

## What's new with APOS theory? A look into levels and Totality

*¿Qué hay de nuevo en la teoría APOE? Una mirada a los niveles y la Totalidad*

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**Abstract** ∞ This paper focusses on developments concerning transitional aspects of learning from the perspective of APOS (Action—Process—Object—Schema) theory. Recent investigations about levels between stages and Totality as a possible new structure are commented on, as well as offering related pedagogical suggestions and ideas for future research.

**Keywords** ∞ APOS theory; Totality; Levels; Transition; Mental structures and mechanisms

**Resumen** ∞ Este artículo se enfoca en los desarrollos que se relacionan con aspectos transicionales del aprendizaje desde la perspectiva de la teoría APOE (Acción—Proceso—Objeto—Esquema). Se comenta sobre investigaciones recientes alrededor de niveles entre etapas y Totalidad como una posible nueva estructura; asimismo se ofrecen sugerencias pedagógicas e ideas para investigaciones futuras.

**Palabras clave** ∞ Teoría APOE; Totalidad; Niveles; Transición; Estructuras y mecanismos mentales

## 1. INTRODUCTION

APOS theory provides a means to explain the learning of mathematical concepts. Its visible history dates back to the 1980's, when Ed Dubinsky, a mathematician well known for his contributions in functional analysis, started to share the principles of a cognitive approach that would later come to be known as APOS—an acronym formed from the initials of the respective components. Resulting from years of study of Piaget's epistemology and theory of cognitive development in children, and the adaptation of notions such as reflective abstraction to undergraduate mathematics, he established the foundations of this theoretical framework.

Later in the 1990's, RUMEC (Research in Undergraduate Mathematics Education Community) was formed. Under the guidance of Dubinsky and funded by Exxon Education Foundation, the members of this body met regularly, produced a considerable amount of research studies and published them in the form of articles as well as monographs. With the discontinuance of financial resources early in the 2000's RUMEC ceased to meet in person and later it was dissolved, although small teams continued to work on APOS-related projects; some of these collaborations led to significant advances in the theory.

In the second decade of the current century, APOS theory was well established in mathematics education research circles as a pioneering cognitive approach to learning. However, several members of the past RUMEC organization were concerned about some studies conducted by other researchers as they displayed substantial differences with the essence of the theory and its intended use in research, that could not be explained only by methods of practice or a divergence of interpretations. They thought that the users of APOS theory in different parts of the world, be it students, instructors, novice researchers or established ones, deserved a didactical resource on which they could rely. María Trigueros led the initiative to start the project of a book that would be named *APOS Theory: A Framework for Research and Curriculum Development in Mathematics Education* (Arnon et al., 2014). The aforementioned volume not only shows the state of research up to that moment, but it also explains the theory in detail with all its components including the methodological aspects as well as its relationship with pedagogy and classwork. It has become the number one reference for citing the ultimate information concerning APOS theory.

Eight years have passed since the publication of the book during which time there has been some progress in both theoretical and empirical areas concerning APOS theory; some of these advancements offer quite interesting prospects. This article has as its aim to give an account of the latest developments related to transitional aspects of learning, specifically about the possible new stage Totality and levels between stages, that occurred or continued after the publication of the book on APOS theory. As the reader progresses in reading this article, I hope it not only will be clear that this theoretical approach is in constant evolution, keeping always its primary intention to improve student learning, but that the paper will also provide a continual link between theoretical considerations, previous results and new findings.

In the remaining sections of this paper I will first give a brief description of APOS theory and its components. Afterwards I will describe both developments separately with examples. Later I will point out to possible routes for continuation of research. Where appropriate, pedagogical suggestions will be mentioned.

## 2. A BRIEF DESCRIPTION OF APOS THEORY

In the article known as the framework paper, Asiala et al. (1996) describe mathematical knowledge as an individual's "tendency to respond to perceived mathematical problem situations by reflecting on problems and their solutions in a social context and by constructing or reconstructing mathematical actions, processes and objects and organizing these in schemas to use in dealing with the situations" (p. 7). As in the description given by this statement, from the viewpoint of APOS theory knowledge construction progresses through the stages of Action, Process, Object and Schema, also known as structures. The first letters of these terms will be written in capitals to distinguish them from the daily usage of the respective words.

An Action is externally driven and leads to a transformation of a previously constructed mental Object. 'External' might refer to a formula, an algorithm or a memorized procedure, where the individual does not have internal control over the notion being studied, has the need to work with specific instances of a concept and does it on a step-by-step manner. An example would be to substitute a number for an unknown in an algebraic expression, which might be considered an Action on the Object of variable.

As the individual repeats the Actions and reflects on them, he or she starts having internal control over those Actions by interiorizing them into a Process. Having constructed this structure implies the ability to think about the concept in general terms as well as perform Actions in one's mind as opposed to having the need to write them down or register them in some way at each step before passing to the next one. For example an individual can think about an algebraic expression as producing different values for different inputs of a variable, without having to calculate those values.

To explain the Object structure let's think of an analogy with an individual who is about to go on a trip and prepares a suitcase to take along. While in the process of getting ready, thinking about the contents of the suitcase and their functions is very important. Once it is ready however, the individual closes the suitcase and now can take it to the airport and give it to the clerk at the check-in counter, without having to worry about the material that is inside or their possible uses. A similar situation occurs when an individual encapsulates a Process to form a mental Object with the purpose of applying Actions on it, as in the case of naming different algebraic expressions with letters and forming a set with them. Sometimes however, it becomes necessary to de-encapsulate an Object to the Process from which it originated, for instance when dividing two expressions one might need to be aware of the values for which the resulting fraction is not defined; for that the Object of algebraic expression in the denominator has to be de-encapsulated to its

original Process, even though performing operations with expressions constitutes an Action. In our analogy, the passenger might remember having written an important piece of information on a paper and having packed it, in which case it would be important to open the suitcase in order to have access to its contents.

A Schema is a coherent collection of different structures related to the concept that is being learned. For example a Schema about algebraic expressions can contain the mental Object of algebraic expression, Process of function, Schema of algebraic operations and the relations between them. As the individual continues constructing her or his mathematical knowledge and studies new concepts, this Schema evolves, admitting new structures and relationships. The coherence of a Schema (Arnon et al., 2014) has to do with the individual's capacity of recognizing a mathematical situation as involving the concept in question, even if the name of the concept is not mentioned. An individual who, given a number pattern, recognizes that an algebraic expression can be used to represent it, might be displaying the coherence of an algebraic expression Schema.

Ideally, a research study from the perspective of APOS theory starts by an initial theoretical analysis called a *genetic decomposition* that explains the construction of a mathematical concept by means of the structures of Action, Process, Object and Schema, together with the mechanisms such as interiorization, encapsulation, coordination, reversal that correspond to ways of passage (i. e. transition) between different mental constructions (see Arnon et al., 2014 for a description of the mental mechanisms). Before obtaining evidence, it is called a preliminary genetic decomposition, and only after having shown its viability by means of empirical evidence, it is accepted as a valid one. If in the process of investigation it becomes necessary to make modifications in it due to discrepancies with data, it is called a refined genetic decomposition after the changes are made. It is a cognitive model that sketches a possible way of constructing a mathematical concept.

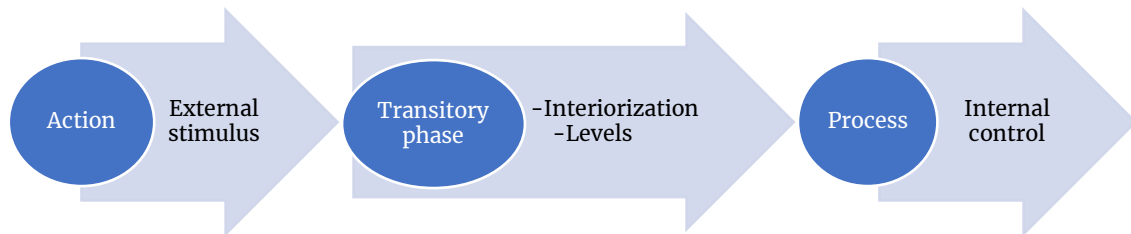
### 3. LEVELS IN APOS THEORY

Each step in the progression  $A \rightarrow P \rightarrow O \rightarrow S$  in terms of the mental stages Action, Process, Object and Schema represents “a change in how the individual thinks about the mathematical concept” in question (Arnon et al., 2014, p. 150); we can even say that each step is a leap in the individual's perspective.

These passages occur by means of mental mechanisms. Let's think about the move from an Action to a Process conception of function, which occurs by means of the interiorization mechanism. With an Action conception, an individual can act on previously constructed Objects such as variables, directed by external stimuli; for example he or she can substitute a specific value in a specific function formula, or repeat previously memorized solutions to problems involving functions. From this stage to another, namely Process, where the individual has internal control over the concept and can think about it in a general manner, can perform the Actions in her/his mind, realizes that to each input there corresponds an output that satisfies certain conditions, there is a huge area to be explored. What happens in the respective transitory phase? How does the interiorization of Actions occur in

that place for an individual who has constructed an Action conception and some of the elements of a Process but has not yet developed a Process conception? This intermediate phase seems to be as important as the Action and Process stages in learning a concept (Figure 1).

**Figure 1.** Transition from Action to Process



Once a Process has been constructed, the mechanism of encapsulation allows the individual to act on that Process and transform it to an Object. The construction of a Schema differs from the other structures in that it is built through the connections of a concept with other notions related through the mathematical experiences of a student. The mechanism of thematization allows the Schema to be converted into an Object so that Actions now can be applied on that Schema (Arnon et al., 2014).

Studies from the perspective of APOS theory mainly focus on the description of stages. Although the mechanisms of passage from one structure to another are included in genetic decompositions, little is known about their nature and how they are carried out. For example how an individual with an Action conception who is on a path towards developing a Process conception thinks about a concept, what kinds of work he or she produces, and what we can observe about this progress are questions that deserve attention if we want to explore how this transition occurs.

Starting from the premise that learning can be explained in terms of transitions (Gueudet et al., 2016) and considering that transitions are at the core of APOS theory to explain mental constructions of mathematical concepts, Oktaç et al. (2021) investigate the understanding of the notion of preimage of a linear transformation. Considering that each transitory phase merits a study of its own, they explore the passage from an Action to a Process conception, making use of *levels* in APOS Theory.

A level appears during the transitory phase between two stages (Figure 1). According to Arnon et al. (2014), in line with the work of Piaget (1975; 1974/1976) and making a reference to the mental constructions of Action, Process and Object, “a level denotes a developmental junction between two of these stages” (p. 139). This notion has been used in very few APOS studies so far (Arnon, 1998; Dubinsky et al., 2013).

Arnon et al. (2014) explain the characteristics of stages and levels as well as the differences between them as follows:

**A stage cannot be skipped.** If it is, the subject's understanding of the concept will lack coherence. Thus, stages are sequential, with each stage necessary for development of successive stages.

**A level may or may not be reflected in the data of a specific subject.** This is because the subject may be able to move to the next level or stage rapidly so that the level is skipped, done very quickly, or is not observable in the already acquired higher level or stage.

**Stages are invariant over topics and are part of the general theory.** Levels will be different for different concepts (Dubinsky et al. 2013). In many works, Piaget gave examples in which the development of different concepts gave rise to different levels. (Arnon et al., 2014, p. 139).

Oktaç et al. (2021) report on the application of the following task (translated from Spanish) to 31 university students as part of a questionnaire, with the intention to investigate the construction of the linear transformation concept from a functional viewpoint as well as related notions such as domain, image and preimage:

Consider the linear transformation  $T: R^2 \rightarrow R^2$  associated to the matrix  $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ .

a) Determine its domain.

b) Determine its image.

c) Does the vector  $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$  belong to the image of the transformation? If the answer is YES find the preimage and graph it. If the answer is NO, justify.

d) Graph the domain of  $T$ .

e) Graph the image of  $T$ .

f) Does the vector  $\begin{bmatrix} 5 \\ 5 \end{bmatrix}$  belong to the image of the transformation? If the answer is YES find the preimage and graph it. If the answer is NO, justify.

Oktaç et al. (2021) identify four levels in the data that they collected on the way from an Action to a Process conception for the preimage concept in the context of linear transformations; they explicitly describe each level in terms of their characteristics as well as the kind of shift that occurs in the student's thinking who evidenced the respective level. These levels represent transitional moments where a student displays that he or she can perform Actions, that they have not yet constructed a Process conception, but shows something more than an Action conception.

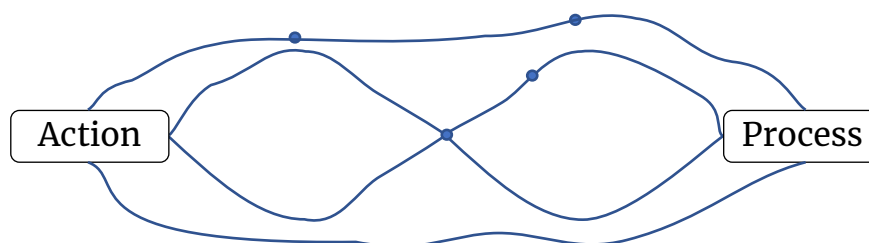
In the particular case of the mentioned study (Oktaç et al., 2021), the observed levels are named as 'preimage as two vectors'; 'preimage as a set of two vectors'; 'preimage as multiple solutions' and 'preimage as infinitely many preimages'. Each of these levels corresponds to an important change in the point of view of the student when thinking about the preimage concept. For example the first level observed, namely 'preimage as two vectors' shows that the student accepts that there

is more than a single vector that satisfies the condition of being a preimage of the given vector in the codomain, which is different from an Action conception where a student offers one vector as a solution; this implies a major shift in thinking about the concept.

Likewise, every level that was observed represents a change in perspective compared to an Action conception and towards a Process conception. When a student shows evidence of the level ‘preimage as a set of two vectors’, besides accepting that there is more than one vector that satisfies the condition of being a preimage, they express the solution as a set. The level ‘preimage as multiple solutions’ implies that the student affirms that there are several preimage vectors without specifying how many, or if there is a finite or an infinite number of them. When they find the solution set as all those vectors that lie on a particular line satisfying a certain condition, they display the level ‘preimage as infinitely many preimages’.

Oktaç et al. (2021) posit that “a level, although not as apparent as in a stage, implies a major shift in thinking about a concept”. They also affirm that as transitional moments between stages, “levels evidenced in different individuals might represent different paths from one conception to another” (Figure 2). This means that an individual might pass through two levels on the way from Action to Process, while another one might pass through only one level and this last one may or may not correspond to any level evidenced by the first person. Some other individual may not present evidence of any levels between Action and Process stages. All these observations also have to do with the reflection that levels are not necessarily sequential although some of them can be considered as closer to one of the stages or other, and that they can be considered as possible stopovers on a path between two stages.

**Figure 2.** Possible learning paths involving levels—represented by points on the paths— (adapted from Oktaç et al., 2021)



Oktaç et al. (2021) underline the importance of task design when working on transitional points in learning. One of the criteria to take into account in designs from the perspective of APOS theory is the opportunity to differentiate between different conceptions, so as to verify the viability of the genetic decomposition and the associated learning path that it anticipates. However, it is difficult to take into account possible levels when designing mathematical situations; unlike stages, levels are volatile and do not even form part of genetic decompositions, since they differ from subject to subject and from person to person. Including them in an *a*

*priori* analysis as part of the methodological strategy and paying attention to details in different elements of the conceptions involved may open the way to observing possible levels between the corresponding stages; when this is done intentionally to search for levels, one may be surprised to observe the richness in student answers and the understanding that they reflect. On the other hand when designing situations, making sure that a student with the starting conception (in this case Action) can deal with it even though he or she cannot solve it completely would improve the probability for observing levels. Also, students who have constructed a conception corresponding to the target stage (in this case Process) should be able to solve the problem; this would allow the identification of the characteristics of different levels observed.

The advantage of focusing on levels between stages in research may be at least twofold. On the one hand, from a theoretical viewpoint, although snapshots between stages, their description can shed light on the nature of the mechanism involved between two conceptions. Also, the observation of levels might indicate that the related mechanism might be getting carried out in different ways by different individuals. More research is needed to explore how mechanisms give rise from one structure to another, and levels can play an important role in this direction.

On the other hand, from a pedagogical point of view levels might be closely related to student difficulties in transiting between stages; in APOS studies it is commonly observed how difficult it is to construct a Process conception, and even more so for an Object conception. The study of levels can offer hints towards appropriate didactical strategies in order to facilitate and motivate the construction of mathematical knowledge.

#### 4. TOTALITY

In order to talk about the notion of Totality from an APOS perspective, let's first have a look at how the Object conception was described in the framework paper:

When an individual reflects on operations applied to a particular process, becomes aware of the process as a totality, realizes that transformations (whether they be actions or processes) can act on it, and is able to actually construct such transformations, then he or she is thinking of this process as an object. In this case, we say that the process has been encapsulated to an object. (Asiala et al., 1996, p. 11)

As can be seen from this description, totality was considered to be a part of the Object structure. According to this explanation, for an Object conception to be constructed, an individual has to show evidence of: 1) becoming aware of the totality of a previously constructed Process, and 2) being able to act on that Process. This description was not questioned for many years, since there had been no evidence showing otherwise, that is, in empirical research, no individual showed evidence of only one of the two aspects of the Object conception; either they were able to do both, or none of the two, or it was assumed to be that way.

First time the term 'totality' was used in a more explicit way and somewhat differing from the description of Object, was in the context of infinite iterative



processes. For example Dubinsky et al. (2008) refer to totality as “part of the transition from process to object” (p. 101) but they do not make reference to a new stage. Brown et al. (2010) describe an Object conception about this particular subject as follows: “Once viewed as a totality...an infinite iterative process might be encapsulated” (p. 10).

Indeed, later when Dubinsky et al. (2013) were investigating student understanding of infinite repeating decimals, their data revealed something that was not anticipated in their initial theoretical analysis, that had to do with what up to then was considered to be the description of an Object conception. Some students showed evidence of having constructed the repeated decimals as a totality but were not able to apply actions on that totality. Students who showed a Process conception considered that the 9’s in  $0.\bar{9}$  kept repeating and never ended. Some other students though, regarded  $0.\bar{9}$  as completed and stated that it is equal to 1. However, some of these students were unable to solve the equation  $0.\bar{9} + x = 1$ , which required acting on this repeating decimal.

The previously described situation led the researchers to consider the possibility of a new stage, Totality, between Process and Object. Arnon et al. (2014) affirm that there is not enough evidence as to the nature of Totality—whether it is a new stage, part of the Process conception or part of the Object conception, or a level between Process and Object. However they express a “strong likelihood” (p. 149) for it to be a separate stage, or that at least it functions as one in the case of infinite repeating decimals, as it “appears to represent a change in how the individual thinks about the mathematical concept” (p. 150). They refer the matter for future research; that is to see if new studies point to its existence in the context of other concepts. Villabona et al. (2022) point out the importance of finding evidence in the context of other mathematical concepts as well as in relation with contexts of infinity other than repeating decimals, in order for Totality to be accepted as a new stage in the theory.

We now turn to one such new study, where an evidence for Totality was found in the context of continuous infinite processes. Villabona Millán (2020) explores the learning of mathematical infinity in a context different from the one studied by Dubinsky et al. (2013) and proposes a *generic genetic decomposition of infinity* which allows the perception and conceptualization of constructs that emerge from infinite iterative and continuous processes. She interviews university instructors through a research instrument composed by two situations that involve tangent lines to a curve and behavior of certain elements at a limit. This instrument was designed specifically in order to be able to differentiate between possible Totality and Object conceptions. Villabona et al. (2022) report the case of one instructor, who during his interview showed evidence of a possible conception Totality.

On the one hand the instructor showed that he can conceive the tangent line as the totality of a process of approaching a point on the curve through secant lines: “when I have finished approaching totally, they will be the same” (referring to the secant and tangent lines, Villabona et al., 2022). But on the other hand, he has difficulty applying Actions on that totality: in another situation involving a circle, two

parallel tangent lines and another line crossing them, he could not conceive of some of the angles that were formed by the figures becoming zero in a limiting case, in which a secant line gets closer to a tangent line by means of approaching some point on the circle to another point on the circle where a tangent line was drawn. This is an indication that he has not yet constructed an Object conception, although he can conceive of the Process as a Totality.

Villabona et al. (2022) posit that if Totality were considered a new stage, the mechanism of encapsulation would correspond to the passage from Totality to Object, and that a new mechanism would be in place for transiting from Process to Totality, named *completez* in Spanish, which can be translated roughly as completeness. This term was first used by Roa Fuentes (2012) to denote a mechanism for construction of a transcendental Object structure in the context of mathematical infinity.

Even though the aforementioned study is still in the context of mathematical infinity, the difference in nature between an iterative infinite process and a continuous one from the cognitive viewpoint speaks to its importance as a further indication for the strong likelihood of a possible new stage of Totality.

We underline the importance of the design of suitable mathematical activities in order to be able to differentiate between Process, possible Totality and Object conceptions. They should be prepared in such a way as to include situations where 'conceiving a Process in its entirety' and 'acting on it' would each correspond to different kinds of student work. It is to be seen if Totality appears in future studies in relation with other mathematical concepts and eventually gets accepted as a new stage, and who knows, changing the name of the theory to APTOS.

## 5. FUTURE RESEARCH

Recent advances about Totality and levels in APOS theory open the way to different paths for continuing research. One possible direction might be choosing a mathematical topic and looking for levels between Action and Process, as well as Process and Object conceptions. Repeating this kind of study with different groups of students and different research instruments would move us towards a better understanding of the mechanisms involved. Furthermore, it would enhance our understanding of the stages, since levels usually entail certain elements of the stage towards which the individual is moving.

Another possibility is to focus on task design with the purpose of searching for evidence for Totality in the context of mathematical concepts other than infinity. The acceptance of this possible structure as a new stage in APOS theory requires its appearance as related to different mathematical concepts. Of course, performing empirical studies using these tasks is needed in order to reach conclusions about this matter.

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## REFERENCIAS

- Arnon, I. (1998). *In the mind's eye: How children develop mathematical concepts—Extending Piaget's theory* (Unpublished doctoral thesis). Haifa University, Israel.
- Arnon, I., Cottrill, J., Dubinsky, E., Oktaç, A., Roa Fuentes, S., Trigueros, M. & Weller, K. (2014). *APOS Theory: A framework for research and curriculum development in mathematics education*. Springer. <https://doi.org/10.1007/978-1-4614-7966-6>
- Asiala, M., Brown, A., DeVries, D., Dubinsky, E., Mathews, D., & Thomas, K. (1996). A framework for research and curriculum development in undergraduate mathematics education. In J. Kaput, A. H. Schoenfeld & E. Dubinsky (Eds.), *Research in Collegiate mathematics education II. CBMS issues in mathematics education* (Vol. 6, pp. 1–32). American Mathematical Society. <https://doi.org/10.1090/cbmath/006/01>
- Brown, A., McDonald, M. & Weller, K. (2010). Step by step: Infinite iterative processes and actual infinity. In F. Hitt, D. Holton y P. W. Thompson (Eds.), *Research in Collegiate Mathematics Education VII CBMS Issues in Mathematics Education*, 16, (pp. 115–141). American Mathematical Society. <https://doi.org/10.1090/cbmath/016/05>
- Dubinsky, E., Weller, K. & Arnon, I. (2013). Preservice teachers' understanding of the relation between a fraction or integer and its decimal expansion: The Case of 0.999... and 1. *Canadian Journal of Science, Mathematics, and Technology Education*, 13(3), 232–258. <https://doi.org/10.1080/14926156.2013.816389>
- Dubinsky, E., Weller, K., Stenger, K. & Vidakovic, D. (2008). Infinite iterative processes: The Tennis Ball Problem. *European Journal of Pure and Applied Mathematics*, 1(1), 99–121.
- Gueudet, G., Bosch, M., DiSessa, A. A., Nam Kwon, O. & Verschaffel, L. (2016). *Transitions in mathematics education*. Springer Nature. <https://doi.org/10.1007/978-3-319-31622-2>
- Oktaç, A., Vázquez Padilla, R., Ramírez Sandoval, O. & Villabona Millán, D. (2021). Transitional points in constructing the preimage concept in linear algebra. *International Journal of Mathematical Education in Science and Technology*. <https://doi.org/10.1080/0020739X.2021.1968523>
- Piaget, J. (1975). Piaget's theory (G. Cellierier & J. Langer, Trans.). In P.B. Neubauer (Ed.), *The process of child development* (pp. 164–212). Jason Aronson.
- Piaget, J. (1976). *The grasp of consciousness* (S. Wedgwood, Trans.). Harvard University Press. (Original work published 1974).
- Roa Fuentes, D. S. (2012). *El infinito: un análisis cognitivo de niños y jóvenes talento en matemáticas* (Unpublished doctoral thesis). Cinvestav-IPN, Mexico.
- Villabona Millán, D. P. (2020). *Construcción de concepciones dinámicas y estáticas del infinito matemático en contextos paradójicos, del cálculo diferencial y de la teoría de conjuntos*. Unpublished doctoral thesis. Cinvestav-IPN, Mexico.

Villabona, D., Roa Fuentes, S. & Oktaç, A. (2022). Concepciones dinámicas y estáticas del infinito: Procesos continuos y sus totalidades. *Enseñanza de las Ciencias*, 40(1), 179-187. <https://doi.org/10.5565/rev/ensciencias.3277>

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Since the publication of the book APOS Theory eight years have passed, during which time some advancements have taken place. Two of these developments have to do with the transitional character of learning. On the one hand, the possibility of integrating a new structure called Totality into the theory would imply some modifications, such as the way we think about the construction of an Object conception. On the other hand, focusing on transitional junctures between stages, namely levels, would enhance our understanding of the mechanisms involved and consequently the structures implied.

Totality as a possible new structure has been proposed in the context of mathematical infinity, more precisely infinite iterative processes. Its integration to the theory as a stage between Process and Object would require empirical evidence in contexts other than repeating infinite decimals where it was originally observed. One such study reports on such evidence in the context of continuous infinite processes involving tangent lines to a curve, increasing hence the likelihood of Totality having the properties of a new structure in APOS Theory.

Levels are developmental junctures between two structures. As opposed to stages, they are not stable and depend on different elements such as the mathematical topic or the person who is constructing her/his knowledge about the mathematical concept in question. Some individuals can evidence some of these levels and others can skip some or all of them revealed by the data as a whole. In a study aimed at explaining student understanding of linear algebra concepts, in particular the preimage notion related to a linear transformation, different levels were observed between Action and Process stages; these levels were described taking into account the shift that they represent from the Action stage and progress towards the Process stage. The characteristics of levels observed in student work can help in designing pedagogical strategies that aim to promote transition towards the target stage. An important conclusion is the possibility that the mechanism of interiorization might take place in different ways for different individuals.

To continue with this line of research, ideas for future studies are presented; pursuing them might lead to new findings and ideas for teaching or contribute to the evolution of the theory.