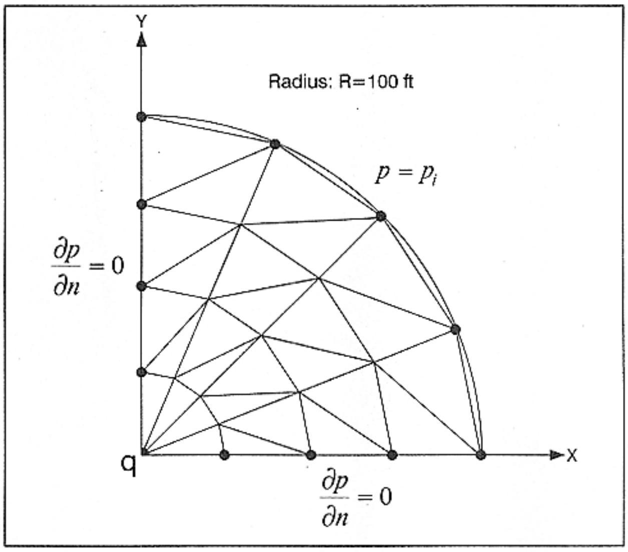

Figure 3. Grid used to simulate radial flow in a circular reservoir with a single well at the center.

point near to the well, r = 5 ft. Therefore, the steady state is not reached in the reservoir after 100 hours of flow.

In general, the results show that the numerical solution behaves in a similar way as an analytical solution. The difference between the two of them increases for the points closer to the well; this is due to the fact that the analytical solution assumes that the radius of the well is very small, while the numerical solution supposes a finite radius.

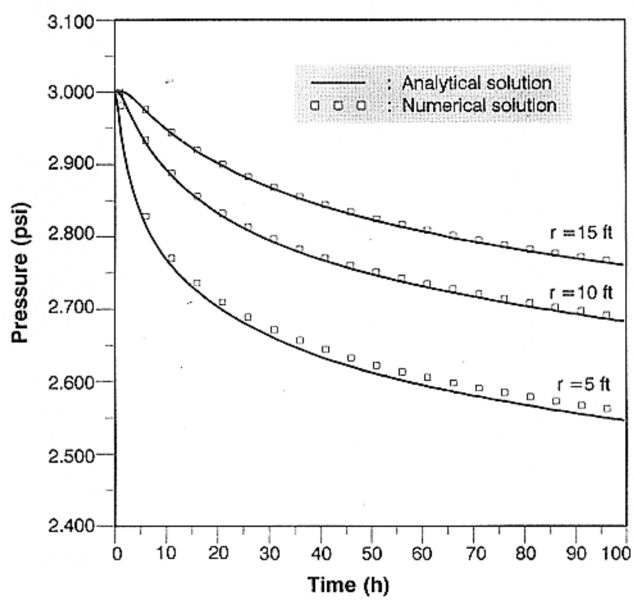

Figure 4. Comparison of the results obtained by the Finite Element Method against the analytical solution for the radial flow system.

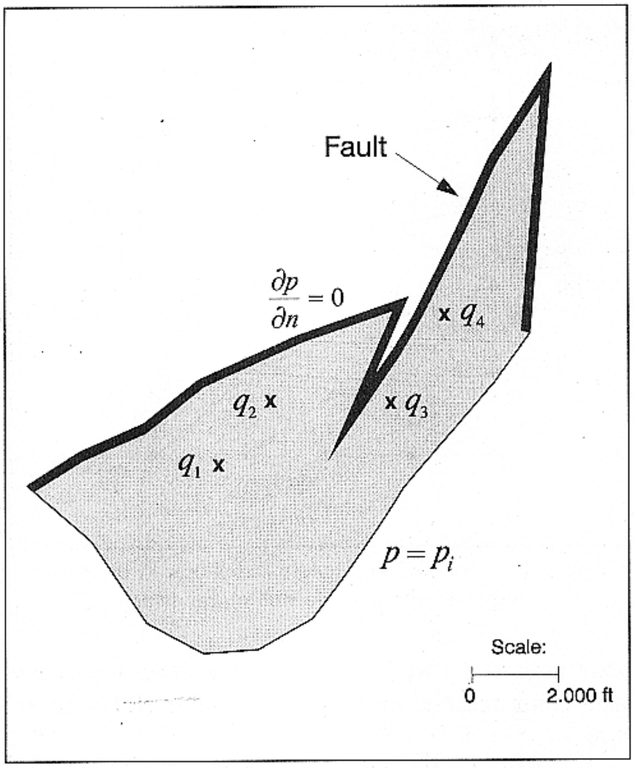

Figure 5. Physical model for reservoir with irregular geometry. Orito's sector containing four roducers wells.

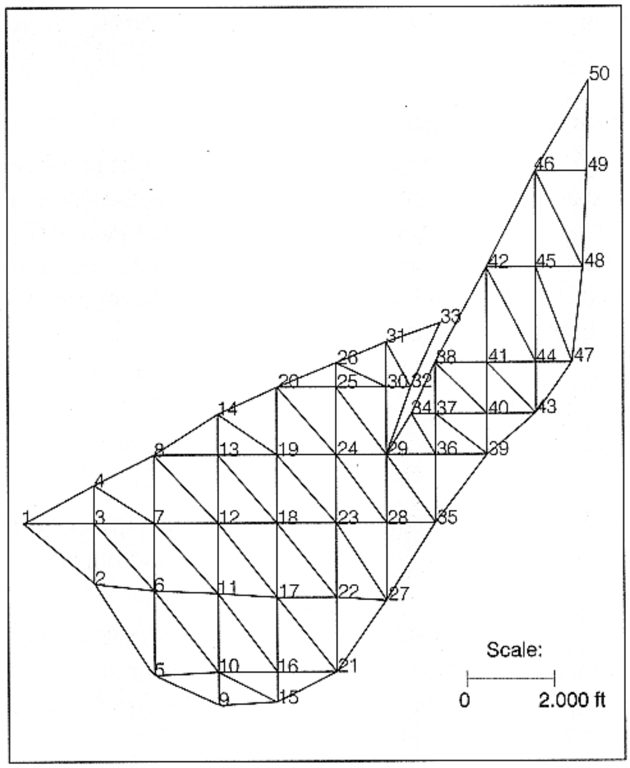

Figure 6. Grid with numbering at the nodes for the irregular geometry system.

Reservoir of Irregular Geometry

The model was applied to model the flow of oil in the Caballos formation Orito field (Putumayo, Colombia). Figure 5 illustrates the geometry and the boundary conditions of the field. In the system, there are four producing wells, a impermeable boundary represented by a fault, and the rest of the boundary is mantained at a constant pressure of 22,7 Mpa (3.293 psi). The fluid and rock properties are: $\phi=0{,}1$, $k=60$ md, $c_t=1{,}5$ Pa ($1{,}5 \times 10^{-5}$ psi), $h=36{,}6$ m (120 ft), $\mu=13$ cp, $B=1{,}0$ RB/STB, $q_1=238{,}4$ m^3d^{-1} (1.500 BPD), $q_2=286{,}1$

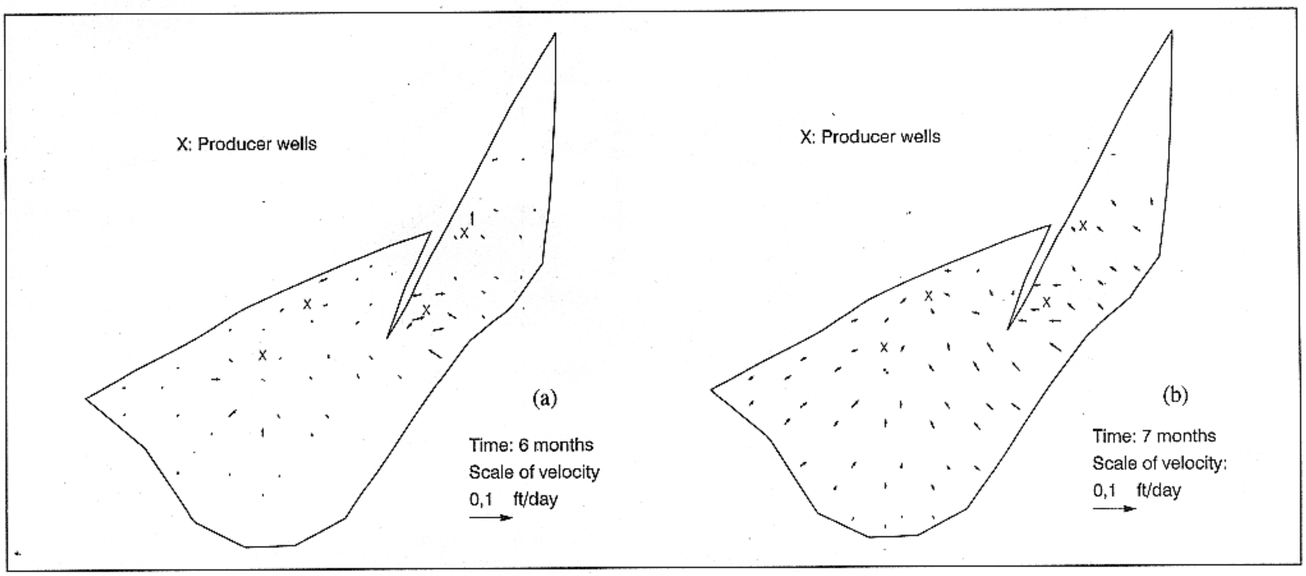

Figure 7. Velocity distribution vectors for Orito's field Sector.

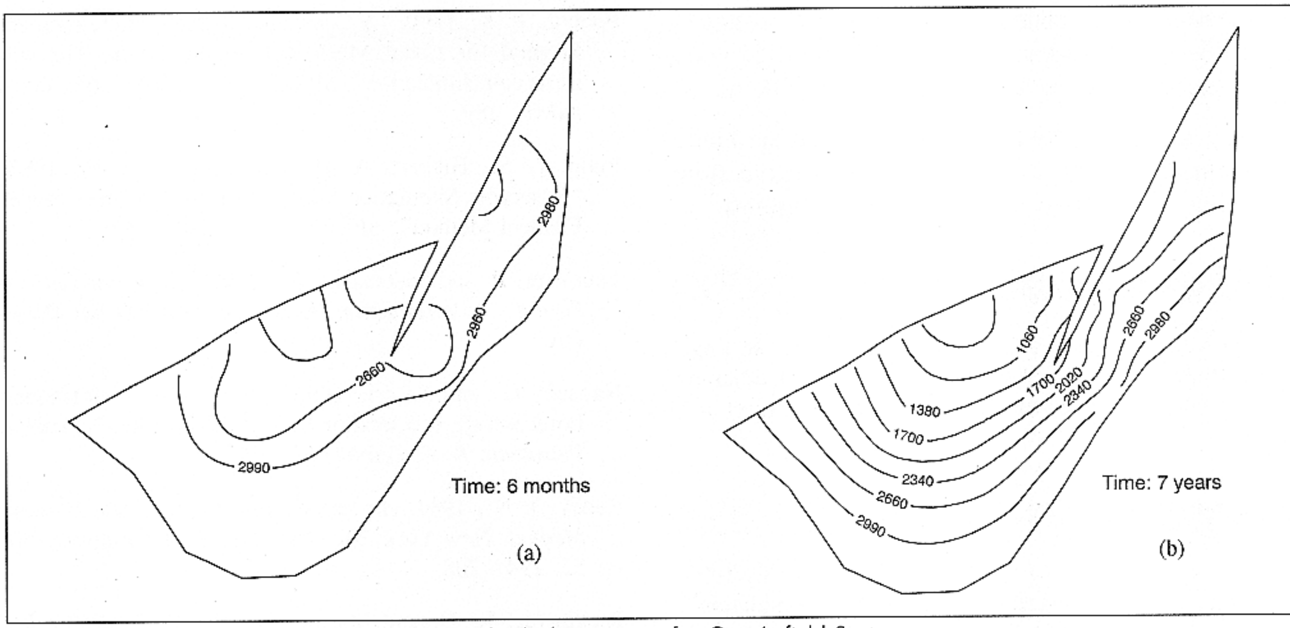

Figure 8. Isobaric maps for Orito's field Sector.

m³d⁻¹ (1.800 BPD), q_3=206,7 m³d⁻¹ (1.300 BPD), q_4=159,0 m³d⁻¹ (1.000 BPD), p_i=22,7 Mpa (3.293 psi).

The area of interest was divided in 70 triangular elements with 50 nodes as shown in Figure 6. The velocities and pressures of the oil are calculated in the centroid of each element for different time steps. The variation of the oil pressure was calculated for 10 years in different points of observation, as well.

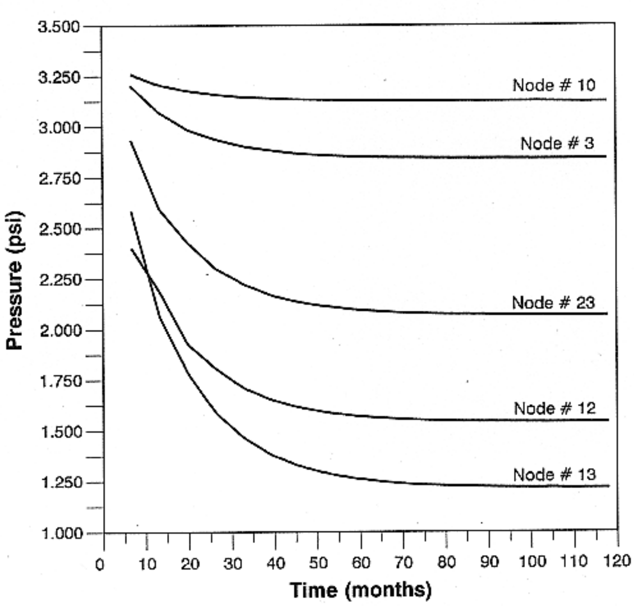

Figure 9. Behavior of pressure against time for differents control points at Orito's field Sector.

Figure 7, shows that the velocity has a radial pattern towards the producing wells. In zones far from the wells, the velocity is minimum, converging to zero near the impermeable boundary. The maximum velocity after 7 years is around 9 mmd⁻¹ (0,03 ft/day).

The isobaric maps presented in Figure 8 indicate that the pressure decreases gradually from 22,7 Mpa (3.293 psi) in the lower boundary to aproximately 1.000 psi in the upper impermeable boundary. In the wells, the pressure lines have a circular tendency, increasing and deforming until it adquires the shape of the boundary.

Figure 9 shows the values of pressure against time for different points of observation in the domain. It is observed that the behavior of the pressure is stable after a period of 60 months.

CONCLUSIONS

- A first aproximation of a numerical reservoir simulator for one single fluid flow in two dimensions, using Finite Elements, has been developed. In the future, the simulator will be improved to consider more complex situations found in multiphase and three dimensional flow.
- The numerical solution for the pressure of an idealized circular reservoir including one well at the center, was acceptable when compared to the analytical solution.

- Pressure values obtained for several control points inside of one sector of Orito's field showed good agreement with the known values at specific times.
- This single fluid flow model showed that the Finite Element Method is an alternative to solve flow problems in porous media in complex systems.

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