ESSENTIAL SPECTRUM OF DISCRETE LAPLACIAN - REVISITED

V. B. Kiran Kumar

Department of Mathematics, Cochin University of Science And Technology, Kochi, Kerala, (India).

E-mail:vbk@cusat.ac.in

ORCID:0000-0001-7643-4436

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ABSTRACT

Consider the discrete Laplacian operator A acting on $l^2(\mathbb{Z})$. It is well known from the classical literature that the essential spectrum of A is a compact interval. In this article, we give an elementary proof for this result, using the finite-dimensional truncations A_n of A. We do not rely on symbol analysis or any infinite-dimensional arguments. Instead, we consider the eigenvalue-sequences of the truncations A_n and make use of the filtration techniques due to Arveson. Usage of such techniques to the discrete Schrödinger operator and to the multi-dimensional settings will be interesting future problems.

KEYWORDS

Essential Spectrum, Discrete Laplacian

1 INTRODUCTION

In this short article, we consider the discrete Laplacian operator A defined on $l^2(\mathbb{Z})$, as follows:

$$A(x(n)) = (x(n-1) + x(n+1)); x = (x(n)) \in l^2(\mathbb{Z}), n \in \mathbb{Z}.$$

This operator arises naturally in many physical situations. For example, when we approximate a partial differential equation by finite differences, such bounded operators come into the picture. This operator is widely used in image processing, particularly in edge detection problems. There are extensions of the discrete Laplacian to various settings, such as multi-dimensional operator (on $\mathbb{Z}^n$) and Laplacian on graphs, etc. An operator close to this example is the discrete Schrödinger operator. This operator can be considered as a perturbation of the discrete Laplacian, defined as follows;

$$H(x(n)) = (x(n-1) + x(n+1) + v(n)x(n)); x = (x(n)) \in l^2(\mathbb{Z}), n \in \mathbb{Z}.$$

Here the sequence $v = v(n)$ is a bounded sequence called the potential.

It is well-known from the classical theory that the spectrum of A is the compact interval $[-2, 2]$. In this article, we use the filtration techniques developed by W. B. Arveson in [1] and some elementary method to give a simple proof of this result. We plan to use such techniques in the computation of the spectrum of the discrete Schrödinger operator. However, the spectrum of the discrete Schrödinger operator can be very complicated, depending on the potential function. For example, if you choose the almost Mathieu potential, the spectrum will be a Cantor-like set (The Ten-Martini Conjecture, see [2] for, eg.).

The article is organized as follows. In the next section, we describe some essential results from [1,3] in connection with the spectral approximation of an infinite-dimensional bounded self-adjoint operator. In the third section, we use these techniques to give an elementary proof of the connectedness of the essential spectrum of A. A possible application to the spectral computation of some special class of discrete Schrödinger operators is mentioned at the end of this article.

2 OPERATORS IN THE ARVESON'S CLASS

"How to approximate spectra of linear operators on separable Hilbert spaces?"is a fundamental question and was considered by many mathematicians. One of the successful methods is to use the finite-dimensional theory in the computation of the spectrum of bounded operators in an infinite dimensional space through an asymptotic way. In 1994, W.B. Arveson identified a class of operators for which the finite-dimensional truncations are helpful in the spectral approximation [1]. We introduce this class of operators here.

Let A be a bounded self-adjoint operator defined on a complex separable Hilbert space $\mathcal{H}$ and $\{e_1, e_2, \ldots\}$ be an orthonormal basis for $\mathcal{H}$. Consider the finite dimensional truncations of A, that is $A_n = P_n A P_n$, where P_n is the projection of $\mathcal{H}$ onto the span of first n elements $\{e_1, e_2, \ldots, e_n\}$ of the basis. We recall the notion of essential points and transient points introduced in [1].

Definition 1. *Essential point: A real number λ is an essential point of A, if for every open set U containing λ, $\lim_{n\to\infty} N_n(U) = \infty$, where $N_n(U)$ is the number of eigenvalues of A_n in U.*

Definition 2. *Transient point: A real number λ is a transient point of A if there is an open set U containing λ, such that $\sup N_n(U)$ with n varying on the set of all natural number is finite.*

Remark 3. *Note that a number can be neither transient nor essential.*

Denote $\Lambda = \{\lambda \in R; \lambda = \lim \lambda_n, \lambda_n \in \sigma(A_n)\}$ and Λ_e as the set of all essential points. The following spectral inclusion result for a bounded self-adjoint operator A is of high importance. Let $\sigma(A), \sigma_{ess}(A)$ denote the spectrum and essential spectrum of A respectively.

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Theorem 4. *[1] The spectrum of a bounded self-adjoint operator is a subset of the set of all limit points of the eigenvalue sequences of its truncations. Also, the essential spectrum is a subset of the set of all essential points. That is,*

$$\sigma(A) \subseteq \Lambda \subseteq [m, M] \quad \textit{and} \quad \sigma_{ess}(A) \subseteq \Lambda_e.$$

W.B Arveson, generalized the notion of band limited matrices in [1], and achieved some useful results in the case of some special class of operators. He used an arbitrary filtration H_n (an increasing subsequence of closed subspaces with the union dense in $\mathcal{H}$) and the sequence of orthogonal projections onto H_n to introduce his class of operators. Here we consider only a special case.

Definition 5. *The **degree** of a bounded operator A on $\mathbb{H}$ is defined by*

$$deg(A) = \sup_{n\geq 1} \ rank(P_nA - AP_n).$$

A Banach $*-$algebra of operators can be defined, which we call **A**rveson's class, as follows.

Definition 6. *A is an operator in the **Arveson's class** if*

$$A = \sum_{n=1}^{\infty} A_n, \quad \textit{where } deg(A_n) < \infty \textit{ for every } n \textit{ and convergence is in the}$$

operator norm, in such a way that

$$\sum_{n=1}^{\infty}(1 + \ deg(A_n)^{\frac{1}{2}})\|A_n\| < \infty$$

The following gives a concrete description of operators in Arveson's class.

Theorem 7. *[1] Let $(a_{i,j})$ be the matrix representation of a bounded operator A, with respect to $\{e_n\}$, and for every $k \in \mathbb{Z}$ let*

$$d_k = \sup_{i\in\mathbb{Z}} |a_{i+k,i}|$$

be the sup norm of the k^{th} diagonal of $(a_{i,j})$. Then A will be in Arveson's class whenever the series $\sum_k |k|^{1/2} d_k$ converges.

Remark 8. *In particular, any operator whose matrix representation $(a_{i,j})$ is band-limited, in the sense that $a_{i,j} = 0$ whenever $|i - j|$ is sufficiently large, must be in Arveson's class. Therefore, the operator under our consideration is in Arveson's class, as we see that its matrix representation is tridiagonal.*

The following result allows us to confine our attention to essential points while looking for essential spectral values for certain classes of operators.

Theorem 9. *[1] If A is a bounded self-adjoint operator in the Arveson's class, then $\sigma_{ess}(A) = \Lambda_e$ and every point in Λ is either transient or essential.*

3 SPECTRUM OF DISCRETE LAPLACIAN

Consider the discrete Laplacian operator A defined on $l^2(\mathbb{Z})$, as follows:

$$A(x(n)) = (x(n-1) + x(n+1)); x = (x(n)) \in l^2(\mathbb{Z}), n \in \mathbb{Z}.$$

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If we use the standard orthonormal basis on $l^2(\mathbb{Z})$, the truncations A_n will have the following matrix representations:

$$A_n = \begin{bmatrix} 0 & 1 & 0 & 0 & & . & & & & \\ 1 & 0 & 1 & 0 & 0 & & . & & & \\ 0 & 1 & 0 & 1 & 0 & 0 & & . & & \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & & . & \\ & 0 & 0 & 1 & 0 & 1 & 0 & 0 & & . \\ . & & 0 & 0 & 1 & 0 & 1 & 0 & 0 & \\ & . & & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ & & . & & 0 & 0 & 1 & 0 & 1 & 0 \\ & & & . & & 0 & 0 & 1 & 0 & 1 \\ & & & & . & & 0 & 0 & 1 & 0 \end{bmatrix}$$

Now we recollect some properties of the discrete Laplacian operator below.

Lemma 1. *$\lambda \in \sigma_{ess}(A)$ if and only if $-\lambda \in \sigma_{ess}(A)$.*

Proof. Notice that this operator A is in Arveson's class, introduced in the last section. Therefore, we have $\lambda \in \sigma_{ess}(A)$ if and only if λ is an essential point. That is exactly when $N_n(U) \to \infty$ for every neighborhood U of λ. The characteristic polynomials of A_n are $P_n(z) = z^n - a_{n-2}z^{n-2} + \ldots \pm 1$, when n is even and $P_n(z) = -z^n + a_{n-2}z^{n-2} - \ldots \pm a_1 z$, when n is odd. Here the coefficients can be computed as follows. $a_k = \frac{(n-k+2)(n-k+4)\ldots(n+k)}{2^k.k!}$

Therefore the eigenvalues of A_n are distributed symmetrically on both sides of 0 in the interval $[-2,2]$. Hence the number of truncated eigenvalues in any neighbourhood of $-\lambda$ and λ are the same if the neighbourhoods are of the same length. We can conclude that $\lambda \in \sigma_{ess}(A)$ if and only if $-\lambda \in \sigma_{ess}(A)$.

Lemma 2. *The operator norm $\|A\| = 2$ and $\pm 2 \in \sigma_{ess}(A)$.*

Proof. For every $x \in l^2(\mathbb{Z})$, we have

$$\|Ax\|^2 = \sum_{-\infty}^{\infty} (x(n-1) + x(n+1))^2 = \sum_{-\infty}^{\infty} \Big[(x(n-1))^2 + (x(n+1))^2 + 2x(n-1)x(n+1)\Big] \leq 4\|x\|^2.$$

Therefore, $\|A\| \leq 2$.

To prove the equality, consider the sequence $x_n = \left(\ldots 0,0,0,\ldots\frac{1}{n},\frac{1}{n},\frac{1}{n}\ldots\frac{1}{n},0,0,0\ldots\right)$, where $\frac{1}{n}$ repeats n^2 times and all other entries are 0. Then x_n has norm 1 and $\|Ax_n\|$ increases to 2. Hence $\|A\| = 2$. Since A is a bounded self-adjoint operator, either $\|A\|$ or $-\|A\|$ is always a spectral value. That is 2 or -2 is in the spectrum, $\sigma(A)$. However, they are not eigenvalues of A, as we see below.

If ± 2 is an eigenvalue of A, then 4 is an eigenvalue of $A^2 = B + 2I$ where B is defined by $B(e_n) = e_{n-2} + e_{n+2}$. (Observe that A^2 is defined as $A^2(e_n) = e_{n-2} + 2e_n + e_{n+2}$). This will imply that 2 is eigenvalue of B. If $Bx = 2x$, for some nonzero x, then $x(n+2) + x(n-2) = 2x(n)$, for all n. Let $x(N) = p \neq 0$, for some N and $x(N-2) = q$. Then $x(n+2k) = (k+1)p - kq$ for every $k\mathbb{Z}$. Such an element x will not be in $l^2(Z)$.

Hence ± 2 is an essential spectral value. By Lemma 1, both ± 2 are in the essential spectrum, $\sigma_{ess}(A)$. Therefore both 2 and -2 are in $\sigma_{ess}(A)$.

Theorem 10. *The spectral gaps of A, if they exist, will appear symmetrically with respect to the origin. That means corresponding to each spectral gap on the positive real axis, there exists a spectral gap on the negative real axis. In particular, A cannot have an odd number of spectral gaps.*

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Proof. We noticed that $A^2 = B + 2I$ where B is defined by $B(e_n) = e_{n-2} + e_{n+2}$. It is worthwhile to notice further that $\|B\| = 2$. This follows easily from the following arguments.

$$\|Bx\|^2 = \sum_{-\infty}^{\infty} (x(n-2) + x(n+2))^2 = \sum_{-\infty}^{\infty} \left[(x(n-2))^2 + (x(n+2))^2 + 2x(n-2)x(n+2) \right] \leq 4\|x\|^2.$$

Therefore we have $\|B\| \leq 2$. Now consider the sequence $x_n = \left(...0, 0, 0, ...\frac{1}{n}, \frac{1}{n}, \frac{1}{n}...\frac{1}{n}, 0, 0, 0...\right)$ where $\frac{1}{n}$ repeats n^2 times and all other entries 0, has norm 1 and $\|Bx_n\|$ increases to 2. Hence $\|B\| = 2$. Also as in the case of A, the truncated eigenvalues are distributed symmetrically on both sides of 0, so that -2 and 2 are in the essential spectrum. Also, since $A^2 = B + 2I$, this implies that 0 is an essential spectral value of A^2, and hence of A. Hence any spectral gap of A can occur either to the right or left side of 0. By Lemma 1, each spectral gap on the right side of 0 will also give a spectral gap on the left side. Hence the proof.

Theorem 11. *The essential spectrum of A is connected. The spectrum and the essential spectrum coincide with the compact interval* $[-2, 2]$.

Proof. First, we show that A has no eigenvalues. This will imply that the spectrum and essential spectrum coincide, as the essential spectrum consists of discrete eigenvalues of finite multiplicity. If $Ax = \lambda x$, for some nonzero x, then

$$x(n+1) + x(n-1) = \lambda x(n), \text{ for all } n.$$

Let $x(N) = p \neq 0$, for some N and $x(N-1) = q$. Then a recursive argument similar to that in the proof of Lemma 2 will show that such a vector will not lie in $l^2(Z)$.

By Theorem 10, spectral gaps can occur symmetrically to the origin. Hence it suffices to show that there is no spectral gap to the right side of the origin. We consider each possible case for the existence of a spectral gap. We rule them out one by one. First consider the case when the spectral gap is of the form $(a, 2)$, with $0 \leq a \leq 1$. In this case, since the interval $(a, 2)$ contains no essential points (as the essential spectrum coincides with the set of all essential points), it will contain at most K eigenvalues of truncations for infinitely many n. Let $\lambda_{n1}, \lambda_{n2}, \lambda_{n3}, \ldots \lambda_{nK}$ be those eigenvalues. From the expression of characteristic polynomials, it is evident that the determinant of $A_n's$ is either 0 or ± 1. Since the eigenvalues are distributed symmetrically to both sides of 0, we have the product of positive eigenvalues equal 1 for n even. Let $s_K := \prod_{i=1}^{K} \lambda_{ni}$. Then $\frac{1}{s_K} > \frac{1}{2^K}$, since $\lambda_{ni} < 2$ for $i = 1, 2, \ldots K$.

But since 0 is in the essential spectrum, it is an essential point, and we can find an N such that the interval $\left(0, \frac{1}{2}\right)$ contains at least $K+1$ eigenvalues of A_n for every $n \geq N$. For such an $n \geq N$, let $\alpha_{n1}, \alpha_{n2}, \alpha_{n3}, \ldots \alpha_{nN-K}$ be the eigenvalues of A_n, in $(0, a)$. Therefore we have,

$$\frac{1}{2^K} < \frac{1}{s_K} = \prod_{i=1}^{N-K} \alpha_{ni} < \prod_{i=1}^{K+1} \alpha_{ni} < \frac{1}{2^{K+1}} < \frac{1}{2^K}.$$

The first equality holds since the product of eigenvalues is 1, and the consequent inequality is because $a \leq 1$ (each additional α_{ni} we multiply will be a positive number below 1. Hence, the product will satisfy this inequality). This contradiction leads to the fact that $(a, 2)$, with $0 \leq a \leq 1$ is not a spectral gap.

Now we see that $(a, 2)$, with $a > 1$ cannot be a spectral gap. For if $(a, 2), a > 1$ is a gap, then it will contain at most K eigenvalues of truncations for infinitely many n. Let $\lambda_{n1}, \lambda_{n2}, \lambda_{n3}, \ldots \lambda_{nK}$ be those eigenvalues. As in the above case, let $s_K := \prod_{i=1}^{K} \lambda_{ni}$. Then $\frac{1}{s_K} > \frac{1}{2^K}$ Find an N such that the interval $\left(0, \frac{1}{2a^N}\right)$ contains at least $K+1$ eigenvalues of A_N. Now let $\alpha_{n1}, \alpha_{n2}, \alpha_{n3}, \ldots \alpha_{nN-K}$ be the eigenvalues of A_N, in $(0, a)$. Therefore we have

$$\frac{1}{s_k} = \prod_{i=1}^{N-K} \alpha_{ni} < \left(\frac{1}{2a^N}\right)^{K+1} a^{N-(2K+1)} < \left(\frac{1}{2}\right)^{K+1} < \frac{1}{2^K}.$$

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The inequality is a consequence of $a > 1$. This contradiction leads to the fact that $(a, 2), a > 1$ is not a gap.

Hence we have seen that there cannot have a spectral gap of the form $(a, 2)$, with a being a non-negative real number. The number 2 does not play any role in the proof. We can easily imitate the proof techniques for intervals of the form (a, b), with $0 \leq a < b \leq 2$.

Remark 12. *We can have a different and straight forward argument to show that* $(a, 2)$ *cannot be a spectral gap. However, this method cannot be extended for arbitrary intervals* (a, b), *with* $0 \leq a < b \leq 2$. *For if* $(a, 2)$ *is a gap, then* 2 *will be an isolated point in the essential spectrum. Since the interval* $(a, 2)$ *contains at most* K *eigenvalues of truncations, but* 2 *is an essential point, we need* 2 *is an eigenvalue of* A_n *for large values of* n *with multiplicity increases to infinity. Nevertheless,* 2 *is not an eigenvalue of* A_n *for any* n. *For if*

$$\begin{bmatrix} 0 & 1 & & & & & & & & \\ 1 & 0 & 1 & & & & & & & \\ & 1 & 0 & 1 & & & & & & \\ & & 1 & 0 & 1 & & & & & \\ & & & 1 & 0 & 1 & & & & \\ & & & & . & . & . & & & \\ & & & & & . & . & . & & \\ & & & & & & . & . & . & \\ & & & & & & & 1 & 0 & 1 \\ & & & & & & & & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ . \\ . \\ . \\ . \\ \\ x_n \end{bmatrix} = 2 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ . \\ . \\ . \\ . \\ \\ x_n \end{bmatrix}$$

then

$x_2 = 2x_1$

$x_3 = 3x_1$

$x_4 = 4x_1, \ldots x_{n-1} = (n-1)x_1, x_n = nx_1$

Also $x_{n-1} = 2x_n = 2nx_1$

But this will hold only when $x_1 = 0$ *which makes* $x = 0$ *and hence 2 is not an eigenvalue. Hence we conclude that* $(a, 2)$ *is not a gap.*

Remark 13. *The eigenvalues of the matrices* A_n *are explicitly calculated to be* $2cos(\pi k/n + 1), k = 1, 2, 3...n$. *We may arrive at some conclusions from that information also.*

Remark 14. *An important question is whether we can approximate the eigenvalues of* A *using the eigenvalues of truncation. Since the operator we considered has no eigenvalues, it is interesting to see from the truncations itself whether the operator has an eigenvalue or not. In general, we observe the following; If* $\lambda = \lim \lambda_n, \lambda_n \in \sigma(A_n)$ *and if the sequence of eigenvectors* x_n *corresponding to* λ_n, *is Cauchy in* $\mathcal{H}$, *then* λ *is an eigenvalue of* A. *This can easily be seen as follows.*

Let x_n *converges to some* x *in* $\mathcal{H}$. $\lambda = \lim \lambda_n, x = \lim x_n$ *together imply* $\lambda x = \lim \lambda_n x_n$ *Also* $\lim A_n x_n = Ax$. *Hence for any* $\epsilon > 0$, *there is an* N *such that* $\|\lambda - \lambda_N x_N\| < \frac{\varepsilon}{2}, \|Ax - A_N x_N\| < \frac{\varepsilon}{2}$ *Hence for any* $\epsilon > 0, \|Ax - \lambda x\| < \varepsilon$. *That is* λ *is an eigenvalue of* A.

4 CONCLUDING REMARKS

We used only elementary tools and the filtration techniques due to Arveson to prove the connectedness of the essential spectrum. When we consider the Discrete Schrödinger operator H defined by

$$H(x(n)) = (x(n-1) + x(n+1) + v(n)x(n)); x = (x(n)) \in l^2(\mathbb{Z}), n \in \mathbb{Z},$$

with the potential sequence $v = v(n)$ being periodic, there will be spectral gaps unless when v is constant (see [4, 5] for example). However, if we write the matrix representation with respect to the standard orthonormal basis, it is tridiagonal; hence, Arveson's techniques are available. Here the spectral analysis depends on the nature of the potential. The spectral gap issues of such operators were studied with the linear algebraic techniques in [6]. The spectral gap issues of arbitrary bounded self-adjoint operators can be found in the literature (see [7, 8] for example).

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Another interesting point is to carry over such techniques to the multi-dimensional case by replacing $\mathbb{Z}$ by $\mathbb{Z}^n$.

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