THE LERAY-SCHAUDER PRINCIPLE IN GEODESIC SPACES

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ABSTRACT

The essential maps introduced by Granas in 1976 is a best tool for proving continuation results for compact maps. Several authors modified this idea to different scenarios. This is a review article, here, we consider the continuation results based on essential maps and the Leray-Schauder principle in the setting of uniquely geodesic spaces [20].

KEYWORDS

$\Gamma-$*uniquely geodesic spaces, Essential maps, Fixed point, Leray Schauder principle.*

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1 INTRODUCTION

This paper is based on the notion of essential maps defined by Granas [9] in 1976. Many studies and extensions of this concept in variety of settings have been done by several authors [2, 8, 16–18]. Essential map techniques are one of the best tools to prove continuation results for compact maps [1].

Compact maps play a vital role in proving the existence and uniqueness of solutions of differential and integral equations. The classical Schauder fixed point theorem proved by Juliusz Schauder in 1930 in the setting of Banach spaces as a generalization of the celebrated Brouwer's fixed point theorem and the famous Leray-Schauder principle are milestones in the theory of fixed points and both of these incredible results are based on compact maps. For many years, researchers were trying to extend the concepts in normed spaces to more general spaces. In that direction, several authors have extended very many important fixed point theorems to various general spaces [3, 15]. Most importantly, the geodesic spaces grasped the attention of many fixed point theorists after the publication of the papers [11, 12] due to Kirk. A version of Schauder fixed point theorem has been proved by Niculescu and Roventa in $CAT(0)$ spaces [15] by assuming the compactness of convex hull of finite number of elements. Followed by this, in [3], Ariza, Li and Lopez have proved the Schauder Fixed point theorem in the setting of $\Gamma-$uniquely geodesic spaces, which includes Busemann spaces, Linear spaces, $CAT(\kappa)$ spaces with diameter less than $D_k/2$, Hyperbolic spaces [19] etc.

Granas introduced essential maps in order to prove continuation results for compact maps in Banach spaces [9]. In his paper, he has proved topological transversality principle and the Leray Schauder principle, using essential maps techniques. The Leray Schauder principle was first proved for compact mappings in Banach spaces and this has been broadly used to obtain fixed points of variety of mappings under different settings and many interesting contributions can be found in the literature [4, 6, 7, 13, 14]. In [9], Granas has also proved several fixed point results for compact maps using essential mappings and homotopical methods. Followed by the results established by Granas, Agarwal and O'Regan [2] have extended the concept of essential maps to a large class of mappings and established several fixed point theorems in 2000. The concept of essential maps has been further extended to $d-$essential maps, and $d-L$-essential maps [18].

In this review article, we consider the continuation results based on essential maps, and the Leray-Schauder principle in the setting of uniquely geodesic spaces.

2 Preliminaries

We recall some basic definitions and results used in this paper.

Definition 1. [5] *Let (X, ρ) be a metric space. A geodesic segment (or geodesic)from $x \in X$ to $y \in X$ is a map γ from a closed interval $[0, l] \subseteq \mathbb{R}$ to X such that $\gamma(0) = u$, $\gamma(l) = v$ and $\rho(\gamma(t_1), \gamma(t_2)) = |t_1 - t_2|$ for all t_1, t_2 in $[0, l]$.*
(X, ρ) is said to be a geodesic space if every two points in X are joined by a geodesic and (X, ρ) is uniquely geodesic if there is exactly one geodesic joining u to v, for all $u, v \in X$.

We denote the set of all geodesic segments in X by Ω. For $A \subseteq X$, closure and boundary of A in X is denoted by $\overline{A}$ and ∂A respectively.

Definition 2. [3] *Let (X, ρ) be a metric space and $\Gamma \subseteq \Omega$ be a family of geodesic segments. We say that (X, ρ) is a $\Gamma-$uniquely geodesic space if for every $x, y \in X$, there exists a unique geodesic in Γ passes through x and y. We denote a unique geodesic segment in Γ joining x and y by $\gamma_{x,y}$.*

Remark 1. [3] *Let (X, ρ) be a $\Gamma-$uniquely geodesic space. Then, the family Γ induces a unique mapping $\bigoplus_\Gamma : X^2 \times [0, 1] \to X$ such that $\bigoplus_\Gamma(x, y, \tau) \in \gamma_{x,y}$ and the following properties hold for each $u, v \in X$:*

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1. $\rho(\bigoplus_\Gamma(x,y,\tau),\bigoplus_\Gamma(u,v,s)) = |\tau - s|\rho(u,v)$ *for all* $\tau, s \in [0,1]$
2. $\bigoplus_\Gamma(u,v,0) = u$ *and* $\bigoplus_\Gamma(u,v,1) = v$

Also, it is enough to consider $\bigoplus_\Gamma(u,v,\tau) = \gamma_{u,v}(\tau\rho(u,v))$, *i.e., it is a point on* γ *at a distance* $\tau\rho(u,v)$ *from* u *and we denote it by* $(1-\tau)u \oplus \tau v$.

Definition 3. [1] *Let* X *and* Y *be two metric spaces. A map* $\zeta : X \to Y$ *is called compact if* $\zeta(X)$ *is contained in a compact subset of* Y.

Definition 4. [1] *Let* T *be a closed convex subset of a Banach space* X *and* S *be a closed subset of* T. *Denote set of all continuous, compact maps* $\zeta : S \to T$ *by* $\mathscr{K}(S,T)$ *and set of all maps* $\zeta \in \mathscr{K}(S,T)$ *with* $x \neq \zeta(x)$ *by* $\mathscr{K}_{\partial S}(S,T)$ *for* $x \in \partial S$.
A map $\zeta \in \mathscr{K}_{\partial S}(S,T)$ *is essential in* $\mathscr{K}_{\partial S}(S,T)$ *if for every map* $\xi \in \mathscr{K}_{\partial S}(S,T)$ *with* $\zeta|_{\partial S} = \xi|_{\partial S}$ *there exists* $x \in intS$ *with* $x = \xi(x)$. *Otherwise* ζ *is inessential in* $\mathscr{K}_{\partial S}(S,T)$, *that is, there exists a fixed point free* $\xi \in \mathscr{K}_{\partial S}(S,T)$ *with* $\zeta|_{\partial S} = \xi|_{\partial S}$.

Definition 5. [1] *Two maps* $\zeta, \xi \in \mathscr{K}_{\partial S}(S,T)$ *are homotopic in* $\mathscr{K}_{\partial S}(S,T)$, *written* $\zeta \simeq \xi$ *in* $\mathscr{K}_{\partial S}(S,T)$, *if there exists a continuous, compact mapping* $H : S \times [0,1] \to T$ *such that* $H_t(x) := H(\cdot,\tau) : S \to T$ *belongs to* $\mathscr{K}_{\partial S}(S,T)$ *for each* $\tau \in [0,1]$ *with* $H_0 = \zeta$ *and* $H_1 = \xi$.

Definition 6. [3] *A* $\Gamma-$*uniquely geodesic space is said to have property (Q) if*

$$\lim_{\varepsilon \to 0} \sup\{\rho((1-\tau)x \oplus \tau y, (1-\tau)x \oplus \tau z) : \tau \in [0,1], \quad x,y,z \in X, \quad \rho(y,z) \leq \varepsilon\} = 0.$$

Definition 7. [3] *Let* S *be a nonempty subset of a* $\Gamma-$*uniquely geodesic space. We say that* S *is* $\Gamma-$*convex if* $\gamma_{x,y} \subseteq S$ *for all* $x,y \in S$

Remark 2. [3] *Let* $(X, \|.\|)$ *be a normed linear space,* Γ_L *the family of linear segments and let* ρ *denote the metric induced by the norm* $\|.\|$. *Then* (X,ρ) *is a* Γ_L*-uniquely geodesic space with property (Q).*

Definition 8. [3] *A metric space* (X,ρ) *is a hyperbolic space (in the sense of Reich-Shafir* [19]*) if* X *is* $\Gamma-$ *uniquely geodesic and the following inequality holds*

$$\rho(\frac{1}{2}x \oplus \frac{1}{2}y, \frac{1}{2}x \oplus \frac{1}{2}z) \leq \frac{1}{2}\rho(y,z).$$

Many authors have developed various extensions, and modifications to Schauder's fixed point theorem [10, 15]. In [3], Schauder-type fixed point theorem in the setting of geodesic spaces with the property (Q) is proved as follows:

Theorem 1. [3] *Let* (X,ρ) *be a* $\Gamma-$ *uniquely geodesic space with property (Q), and all balls are* Γ*-convex. Let* K *be a nonempty, closed,* $\Gamma-$*convex subset of* (X,ρ). *Then, any continuous mapping* $T : K \to K$ *with compact range* $\overline{T(K)}$ *has at least one fixed point in* K.

Throughout this paper, we denote a closed $\Gamma-$convex subset of the geodesic space X by T and interior of S by int S, where S is a closed subset of T. We denote the set of all continuous, compact maps $\zeta : S \to T$ by $\mathscr{K}(S,T)$ and set of all maps $\zeta \in \mathscr{K}(S,T)$ with $x \neq \zeta(x)$ by $\mathscr{K}_{\partial S}(S,T)$ for $x \in \partial S$.

3 RESULTS

Theorem 2. [20] *Let* (X,ρ) *be a* Γ*-uniquely geodesic space satisfying property (Q) and* $\zeta, \xi \in \mathscr{K}_{\partial S}(S,T)$. *Suppose that for all* $(u,\tau) \in \partial S \times [0,1]$,

$$u \neq (1-\tau)\zeta(u) \oplus \tau\xi(u)$$

i.e., geodesic segment joining $\zeta(u)$ *and* $\xi(u)$ *does not contain* u. *Then* $\zeta \simeq \xi$ *in* $\mathscr{K}_{\partial S}(S,T)$.

Proof. Let $H(u,\tau) = H_t(u) = \bigoplus_\Gamma(\zeta(u),\xi(u),\tau) = (1-\tau)u \oplus \tau\xi u$. Clearly, H is continuous. We show that $H : S \times [0,1] \to T$ is a compact map.
Let $\{x_i\}$ be a sequence in S. Since $\zeta, \xi : S \to T$ are compact maps, $\zeta(x_i) \to u$ and $\xi(x_i) \to v$ as $i \to \infty$ for some subsequence $\mathbb{S}$ of natural numbers and $u, v \in T$. Let $\tau \in [0,1]$ be such that τ is the limit of some sequence $\tau_i \in [0,1]$. Now using Remark 1,

$$\begin{aligned}
\rho\big(H(x_i,\tau_i),(1-\tau)u \oplus \tau v\big) &= \rho\big(\bigoplus_\Gamma(\zeta(x_i),\xi(x_i),\tau_i),(1-\tau)u \oplus \tau v\big) \\
&\leq \rho\big((1-\tau_i)\zeta(x_i) \oplus \tau_i\xi(x_i),(1-\tau)\zeta(x_i) \oplus \tau\xi(x_i)\big) \\
&\quad + \rho\big((1-\tau)\zeta(x_i) \oplus \tau\xi(x_i),(1-\tau)u \oplus \tau v\big) \\
&\leq |\tau_i - \tau|\rho(\zeta(x_i),\xi(x_i)) \\
&\quad + \rho((1-\tau)\zeta(x_i) \oplus \tau\xi(x_i),(1-\tau)\zeta(x_i) \oplus \tau v) \\
&\quad + \rho((1-\tau)\zeta(x_i) \oplus \tau v,(1-\tau)u \oplus \tau v). \qquad (1)
\end{aligned}$$

Since $\tau_i \to \tau$, we get

$$|\tau_i - \tau|\rho(\zeta(x_i),\xi(x_i)) \to 0. \qquad (2)$$

Since $\xi(x_i) \to v$, we have $\rho(\xi(x_i),v) \leq \epsilon$ for every $\epsilon > 0$. Thus using property (Q), we get,

$$\rho\big((1-\tau)\zeta(x_i) \oplus \tau\xi(x_i),(1-\tau)\zeta(x_i) \oplus \tau v\big) \to 0. \qquad (3)$$

Similarly,

$$\rho\big((1-\tau)\zeta(x_i) \oplus \tau v,(1-\tau)u \oplus \tau v\big) \to 0. \qquad (4)$$

Using (2), (3) and (4) in (1), we have

$$\{H(x_i,\tau_i)\} \to (1-\tau)u \oplus \tau v, \quad \text{for } t_i \in [0,1]. \qquad (5)$$

Since T is $\Gamma-$convex, $(1-\tau)u \oplus \tau v \in T$. Hence H is compact.
But it is given that $u \neq (1-\tau)\zeta(u) \oplus \tau\xi(u)$ for $(u,\tau) \in \partial S \times [0,1]$. Hence $H_\tau(u) \in \mathscr{K}_{\partial S}(S,T)$, $H(u,0) = \zeta(u)$ and $H(u,1) = \xi(u)$. Therefore $\zeta \simeq \xi$ in $\mathscr{K}_{\partial S}(S,T)$.

Following result in [1] is a corollary to our theorem.

Corollary 1. [1, Theorem 6.1] *Let X be a Banach space, T a closed, convex subset of X, S a closed subset of T and $\zeta, \xi \in \mathscr{K}_{\partial S}(S,T)$. Suppose that for all $(u,\lambda) \in \partial S \times [0,1]$,*

$$u \neq (1-\lambda)\zeta(u) + \lambda\xi(u).$$

Then $\zeta \simeq \xi$ in $\mathscr{K}_{\partial S}(S,T)$.

Proof. Using Remark 2, X is a Γ_L-uniquely geodesic space. Hence by Theorem 2, the result follows.

Next result is a characterization of inessential maps in $\mathscr{K}_{\partial S}(S,T)$ in $\Gamma-$uniquely geodesic spaces.

Theorem 3. [20] *Let (X,ρ) be a Γ-uniquely geodesic space satisfying property (Q) and let $\zeta \in \mathscr{K}_{\partial S}(S,T)$. Then ζ is inessential in $\mathscr{K}_{\partial S}(S,T)$ if and only if there exists $\xi \in \mathscr{K}_{\partial S}(S,T)$ with $\xi(u) \neq u$ for all $u \in S$ and $\zeta \simeq \xi$ in $\mathscr{K}_{\partial S}(S,T)$.*

Proof. Assume that ζ is inessential in $\mathscr{K}_{\partial S}(S,T)$. Hence there exists a map $\xi \in \mathscr{K}_{\partial S}(S,T)$ such that $\xi(u) \neq u$ for all $u \in S$ and $\zeta|_{\partial S} = \xi|_{\partial S}$ by definition. Suppose there exists $(u,\tau) \in \partial S \times [0,1]$ such that $u = (1-\tau)\zeta(u) \oplus \tau\xi(u)$. Since $\zeta|_{\partial S} = \xi|_{\partial S}$, it follows that $u = \xi(u)$, which is a contradiction to the fact that $\xi \in \mathscr{K}_{\partial S}(S,T)$. Hence $u \neq (1-\tau)\zeta(u) \oplus \tau\xi(u)$, for each $(u,\tau) \in \partial S \times [0,1]$. Hence by using Theorem 2, we have $\zeta \simeq \xi$. Thus we have $\xi u \neq u$ and $\zeta \simeq \xi$.
Conversely assume that there exists $\xi \in \mathscr{K}_{\partial S}(S,T)$ with $\xi(u) \neq u$ for all $u \in S$ and $\zeta \simeq \xi$ in $\mathscr{K}_{\partial S}(S,T)$. Let $H : S \times [0,1] \to T$ with $H_t \in \mathscr{K}_{\partial S}(S,T)$ for all $\tau \in [0,1]$ be a continuous compact map with $H_0 = \zeta$ and $H_1 = \xi$. Consider

$$M = \Big\{u \in S : u = H(u,\tau) \text{ for some } \tau \in [0,1]\Big\}.$$

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There arise two cases.

Case-1: $M = \emptyset$.
If $M = \emptyset$ then $H_t(u) \neq u$ for all $u \in S$ and $\tau \in [0,1]$. In particular, $\zeta(u) = H_0(u) \neq u$ for all $u \in S$. Hence the inessentiality of ζ in $\mathscr{K}_{\partial S}(S,T)$ follows.

Case-2: $M \neq \emptyset$.
Let $\{x_i\} \in M$ such that $\{x_i\} \to u$. Then $x_i = H(x_i, \tau_i)$. Using the continuity of H, we get $u = H(u,\tau)$. Thus M is a closed subset of S.
Now, suppose $M \bigcap \partial S \neq \emptyset$. If $u \in M \bigcap \partial S$, then $u \in M$ and $u \in \partial S$, which implies $u = H_t(u)$ for $u \in \partial S$, a contradiction since $H_t \in \mathscr{K}_{\partial S}(S,T)$. Hence by the Urysohn's lemma, there exist $\xi : S \to [0,1]$ continuous, with $\xi(M) = 1$ and $\xi(\partial S) = 0$.
Define $f : S \to T$ by $f(u) = H(u, \xi(u))$. Clearly, f is a continuous compact map. If $u \in \partial S$, $f(u) = H(u,0) = \zeta(u)$. Thus $f|_{\partial S} = \zeta|_{\partial S}$.
If $u = f(u)$ for some $u \in S$ then, $u = f(u) = H(u,\xi(u))$. Thus $u \in M$ and hence $\xi(u) = 1$. Hence $u = f(u) = H(u,\xi(u)) = H(u,1) = \xi(u)$. Thus $f \in \mathscr{K}_{\partial S}(S,T)$ with $u = f(u)$ with $f|_{\partial S} = \zeta|_{\partial S}$. Therefore ξ has a fixed point, which is a contradiction to our assumption. Thus $u \neq f(u)$. Hence ζ is inessential in $\mathscr{K}_{\partial S}(S,T)$, by Definition 4. Hence the proof.

As a consequence of Theorem 3, we obtain the following corollary.

Corollary 2. *Let X be a Hyperbolic space (in the sense of [19]), T a closed, convex subset of X, S a closed subset of T and $\zeta \in \mathscr{K}_{\partial S}(S,T)$. Then the following are equivalent:*

(i) ζ is inessential in $\mathscr{K}_{\partial S}(S,T)$.

(ii) There exists $\xi \in \mathscr{K}_{\partial S}(S,T)$ with $\xi(u) \neq u$ for all $u \in S$ and $\zeta \simeq \xi$ in $\mathscr{K}_{\partial S}(S,T)$.

Proof. Every Hyperbolic space is a Γ-uniquely geodesic space with property (Q). Hence the result follows from Theorem 3.

Theorem 4. *[20] Let (X,ρ) be a Γ-uniquely geodesic space satisfying property (Q). Suppose that $\zeta, \xi \in \mathscr{K}_{\partial S}(S,T)$ with $\zeta \simeq \xi$ in $\mathscr{K}_{\partial S}(S,T)$. Then ζ is essential in $\mathscr{K}_{\partial S}(S,T)$ if and only if ξ is essential in $\mathscr{K}_{\partial S}(S,T)$.*

Proof. Suppose ζ is inessential in $\mathscr{K}_{\partial S}(S,T)$. Then from Theorem 3, there exists $T \in \mathscr{K}_{\partial S}(S,T)$ with $\zeta \simeq T$ in $\mathscr{K}_{\partial S}(S,T)$ such that $T(u) \neq u$ for all $u \in S$. Hence, $\xi \simeq T$ in $\mathscr{K}_{\partial S}(S,T)$. Thus $\zeta \simeq \xi$ and $\xi \simeq T$ implies $\xi \simeq T$ in $\mathscr{K}_{\partial S}(S,T)$. Therefore by Theorem 3, ξ is inessential in $\mathscr{K}_{\partial S}(S,T)$. Hence the proof.

Corollary 3. *[20] Let X be a Hyperbolic space (in the sense of [19]), S be a closed, convex subset of X, S be a closed subset of T and $\zeta, \xi \in \mathscr{K}_{\partial S}(S,T)$ with $\zeta \simeq \xi$ in $\mathscr{K}_{\partial S}(S,T)$. Then ζ is essential in $\mathscr{K}_{\partial S}(S,T)$ if and only if ξ is essential in $\mathscr{K}_{\partial S}(S,T)$.*

The above corollary is a special case of Theorem 4.

Theorem 5. *[20] Let (X,ρ) be a Γ-uniquely geodesic space with all balls are $\Gamma-$convex and satisfies property (Q). Let $w \in \operatorname{int} S$. Then the map $\zeta(S) = w$ is essential in $\mathscr{K}_{\partial S}(S,T)$.*

Proof. Consider the continuous compact map $\xi : S \to T$ which agrees with ζ on ∂S. It is enough to show that $\xi(u) = u$ for some $u \in \operatorname{int} S$. Let $\zeta : T \to T$ be given by

$$\zeta(u) = \begin{cases} \xi(u), & u \in S; \\ w, & u \in T \setminus S. \end{cases}$$

Then ζ is continuous and compact (since ξ and w are continuous and compact and on ∂S, $\zeta(u) = \xi(u) = w$). Thus by Theorem 1, $\zeta(u) = u$ for some $u \in S$. If $u \in T \setminus S$, we get $\zeta(u) = w$. Hence

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$\zeta(w) = w$, which is a contradiction, since $w \in \text{int}S$. Clearly $u \in \text{int } S$. Hence x is a fixed point of ξ. Therefore ζ is essential in $\mathscr{K}_{\partial S}(S, S)$.

The following result in [1] is a corollary to our theorem.

Corollary 4. [1, Theorem 6.5] *Let X be a Banach space, T a closed, convex subset of X, S a closed subset of T and $u \in$ int S. Then the constant map $\zeta(S) = w$ is essential in $\mathscr{K}_{\partial S}(S, T)$.*

Proof. We know that all balls in Banach spaces are convex. Hence the result follows from theorem 5.

As a consequence of the above theorems, authors proved the Leray Schauder principle in $\Gamma-$uniquely geodesic spaces in [20], which generalizes the Leray Schauder principle in Hyperbolic spaces proved in [3].

Theorem 6. [20] *Let (X, ρ) be a Γ-uniquely geodesic space satisfying property (Q), and all balls are $\Gamma-$convex. Suppose that $\zeta : S \to T$ is a continuous compact map. Then either*

(i) ζ has a fixed point in S, or

(ii) There exists $(x_0, \tau) \in \partial S \times [0, 1]$ such that $x_0 = (1 - \tau)u \oplus \tau\zeta(x_0)$.

Proof. Suppose that (ii) does not hold and $\zeta(u) \neq u$ for all $u \in \partial S$. Define $\xi : S \to T$ by $\xi(u) = w$ for all $u \in S$. Consider the map $H : S \times [0, 1] \to T$ defined by

$$H(u, \tau) := (1 - \tau)w \oplus \tau\zeta(u),$$

which is continuous and compact. Also, for all $u \in \partial S$, $H_t(u) = (1 - \tau)w \oplus \tau\zeta(u) \neq u$ for a fixed τ (since we assumed that condition (ii) does not hold). Hence by Theorem 2, $\zeta \simeq w$ in $\mathscr{K}_{\partial S}(S, T)$. But we know that w is essential in $\mathscr{K}_{\partial S}(S, T)$ by theorem 5. Hence ζ is essential in $\mathscr{K}_{\partial S}(S, T)$ by Theorem 4. Now using Definition 4, $\zeta(u) = u$ for some $u \in$ int S.

The following Theorem is a consequence of Theorem 6.

Corollary 5. [3, Theorem 23] *Let X be a Hyperbolic space(in the sense of [19]), $x_0 \in X$ and $r > 0$. Suppose that $T : B[x_0, r] \to X$ be a continuous mapping with $\overline{T(B[x_0, r])}$ compact. Then either*

(i) T has at least one fixed point in $B[x_0, r]$, or

(ii) There exists $(u, \lambda) \in \partial B[x_0, r] \times [0, 1]$ with $u = (1 - \lambda)x_0 \oplus \lambda T(u)$.

Proof. Hyperbolic spaces are geodesic spaces with property (Q), and balls are Γ-convex. Hence the result follows from the above theorem.

4 CONCLUSIONS

In this review article, we considered continuation results in Γ-uniquely geodesic spaces and the Leray-Schauder principle in $\Gamma-$uniquely geodesic spaces proved in [20]. Geodesic spaces having the property (Q) include Busemann spaces, linear spaces, $CAT(\kappa)$ spaces with diameters smaller than $D_k/2$, hyperbolic spaces (in the sense of [19], etc., and balls are Γ-convex in these spaces. All the results in this paper are thus applicable to these spaces as well. One can attempt to develop similar results for multivalued mappings in geodesic spaces and establish similar results for d-essential maps, d-L essential maps, and other general classes of maps [2, 18].

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