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A progression of student symbolizing: Solutions to systems of linear equations

Una progresión de la simbolización de estudiantes: Soluciones de un sistema de ecuaciones lineales

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Abstract ∞ Systems of linear equations (SLE) comprise a fundamental concept in linear algebra, but there is relatively little research regarding the teaching and learning of SLE, especially students' conceptions of solutions. It has been shown that solving systems with no or infinitely many solutions tends to be less intuitive for students, pointing to the need for more research on the teaching and learning of the topic. We interviewed two mathematics majors who were also preservice teachers in a paired teaching experiment to see how they reasoned about solutions to SLE in $\mathbb{R}^3$. We present findings focused on the progression of students' reasoning about solutions to SLE through the lens of symbolizing. We document their progression of reasoning as an accumulation of coordinated numeric, algebraic, and graphical meanings and symbolizations for solution sets.

Keywords ∞ Systems of linear equations; Linear algebra; Symbolizing; Student reasoning; Realistic Mathematics Education

Resumen ∞ Los sistemas de ecuaciones lineales (SEL) corresponden a un concepto fundamental del álgebra lineal, pero hay relativamente poca investigación, pero hay relativamente poca investigación acerca de la enseñanza y el aprendizaje de los SEL, particularmente de las concepciones de los estudiantes acerca de sus soluciones. Se ha encontrado que la resolución de sistemas con un número infinito de soluciones o sin solución tiende a ser menos intuitivo para los estudiantes, lo cual indica la necesidad de más investigación en la enseñanza y aprendizaje de este tema. Entrevistamos a dos estudiantes de matemáticas que eran también maestros en formación a través de un experimento de enseñanza por parejas para mirar cómo razonaban acerca de las soluciones de SEL en $\mathbb{R}^3$. Presentamos los resultados enfocando en la progresión del razonamiento de los estudiantes sobre las soluciones de los SEL a través del lento de simbolización. Documentamos la progresión de su razonamiento como una acumulación de significados numéricos, algebraicos y gráficos coordinados y las simbolizaciones de sus conjuntos solución.

Palabras clave ∞ Sistemas de ecuaciones lineales; Álgebra lineal; Simbolización; Razonamiento de los estudiantes; Educación Matemática Reaística

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1. Introduction

Systems of linear equations (SLE) are a core concept in linear algebra. Systems of equations are one way to model relationships among multiple quantities (Smith & Thompson, 2007). Students from applied science, technology, engineering, and mathematics make up a dominant portion of those enrolled in linear algebra in the U.S. and Canada, and applications related to linear systems are an important component of their learning in these courses (Andrews-Larson et al., 2022). Having a deep understanding of systems of equations helps in learning these applications, as well as other topics in linear algebra.

SLE that have no solution or infinitely many solutions are often discussed as nonstandard cases. Instruction usually foregrounds solving methods such as eliminating variables focus more on unique solution cases than non-unique cases, which may cause students to conceive unique solutions as standard (Huntley et al., 2007; Oktaç, 2018). However, in applied contexts, there are often SLE of no solution or infinitely many solutions. To give students more experience with SLE that have no or infinitely many solutions, we designed a task sequence that focuses on student reasoning about these SLE. Members of our author team (Larson & Zandieh, 2013; Zandieh & Andrews-Larson, 2019; Zandieh et al., 2017) have previously examined students' use of symbolizing to better understand students' reasoning about SLE and their solutions. In this paper, we examine the progression of students' ways of symbolizing to see the progression of their reasoning about solutions to SLE.

2. Background and theoretical framework

2.1. Literature review

Research regarding SLE is limited but growing. In this small pool of research, some regard solutions to SLE, whether through task development (e.g., Possani et al., 2010; Sandoval & Possani, 2016) or examining students' understanding of solutions algebraically and geometrically (e.g., Harel, 2017; Huntley et al., 2007; Oktaç, 2018; Zandieh & Andrews-Larson, 2019). Possani et al. (2010) developed a task about a set of streets, asking students which streets could be closed for road work without disrupting the flow of traffic. Major parts of this task were modeling the context, using variables to represent important quantities, and then writing an SLE based on those understandings. Sandoval and Possani (2016) developed a task to see when moving between representations in $\mathbb{R}^3$ (e.g., graphs versus equations of vectors or planes) makes sense for students. They found students could naturally interpret geometric representations of solutions to SLE, but connecting geometric representations to algebraic representations of solutions to be less natural.

Students at a variety of grade levels have exhibited more success solving SLE with unique solutions as compared to solving inconsistent SLE or SLE with an infinite number of solutions (Harel, 2017; Huntley et al., 2007; Oktaç, 2018). Huntley et al. (2007) and Oktaç (2018) found that the cases which resulted in Identity (e.g., $0 = 0$), and Contradiction (e.g., $0 = 1$) equations were not intuitive for students to interpret. Oktaç (2018) noted that when interpreting intersections of graphs as geometric solutions, students found lines formed in a triangle to have three solutions in the plane.

Several authors found that students emphasized algorithmic methods to solve SLE (Harel, 2017; Oktaç, 2018; Zandieh & Andrews-Larson, 2019). Harel (2017) noted that students tended to view finding solutions to SLE as an activity to apply a procedure, rather than to recognize it as a task constituted by intellectual need to find common solutions satisfying a set of quantitative constraints. It has more recently been documented that though students are relatively successful at rewriting SLE as augmented matrices and row reducing with technological assistance, many students experience a disconnect in reinterpreting solutions in relation to the original SLE (Zandieh & Andrews-Larson, 2019). Thus, research points to a need for tasks that help students to reason about SLE algebraically and geometrically with no or infinitely many solutions, especially those in $\mathbb{R}^3$.

2.2. Theoretical framework

In this paper, we focus on students' reasoning about solutions to SLE, and the way in which this reasoning progressed in the context of a particular sequence of tasks designed to support their learning. Mathematical progression can be described in multiple ways. For example, Rasmussen et al. (2015) describe two individual and collective ways of analyzing mathematical progression. One of their collective lenses is disciplinary practices, which describes student engagement in a mathematical practice such as defining, proving, algorithmatizing (creating algorithms) or symbolizing.

We use the disciplinary practices of symbolizing as an analytic lens through which to examine students' reasoning and characterize its progression. We agree with Rasmussen et al. (2005) who describe symbolizing as a form of activity in which students shift from "recording and communicating their thinking to using their symbolizations as inputs for further mathematical reasoning and conceptualization" (p. 57). We include graphical inscriptions as well as algebraic and numeric inscriptions as ways students may record and further their mathematical reasoning.

We leverage two previous papers that focus on student symbolizing in the context of linear algebra. Zandieh et al. (2017) highlighted four different types of symbolization that occurred as students progressed through a set of activities created to help students reinvent the diagonalization of square matrices. The three of these that are most relevant to our study are notating calculations or processes, articulating relationships between mathematical objects, and expressing connections between representations. The second paper (Zandieh & Andrews-Larson, 2019) extends this work to symbolizations of augmented matrices and discusses how the solution(s) to SLE may be expressed using both implicit and explicit notation. For example, the statement of an SLE to be solved is already an implicit description of the solution set, but after processes such as algebraic manipulation, solutions can be expressed more explicitly as a parametrized equation. We draw on both studies to examine how students' symbolizing about SLE and their solutions progressed as the students engaged in our Realistic Mathematics Education task sequence. We used the following research question to guide our analysis: *How did students' reasoning about solutions progress in the context of an RME task sequence about systems of equations?*

3. METHODS

3.1. Study Context: Task Sequence

Based on our review of the literature, it seemed to us that students needed instruction that specifically supported their conceptualizing and representation of solution sets. To do this, we drew on the instructional design heuristics of Realistic Mathematics Education (RME) to design a sequence of tasks with this goal. In doing so, we draw on Freudenthal's (1973, 1991) framing of mathematics as a human activity. One important heuristic in RME is the use of experientially real starting points (Gravemeijer & Doorman, 1999), an aspect of didactical phenomenology. These starting points allow students to engage in a setting that is familiar to them and from which the mathematical ideas of the task sequence may emerge.

In addition to didactical phenomenology, RME has the heuristics of guided reinvention and emergent models. Guided reinvention describes the process by which an instructor interacts with the students to help them reinvent aspects of the mathematics. The interview excerpts in our findings include examples of the interviewer serving in the role of the teacher who is guiding the reinvention. Lastly, emergent models describe the process of students progressing from context-specific reasoning to more general and formal reasoning as they work through the task sequence. Gravemeijer (1999) delineates four levels of activity that span this process. We will highlight *situational activity in the task setting*, *referential activity*, and *general activity*. (The fourth, formal activity, involves using the results of general activity in a new task setting and is beyond the scope of our data.)

In our sequence, students begin by engaging with a meal plan task which the students symbolize as a system of two linear equations (*situational activity in the task setting*). The students work with the SLE in a way that refers back to the meal plan scenario as a support for their reasoning (*referential activity*) and then engage in more abstract tasks in which they make conjectures about the nature of solutions to systems and their associated graphs (*general activity*). In the next sections, we provide more details about the task sequence.

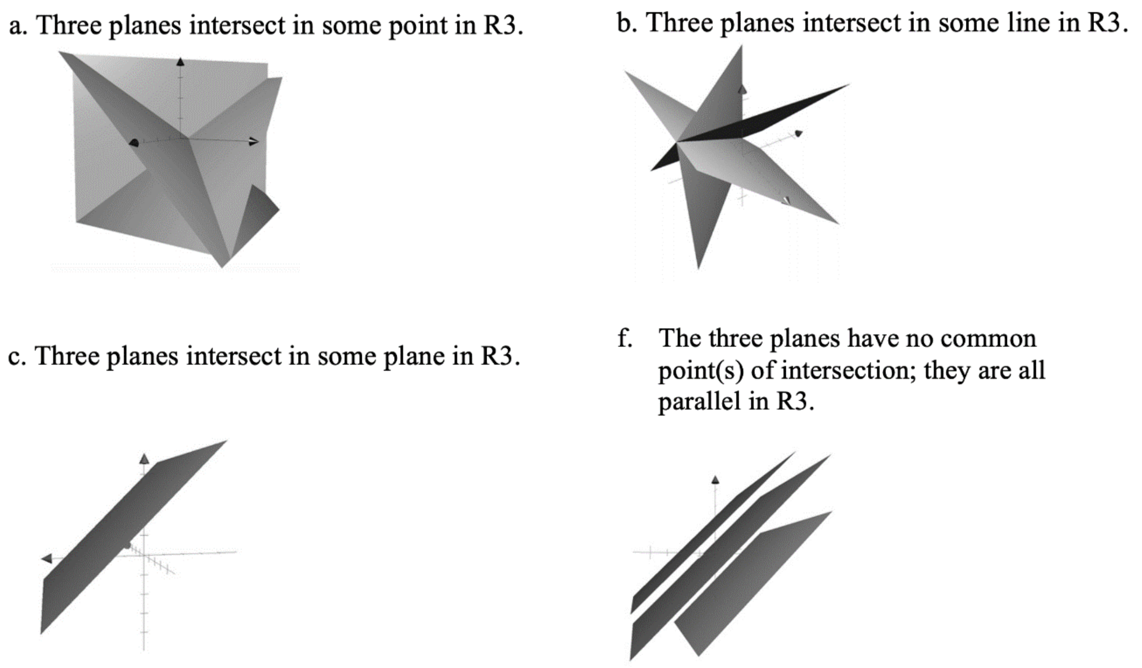
Task 1: Finding solutions to the number and cost of meals equations. We intend for students to engage in situational activity in the task setting by considering a constraint regarding the number of meals that can be purchased in the context of the meal plans (Table 1, Part 1). They are asked to list a few different choices for the 210-meal plan and estimate the total number of possible 210-meal plans. The goal for this part of the task is to have the students reason about and organize a large solution set (not to treat it as a combinatorics problem). Students are then given an additional constraint related to the cost of meals (Table 1, Part 2), and asked to write equations corresponding to each constraint. They then work to identify solutions that satisfies the constraints for the number of meals but not the cost (and vice versa) as well as solutions that simultaneously satisfies both constraints.

Task 2: Representing solutions geometrically using physical models. We intend that students continue with their situational activity in the task setting and are asked to predict what it would look like if all of the solutions to the number of meals constraint ($b + l + d = 210$) were graphed in three dimensions, and then to try to represent these solutions using the corner of a box as a representational aid. (Note: here b, l, and d are the number of breakfasts, lunches, and dinners, respectively.) They are then asked to find a way to represent all of the "no dinner" meal plans (ignoring the 105-meal maximum from day 1), and to make similar predictions corresponding to the cost constraint equation ($5b + 7l + 10d = 1500$).

Table 1. The Meal Plans Context

Part 1: Number of Meals. A university meal plan called the "210-meal plan" requires that a student purchase exactly 210 meals for a 15–week semester (i.e., on average 2 per day for 15 x 7 = 105 days).
Part 2: Costs of Meals. You just made an estimate of how many different choices would fit the requirements of the 210-meal plan. As you read the brochure more carefully, you noticed that the cost of your 210 meals must add up to exactly \$1500. Breakfasts cost \$5 each, lunches cost \$7 each, and dinners cost \$10 each.

Figure 1. Examples of options in the Intersections of Three Planes task (Wawro et al., 2013).



Task 3: Using GeoGebra to represent systems and their solution sets. We intend that in this task, students engage in referential activity. They begin by using GeoGebra to check their predictions about the graph of the SLE in the Meal Plans task. They are then given a new SLE of linear equations where the third equation is the sum of the first two equations and asked to predict which of six options could correspond to the SLE (sample options are shown in Figure 1). In the last part of this task, students use GeoGebra as an aid to try to construct SLE that look the different options.

Task 4: Constructing systems with particular solution types. In this task, we intend that students engage in general activity and work to generate examples of SLE with specified numbers of equations, unknowns, and solutions. In this paper, we use abbreviations to report these more succinctly. For example, 3E2U refers to 3 equations and 2 unknowns. Finally, students are asked to make generalizations about SLE regarding the number of solutions, equations, and unknowns.

3.2. Participants, Data Sources, and Methods of Analysis

This study is part of a broader NSF-funded project (1914793, 1914841, 1915156) focused on the development of curricular materials in inquiry-oriented linear algebra and extends our findings presented in Smith et al. (2021). This particular task sequence was designed to help students create or reinvent ways of thinking about solutions to SLE, especially infinite solution sets. Participants were two undergraduate students at a large public university in the Southeastern United States. Both were mathematics majors and preservice secondary mathematics teachers. Both students were white; one was a woman ("R"), and one was a man ("L"). The students were recruited by asking faculty in a local secondary teacher preparation program to identify two math majors preparing to be secondary math teachers who had taken Calculus I but not linear algebra, were neither atypically "strong" nor "weak" in terms of their mathematical preparation, and would be willing to explain their reasoning.

Two of the authors interviewed the students across four consecutive days on Zoom, working through as much of the designed task sequence as possible. We used a paired teaching experiment (PTE) to see how students reasoned with the tasks (Steffe & Thompson, 2000) and to allow for discussion between the students and the interviewer. When working through the task sequence, students briefly worked individually before discussing their reasoning. The interviewer prompted the students to think about a problem and explain their ideas, asking questions to further their thinking at some points.

Our data sources include Zoom video recordings, students' written work, and field notes. In order to document the progression of students' reasoning about solutions over the course of this four-day PTE, the research team analyzed the data in four phases. In the first phase of analysis, all four authors met together to watch and discuss videos of each day, making notes on students' reasoning related to SLE and their solution sets. In the second phase of analysis, two of the authors engaged in holistic coding (Saldaña, 2021) to identify focal exchanges that offer insight into student reasoning about solutions and selectively transcribed these exchanges. In the third phase of analysis, the author team generated analytic memos for each focal exchange and characterized shifts in student reasoning about solutions observed in the selected data. In the fourth phase of analysis, the author team identified symbolizing as a useful theoretical lens for characterizing shifts in students' reasoning about solutions. The authors knew from having previously watched the videos, but also anticipated from the literature (e.g., Larson & Zandieh, 2013), that many of the shifts would involve changing between algebraic, numeric, and graphical ways of symbolizing. They wanted to see what students were orienting to while working with these different ways of symbolizing. They triangulated raw video data, transcript, researcher notes from phase one with the narrative accounts of the selected exchanges, made any needed elaborations or corrections, and reinterpreted the findings through the lens of symbolizing in alignment with Larson and Zandieh (2013). To account for the progression of students' symbolization and, thus, reasoning, the authors examined instances when students used new ways of symbolizing. For example, if students began by only symbolizing algebraically and later symbolized graphically, they counted this as a progression in students' ways of symbolizing. This is consistent with Saldaña's (2021) characterization of elaborative coding, which importantly relies on extending findings from earlier studies. Note that this progression can be due to the design of the task sequence, a prompt from the interviewer, or students' own exploration.

4. Findings

In this section, we begin by describing students' initial ways of representing solutions, how the interviewer leveraged these initial ideas, and students' various ways of symbolizing solutions throughout tasks 1 and 2. Then, we report how students applied and extended these ways of symbolizing solutions when constructing SLE in tasks 3 and 4. We will describe the students' progression of reasoning through the accumulation of ways of symbolizing solutions.

4.1. Task 1: Finding solutions to the number and cost of meals equations

4.1.1. Solutions as ordered triples: Numerical symbolization of solutions

In the Meal Plans task, students were asked to list and estimate the number of solutions to the constraint that there were 210 meals spread across breakfast, lunch, and dinner ($b + l + d = 210$). Students began proposing numbers for each meal, such as saying there could be 70 breakfasts, 70 lunches, and 70 dinners. The interviewer suggested listing these as *ordered* triplets (e.g., (70,70,70)). This move is what Rasmussen and Marrongelle (2006) call a *transformational record*, in that the interviewer symbolized what the students were discussing while connecting it to a standard symbolization of the mathematics community.

As the students discussed how many different meal plan options were possible, student R asked, "Are we thinking about (105, 105, 0) being different from (105, 0, 105)?" Student L replied, "Does it matter?" They decided it made sense to count (105, 105, 0) and (105, 0, 105) differently since the order indicated which meal was which. This shift indicated an adaptation to their symbolization of solution, in that they came to explicitly think of the 3-tuple as an *ordered* triple where the order carried meaningful information. In the context of this task, the students came to symbolize solutions as ordered triples with the development of a shared notation with the convention that the number of breakfasts is the first value, lunches the second, and dinners the third.

4.1.2. Solutions notated in terms of a variable: Algebraic symbolization of solutions

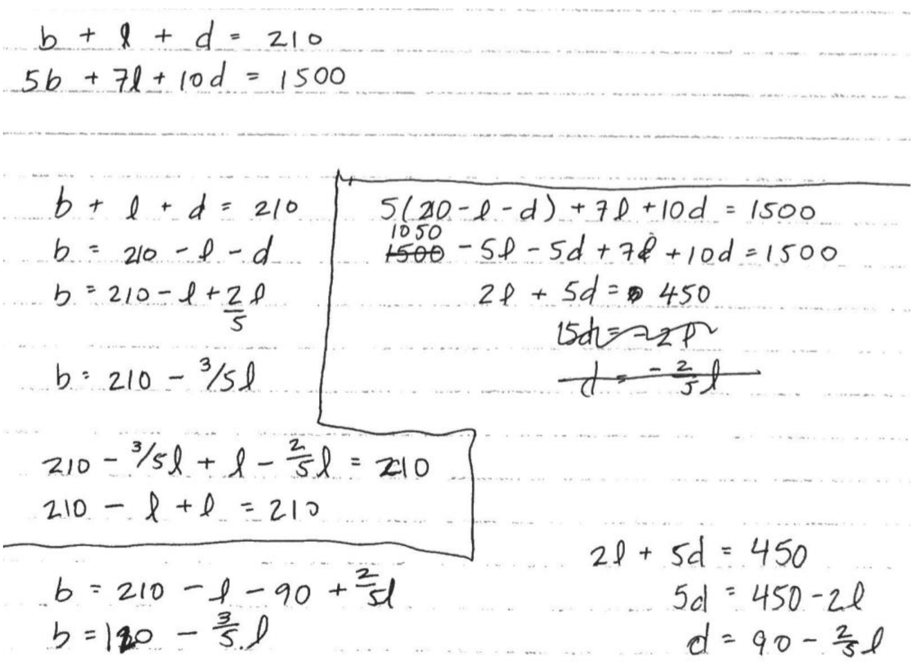
After estimating the total number of possible options satisfying the constraint on the number of meals, students were given an additional constraint on the cost of meals in the plan (mathematically equivalent to $5b + 7l + 10d = 1500$). After quickly symbolizing the two constraints algebraically, the students treated these as an SLE and began to use algebraic manipulation methods, primarily substitution (see Figure 2).

> L: I'm trying to have b equal to something and have b and l equal to whatever just based on the first equation. b + l + d = 210, b = 210 − l - d. I tried to plug it into the second equation.
>
> R: Something went wrong.
>
> L: It's because there are three variables, it's really messing it up.
>
> R: There's no variable anymore. Like it was 0 = 0. I got 210 − l + l = 210 which is not helping me with finding out what the variable equals. I'm pretty sure I just did the math wrong somewhere.
>
> I: What sort of thing are you hoping to get at the end?
>
> L: Probably going to be a positive number, the solution that satisfies the b, l, d.
>
> R: I'm realizing ... we're going to get an answer that satisfies both requirements. There's probably more than one. ... It feels like there would be more than one option or more than one solution for an equation that has this many variables in it.

Following Student R's prediction that there would be more than one solution, the students spent a few more minutes working on their algebraic manipulations and expressing frustration that the variables cancel. Student L remarked, "I'm just trying to get all the variables in terms of one variable like [R] was doing, but every time I try it I guess they'll either cancel or like I'll run into another issue where I just can't. Pretty much what I'm trying to do is try to relate all the quantities to like one specific variable."

The interviewer reminded the students of the many solutions they had previously listed, and asked them to think about whether any of those might work. Student L suggested the "easiest one is to just plug in 70." Student R noted that (70,70,70) satisfies the first equation but not the second, and observed that if you plug 70 into the cost equation, "7 times 70 is 490 so I have left $1010." Student R began to try values for *b* that might work for this, and the interviewer reminded her she had previously found an expression for *d* in terms of *l*.

Figure 2. Student R's algebra



R: [Shows work] Okay, so I have the d = 90 - 2/5l and I plugged in 70 and I got d = 62. So then I plugged that into the price equation with my 70 times 7 and I got 5b + 1110 = 1500. And then I got b = 78, which is how many breakfasts there are. So then I added 78 + 62 + 70 and that equals 210.

I: And then did it satisfy the 1500? Or I guess you plugged into the 1500 one, didn't you?

R: Right.

I: Okay, so you found one that works.

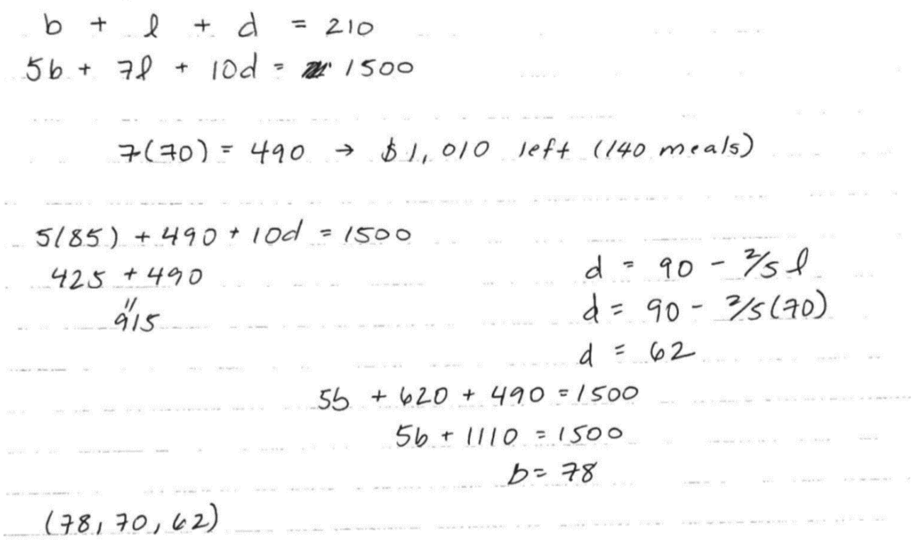
R: Yes. The ordered triple is (78, 70, 62). Which is a very random number. So, how are we supposed to find more? So does it work like that, well obviously every number doesn't work. So do you have to like get lucky with the number you choose to plug in?

I: What do you mean every number doesn't work?

R: Because we know that some, like (70, 70, 70) doesn't make both equations true but then when lunch is 70, there is a solution. So like, if I just like plugged in 75 for lunches like, how are we supposed to know which number works or is it just guess and check?

By plugging 70 in for lunch into an equation for d in terms of l (Figure 3) and using the second equation to find b, she found that (78, 70, 62) was a solution for both equations. Through this process, Student R was able to generate many ordered triples satisfying the two constraints, or solutions to the SLE. In working to solve two equations in three unknowns, the students initially anticipated finding a value for each variable. When their substitution methods did not yield this result, they initially thought they had made an error but eventually speculated there might be multiple solutions that could be expressed in terms of one of the variables. Even after finding correct expressions for b and d parameterized in terms of l, it was not immediately obvious to students how these could be used to find specific values that satisfied each equation. Students' drew on their initial interpretations for solutions as values that satisfy given constraints by making an equation a true statement. Using their parameterized expressions for b and d to find particular values for each variable that satisfied both equations helped students link their parameterized expressions for solutions with their initial interpretation of solutions.

Figure 3. Student R finds an ordered triple that satisfies the two given constraints



4.2. Task 2: Representing solutions geometrically using models

4.2.1. Solutions as collections of points and lines: Coordinating numeric and graphical symbolization

In the second session, the students were asked to think about the graph of the equations constraining the number and cost of meals in the meal plan from the previous day, considering the special case of a 'no dinners' line. Both drew on their prior knowledge of graphing a line in the xy-plane (Figure 4).

Figure 4. Student R's work in the plane

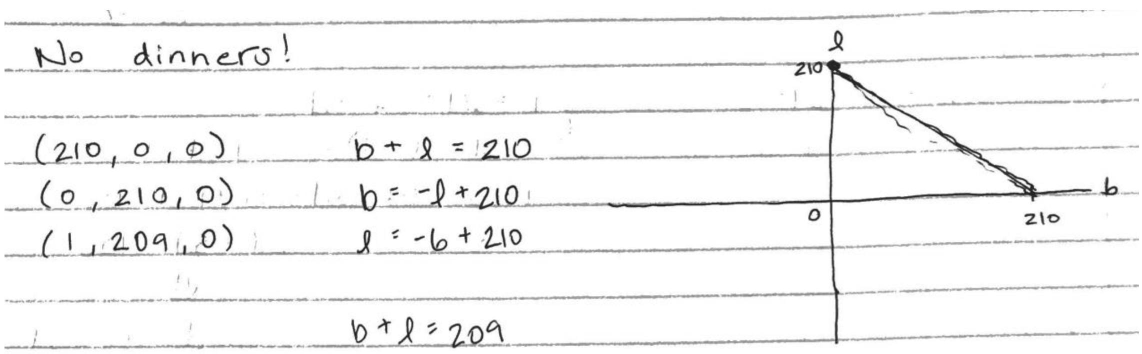
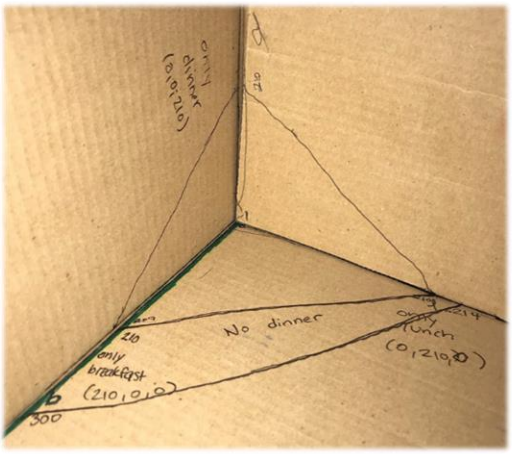


Figure 5. Student R's work on the box



> R: I first started listing out some of the triples with d equaling 0 but then I was like wait, I think there is an easier way to do this ... b + l is 210 ... and then I just graphed that which was a line with the y and x intercept both being 210 and connected them.
>
> L: I kind of did the same thing where I just plugged in 0 in the equation and I got b + l = 210, but then from there I wasn't really sure what to do. Obviously, b or l can be any number between 0 and 210 including 0 and 210. ... But ... I'm not really sure how to even display that geometrically, either.

Student R showed Student L her graph (see Figure 4) to which he replied, "Ok, so just a line." In trying to reason with a graph in $\mathbb{R}^3$, students were asked to generate a graph using the corner of a cardboard box (see Figure 5), noting that also includes the no-dinners line for the cost constraint that was added later).

After the interviewer oriented students to how they could set up a coordinate system in the corner of a box, the students continued using their knowledge of lines in two dimensions to draw the 'no dinners,' 'no lunches,' and 'no breakfasts' lines for the constraint on the number of meals (see Figure 5). The students then wondered what shape would be formed if all the solutions to the $b + l + d = 210$ equation were plotted. They considered a pyramid or a triangle and whether it would be hollow. Student L suggested that "the d would get higher, but the b and the l would get closer in" and Student R recognized that if we considered all real numbers (not just integers) then "it would be flat and all the points would be connected." Here, the students physically graphed portions of the constraints corresponding to $b + l + d = 210$ that intersected with the b - l, b - d, and l - d planes in $\mathbb{R}^3$ by having each variable as the axis of the box and using their prior knowledge about lines in two dimensions. Students' connection of ordered triples with coordinates on a graph in the box indicated they came to view points and lines on a graph as a way to symbolize a set of numerical solutions (ordered triples). They interpreted intersections as solutions to multiple constraints; in this way they coordinated numeric, algebraic, and geometric interpretations of solutions. We interpret students' symbolization of solutions by plotting coordinates and lines in a physical box as adding to their accumulation of symbolizations, while building on their previous way of symbolizing solutions as ordered triples satisfying the given equation.

4.2.2. Solutions as intersections because of numbers satisfying both equations: Coordinating numeric and graphical symbolizations

After the students had described their view of the set of points satisfying the constraint on the number of meals, the interviewer prompted the students to consider the cost equation starting again with the 'no dinners' line for that equation. Student R added this to her box (see Figure 5) and noted that the triangle is no longer equilateral. The interviewer asked, "What do you think it means that the lines don't intersect?" Student R initially noted "the first equation is inside the graph of the second equation. So, in that way they do intersect, or they overlap." When the interviewer asked for clarification, Student L stated a different perspective, then analogized to equations of lines.

> L: I would think that eventually they would intersect because we found that there are numbers that will satisfy both equations.
>
> R: Hmm. That's true.
>
> I: Why would that tell you that they would meet?
>
> L: Because if there's two lines that intersect at a certain point, then that point is a solution ... then that x value that they hit at will be able to be plugged in and you get the same y for both of them.
>
> R: Yes. At first I was thinking that because the smaller shape was inside the bigger shape that where they like overlap is a solution "overlap" but then I realized that that would mean that the whole smaller shape would be a solution, and we know that that's not true, that every number in that is not a solution to both equations. So I went back to the equation that we were looking at and I solved it for d this time in terms of b. And when I did that I got 150-½ b which means that the d intercept would be at 150 which would mean that the two shapes like intersected [shows one of her hands intersecting the other].
>
> I: It's going to hit before you get to the 210?
>
> R: Right. So, I agree that where they intersect is where the solutions for both equations are located.

Here, the students reasoned about whether the graphs intersect or not in relation to the existence of solutions to the SLE. The students noted that the graphs should intersect because they had already found solutions previously. This, again, points to students connecting solutions as ordered triples satisfying the constraints to solutions as ordered triples in the intersection of the graphs. In working to plot solutions in the physical box, the students made substantial progress in building their geometric understanding of 3D solution sets using the box.

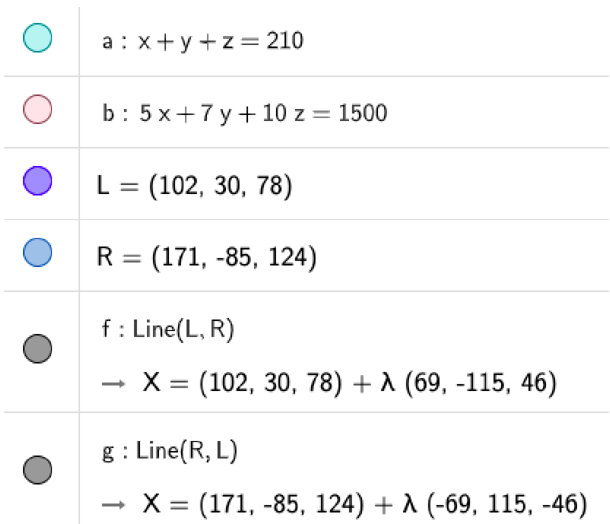
4.3. Task 3: Using GeoGebra to Represent Systems and Their Solution Sets

4.3.1. Solutions as a parametric vector equation in GeoGebra - Algebraic symbolization of solutions

In the third session, students focused on using GeoGebra as a tool for exploring algebraic and graphical representations of SLE and their solution sets. Students started by graphing the SLE corresponding to the Meal Plans context in GeoGebra. The interviewer then asked them to use the parameterized expressions they had found on the first day to find and plot a couple of solutions that satisfied both equations. Student R found (171, -85, 124) and Student L found (102, 30, 78). They plotted the two points on the graph with the SLE (in GeoGebra), and Student L noted "They are on the same line. It looks like, at least." The interviewer asked if this was surprising, Student L said, "No, it doesn't surprise me. I figured that would happen." Student R added, "Because we both used the same equations." This numerical to graphical connections mirrors their discussion the previous day, providing additional evidence that students view that solutions represented as ordered triples correspond to points that lie on the line of intersection between the two planes corresponding to the set of solutions to the two equations, respectively.

The interviewer then prompted the students to have GeoGebra construct the line connecting those two points and brought their attention to the way GeoGebra writes the line equation, $(102, 30{,}87) + \lambda(69, -115, 46)$, shown in Figure 6.

Figure 6. Students' input in GeoGebra



When asked what they noticed about the equation, Student L noted that the first set of parentheses is just the point L and that the second set of parentheses is R minus L, "Well, the first set of parentheses is just L. And then the second set is R minus L. So, 171 minus 102, and then negative 85, minus 30. And then 124 minus 78." The interviewer then prompted the students to now draw a line from L to R (see Figure 6). Student L pointed out that it looks like the same line on the graph even though the equation changed, to which Student R reasoned that "the fact that the lines are the same means that the equations that it comes up with are equal to each other."

Student L had previously mentioned calculating the slope of the line of intersection. The interviewer revisited this, asking what part of the equation looks like the slope. Student L drew on his knowledge of lines in $\mathbb{R}^2$, stating "we're taught slope, it's rise over run. So it's how tall you go up and then versus how long you go. But that's normally on an xy-plane. That's two dimensions. But with this z-axis, I'm not sure." Student R looked at the equation and observed, "it kind of looks like it's in slope-intercept form, in a way. Kind of looks like y equals b plus mx and the m, the slope, maybe is **λ**." Later, the interviewer asked the students to change the **λ** in the equation to a k and create a slider to more easily change the value of k.

> I: So, what do you guys think is happening right now?
>
> L: Well, as you're changing **λ** or k, this is making it go further along the curve or less along the curve, depending on what you enter, higher up or lower up on the curve.
>
> R: So that makes me think that λ isn't like a slope in the way that I thought it was before. Because if you we're changing that, then the point would move differently than just like along the same line.
>
> I: Like if we're changing the slope?
>
> R: Yeah, if you're changing the slope, it would be all over the place.

During this session, students examined a parametric vector equation of a line in $\mathbb{R}^3$ generated by GeoGebra, specifically the line representing solutions to an SLE in relation to its graph. Students had not seen an equation written like this before, so they used their prior knowledge to reason about what each component of the equation does. This was the first time the students explicitly attended to the slope of a line in the interview sessions. Ultimately, Student R used her knowledge that the slope of a line should not change when moving points along that line, to decide that **λ** could not represent the slope. The students reasoned about the form of a parametric vector equation of a line in 3 dimensions. While this line corresponded to the solution set they had previously considered, their reasoning was primarily to understand the relationship between the form and the points being traced out on the line through algebraic and graphical symbolizations.

4.3.2. Solution sets involving parallel lines and planes: Constructing parallel lines and planes to design for infinite and empty solution sets

Students were again given the images in the Intersection of Three Planes Task and asked to generate SLE that corresponded to any of the given images (using GeoGebra as an aid). They developed ways of reasoning about parallel planes that they heavily leveraged in the subsequent Example Generation task. Student R began by positing that it would likely be easier to construct one of the SLE with parallel planes because "if you just double the values, then it'll be the same plane. But maybe if you manipulate the number that it's set equal to, to not be double, if you doubled the coefficients of the variables ... [then] I think we can make the planes parallel." She pointed out that every value cannot be multiplied by the same number, but parallel planes might be achievable by changing the constant by a different factor from the coefficients. While working on making three parallel planes, Student R referred to the "slopes" of the equations in terms of the coefficients in each. She explained, "If we want the slope to be the same, they all have to be multiplied by the same ... If you change the coefficient of one number, it's no longer gonna be parallel because then the slopes of the equations aren't the same anymore."

Both students pointed out that if one coefficient is not multiplied, then the plane will no longer be parallel to the original. They then used the parallel planes they had just developed to recreate an SLE consisting of two parallel planes with a third plane intersecting both. When asked about the number of solutions to the SLE, Student R said "Zero. Because two of the planes are parallel. And those are never going to have a solution that makes them both true." Student L came to the same conclusion but by reasoning numerically, stating, "And also because you have x + y + z is two, and x + y + z is eight. So that doesn't really make sense either, because you can't have three numbers that equal two and also equal eight when you add them together." In this portion of the task sequence, the students developed initial reasoning for parallel planes that coordinated reasoning about solutions symbolized graphically as intersections, and numerically as sets of values that satisfy each equation in the system. This provided them with ways of reasoning about systems with infinitely many and no solutions that they heavily leveraged in the final task of the sequence.

4.4. Task 4: Constructing Systems with Particular Solution Types

4.4.1. Constructing SLE with no solution and infinitely many solutions

The students' work on this portion of the Example Generation task leveraged their reasoning and ways of symbolizing in the Intersections of Three Planes task. They both manipulated the coefficients in the equations to create parallel lines and planes to create SLE with no solutions. For example, the students started with one equation, $x + y = 3$, and created another equation by multiplying each coefficient by two, $2x + 2y = 3$. When creating SLE with infinitely many solutions, the students again drew on their previous ways of reasoning and symbolizing by creating equations representing the same lines and planes through coefficient manipulation. Students did not accumulate any new ways of symbolizing, but rather drew on symbolizations and reasoning they had previously developed.

4.4.2. Attempting to construct SLEs with unique solutions

In working to construct an SLE of 2E2U with a unique solution, Student L used a strongly numeric approach. He started with one equation, found a viable solution to that equation, then created a second equation in which the previously viable solution satisfied. In other words, Student L started with the equation $x + y = 3$ and a solution of $x = 2, y = 1$ and developed $2x + y = 5$ as a second equation. He explained, "So I just used x + y is equal to three. And then I said, okay, x is equal to two y is equal to one. So, then I was just like, well, two times two is four plus that one is five.

So that should give us the one solution. It'll be two, one." The students only referenced the geometry of this SLE by describing (2, 1) as the point of intersection. In moving to 3E2U, Student L used the same process to develop a third equation with the same unique solution. This presented a new way of symbolizing through numeric manipulation to develop an SLE with a unique solution.

In shifting to the case with 2E3U, Student L stuck with his numerical approach, presumably because it had been successful previously. Student R thought more graphically, arguing that it was impossible for the SLE to have a single solution because two planes cannot intersect at a single point. She explained:

> When we were doing the algebra part of it, I was more like maybe there is only one solution. But because when you look at it, it's hard to just tell from the numbers. There's no way just to look at it and be like there's a million solutions. I know what they all are. But when you look at the graph, it becomes clear that there's no way that these two planes are going to ever intersect in just a point because they're planes. And there's only two of them.

Student L tried a guess-and-check method, but he later accepted Student R's justification because he could not find an SLE with 2 equations and 3 unknowns that had only one solution.

In the case of 3E3U, the students questioned whether it was possible to have a single solution, thus revisiting their previous query. Their intuition told them that it was possible. They constructed the SLE in GeoGebra by creating two planes that intersect in a line and then added a plane that intersected that line at one point. In both situations with three unknowns, the students found that new graphical symbolizations (e.g., creating a system of three plane equations with a unique solutions in GeoGebra) lent to developing SLE more easily.

4.4.3. Making Generalizations

At the end of the final session, the interviewer asked students for generalizations about SLE and their solutions. Both students attended to ideas they developed when reasoning about parallel planes, such as looking at coefficients and constants, to help determine the nature of the solution set. For example, Student R said, "I would say pay attention to the coefficients … Parallel will tell you that there would be no solutions for the system." Student L extended on her reasoning by explaining that when they saw SLE with "no solutions, we know that the constant is just going to stay the same, because the lines are parallel." Student R also considered the number of equations and unknowns, something that had not been deeply discussed previously. She stated:

> In the beginning, we were trying to find a solution to that system of equations that had only two equations and three unknowns. Now I know in the future, don't waste your time looking, doing substitution. It's not gonna work … Maybe because there's more unknowns than equations, you don't have enough information to use substitution or anything because there's not going to be a single solution. I don't know if that's true.

It could be that the design of the Example Generation task led Student R to connect that the reason two planes could not have one solution was that there were fewer equations than unknowns (along with her geometric symbolization), leading her to conjecture that there can never be one solution to an SLE with that trait.

5. Discussion

We have detailed how student reasoning and symbolization progressed in the context of our RME task sequence, with an emphasis on accumulation of representations. In the portions of the task sequence designed for *situational activity in the task setting*, we observed students symbolizing solutions as values satisfying the given constraints, as ordered triples, and as ordered triples that result from manipulation of equations where all variables are expressed in terms of one. Within this type of situational activity, students also progressed to symbolizing solutions graphically as collections of points and lines and intersections representing ordered triples satisfying an SLE. When reasoning about solutions to a second equation, they came to reason about each plane as corresponding to solutions of a single equation so that the intersection of the planes are those ordered triples satisfying both equations. Both of these allowed for the accumulation of new ways of symbolizing. In their *referential activity*, students used GeoGebra to coordinate algebraic, numeric, and graphical ways of reasoning, examining intersections of graphs, reasoning about parametric vector equations of lines, and coordinating algebraic, numeric, and graphical representations of parallel lines and planes to avoid intersections of graphs.

Lastly, students created systems with particular types of solution sets as they engaged in *general activity*. They used their previous ways of symbolizing and reasoning to develop systems with no solutions (parallel lines and planes) and infinitely many solutions (same line/same plane). Creating systems with unique solutions presented opportunities for students to reason both numerically and graphically. In this activity, students' symbolizing progressed to incorporate the different ways they had previously developed with regard to reasoning about solutions but in a new setting. For example, they drew on their reasoning about parallel lines and planes as a way to achieve no intersections, but now were thinking of this more explicitly as a way of creating SLE with no solution. Students eventually linked the format of the equations to the geometry of the SLE in order to predict possible numbers of solutions (leveraging parallel lines and planes), predicting that there would not be a (unique) solution when there are more unknowns than equations. Throughout the task sequence, the students accumulated a variety of ways of symbolizing and reasoning about solutions. This progression related to the prompts in the task sequence or those from the interviewer (whether that be bringing students' attention to a particular idea or assisting students in moments of unhelpful confusion), but these allowed new opportunities for students' ways of symbolizing and reasoning.

Drawing on the instructional design heuristics of RME, our Meal Plan task initiated SLE as a set of quantitative constraints, not merely as a set of equations. By extending SLE to have more equations than unknowns and vice versa, our task sequence gives students an opportunity to broaden their thinking beyond the setting of an SLE having as many equations as unknowns and one unique solution. We note one limitation of this study: the task only elicited reasoning about scalar multiples of equations. We understand that linear combinations are also an important aspect of SLE, and this presents a next step in terms of task sequence refinement. In this task sequence, students were able to explore and reason about various solution sets through their own symbolizations, allowing for their building intuition for solutions. We posit that allowing students to see different types of solution sets and the ways those can be symbolized *before* learning procedures like row reduction can be helpful in understanding and interpreting the results of those procedures. In this paper, we have presented what building that understanding might look like.

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A progression of student symbolizing: Solutions to systems of linear equations

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Understanding the structure of solution sets to systems of linear equations (SLE) is of central importance to a variety of related concepts in linear algebra courses. In this paper, we analyze the progression of two students' symbolization and reasoning as they worked through a task sequence informed by Realistic Mathematics Education instructional design heuristics. The task sequence aims to support students in building initial intuitions about infinite solution sets by working in the context of a meal plan, where students reason about viable meal plan options given constraints on the cost and number of meals. After symbolizing these solution sets in multiple ways, students developed SLE given graphs featuring various kinds of solution sets. The participants were math majors and preservice secondary teachers who had not yet taken linear algebra. We examined their symbolization through their use of and attention to particular inscriptions: components of the SLE (e.g., the coefficients), aspects of the graph of equations or the system, algebraic manipulation and its results, and more. We consider the progression of students' symbolizing activity in terms of an *accumulation* of symbolizations. In other words, if students expanded on a previous or added a new way of symbolizing, we considered this to be evidence of progression in their symbolizing activity. Students initially symbolized solutions as values satisfying the given constraints, as ordered triples, and ordered triples that result from manipulation of equations where all variables are expressed in terms of one. Students then progressed to symbolizing solutions graphically as collections of points and lines and intersections representing ordered triples satisfying an SLE. When reasoning about solutions to a second equation, they came to reason about each plane as corresponding to solutions of a single equation so that the intersection of the planes is those ordered triples satisfying both equations. Third, students used GeoGebra to coordinate algebraic, numeric, and graphical ways of reasoning, examining intersections of graphs, reasoning about parametric vector equations of lines, and coordinating algebraic, numeric, and graphical representations of parallel lines and planes to avoid intersections of graphs. Lastly, students created systems with particular types of solution sets using their previous ways of symbolizing and reasoning to develop systems with no solutions (parallel lines and planes) and infinitely many solutions (same line/same plane). Students eventually linked the format of the equations to the geometry of the SLE in order to predict possible numbers of solutions (leveraging parallel lines and planes), predicting that there would not be a (unique) solution when there are more unknowns than equations. We postulate that allowing students to symbolize large solution sets in multiple ways *before* abstracting the procedures that yield these sets (such as row reduction) helps students to develop intuition for solution sets and their graphical representations.