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# Importancia de la clase de fuentes armónicas en la identificación de fuentes en el problema inverso electroencefalográfico

# Importance of the class of harmonic sources in the identification of sources in the inverse electroencephalographic problem

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**Resumen**

**Introducción**: En este trabajo se estudiará la importancia de las fuentes armónicas en el volumen cerebral que reproducen una distribución de potencial en el cuero cabelludo determinada, las cuales además de ser clase de unicidad, tienen una importancia fundamental para la solución del problema inverso de identificación de fuentes en cualquier otra clase de fuentes, para reproducir la misma medición.

**Método**: Se propone el modelo de medio conductor para relacionar las mediciones con las fuentes que las reproducen. Se reduce el problema a un planteamiento operacional que nos permitirá caracterizar las llamadas mediciones admisibles con respecto a la clase de fuentes que se considere.

**Resultados:** Se caracteriza el conjunto de datos admisibles para la clase de fuentes armónicas en el volumen del cerebro y se ve la importancia de la clase de fuentes armónicas para la identificación de fuentes en otras clases de fuentes, lo cual se ejemplifica con la clase de fuentes armónicas en una vecindad de la corteza cerebral. También se muestra la importancia de la clase

de fuentes armónicas en la aplicación del método de datos admisibles (MDA) a un esquema general de regularización para el problema de identificación de fuentes pertenecientes a clases de unicidad.
**Conclusión**: Se propone una metodología general de resolución del problema inverso electroencefalográfico de identificación de fuentes, haciendo uso de la clase de fuentes armónicas en el volumen cerebral. Dada una clase arbitraria $\mathcal{F}$ de fuentes con la propiedad de unicidad para la solución del problema inverso, se desarrolla un método general para la identificación de la fuente en $\mathcal{F}$ que mejor aproxima una medición del potencial sobre el cuero cabelludo.
**Palabras clave:** modelo de medio conductor; fuentes armónicas; fuentes equivalentes; problema inverso electroencefalográfico; método de datos admisibles

**Abstract**
**Introduction:** In this work we discuss the relevance of the harmonic sources on the brain volume, which reproduce a given potential distribution on the scalp. These sources, apart from being a unicity class, they play a fundamental role in the resolution of the inverse problem of source identification with respect to any other sources class.
**Method:** We make use of the volume conductor model for the head, in order to relate sources and reproduced measurements. The problem is rewritten as an operational formulation which allows to characterize the admissible measurements with respect to any considered sources class.
**Results:** The admissible data set is characterized for the harmonic sources class on the brain volume. Also, the importance of this class in the context of the source estimation problem, with respect to any sources class, is shown. This is specifically illustrated considering the class of harmonic sources on a neighborhood of the cortex. Moreover, it is also shown the role the harmonic sources class on the brain plays when applying the Admissible Data Method (ADM) in order to get a general regularization scheme for the source estimation problem with respect to a unicity sources class.
**Conclusion:** A general resolution methodology for the source estimation problem in the context of the inverse electroencephalographic problem is proposed, in which the harmonic sources class on the brain volume is crucial. Namely, given an arbitrary sources unicity class (for this inverse problem), a general method is developed for identifying the source in this class whose reproduced

potential distribution best approximates a given potential measurement on the scalp. We consider sources classes in connection with the electrical activity near the cortex.
**Keywords:** volume conductor model; harmonic sources; equivalent sources; inverse electroencephalographic problema; admissible data method

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# Introduction

There are many works in the specialized literature dealing with the electrical activity source estimation problem in the brain, starting from potential distribution measurements on the scalp, corresponding to instantaneous electroencephalographic (EEG) measurements (Munck, Van Dik, & Spekreijse, 1988), (Amir, 1994), (El Badia & Ha Duong, 1998), (Fraguela Collar, Morín Castillo, & Oliveros Oliveros, 2008), (Morín Castillo, *et al*., 2013), (Fraguela Collar, Oliveros Oliveros, Morín Castillo, & Conde Mones, 2015) and references therein. Roughly speaking, this problem consists on identifying or estimate the source on the brain volume, including location and description, which yields the measured electric potential in form of electroencephalographic signal (EEG). These works make use of the volume conductor model, justified in (Sarvas, 1987), (Plonsey & Fleming, 1969). From a mathematical point of view, some drawbacks arise from the kind of models, partially but not completely solved in the literature.

This current work is mainly devoted to highlight the importance of the harmonic sources on the brain volume in the context of the previously mentioned source estimation problem. This kind of sources has no physiological meaning, since they are distributed over the whole brain volume and it is a well-known fact that EEG basically reflects the electrical activity close to the cortex at a macro spatial scale (regardless of whether or not it is influenced by any other inner sources) (Nunez, Nunez, & Srinivasan, 2019). However, from a mathematical perspective, this sources class plays a fundamental role in the sources characterization with respect to any other sources class, especially the ones relevant from a physiological point of view, as the multipoles, the sources concentrated in the cortex, or the spatially piecewise constant sources. It is worth to mention that these classes, commonly used in this context, have empty interior, which turns to be a serious drawback in the application of optimization methodologies.

We make emphasis on the unicity property for the sources class, an important requirement in the identification problem, which seems to be overlooked in previous related works. On the other hand, a big part of this manuscript is dedicated to show how the harmonic sources class on the brain volume is used to identify sources (belonging to a unicity class) which reproduce a potential measurement on the scalp.

In spite of an EEG measurement is given on a finite number of electrodes, we consider the potential measurement is known on the scalp. The interpolation problem consisting in extend this kind of data to the whole scalp is out of the scope of the present article.

Here we focus on showing the general methodology. Therefore, we choose a simple geometric model for the head, consisting in two concentric spheres modeling the splitting surfaces between the brain and the rest of the head. In this case, this makes possible the use of explicit analytic expressions for the solutions of certain contour problems. In case of requiring more realistic models, more complex geometries (with more layers and more involved surfaces) are mandatory, and the contour problems need to be numerically solved, in general.

Finally, since the source estimation problem is not well posed, a regularization algorithm is required in order to minimize the error sensitivity on the potential measurement (the problem of lack of uniqueness is easily solved by choosing appropriate unicity sources classes) (Kirsch, 2013). Usually, iterative methods are used in this context. However, we decided to use a different approach: the “Admissible Data Method” (Hernandez Montero, Fraguela Collar, & Henry, 2019). This methodology allows to clarify a priori if a certain sources class is appropriate to identify a given measurement.

We apply the previously mentioned identification methodology to the specific case of sources supported and harmonic on a neighborhood of the cortex, which turn to be a more natural and convenient class than the above cited harmonic on the whole brain volume class. This should be understood in the following sense: by natural we mean an equivalent mathematical source which both reproduces the measurement and it is concentrated at the biological active zone. As it has been said above, the EEG reflects synaptic activity occurring near the cortex (Nunez, Nunez, & Srinivasan, 2019).

# Method

## The model and the Inverse Electroencephalographic Problem

In the simple volume conductor model (studied in (Sarvas, 1987) based on the results in (Geselowitz, 1967)), the brain is considered as conducting medium of electrical current, in which there is also a generating mechanism of other biological currents produced by neuronal activity, called impressed currents.

We will denote by $\Omega_1$ the region occupied by the brain, and by $\partial\Omega_1 = S_1$ its border, corresponding to the cortex. $\sigma_1$ denotes the Ohmic current conductivity (in a normal brain is considered to be constant and equal to the one corresponding to the saturated salt water (Nunez & Srinivasan, 2006)) and $\mu$ denotes the magnetic permeability. From the perspective of a macro spatial scale, the EEG measurements detect average synaptic source activity (Nunez, Nunez, & Srinivasan, 2019), so the constant conductivities reflect the average effect of microscopic spatial fluctuations. Additionally, we suppose that the electric field generated in the brain is due to their conductive properties as physical medium, and electrical sources originated by neuronal activity. Finally, $u_1$ will be the electric potential in the brain volume $\Omega_1$. On the other hand, the rest of the head is considered also as a homogeneous conducting medium $\Omega_2$, with outer border $\partial\Omega_2 = S_2$ corresponding to the scalp. Its average conductivity, considered constant, is $\sigma_2$. In this region there are no sources of electrical activity (see Fig. 1). Analogously, $u_2$ will be the electric potential distribution in $\Omega_2$. We set $\Omega = \Omega_1 \cup \Omega_2$.

In this way, other layers such as the skull or the spinal brain fluid, among others, are not considered. The outer medium (outside the head) is supposed to have a vanishing conductivity.

Hence, the simple quasi-static model describes the behavior of potentials $u_1$ and $u_2$, in the following way:

$$-\sigma_1 \Delta u_1 = f \text{ in } \Omega_1, \tag{1}$$

$$\Delta u_2 = 0 \text{ in } \Omega_2, \tag{2}$$

$$u_1 = u_2 \text{ on } S_1, \tag{3}$$

$$\sigma_1 \frac{\partial u_1}{\partial n_1} = \sigma_2 \frac{\partial u_2}{\partial n_1} \text{ on } S_1, \tag{4}$$

$$\left.\frac{\partial u_2}{\partial n_2}\right|_{x \in \partial\Omega_2} = 0 \text{ on } S_2, \tag{5}$$

Where $\frac{\partial u_i}{\partial n_j}$ denotes the normal derivative of $u_i$ on $S_j$ with respect to the normal unitary vector $n_j$, outer to $\Omega_j$, $i, j = 1,2$. Here, the forward problem consists on, given a source $f$, finding the potentials $u_1$ and $u_2$. Thus, the corresponding inverse problem consists on, given a measurement $V$, finding an equivalent source $f$ satisfying ( 1 ) - ( 5 ) in such a way that the corresponding potential simultaneously fulfills the condition $u_2|_{S_2} = V$.

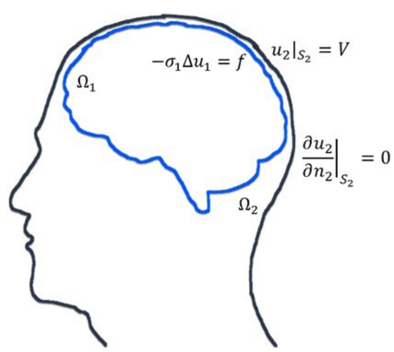


**Fig. 1.** Simple schematic representation of the volume conductor model.
**Source**: Own elaboration.

We consider the Hilbert spaces $L_2(\Omega_i)$, $L_2(S_i)$ and $L_2(\Omega)$ of square summable functions defined on $\Omega_i$, $S_i$ and $\Omega$; respectively $(i = 1,2)$. Analogously, we will denote by $H_1(\Omega_i)$ and $H_1(\Omega)$ the corresponding Sobolev spaces $(i = 1,2)$. We will denote also by $H_1(\mathrm{S}_i)$ the subspaces of $L_2(S_i)$ given by the traces in $S_i$ of functions in $H_1(\Omega_i)$ $(i = 1,2)$.

The superindex (1) will indicate subspace corresponding to functions orthogonal to constants (with respect to the appropriate inner product). Namely, if $W$ is a function Hilbert space with inner product $\langle , \rangle_W$, then $W^{(1)} = \{w \in W: \langle w, 1\rangle_W = 0\}$, where 1 is the indicator function of the corresponding domain. Also, the superindex $\perp$ denotes the orthogonal complement in $W$ of a set $F$ in $W$, that is, $F^{\perp} = \{w \in W: \langle w, f\rangle_W = 0, \forall f \in F\}$.

In the neuronal activity source estimation problem, via model ( 1 ) - ( 5 ), the operative formulation requires to find an operator $\mathcal{A}$ which associates to each neuronal activity source $f$ (in a certain class $\mathcal{F}$) the measurement $V$. Namely, the inverse source estimation problem reduces to solve the operational equation

$$\mathcal{A}(f) = V, \tag{6}$$

Where $V = u_2|_{S_2}$ is the potential distribution measurement on $S_2$.

Hence, solving the inverse source estimation problem means that, starting from the instantaneous electroencephalographic measurement $V = u_2|_{S_2}$, a source $f$ "reproducing this measurement" could be find, i.e., substituting $f$ in the model ( 1 ) - ( 5 ), there exists a unique solution $\left(u_1^{(f)}, u_2^{(f)}\right)$ satisfying $u_2^{(f)}\Big|_{S_2} = V$.

Given a class $\mathcal{F}$ of functions defined on the brain, the image of $\mathcal{F}$ by the operator $\mathcal{A}$ will be called admissible data set. This is exactly the set of measurements which could be reproduced by sources in the class $\mathcal{F}$. Furthermore, the class $\mathcal{F}$ will be said to be a unicity class if operator $\mathcal{A}$ restricted to $\mathcal{F}$ is injective. In what follows the admissible data set associated to a sources class $\mathcal{F}$ will be denoted by $\mathcal{M}[\mathcal{F}]$:

$$\mathcal{A}[\mathcal{F}] = \mathcal{M}[\mathcal{F}]. \tag{7}$$

In (Fraguela Collar, Oliveros Oliveros, Morín Castillo, & Conde Mones, 2015) was proved that operator $\mathcal{A}$ is compact from $L_2^{(1)}(\Omega_1)$ to $L_2^{(1)}(S_2)$. In addition, in Theorem 2.3 was shown that

$(\text{Ker}\mathcal{A})^{\perp} = \mathcal{H}^{(1)}(\Omega_1)$, where $\mathcal{H}(\Omega_1)$ is the closed subspace in $L_2(\Omega_1)$ of harmonic functions. From this result yield the following conclusions, which will be important in what follows:

a) If the desired source is required to be in $\mathcal{H}^{(1)}(\Omega_1)$, then the unicity theorem for the solution of the inverse problem corresponding to the operational equation ( 6 ) is fulfilled.
b) If the source $f$ which reproduces a given potential distribution $V$ is required to be in a certain class $\mathcal{F}$ which satisfies the unicity condition for the solution of the inverse problem, and $V$ can also be reproduced by $h \in \mathcal{H}^{(1)}(\Omega_1)$ (note that each one uniquely reproduces $V$ in their own class. Certainly, there is no unique representation in $L_2(\Omega_1)$), then $h$ is the orthogonal projection of $f$ on $\mathcal{H}^{(1)}(\Omega_1)$, independently of the chosen class $\mathcal{F}$.

We note that another unicity class for the source estimation problem was obtained in (Fraguela Collar, Oliveros Oliveros, Morín Castillo, & Conde Mones, 2015) consisting on certain kind of piecewise constant sources.

## Operational formulation of the inverse problem

In order to solve the source estimation problem for certain unicity classes in the context of the volume conductor model ( 1 ) - ( 5 ), we need to reduce the inverse problem to an equivalent operational formulation. In general, solving this inverse problem in a realistic geometry is a quite involved mathematical problem. For the sake of simplicity, in order to explain our methodology more easily, we will consider a simple geometric model for the head.

We consider two concentric spheres. The interior of the inner sphere $S_1$, of radius $R_1 > 0$ corresponds to the brain volume $\Omega_1$, and the interior of the outer sphere $S_2$, of radius $R_2 > R_1$, corresponds to the whole head $\Omega$ and outlines the region $\Omega_2$ corresponding to the spherical crown (see Fig. 1 and Fig. 2).

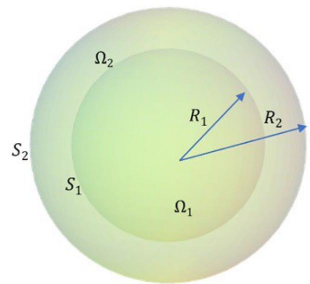


**Fig. 2**. Simple scheme of the head and brain.
**Source**: Own elaboration.

This simple spherical model allows us to build explicit analytical expressions for the solutions of the volume conductor model ( 1 ) - ( 5 ) in terms of the Fourier series with respect to classical orthonormales bases, which eases the qualitative analysis, and also constitutes the basis of an algorithm for the numerical resolution of the forward and inverse problems.

We start from assuming that the source is in a unicity class $\mathcal{F}$ in $L_2^{(1)}(\Omega_1)$. Note that, in $L_2^{(1)}(\Omega_1)$, the forward problem ( 1 ) - ( 5 ) has an unique solution in the Sobolev space $H_1^{(1)}(\Omega)$, and hence its restriction to the border $S_2$ could be considered in the sense of Sobolev traces on $S_2$, (Mijailov, 1978).

Next, we introduce some contour problems, which are auxiliary in solving the main inverse problem of source estimation, and associated to the corresponding solution. Note that these contour problems have no physical meaning; we use them for analyzing the operational formulation of the inverse problem. Their solutions will be interpreted in a weak sense. Fix a source $f \in L_2^{(1)}(\Omega_1)$ and $\psi \in L_2^{(1)}(S_1)$.

$$\sigma_1 \Delta w_1 = f \text{ in } \Omega_1, \qquad (8)$$
$$\frac{\partial w_1}{\partial n_1} = 0 \text{ on } S_1,$$

$$\Delta v_1 = 0 \text{ in } \Omega_1, \qquad (9)$$
$$\sigma_1 \frac{\partial v_1}{\partial n_1} = \psi \text{ on } S_1,$$

$$\varDelta v_2 = 0 \text{ in } \Omega_2, \qquad (10)$$

$$\sigma_2 \frac{\partial v_2}{\partial n_1} = \psi \text{ on } S_1,$$

$$\frac{\partial v_2}{\partial n_2} = 0 \text{ on } S_2.$$

**Definition 1.** The functions $w_1, v_1 \in H_1(\varOmega_1), v_2 \in H_1(\varOmega_2)$, satisfying

$$\sigma_1 \int_{\Omega_1} \nabla w_1 \cdot \nabla \omega_1 dx = \int_{\Omega_1} f \omega_1 dx, \forall \omega_1 \in H_1(\Omega_1), \qquad (11)$$

$$\sigma_1 \int_{\Omega_1} \nabla v_1 \cdot \nabla \omega_1 dx = \int_{S_1} \psi \omega_1 dx, \forall \omega_1 \in H_1(\Omega_1), \qquad (12)$$

$$\sigma_2 \int_{\Omega_2} \nabla v_2 \cdot \nabla \omega_2 dx = -\int_{S_1} \psi \omega_2 ds, \forall \omega_2 \in H_1(\Omega_2), \qquad (13)$$

will be called weak solutions of problems ( 8 ), ( 9 ) and ( 10 ), respectively.

The following result holds.

**Theorem 1.** Problems ( 8 ), ( 9 ) and ( 10 ) have weak solution if and only if $f \in L_2^{(1)}(\varOmega_1)$ and $\psi \in L_2^{(1)}(S_1)$. In this case, the weak solutions are unique in their respective spaces $H_1^{(1)}(\varOmega_1)$ and $H_1^{(1)}(\varOmega_2)$, and the following inequalities hold:

$$\|w_1\|_{H_1(\Omega_1)} \leq C_1 \|f\|_{L_2(\Omega_1)}, \qquad (14)$$

$$\|v_1\|_{H_1(\Omega_1)} \leq K_1 \|\psi\|_{L_2(S_1)}, \qquad (15)$$

$$\|v_2\|_{H_1(\Omega_2)} \leq C_2 \|\psi\|_{L_2(S_1)}, \qquad (16)$$

Where constants $C_1$, $K_1$ and $C_2$ do not depend on $f$ and $\psi$, (see Mijailov, 1978).

Consequently, the following operators are well defined by using the weak solutions of problems ( 8 ), ( 9 ) and ( 10 ):

$$A_0: L_2^{(1)}(\Omega_1) \longrightarrow L_2(S_1),\ A_0 f = w_1|_{S_1}, \tag{17}$$

$$B_0: L_2^{(1)}(S_1) \longrightarrow L_2(S_1),\ B_0 \psi = v_1|_{S_1}, \tag{18}$$

$$C_0: L_2^{(1)}(S_1) \longrightarrow L_2(S_1),\ C_0 \psi = v_2|_{S_1}, \tag{19}$$

$$D_0: L_2^{(1)}(S_1) \longrightarrow L_2(S_2),\ D_0 \psi = v_2|_{S_2}. \tag{20}$$

Making use of previous results and the compactness of the trace operators from $H_1(\Omega_1)$ to $L_2(\mathrm{S}_1)$, and from $H_1(\Omega_2)$ to $L_2(S_1)$ and to $L_2(S_2)$, we conclude the compactness of the above operators, $A_0$, $B_0$, $C_0$ and $D_0$. It is convenient to consider the projection operators $P_i: L_2(S_i) \longrightarrow L_2^{(1)}(\mathrm{S}_i)$, defined by:

$$P_i \varphi_i = \varphi_i - \frac{1}{|S_i|} \int_{S_i} \varphi_i ds\,, i = 1{,}2, \tag{21}$$

Where $|S_i|$ denotes the Lebesgue measure of $S_i$. These averages could be considered as reference potentials, and these substractions are required in order to assure existence of the corresponding solutions. Thus we define the following operators:

$$A = P_1 \circ A_0: L_2^{(1)}(\Omega_1) \longrightarrow L_2^{(1)}(S_1), \tag{22}$$

$$B = P_1 \circ B_0: L_2^{(1)}(S_1) \longrightarrow L_2^{(1)}(S_1), \tag{23}$$

$$C = P_1 \circ C_0: L_2^{(1)}(S_1) \longrightarrow L_2^{(1)}(S_1), \tag{24}$$

$$D = P_2 \circ D_0: L_2^{(1)}(S_1) \longrightarrow L_2^{(1)}(S_2). \tag{25}$$

By using these operators $A, B, C$ and $D$, the solution of the inverse problem associated to the contour problem ( 1 ) - ( 5 ) can be obtained by solving the following system of operational equations:

$$Af + B\psi = C\psi, \tag{26}$$

$$D\psi = V. \tag{27}$$

Actually, equation ( 27 ) is equivalent to the Cauchy problem in region $\Omega_2$ for the Laplace operator with respect to the Cauchy data $\frac{\partial u_2}{\partial n_2} = 0$ on $S_2$ and $u_2|_{S_2} = V$ , where $V \in L_2^{(1)}(S_2)$. Once $\psi$ is known, it can be substituted in ( 26 ), obtaining the operational equation:

$$Af = -(B - C)\psi. \tag{28}$$

Thus, equation ( 28 ) give us $f$, so system ( 26 ) - ( 27 ) is equivalent to inverse source estimation problem. Finally, the operator which relates the source $f$ with the measurement $V$ is given by:

$$\mathcal{A} = D(-(B - C)^{-1}A) = -D(B - C)^{-1}A. \tag{29}$$

The following result can be found in (Fraguela Collar, Oliveros Oliveros, Morín Castillo, & Conde Mones, 2015).

**Theorem 2.** One has

$$[Ker\ \mathcal{A}]^{\perp} = \mathcal{H}^{(1)}(\Omega_1).$$

This theorem justifies remarks a) and b) above. An important corollary follows also from it: if $f \in \mathcal{F}$ reproduces a measurement $V$, and the harmonic function $h_0$ is the unique harmonic source on $\Omega_1$ which also reproduces $V$, then $h_0$ is the orthogonal projection of $f$ on $\mathcal{H}^{(1)}(\Omega_1)$, independently of the chosen class $\mathcal{F}$, and $f - h_0$ is the component of $f$ in Ker $\mathcal{A}$.

Hence, for the class $\mathcal{F}$ to be an unicity class is necessary and sufficient that

$$(\mathcal{F} - \mathcal{F}) \cap Ker\ \mathcal{A} = \{0\},$$

Where $\mathcal{F} - \mathcal{F} = \{f_1 - f_2 : f_1, f_2 \in \mathcal{F}\}$ and 0 is the vanishing function. Note that the class $\mathcal{F}$ need not to be a linear space. Indeed, $\mathcal{F}$ is a unicity class if $\mathcal{A}$ (or A) restricted to $\mathcal{F}$ is injective. This happens if for any $f_1, f_2 \in F$ with $\mathcal{A}(f_1) = \mathcal{A}(f_2)$ one gets $f_1 = f_2$, but $\mathcal{A}(f_1 - f_2) = \mathcal{A}(f_1) - \mathcal{A}(f_2)$, so $\mathcal{A}(f_1) = \mathcal{A}(f_2)$ is equivalent to $f_1 - f_2 \in Ker\mathcal{A}$, which has to be trivial.

## Admisible data method

The "Admissible Data Method" (ADM), as a regularization strategy (Hernandez Montero, Fraguela Collar, & Henry, 2019), can be applied to any sources unicity class $\mathcal{F}$. Concretely, in the context of the inverse source estimation problem linked to the instantaneous contour problem ( 1 ) - ( 5 ), when the admissible data set $\mathcal{M}[\mathcal{F}]$ is contained into the admissible data set corresponding to the harmonic sources $\mathcal{H}^{(1)}(\Omega_1)$, consists of the following steps:

1. Choose, if possible, a closed vector subspace or compact convex subset $\mathcal{M}_0$ of $\mathcal{M}[F]$ in such a way that the minimum distance problem to $\mathcal{M}_0$, would be well posed in $L_2(S_2)$.
2. Assuming fixed a measurement error and a measurement $\tilde{V} \in L_2(S_2)$ (with error), find the admissible data $V_0 \in \mathcal{M}_0$ which attains the minimum distance to $\tilde{V}$, and check that this distance has the magnitude order of the measurement error.
3. Compute the harmonic source $h_0$ which reproduces the measurement $V_0$.
4. Finally, use $h_0$ to characterize the source $f \in \mathcal{F}$ which best reproduces the measurement $V_0$ in the class $\mathcal{F}$. In order to do that, one makes use of remark after Theorem 2: $f - h_0$ is orthogonal to any harmonic function.

This methodology will be illustrated in the last section, in the case of the sources supported and harmonic in a neighborhood of the cortex.

# Results

## Reduction of the instantaneous source estimation problem to a Cauchy data problem on cortex

In order to find the restriction to $S_1$ of the solution $u_2$ on $\Omega_2$, we will study the following contour problem, which is a Cauchy problem for the Laplace equation on $\Omega_2$, with vanishing Neumann contour condition and Dirichlet data $V$:

$$\Delta u_2 = 0 \text{ in } \Omega_2 , \tag{30}$$

$$\frac{\partial u_2}{\partial n_2} = 0 \ \text{ on } S_2, \tag{31}$$

$$u_2\big|_{S_2} = V, \tag{32}$$

Where $V$ is the potential distribution electroencephalographic measurement $V$ in $S_2$ (the whole scalp) and $u_2$ is the electric potential in $\Omega_2$, at a given time instant. We proceed by computing formally the solution and studying the convergence of the obtained series (Mijailov, 1978).

We consider the normalized spherical harmonics (Tijonov & Samarsky, 1980, pp. 765-778):

$$Y_{nm}(\theta,\phi) = \sqrt{\frac{2n+1}{4\pi}\frac{(n-|m|)!}{(n+|m|)!}} P_n^{|m|}(cos\theta)e^{im\phi}, \tag{33}$$

Where $P_n^{(m)}(\theta,\phi)$ are the associated Legendre polynomials, and $\theta \in [0,\pi], \phi \in [0,2\pi], 1 \leq n < \infty, -n \leq m \leq n$. Thus, $V$ is given by

$$V(\theta,\phi)=\sum_{n=0}^{\infty}\sum_{m=-n}^{n} V_{nm}\,Y_{nm}(\theta,\phi), \tag{34}$$

Where $V_{nm}$ are the Fourier coefficients of $V$. It can be checked that the corresponding solution to problem ( 30 ) - ( 32 ) is given by

$$u_2(r,\theta,\phi)=\sum_{n=1}^{\infty}\sum_{m=-n}^{n}\left(\frac{n+1}{2n+1}\left(\frac{r}{R_2}\right)^{n}+\frac{n}{2n+1}\left(\frac{R_2}{r}\right)^{n+1}\right)V_{nm}Y_{nm}(\theta,\phi), \tag{35}$$

With convergence in $L_2(\Omega_2)$. The solution of the Cauchy problem corresponds to restrict $u_2$ to $S_1$, which means to evaluate ( 35 ) at $r=R_1$, obtaining

$$u_2(R_1,\theta,\phi)=\sum_{n=1}^{\infty}\sum_{m=-n}^{n}\left(\frac{n+1}{2n+1}\left(\frac{R_1}{R_2}\right)^{n}+\frac{n}{2n+1}\left(\frac{R_2}{R_1}\right)^{n+1}\right)V_{nm}Y_{nm}(\theta,\phi). \tag{36}$$

In order to assure that the above expression for $u_2(R_1,\theta,\phi)$ to be the trace in $S_1$ of a function in $H_1(\Omega_1)$, the series composed with the squares of Fourier coefficients of ( 36 ), multiplied by $n$, must converge (Mijailov, 1978, p. 221). Then, by using an elementary result on convergence of numerical series[1] we obtain the following condition on the Fourier coefficients of $V$:

$$\sum_{n=1}^{\infty} n\left(\frac{R_2}{R_1}\right)^{2n}\sum_{m=-n}^{n}|V_{nm}|^2<\infty. \tag{37}$$

Next we find out what conditions on $V$ must be imposed in order for $\psi=\frac{\sigma_2}{\sigma_1}\frac{\partial u_2}{\partial r}\Big|_{r=R_1}$ to be $L_2^{(1)}(S_1)$[2]. It is easy to see that

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[1] If $x_n$ and $y_n$ are nonnegative term sequences, and $\lim_{n\to\infty}\frac{x_n}{y_n}=\alpha\in(0,\infty)$, then these sequences are said to be "equivalent", and then $\sum_{n=1}^{\infty}x_n<+\infty$ iff $\sum_{n=1}^{\infty}y_n<+\infty$

[2] This is a consequence of the necessary and sufficient condition for the existence of solution Neumann problem for the Laplace equation on $\Omega_1\cup\Omega_2$ and on $\Omega_1$.

$$\psi = \frac{\sigma_2}{\sigma_1}\frac{\partial u_2}{\partial r}\bigg|_{r=R_1} = \frac{\sigma_2}{\sigma_1}\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\frac{n(n+1)}{2n+1}\left(\frac{R_1^{n-1}}{R_2^{n}} - \frac{R_2^{n+1}}{R_1^{n+2}}\right)V_{nm}Y_{nm}(\theta,\phi). \quad (38)$$

Consequently, ( 38 ) must converge in $L_2(S_1)$. Proceeding as before, we get the following result:

**Theorem 3.** One has $\psi \in L_2^{(1)}(S_1)$ if and only if the Fourier coefficients $V_{nm}$ satisfy

$$\sum_{n=1}^{\infty} n^2 \left(\frac{R_2}{R_1}\right)^{2n} \sum_{m=-n}^{n} |V_{nm}|^2 < \infty. \quad (39)$$

All of this machinery allows us to convert the source estimation problem in the head into a source estimation problem in the brain from Cauchy data on the cortex. From now on we will suppose that the Fourier coefficients $V_{nm}$ of $V$ satisfy condition ( 39 ). From an operational point of view, it makes sense to solve equation ( 28 ) from $\psi$ given by ( 38 ). That is

$$Af = C\psi - B\psi = g, \quad (40)$$

Which could be viewed as the identification of the source $f$, which reproduces the potential $g$ on the cortex $S_1$, as it would be artificially measured.

In order to obtain an explicit expression for $g = C\psi - B\psi$, we make use of ( 38 ). Since operators $B$ and $C$ linear and continuous, it is enough to compute $B(Y_{nm}(\theta,\phi))$ and $C(Y_{nm}(\theta,\phi))$. For this purpose, in view of ( 18 ) and ( 19 ), we solve problems ( 9 ) and ( 10 ) with contour data $Y_{nm}(\theta,\phi)$.

For $B(Y_{nm}(\theta,\phi))$, we look for the solution $v_1$ of the contour problem ( 9 ) of the form

$$v_1(r,\theta,\varphi) = Ar^nY_{nm}(\theta,\varphi), \quad (41)$$

And for $C(Y_{nm}(\theta,\phi))$, we seek the solution $v_2$ of the contour problem ( 10 ) of the form

$$v_2(r,\theta,\phi) = \left(A_n r^n + B_n r^{-(n+1)}\right) Y_{nm}(\theta,\phi). \tag{42}$$

In this way we get

$$B(\psi) = \frac{\sigma_2}{\sigma_1^2} \sum_{n=1}^{\infty} \frac{n+1}{2n+1} \left(\frac{R_2}{R_1}\right)^{n+1} \left(\left(\frac{R_1}{R_2}\right)^{2n+1} - 1\right) \sum_{m=-n}^{n} V_{nm} Y_{nm}(\theta,\phi), \tag{43}$$

$$C(\psi) = \frac{1}{\sigma_2} \sum_{n=1}^{\infty} \frac{n(n+1)}{(2n+1)} \left(\frac{1}{n}\left(\frac{R_1}{R_2}\right)^{n} + \frac{1}{n+1}\left(\frac{R_2}{R_1}\right)^{n+1}\right) \sum_{m=-n}^{n} V_{nm} Y_{nm}(\theta,\phi). \tag{44}$$

From this expressions one easily gets

$$g = \frac{1}{\sigma_1} \sum_{n=1}^{\infty} \frac{n+1}{2n+1} \left\{ \left(\frac{R_1}{R_2}\right)^{n} \left(1 - \frac{\sigma_2}{\sigma_1}\right) + \left(\frac{R_2}{R_1}\right)^{n+1} \left(\frac{n}{n+1} + \frac{\sigma_2}{\sigma_1}\right) \right\} \sum_{m=-n}^{n} V_{nm} Y_{nm}(\theta,\phi). \tag{45}$$

In summary, we conclude that, if the Fourier coefficients $V_{nm}$ of measurement $V$ satisfy condition ( 39 ), then the source estimation problem turns to solve equation $Af = g$, where $g$ is given by ( 45 ). This problem can be interpreted as a source estimation problem in the brain, assuming it is isolated, and starting from the "measurement" $g$ on the cortex. In what follows, we will apply this conclusion.

## Identification of harmonic sources in the brain

In this section we consider a given instantaneous potential distribution $V$ on the scalp, and we study under what conditions there exists an (unique) harmonic source $f$ in the brain which reproduces $V$. By the way, we will find an explicit expression for $f$.

We start from the fact that any harmonic source $f$ in $\Omega_1$ should take the following form:

$$f(r,\theta,\phi)=\sum_{n=1}^{\infty}\sum_{m=-n}^{n}F_{nm}\sqrt{\frac{2n+3}{R_1^{2n+3}}}r^nY_{nm}(\theta,\phi), \tag{46}$$

Which is a development with respect to the orthonormal basis of spherical harmonics $\sqrt{\frac{2n+3}{R_1^{2n+3}}}r^nY_{nm}(\theta,\phi)$. Hence, the convergence of this series (in the $L_2(\Omega_1)$ sense) is equivalent to the condition

$$\sum_{n=1}^{\infty}\sum_{m=-n}^{n}|F_{nm}|^2<\infty. \tag{47}$$

Since operator $A$ is linear and continuous, we get

$$Af=\sum_{n=1}^{\infty}\sum_{m=-n}^{n}F_{nm}\sqrt{\frac{2n+3}{R_1^{2n+3}}}A\big(r^nY_{nm}(\theta,\phi)\big). \tag{48}$$

In order to compute

$$A\big(r^nY_{nm}(\theta,\phi)\big), \tag{49}$$

In view of ( 17 ), it is required to solve the associated problem

$$-\sigma_1\Delta w_1(r,\theta,\phi)=r^nY_{nm}(\theta,\phi)\text{ en }\Omega_1, \tag{50}$$

$$\left.\frac{\partial w_1}{\partial n_1}\right|_{S_1}=0\text{ sobre }S_1. \tag{51}$$

Thus we look for a solution of the form

$$w_1=v(r)Y_{nm}(\theta,\phi), \tag{52}$$

Where $v(r)$ satisfies the equation

$$\frac{d}{dr}\left(r^2\frac{d\big(v(r)\big)}{dr}\right) - v(r)n(n+1) = -\frac{r^{n+2}}{\sigma_1}. \tag{53}$$

The general solution of this equation takes the form

$$v(r) = v_0(r) + v_p(r), \tag{54}$$

Where $v_0$ is solution for the associated homogeneous equation and $v_p$ is a particular solution. We propose a particular solution of the form

$$v_p(r) = Lr^k. \tag{55}$$

From ( 53 ) we obtain

$$r^k\, L\big(k(k+1) - n(n+1)\big) = -\frac{1}{\sigma_1}r^{n+2}. \tag{56}$$

Thus we get the condition $k = n + 2$, so

$$L(n^2 + 5n + 6 - n^2 - n) = -\frac{1}{\sigma_1}, \tag{57}$$

And

$$L = -\frac{1}{2\sigma_1(2n+3)}. \tag{58}$$

The function $v_0$ satisfies the homogeneous equation

$$\frac{d}{dr}\left(r^2\frac{dv_0(r)}{dr}\right) - v_0(r)n(n+1) = 0, \tag{59}$$

And must be bounded at $r = 0$, so takes the form

$$v_0(r) = Br^n. \tag{60}$$

From the contour condition ( 51 ) we get

$$\frac{dv_0(R_1)}{dr} + \frac{dv_p(R_1)}{dr} = 0, \tag{61}$$

So

$$B = \frac{(n+2)}{\sigma_1 2n(2n+3)} R_1^2. \tag{62}$$

In this way we have finally found that

$$Af(r,\theta,\phi) = \sum_{n=1}^{\infty}\sum_{m=-n}^{n} F_{nm}\frac{\sqrt{(2n+3)R_1}}{\sigma_1 n(2n+3)} Y_{nm}(\theta,\phi). \tag{63}$$

From ( 40 ) we must equate expressions ( 63 ) and ( 45 ). From the unicity of the coefficients, we obtain the Fourier coefficients $F_{nm}$ in terms of the corresponding $V_{nm}$. Concretely, the final expression for $f$ is:

$$f = \sum_{n=1}^{\infty}\sum_{m=-n}^{n} \sqrt{\frac{1}{(2n+3)R_1}} \frac{n(n+1)(2n+3)}{2n+1}\left\{\left(\frac{R_1}{R_2}\right)^n\left(1-\frac{\sigma_2}{\sigma_1}\right) + \left(\frac{R_2}{R_1}\right)^{n+1}\left(\frac{n}{n+1}+\frac{\sigma_2}{\sigma_1}\right)\right\} V_{nm}\sqrt{\frac{2n+3}{R_1^{2n+3}}} r^n Y_{nm}(\theta,\phi), \tag{64}$$

Where the convergence must be in $L_2(\Omega_1)$. Comparing expressions ( 48 ) and ( 64 ) we conclude the following result.

**Theorem 4.** There exists a biunivocal correspondence between the set of harmonic sources $f$ in the brain $\Omega_1$ and the set of measurements $V$ on the scalp (in $L_2(S_2)$) whose Fourier coefficients $V_{nm}$ with respect to the orthonormalized spherical harmonics $Y_{nm}(\theta,\phi)$ satisfy:

$$\sum_{n=1}^{\infty} n^3 \left(\frac{R_2}{R_1}\right)^{2n} \sum_{m=-n}^{n} |V_{nm}|^2 < \infty. \qquad (65)$$

Furthermore, given a measurement $V$ satisfying condition ( 65 ), the unique harmonic source in $\Omega_1$ which reproduces it is given by ( 64 ).

We recall that from the previous theorem it follows that $\mathcal{H}^{(1)}(\Omega_1)$ is a sources unicity class, and the admissible data set corresponding to this class of harmonic sources orthogonal to the constants, denoted by $\mathcal{M}\left[\mathcal{H}^{(1)}(\Omega_1)\right]$ following notation ( 7 ), turns to be the set of measurements $V$ satisfying ( 65 ).

Note that, from Theorem 2 and 4 it follows that for any sources class $\mathcal{F}$ in $L_2(\Omega_1)$ the corresponding admissible data set $\mathcal{M}[\mathcal{F}]$ is contained into the set of measurements $V$ in $L_2(S_2)$ which satisfy condition ( 65 ). Note also that this condition ( 65 ) is a quite strong smooth requirement on the spatial distribution of the measurements $V$, for each time instant.

## Importance of the class of harmonic sources in the sources identification methodology on arbitrary unicity classes. The particular case of harmonic sources in a neighbourhood of the cortex

In this section we assume as before a given instantaneous potential distribution $V$ on the scalp, and we study the existence of a source in the brain reproducing it. For this source $f$ we assume it is supported and harmonic in a neigbourhood of the cortex. We stablish conditions on $V$ for its existence, and we will find the admissible data set for this class.

Let us first clarify what will be understood for "a neighbourhood of the cortex" (see Fig. 3): Fix $R_0$ satisfying $0 < R_0 < R_1$. $\Omega_1^{(0)} = \{0 < r \leq R_0\}$ represents the subcortical part of the brain, $\Omega_1^{(1)} = \{R_0 < r \leq R_1\}$ stands for a certain spherical neighbourhood of the cortex, and $\Omega_2 = \{R_1 < r \leq R_2\}$ is the rest of the head as before. Analogously, the sphere $S_0 = \{r = R_0\}$ is the border of $\Omega_1^{(0)}$, the sphere $S_1 = \{r = R_1\}$ stands for the cortex and is the common border of $\Omega_1^{(0)}$ and $\Omega_1^{(1)}$, and $S_2 = \{r = R_2\}$ stands for the scalp, where the measurements are taken, which is the outer border of $\Omega_2$.

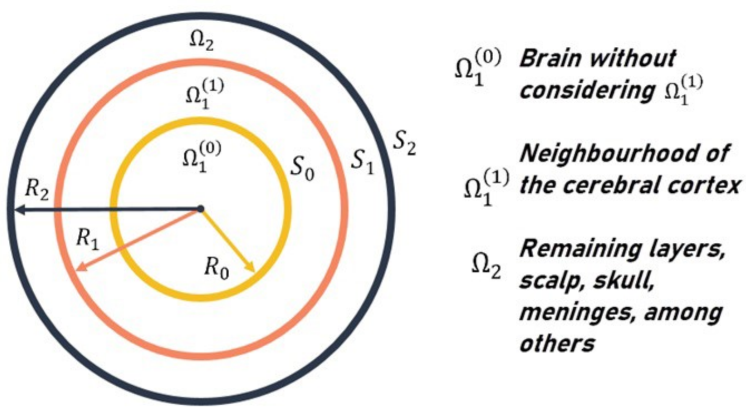


**Fig. 3** Schematic figure of volume conductor model, where $\Omega_1^{(1)}$ denotes the cerebral cortex $\Omega_1^{(0)}$ it represents the remaining layers such as the scalp, skull, meninges, among others.
**Source:** Own elaboration.

It will be convenient to denote

$$\hat{r} = \begin{cases} 0 & if \ 0 < r \leq R_0 \\ r & if \ R_0 < r \leq R_1 \end{cases}. \qquad (66)$$

We consider the class $\mathcal{H}^{(1)}\left(\Omega_1^{(1)}\right)$ of sources in the brain which are supported and harmonic in $\Omega_1$ (vanishing outside). It is easy to see that this sources $f$ can be represented in the form (Tijonov & Samarsky, 1980, p. 772):

$$f(r,\theta,\phi) = \sum_{n=1}^{\infty}\sum_{m=-n}^{n}\left(a_{nm}\sqrt{\frac{2n+3}{R_1^{2n+3}-R_0^{2n+3}}}\hat{r}^n + b_{nm}\sqrt{\frac{2n-1}{R_0^{-2n+1}-R_1^{-2n+1}}}\hat{r}^{-(n+1)}\right)Y_{nm}(\theta,\phi), \quad (67)$$

Where the corresponding series of Fourier coefficients

$$\sum_{n=1}^{\infty}\sum_{m=-n}^{n}|a_{nm}|^2, \sum_{n=1}^{\infty}\sum_{m=-n}^{n}|b_{nm}|^2, \quad (68)$$

should converge.

Next, we solve the auxiliary contour problem ( 8 ), which in view of ( 17 ) means compute $A(f)$. Since operator $A$ is linear and continuous, it is enough to compute $A(\hat{r}^n Y_{nm}(\theta,\phi))$ and $A\left(\hat{r}^{-(n+1)}Y_{nm}(\theta,\phi)\right)$. In order to find:

$$A(\hat{r}^n Y_{nm}(\theta,\phi)), \quad (69)$$

it is necessary to solve the associated contour problem

$$-\sigma_1 \Delta w_1(r,\theta,\phi) = \hat{r}^n Y_{nm}(\theta,\phi) \text{ en } \Omega_1, \quad (70)$$

$$\left.\frac{\partial w_1}{\partial n_1}\right|_{S_1} = 0 \text{ on } S_1, \quad (71)$$

whose solution is imposed to take the form

$$w_1 = a(r)Y_{nm}(\theta, \phi). \tag{72}$$

It is easy to see that $a(r)$ satisfies the equation

$$\frac{d}{dr}\left(r^2 \frac{d\big(a(r)\big)}{dr}\right) - a(r)n(n+1) = -\frac{\hat{r}^{n+2}}{\sigma_1}. \tag{73}$$

If we denote by $a_0(r)$, $w_1^{(0)}$ and $a_1(r)$, $w_1^{(1)}$ the corresponding restrictions of $a(r)$ and $w_1$ to $\Omega_1^{(0)}$ and $\Omega_1^{(1)}$, respectively, and consider the compatibility conditions on $S_0$,

$$w_1^{(0)}\Big|_{r=R_0} = w_1^{(1)}\Big|_{r=R_0} \text{ on } S_0, \tag{74}$$

$$\frac{\partial w_1^{(0)}}{\partial r}\bigg|_{r=R_0} = \frac{\partial w_1^{(1)}}{\partial r}\bigg|_{r=R_0} \text{ on } S_0, \tag{75}$$

we arrive to functions $a_0(r)$ and $a_1(r)$ satisfy

$$\frac{d}{dr}\left(r^2 \frac{d\big(a_0(r)\big)}{dr}\right) - a_0(r)n(n+1) = 0, \text{ for } 0 < r \le R_0, \tag{76}$$

$$\frac{d}{dr}\left(r^2 \frac{d\big(a_1(r)\big)}{dr}\right) - a_1(r)n(n+1) = -\frac{r^{n+2}}{\sigma_1}, \text{ for } R_0 < r \le R_1, \tag{77}$$

and the contour conditions at $r = R_0$

$$a_0(R_0) = a_1(R_0), \tag{78}$$

$$\frac{da_0}{dr}(R_0) = \frac{da_1}{dr}(R_0). \tag{79}$$

Also, the following additional conditions must be added:

$$|a_0(0)| < \infty, \tag{80}$$

$$\frac{da_1}{dr}(R_1) = 0. \tag{81}$$

Solving this problem for $a_0(r)$ and $a_1(r)$ gives us

$$a_0(r) = \frac{1}{\sigma_1}\left(-\frac{n+1}{n(2n+1)(2n+3)}\frac{R_0^{2n+3}}{R_1^{2n+3}} + \frac{(n+2)R_1^2}{2n(2n+3)} - \frac{R_0^2}{2(2n+1)}\right)r^n, \tag{82}$$

$$\begin{aligned} a_1(r) = {} & \left(-\frac{n+1}{\sigma_1 n(2n+1)(2n+3)}\frac{R_0^{2n+3}}{R_1^{2n+3}} + \frac{(n+2)R_1^2}{2\sigma_1 n(2n+3)}\right)r^n \\ & - \left(\frac{R_0^{2n+3}}{\sigma_1(2n+1)(2n+3)}\right)r^{-(n+1)} - \frac{1}{\sigma_1 2(2n+3)}r^{n+2}. \end{aligned} \tag{83}$$

Thus, we finally get

$$A\big(\hat{r}^n Y_{nm}(\theta,\phi)\big) = \left(-\frac{\big(1+n(1+R_1^2)\big)R_0^{2n+3}}{\sigma_1 n(2n+1)(2n+3)R_1^{n+3}} + \frac{R_1^{n+2}}{\sigma_1 n(2n+3)}\right)Y_{nm}(\theta,\phi). \tag{84}$$

In a similar fashion one obtains

$$A\left(\hat{r}^{-(n+1)}Y_{nm}(\theta,\phi)\right) = \left(\frac{R_1^{-n+1} - R_0^2 R_1^{-n-1}}{2\sigma_1 n}\right)Y_{nm}(\theta,\phi). \tag{85}$$

Equipped with ( 84 ) and ( 85 ), we can easily write the resulting formula for $A(f)$:

$$Af = \sum_{n=1}^{\infty}\sum_{m=-n}^{n}\left(a_{nm}\sqrt{\frac{2n+3}{R_1^{2n+3}-R_0^{2n+3}}}\left(-\frac{R_0^{2n+3}\left(1+n(1+R_1^2)\right)}{\sigma_1 n(2n+3)(2n+1)R_1^{n+3}} + \frac{R_1^{n+2}}{\sigma_1 n(2n+3)}\right) + b_{nm}\sqrt{\frac{2n-1}{R_0^{-2n+1}-R_1^{-2n+1}}}\left(\frac{R_1^{-n+1}-R_0^2R_1^{-n-1}}{2\sigma_1 n}\right)\right)Y_{nm}(\theta,\phi). \tag{86}$$

Finally, we must equate (86) and ( 45 ) and apply the coefficients unicity in order to obtain

$$\left(a_{nm}\sqrt{\frac{2n+3}{R_1^{2n+3}-R_0^{2n+3}}}\left(-\frac{\left(1+n(1+R_1^2)\right)R_0^{2n+3}}{n(2n+1)(2n+3)R_1^{n+3}} + \frac{R_1^{n+2}}{n(2n+3)}\right) + b_{nm}\sqrt{\frac{2n-1}{R_0^{-2n+1}-R_1^{-2n+1}}}\left(\frac{(R_1^{-n+1}-R_0^2R_1^{-n-1})}{2\sigma_1 n}\right)\right) = \frac{(n+1)}{(2n+1)}\left\{\left(\frac{R_1}{R_2}\right)^n\left(1-\frac{\sigma_2}{\sigma_1}\right) + \left(\frac{R_2}{R_1}\right)^{n+1}\left(\frac{n}{n+1}+\frac{\sigma_2}{\sigma_1}\right)\right\}V_{nm}. \tag{87}$$

Let us define $\alpha_n$, $\beta_n$, and $\gamma_n$ by

$$\alpha_n = \sqrt{\frac{2n+3}{R_1^{2n+3}-R_0^{2n+3}}}\left(\frac{R_1^{n+2}}{n(2n+3)} - \frac{\left(1+n(1+R_1^2)\right)R_0^{2n+3}}{n(2n+3)(2n+1)R_1^{n+3}}\right), \tag{88}$$

$$\beta_n = \frac{(R_1^2-R_0^2)\sqrt{2-\frac{1}{n}}}{2\sqrt{n}R_1\sqrt{R_1}\sqrt{\left(\frac{R_1}{R_0}\right)^{2n-1}-1}}, \tag{89}$$

$$\gamma_n = \frac{n+1}{2n+1}\left\{\left(\frac{R_1}{R_2}\right)^n\left(1-\frac{\sigma_2}{\sigma_1}\right) + \left(\frac{R_2}{R_1}\right)^{n+1}\left(\frac{n}{n+1}+\frac{\sigma_2}{\sigma_1}\right)\right\}. \tag{90}$$

It is easy to see that

$$\lim_{n\to\infty} \frac{\alpha_n}{n^{-\frac{3}{2}}} = \frac{\sqrt{R_1}}{\sqrt{2}}, \tag{91}$$

$$\lim_{n\to\infty} \frac{\beta_n}{n^{-\frac{1}{2}}\left(\frac{R_0}{R_1}\right)^n} = \frac{R_1^2 - R_0^2}{R_1\sqrt{2R_0}}, \tag{92}$$

$$\lim_{n\to\infty} \frac{\gamma_n}{\left(\frac{R_2}{R_1}\right)^n} = \frac{1}{2}\left(\frac{R_2}{R_1}\right)\left(1 + \frac{\sigma_2}{\sigma_1}\right). \tag{93}$$

Let us write ( 87 ) in the form

$$\alpha_n a_{nm} + \beta_n b_{nm} = \gamma_n V_{nm}, \tag{94}$$

or equivalently

$$a_{nm} + \frac{\beta_n}{\alpha_n} b_{nm} = \frac{\gamma_n}{\alpha_n} V_{nm}. \tag{95}$$

This yields

$$a_{nm}^2 + b_{nm}^2 \frac{\beta_n^2}{\alpha_n^2} + 2a_{nm}b_{nm}\frac{\beta_n}{\alpha_n} = \frac{\gamma_n^2}{\alpha_n^2} V_{nm}^2. \tag{96}$$

Finally, have the sum for any positive integer $n$ and $m$ from $-n$ to $n$. If we take into account that $f$ is harmonic in $\Omega_1$, we have the following conclusions:

1. $\sum_{n=1}^{\infty} \sum_{m=-n}^{n} |a_{nm}|^2$ converges, since $a_{nm}$ are the Fourier coefficients of $f$.
2. Consider the series with general term the second term of the left side of ( 96 ), which is equivalent to $n^2 \left(\frac{R_0}{R_1}\right)^{2n} |b_{nm}|^2$. Since $n^2 \left(\frac{R_0}{R_1}\right)^{2n} \underset{n\to\infty}{\rightarrow} 0$ and $b_{nm}$ are Fourier coefficients, this series converges.

3. Then the series with general term the right side of ( 96 ) also converges, whose terms are equivalent to $n^3 \left(\frac{R_2}{R_1}\right)^{2n} |V_{nm}|^2$.

In this way, we arrive to the equivalent condition:

$$\sum_{n=1}^{\infty} n^3 \left(\frac{R_2}{R_1}\right)^{2n} \sum_{m=-n}^{n} |V_{nm}|^2 < +\infty, \tag{97}$$

which turns to be the necessary and sufficient condition previously obtained for $V$ to be reproducible by an harmonic source defined in $\Omega_1$. That is, if there is an harmonic source $f \in \mathcal{H}^{(1)}\left(\Omega_1^{(1)}\right)$ which reproduces $V$, then ( 97 ) is fulfilled. In other words, ( 97 ) is a necessary condition.

Next, we will characterize the admissible data set corresponding to harmonic sources in a neighborhood of the cortex. By Theorem 2 we have that, if $f \in \mathcal{H}^{(1)}\left(\Omega_1^{(1)}\right)$ and $h_0 \in \mathcal{H}^{(1)}(\Omega_1)$ both reproduce $V$, then $f - h_0 \in \mathcal{H}^{(1)}(\Omega_1)^{\perp}$, so

$$\langle f - h_0, r^n Y_{nm}(\theta, \phi) \rangle = 0, \tag{98}$$

for $n \geq 1, -n \leq m \leq n$ , where $\langle , \rangle$ denotes the inner product in $L_2(\Omega_1)$. Now, from ( 67 ) and since $\hat{r}^n Y_{nm}(\theta, \phi)$ and $\hat{r}^{-(n+1)} Y_{nm}(\theta, \phi)$ are orthogonal in $L_2\left(\Omega_1^{(1)}\right)$, it follows

$$\langle f, r^n Y_{nm}(\theta, \phi) \rangle = a_{nm} \sqrt{\frac{R_1^{2n+3} - R_0^{2n+3}}{2n+3}}. \tag{99}$$

On the other hand, we have

$$h_0 = \sum_{n=1}^{\infty}\sum_{m=-n}^{n}\sqrt{\frac{1}{(2n+3)R_1}}\frac{n(n+1)(2n+3)}{(2n+1)}\left\{\left(\frac{R_1}{R_2}\right)^n\left(1-\frac{\sigma_2}{\sigma_1}\right) + \left(\frac{R_2}{R_1}\right)^{n+1}\left(\frac{n}{n+1}+\frac{\sigma_2}{\sigma_1}\right)\right\}V_{nm}\sqrt{\frac{2n+3}{R_1^{2n+3}}}r^nY_{nm}(\theta,\phi), \tag{100}$$

So

$$\langle h_0, r^nY_{nm}(\theta,\phi)\rangle = \frac{R_1^{n+1}n(n+1)}{(2n+1)}\left(\left(\frac{R_1}{R_2}\right)^n\left[1-\frac{\sigma_2}{\sigma_1}\right]+\left(\frac{R_2}{R_1}\right)^{n+1}\left[\frac{n}{n+1}+\frac{\sigma_2}{\sigma_1}\right]\right)V_{nm}. \tag{101}$$

Thus, from ( 98 ), ( 99 ) and ( 101 ) we get

$$a_{nm} = \sqrt{\frac{2n+3}{R_1^{2n+3}-R_0^{2n+3}}}\frac{n(n+1)}{(2n+1)}R_1^{n+1}\left(\left(\frac{R_1}{R_2}\right)^n\left[1-\frac{\sigma_2}{\sigma_1}\right]+\left(\frac{R_2}{R_1}\right)^{n+1}\left[\frac{n}{n+1}+\frac{\sigma_2}{\sigma_1}\right]\right)V_{nm}. \tag{102}$$

also, from ( 94 ) we get

$$b_{nm} = \frac{\gamma_n}{\beta_n}V_{nm} - \frac{\alpha_n}{\beta_n}a_{nm}. \tag{103}$$

Making the computations, from ( 102 ) and ( 103 ) we get that $|a_{nm}|$ and $|b_{nm}|$ are equivalent to $n^{\frac{3}{2}}\left(\frac{R_2}{R_1}\right)^n|V_{nm}|$ and $\sqrt{n}\left(\frac{R_0}{R_1}\right)^n\left(\frac{R_2}{R_1}\right)^n|V_{nm}|$, respectively, and then $|a_{nm}|^2+|b_{nm}|^2$ is equivalent to $n^3\left(\frac{R_2}{R_1}\right)^{2n}\left[1+\frac{1}{n^2}\left(\frac{R_0}{R_1}\right)^{2n}\right]|V_{nm}|^2$. Consequently, ( 97 ) is necessary and sufficient for $V$ to be reproducible by a source in $\mathcal{H}^{(1)}\left(\Omega_1^{(1)}\right)$.

However, these classes $\mathcal{H}^{(1)}\left(\Omega_1^{(1)}\right)$ are not unicity classes. In fact, if $f_1$ and $f_2$ are sources in this class generating the same measurement $V$ in $\mathcal{M}\left(\mathcal{H}^{(1)}\left(\Omega_1^{(1)}\right)\right)$, thus $f_1 - f_2 \in KerA$, so

$$\langle f_1 - f_2, r^n Y_{nm} \rangle = 0, 1 \leq n < \infty, -n \leq m \leq n. \tag{104}$$

Then, we get $a_{nm}^{(1)} = a_{nm}^{(2)}$, but nothing can be said about $b_{nm}^{(1)}$ and $b_{nm}^{(2)}$. From this it can be deduced that an unicity subclass contained in $\mathcal{H}^{(1)}\left(\Omega_1^{(1)}\right)$ is obtained considering harmonic functions given by series with respect only the first terms; i.e., those of the following form:

$$f = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} a_{nm} \sqrt{\frac{2n+3}{R_1^{2n+3} - R_0^{2n+3}}} \hat{r}^n Y_{nm}(\theta, \phi). \tag{105}$$

Starting from this development, and computing their coefficients via the equation $Af = g$ as before, we get

$$f = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \frac{\left(1+\frac{1}{n}\right)\sqrt{2+\frac{3}{n}}\sqrt{R_1\left(1-\left(\frac{R_0}{R_1}\right)^{2n+3}\right)}}{\left[2+\frac{1}{n}-\frac{1}{n}\left(\frac{1}{R_1^2}[1-n]-1\right)\left(\frac{R_0}{R_1}\right)^{2n+3}\right]} n^{\frac{3}{2}} \left\{\left(\frac{R_1}{R_2}\right)^n \left(1-\frac{\sigma_2}{\sigma_1}\right) + \left(\frac{R_2}{R_1}\right)^{2n+1} \left(\frac{n}{n+1}+\frac{\sigma_2}{\sigma_1}\right)\right\} \sqrt{\frac{2n+3}{R_1^{2n+3} - R_0^{2n+3}}} \bar{r}^n V_{nm} Y_{nm}(\theta, \phi). \tag{106}$$

From now, this subclass of harmonic sources will be denoted by

$$\mathcal{H}_0^{(1)}\left(\Omega_1^{(1)}\right). \tag{107}$$

Note that sources class ( 107 ) precisely matches the set of sources that equal harmonic sources in $\Omega_1$ when restricted at $\Omega_1^{(1)}$, and vanish at $\Omega_1^{(0)}$.

The following result summarizes these results.

**Theorem 5.** If $0 < R_0 < R_1$, then the admissible data sets $\mathcal{M}\left(\mathcal{H}^{(1)}\left(\Omega_1^{(1)}\right)\right)$ and $\mathcal{M}\left(\mathcal{H}_0^{(1)}\left(\Omega_1^{(1)}\right)\right)$ coincide with the admissible data set $\mathcal{M}\left(\mathcal{H}^{(1)}(\Omega_1)\right)$. These sets are the potential distributions measurements $V$ on the scalp whose Fourier coefficients satisfy

$$\sum_{n=1}^{\infty} n^3 \left(\frac{R_2}{R_1}\right)^{2n} \sum_{m=-n}^{n} |V_{nm}|^2 < \infty. \tag{108}$$

In addition, the class $\mathcal{H}_0^{(1)}\left(\Omega_1^{(1)}\right)$ is an unicity class, while $\mathcal{H}^{(1)}\left(\Omega_1^{(1)}\right)$ is not.

This last statement could result a little bit confusing; let us clarify this point. By making use of the unicity of extension of harmonic functions, one can identify a harmonic source in $\mathcal{H}_0^{(1)}\left(\Omega_1^{(1)}\right)$ with its harmonic extension to $\Omega_1$. In this sense, the classes $\mathcal{H}^{(1)}(\Omega_1)$ and $\mathcal{H}_0^{(1)}\left(\Omega_1^{(1)}\right)$ coincide as sets, while that $\mathcal{H}^{(1)}\left(\Omega_1^{(1)}\right)$ strictly contains $\mathcal{H}^{(1)}(\Omega_1)$.

# Conclusions

In this article, a methodology for solving the inverse bioelectric source estimation problem in the brain, starting from instantaneous electroencephalographic measurements on the whole scalp, is proposed. We make use of a volume conductor model, in which the head and the brain are represented by means of concentric spheres outlining conducting layers with different but constant conductivities.

The main product of this work is the proposal of a general methodology for solving the inverse electroencephalographic source estimation problem. Given an arbitrary sources class $\mathcal{F}$ satisfying unicity condition for the inverse problem, a general method for determining the source in $\mathcal{F}$ which best approximates a potential measurement on the scalp is developed.

The process is basically as follows. First of all, one obtains the harmonic source which reproduces approximately the measurement, via the Admissible Data Method. Later, the specific source in a given class which best approximates the measurement is determined by using additional information provided by the harmonic source. The representing functions sets generating series development solutions and the computational approach depend on the geometric head model.

All this methodology can be extended to the case of time-dependent measurements (EEG), and provides algorithms for directly identifying time-dependent sources which do not depend on previous time discretizations. For the sake of clarity, this is out of the scope of the current work.

Finally, we want to make emphasis on the importance of Theorem 5. As a consequence, we find that any measurement $V$ in $S_2$ can be reproduced by a harmonic source concentrated as close as wanted to the cortex $S_1$ . Furthermore, although these sources extend in a unique way to $\Omega_1$, these specific extensions need not to reproduce the same measurement $V$, when the corresponding support is contained into a different neighborhood (of the cortex) from that corresponding to the original source.

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