LIMIT CYCLES OF PERTURBED GLOBAL ISOCHRONOUS CENTER

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ABSTRACT

We apply the averaging method of first order to study the maximum number of limit cycles of the ordinary differential systems of the form

$$\begin{cases} \ddot{x} + x = \varepsilon\left(f_1(x,y)y + f_2\left(x,y\right)\right), \\ \ddot{y} + y = \varepsilon\left(g_1(x,y)x + g_2\left(x,y\right)\right), \end{cases}$$

where $f_1(x,y)$ and $g_1(x,y)$ are real cubic polynomials; $f_2(x,y)$ and $g_2(x,y)$ are real quadratic polynomials. Furthermore ε is a small parameter.

KEYWORDS

Limit Cycles, Averaging Method, Ordinary Differential Systems

1 INTRODUCTION AND STATEMENT OF THE MAIN RESULT

At the Paris International Congress of Mathematics in 1900, Hilbert presented twenty-three problems in mathematics. Some problems are still unsolved so far, they were a challenge for all mathematicians of that era. The second part of the well-known Hilbert's 16th problem is to find the maximum number of limit cycles and their position for an ordinary differential planar system of degree n of the form

$$\begin{cases} \dot{x} = \psi(x, y), \\ \dot{y} = \eta(x, y), \end{cases} \tag{1}$$

where n is a positive integer, the dot above the variables represents the first derivative with respect to the variable t, $\psi(x, y)$ and $\eta(x, y)$ are real polynomials, see for instance [13, 14, 17]. This problem has so far remained unresolved, for $n \geq 2$. Let us denote by $\mathcal{H}(n)$ the maximum number of limit cycles of differential system (1) which is usually called a Hilbert number. For example, Chen and Wang in [3], Shi in [22] gave the best result up to now about the lower bounds of $\mathcal{H}(2)$, which is $\mathcal{H}(2) \geq 4$. Li, Liu, and Yang in [15] proved $\mathcal{H}(3) \geq 13$, Li and Li in [16] proved $\mathcal{H}(3) \geq 11$, also Han, Wu and Bi in [10] and Han, Zhang and Zang in [12] proved $\mathcal{H}(3) \geq 11$. For more results about the Hilbert number, see, for example, the paper [9] and the references therein.

There are many papers that studied number of limit cycles using several methods, including the Poincaré–Melnikov integrals, see for instance [11]; the Poincaré return map, see for example [1]; the Abelian integrals, see for example [4]; the averaging method, see [5, 6]; the inverse integrating factor, see for instance [8].

In [19], Llibre and Teixeira used the averaging method of first-order for study the existence of limit cycles of the system of second-order differential equations

$$\begin{cases} \ddot{x} + x = \varepsilon f(x, y), \\ \ddot{y} + y = \varepsilon g(x, y), \end{cases}$$

where $f(x, y)$ and $g(x, y)$ are real cubic polynomials and ε is a small parameter.

In this paper, we apply the averaging method of first-order for study the existence of limit cycles of system of second-order differential equations

$$\begin{cases} \ddot{x} + x = \varepsilon \left(f_1(x, y)y + f_2(x, y)\right), \\ \ddot{y} + y = \varepsilon \left(g_1(x, y)x + g_2(x, y)\right), \end{cases} \tag{2}$$

where $f_1(x, y)$ and $g_1(x, y)$ are real cubic polynomials, $f_2(x, y)$ and $g_2(x, y)$ are real quadratic polynomials such that $f_i(0, 0) = g_i(0, 0) = 0$, for $i = 1, 2$, and ε is a small parameter. These polynomials are expressions of the form

$$\begin{aligned} f_1(x, y) &= a_1x + a_2y + a_3x^2 + a_4xy + a_5y^2 + a_6x^3 + a_7x^2y + a_8xy^2 + a_9y^3, \\ f_2(x, y) &= A_1x + A_2y + A_3x^2 + A_4xy + A_5y^2, \\ g_1(x, y) &= b_1x + b_2y + b_3x^2 + b_4xy + b_5y^2 + b_6x^3 + b_7x^2y + b_8xy^2 + b_9y^3, \\ g_2(x, y) &= B_1x + B_2y + B_3x^2 + B_4xy + B_5y^2. \end{aligned}$$

The system of second order differential equations (2) can be expressed as the following system of first-order differential equations in the usual way:

$$\begin{cases} \dot{x} = u, \\ \dot{u} = -x + \varepsilon \left(f_1(x, y)y + f_2(x, y)\right), \\ \dot{y} = v, \\ \dot{v} = -y + \varepsilon \left(g_1(x, y)x + g_2(x, y)\right). \end{cases} \tag{3}$$

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Note that system (3) when $\varepsilon = 0$, is

$$\begin{cases} \dot{x} = u, \\ \dot{u} = -x, \\ \dot{y} = v, \\ \dot{v} = -y. \end{cases} \tag{4}$$

We notice that the system (4) have a global isochronous center at the the origin, i.e. all orbits different from the origin are 2π-periodic, for more detail see [18].

The main result of our work is the following.

Theorem 1. *Using the first-order averaging method, system of second-order differential equations* (3)*, where $f_1(x, y)$ and $g_1(x, y)$ are real cubic polynomials; $f_2(x, y)$ and $g_2(x, y)$ are real quadratic polynomials, has at most four limit cycles bifurcating from the periodic orbits of the linear center $\dot{x} = u, \dot{u} = -x, \dot{y} = v, \dot{v} = -y$. Here ε is a small parameter. Moreover if $a_4 = 0$ and $a_3 = 0$, the system* (3) *has at most two periodic solutions.*

The first-order averaging method theory, that we summarize in the sequel, can be found in a more extended way in [2]. Similar works where the perturbations via polynomials play an important role are for instance [7] and [20].

2 THE FIRST-ORDER AVERAGING METHOD FOR COMPUTING PERIODIC ORBITS

Theorem 2. *We consider the following two problems*

$$\dot{x}(t) = \varepsilon F(t, x(t)) + \varepsilon^2 R(t, x(t), \varepsilon), \tag{5}$$

and

$$\dot{y}(t) = \varepsilon \mathrm{f}(y(t)), \tag{6}$$

where $t \in [0, +\infty)$, x and y in some open D of $\mathbb{R}^n$ and $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ is a small parameter. Moreover, we suppose that the vector functions $F(t, x)$ and $R(t, x, \varepsilon)$ are $T-$periodic in the first variable and we consider the first-order averaging function

$$\mathrm{f}(y) = \frac{1}{T}\int_0^T F(s, y)ds.$$

Suppose that F, R, $\mathrm{D}_x F$, and $\mathrm{D}_x^2 F$ are continuous and bounded by a constant M in $[0, \infty) \times D \times (-\varepsilon_0, \varepsilon_0)$ where M is independent of ε. Then, the statements (I) and (II) satisfied:

(I) *If there exists an equilibrium point $\alpha \in D$ of* (6) *such that*

$$\det \left.\frac{\partial\left(\mathrm{f}\left(y\right)\right)}{\partial y}\right|_{y=p} \neq 0,$$

then, for $\varepsilon > 0$ sufficiently small, there exists an isolated $T-$periodic solution $\phi_\varepsilon(t)$ of system (5) *such that $\phi_\varepsilon(t) \to 0$ when $\varepsilon \to 0$.*

(II) *If $y = \alpha$ (the equilibrium point) of* (6) *is hyperbolic. Then, for $\varepsilon > 0$ sufficiently small, the corresponding periodic solution of system* (5) *is unique, hyperbolic and of the same stability type as α.*

The proof of this theorem can be seen in [21, 23].

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3 PROOF OF THEOREM 1

In this work, we consider $\rho > 0$, $s > 0$ and writing differential system (3) in the new variables $(\theta, \rho, s, \omega)$ given by

$$\begin{aligned} x &= \rho\cos(\theta), \\ u &= \rho\sin(\theta), \\ y &= s\cos(\theta+\omega), \\ v &= s\sin(\theta+\omega). \end{aligned}$$

we get

$$\begin{cases} \dot{\theta} = -1 + \varepsilon \mathrm{G}(\theta, \rho, s, \omega), \\ \dot{\rho} = \varepsilon \mathrm{F}_1(\theta, \rho, s, \omega), \\ \dot{s} = \varepsilon \mathrm{F}_2(\theta, \rho, s, \omega), \\ \dot{\omega} = \varepsilon \mathrm{F}_3(\theta, \rho, s, \omega), \end{cases}$$

where

$$\begin{aligned} \mathrm{G}(\theta, \rho, s, \omega) =& \frac{1}{\rho}\cos(\theta)\Big[A_1\rho\cos(\theta) + A_3\rho^2\cos^2(\theta) \\ & + s\cos(\theta+\omega)\Big(A_2 + (a_1 + A_4)\rho\cos(\theta) + a_3\rho^2\cos^2(\theta) + a_6\rho^3\cos^3(\theta)\Big) \\ & + s^2\cos^2(\theta+\omega)\Big(a_2 + A_5 + a_4\rho\cos(\theta) + a_7\rho^2\cos^2(\theta)\Big) \\ & + s^3\cos^3(\theta+\omega)\Big(a_5 + a_8\rho\cos(\theta)\Big) \\ & + s^4\cos^4(\theta+\omega)a_9\Big] \end{aligned}$$

$$\begin{aligned} \mathrm{F}_1(\theta, \rho, s, \omega) =& \sin(\theta)\Big[A_1\rho\cos(\theta) + A_3\rho^2\cos^2(\theta) \\ & + s\cos(\theta+\omega)\Big(A_2 + (a_1 + A_4)\rho\cos(\theta) + a_3\rho^2\cos^2(\theta) + a_6\rho^3\cos^3(\theta)\Big) \\ & + s^2\cos^2(\theta+\omega)\Big(a_2 + A_5 + a_4\rho\cos(\theta) + a_7\rho^2\cos^2(\theta)\Big) \\ & + s^3\cos^3(\theta+\omega)\Big(a_5 + a_8\rho\cos(\theta)\Big) \\ & + s^4\cos^4(\theta+\omega)a_9\Big] \end{aligned}$$

$$\begin{aligned} \mathrm{F}_2(\theta, \rho, s, \omega) =& \sin(\theta+\omega)\Big[B_1\rho\cos(\theta) + (b_1 + B_3)\rho^2\cos^2(\theta) + b_3\rho^3\cos^3(\theta) + b_6\rho^4\cos^4(\theta) \\ & + s\cos(\theta+\omega)\Big(B_2 + (b_2 + B_4)\rho\cos(\theta) + b_4\rho^2\cos^2(\theta) + b_7\rho^3\cos^3(\theta)\Big) \\ & + s^2\cos^2(\theta+\omega)\Big(B_5 + b_5\rho\cos(\theta) + b_8\rho^2\cos^2(\theta)\Big) \\ & + s^3\cos^3(\theta+\omega)b_9\rho\cos(\theta)\Big] \end{aligned}$$

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$$
\begin{aligned}
\mathrm{F}_3(\theta,\rho,s,\omega) = -\frac{\cos(\theta)}{\rho}\Big[& A_1\rho\cos(\theta) + A_3\rho^2\cos^2(\theta) \\
& + s\cos(\theta+\omega)\Big(A_2 + (a_1+A_4)\rho\cos(\theta) + a_3\rho^2\cos^2(\theta) + a_6\rho^3\cos^3(\theta)\Big) \\
& + s^2\cos^2(\theta+\omega)\Big(a_2 + A_5 + a_4\rho\cos(\theta) + a_7\rho^2\cos^2(\theta)\Big) \\
& + s^3\cos^3(\theta+\omega)\Big(a_5 + a_8\rho\cos(\theta)\Big) \\
& + s^4\cos^4(\theta+\omega)a_9\Big] \\
+\frac{\cos(\theta+\omega)}{s}\Big[& B_1\rho\cos(\theta) + (b_1+B_3)\rho^2\cos^2(\theta) + b_3\rho^3\cos^3(\theta) + b_6\rho^4\cos^4(\theta) \\
& + s\cos(\theta+\omega)\Big(B_2 + (b_2+B_4)\rho\cos(\theta) \\
& \qquad + b_4\rho^2\cos^2(\theta) + b_7\rho^3\cos^3(\theta)\Big) \\
& + s^2\cos^2(\theta+\omega)\Big(B_5 + b_5\rho\cos(\theta) + b_8\rho^2\cos^2(\theta)\Big) \\
& + s^3\cos^3(\theta+\omega)b_9\rho\cos(\theta)\Big]
\end{aligned}
$$

The previous differential system in the new independent variable θ becomes as follows

$$
\begin{cases}
\dfrac{d\rho}{d\theta} = -\varepsilon\mathrm{F}_1(\theta,\rho,s,\omega) + \varepsilon^2\mathrm{R}_1(\theta,\rho,s,\omega,\varepsilon), \\
\dfrac{ds}{d\theta} = -\varepsilon\mathrm{F}_2(\theta,\rho,s,\omega) + \varepsilon^2\mathrm{R}_2(\theta,\rho,s,\omega,\varepsilon), \\
\dfrac{d\omega}{d\theta} = -\varepsilon\mathrm{F}_3(\theta,\rho,s,\omega) + \varepsilon^2\mathrm{R}_3(\theta,\rho,s,\omega,\varepsilon),
\end{cases} \tag{7}
$$

We observe that this system is into the normal form of the averaging method (5), with $t = \theta$ and $x = (\rho, s, \omega)$ that the all assumptions of the Theorem 2 are satisfied for the system (7). We compute the average functions of the first-order associated with the system (7)

$$
\mathrm{f}_i(\rho,s,\omega) = \frac{1}{2\pi}\int_0^{2\pi}\mathrm{F}_i(\theta,\rho,s,\omega)d\theta,
$$

for $i = 1, 2, 3$, we obtain

$$
\begin{cases}
\mathrm{f}_1(\rho,s,\omega) = \dfrac{s\sin(\omega)}{8}\Big(3\,s^2a_5 + 4\,A_2 + 2\,sa_4\rho\,\cos(\omega) + a_3\rho^2\Big), \\
\mathrm{f}_2(\rho,s,\omega) = -\dfrac{\rho\sin(\omega)}{8}\Big(2\,\rho\,b_4s\cos(\omega) + 4\,B_1 + 3\,\rho^2b_3 + b_5s^2\Big), \\
\mathrm{f}_3(\rho,s,\omega) = -\dfrac{1}{8\rho s}\Big(-2\,s^3a_4\rho\,\cos^2(\omega) + \rho^3b_4s - 3\,s^4a_5\cos(\omega) \\
\qquad + 4\,\rho^2B_1\cos(\omega) + 2\,\rho^3b_4s\cos^2(\omega) - 4\,s^2A_2\cos(\omega) \\
\qquad + 3\,\rho^2b_5s^2\cos(\omega) - 3\,s^2a_3\rho^2\cos(\omega) + 3\,\rho^4b_3\cos(\omega) \\
\qquad + 4\,\rho\,B_2s - 4\,sA_1\rho - s^3a_4\rho\Big).
\end{cases}
$$

If (ρ_0, s_0, ω_0) is a zero of the system

$$
\mathrm{f}_i(\rho,s,\omega) = 0, \ \text{for } i = 1,2,3, \tag{8}
$$

such that

$$
\det\left.\frac{\partial(\mathrm{f}_1,\mathrm{f}_2,\mathrm{f}_3)}{\partial(\rho,s,\omega)}\right|_{(\rho_0,s_0,\omega_0)} \neq 0, \tag{9}
$$

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then Theorem 2 assures that the system (3) has a periodic solutions. So, in particular a zero of (8) must be isolated in the set of all zeros of (8). We note that the zeros of (8) having $\sin(\omega) = 0$ are non-isolated, so we cannot apply to them the averaging theory for obtaining limit cycles. Moreover, since the differential system (7) is only well defined when $s > 0$ and $\rho > 0$, in the rest of this section we will assume that $\rho > 0$, $s > 0$ and $\sin(\omega) \neq 0$, and consequently we can restrict to look for the zeros of

$$\begin{cases} \xi_1(\rho, s, \omega) = 0, \\ \xi_2(\rho, s, \omega) = 0, \\ \xi_3(\rho, s, \omega) = 0, \end{cases} \tag{10}$$

satisfying (9), where

$$\begin{cases} \xi_1(\rho, s, \omega) = \dfrac{8\mathrm{f}_1}{s\sin(\omega)}, \\ \xi_2(\rho, s, \omega) = \dfrac{-8\mathrm{f}_2}{\rho\sin(\omega)}, \\ \xi_3(\rho, s, \omega) = -8\rho s\mathrm{f}_3. \end{cases}$$

The rest of the proof of Theorem 1 is divided into the following cases and subcases.

Case 1 If $a_4 \neq 0$. Then, by solving the first equation $\xi_1 = 0$ with respect to $\cos(\omega)$ we get

$$\cos(\omega) = -\frac{a_3\rho^2 + 4\,A_2 + 3\,s^2 a_5}{2sa_4\rho}.$$

Substituting the expression of $\cos(\omega)$ in the second equation, $\xi_2 = 0$, we obtain

$$-b_4a_3\rho^2 - 4\,b_4A_2 - 3\,b_4s^2a_5 + 4\,B_1a_4 + 3\,b_3\rho^2a_4 + b_5s^2a_4 = 0. \tag{11}$$

Subcase 1.1 If $b_4a_3 - 3\,b_3a_4 \neq 0$. Then, from (11) we get

$$\rho = \sqrt{\frac{4\,B_1a_4 - 4\,b_4A_2 - 3\,b_4s^2a_5 + b_5s^2a_4}{b_4a_3 - 3\,b_3a_4}}.$$

Substituting the expressions of ρ and $\cos(\omega)$ in the third equation, $\xi_3 = 0$, we obtain an equation of the form

$$\frac{s(\mathbf{A} + \mathbf{B}s^2 + \mathbf{C}s^4)}{\sqrt{(b_4a_3 - 3b_3a_4)^2\big((b_5a_4 - 3b_4a_5)s^2 - 4b_4A_2 + 4B_1a_4\big)}} = 0,$$

where $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are constants. From now onwards, we are going to denote by $\mathbf{A}, \mathbf{B}, \mathbf{C}$ this kind of generic constants. As $\mathbf{A} + \mathbf{B}s^2 + \mathbf{C}s^4$ is a quadratic polynomial in s^2, which can have at most two positive solutions for s, in this subcase we get two values for ρ, s and $\cos(\omega)$. Observe that each value of (ρ, s) provides at most two solutions for ω. Hence, assuming that in these four solutions the determinant (9) is not zero, by Theorem 2, it follows that in this subcase we have at most four periodic solutions of system (3).

Example 1. *Let us consider the system of differential equations:*

$$\begin{cases} \dot{x} = u, \\ \dot{u} = -x + \varepsilon\left(\left(\dfrac{5}{2}x + x^2 + \dfrac{1}{2}xy - \dfrac{1}{2}y^2 + 5x^3 - \dfrac{3}{2}x^2y + y^3\right)y \right. \\ \qquad\qquad \left. + x + x^2 + \dfrac{1}{2}y^2\right), \\ \dot{y} = v, \\ \dot{v} = -y + \varepsilon\left(\left(\dfrac{3}{4}y + x^2 - \dfrac{1}{2}xy - \dfrac{3}{2}y^2 + x^3 - xy^2 - y^3\right)x \right. \\ \qquad\qquad \left. + 2y + x^2 - xy + y^2\right). \end{cases} \tag{12}$$

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Then, it can be checked that

$$(\rho, s, \omega) = \Big(2\sqrt{\frac{6}{37}}, 4\sqrt{\frac{2}{37}}, \frac{\pi}{6}\Big)$$
$$(\rho, s, \omega) = \Big(2\sqrt{\frac{6}{37}}, 4\sqrt{\frac{2}{37}}, \frac{11\pi}{6}\Big)$$

are zeros of system (12) *with determinant* (9) *equals to* $\pm\dfrac{6\sqrt{3}}{1369}$, *respectively. So, this system has two periodic solutions coming from periodic orbits of the center* (4) .

Subcase 1.2 If $b_4a_3 - 3\,b_3a_4 = 0$, then $b_3 = a_3b_4/(3a_4)$, and we need to consider the following subcases.

Subcase 1.2.1 If $b_5a_4 - 3\,b_4a_5 \neq 0$, then, from (11) we obtain that

$$s = 2\sqrt{\frac{-B_1a_4 + b_4A_2}{b_5a_4 - 3\,b_4a_5}}.$$

We must consider that $-B_1a_4 + b_4A_2 \neq 0$, otherwise $s = 0$ and we cannot get periodic solutions. If we substitute the expressions of $\cos(\omega)$ and s in $\xi_3 = 0$, we get an equation of the form $\rho(\mathbf{A} + \mathbf{B}\rho^2 + \mathbf{C}\rho^4) = 0$. Since ρ must be positive, again in this subcase we get two values for ρ, s and $\cos(\omega)$; and consequently at most four periodic solutions of system (3).

Subcase 1.2.2 $b_5a_4 - 3\,b_4a_5 = 0$. Therefore, from (11) we must have that $-B_1a_4 + b_4A_2 = 0$, otherwise we do not have solutions. That is, $b_5 = 3\,b_4a_5/a_4$. Substituting now $\cos(\omega)$ in $\xi_3 = 0$, we get a continuum of solutions for ρ and s. So, in this case we cannot apply Theorem 2.

Case 2 $a_4 = 0$. Again we need to consider the following subcases.

Subcase 2.1 $a_3 \neq 0$. Therefore, from the first equation, $\xi_1 = 0$, we get

$$\rho = \frac{\sqrt{-a_3\,(3\,s^2a_5 + 4\,A_2)}}{a_3}.$$

Of course we suppose $-a_3\,\big(3\,s^2a_5 + 4\,A_2\big) \neq 0$, otherwise $\rho = 0$. Now, we substitute the expression of ρ in the second equation $\xi_2 = 0$.

Subcase 2.1.1 $b_4 \neq 0$. Therefore, from the second equation, $\xi_2 = 0$, we get that

$$\cos(\omega) = -\frac{(b_5a_3 - 9\,b_3a_5)\,s^2 - 12\,b_3A_2 + 4\,B_1a_3}{2b_4s\sqrt{-a_3\,(3\,s^2a_5 + 4\,A_2)}}.$$

Substituting the expressions of ρ and $\cos(\omega)$ in $\xi_3 = 0$, we get an equation of the form

$$\frac{s}{a_3b_4\sqrt{-a_3\,(3\,s^2a_5 + 4\,A_2)}}(\mathbf{A} + \mathbf{B}s^2 + \mathbf{C}s^4) = 0.$$

Since the first factor cannot be zero, as in the previous subcases, we can get at most four periodic solutions of system (3).

Subcase 2.1.2 $b_4 = 0$.

Subcase 2.1.2.1 If $b_5a_3 - 9\,a_5b_3 \neq 0$, then, from the second equation, $\xi_2 = 0$, we obtain

$$s = 2\sqrt{\frac{3\,b_3A_2 - B_1a_3}{b_5a_3 - 9\,a_5b_3}},$$

substituting the expressions of ρ and s in $\xi_3 = 0$, we arrive to an equation of the form $\mathbf{A} + \mathbf{B}\cos(\omega) + \mathbf{C}\cos^2(\omega) = 0$. So, once again, we can obtain at most four solutions for ρ, s and ω, and, consequently, we obtains at most four periodic solutions for system (3).

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Subcase 2.1.2.2 If $b_5 a_3 - 9\, a_5 b_3 = 0$, then, $b_5 = \dfrac{9\, a_5 b_3}{a_3}$. Now, from $\xi_2 = 0$, it follows that $B_1 a_3 - 3\, b_3 A_2 = 0$, otherwise we have no solutions. Therefore, $B_1 = \dfrac{3 b_3 A_2}{a_3}$. Substituting the expression of ρ in $\xi_3 = 0$, we get a continuum of solutions. So, again, we are not in the assumptions of Theorem 2.

Subcase 2.2 $a_3 = 0$. Looking at equation $\xi_1 = 0$, we see that a_5 cannot be zero, otherwise $\xi_1 = 0$ reduces to $A_2 = 0$, and either we do not have solutions or we have a continuum of solutions. Then, from $\xi_1 = 0$ we get

$$s = 2\sqrt{-\frac{A_2}{3a_5}}.$$

Substituting the expression of s in the second equation $\xi_2 = 0$, we must consider the subcases:

Subcase 2.2.1 If $b_4 \neq 0$, then, by solving the equation $\xi_2 = 0$ with respect to $\cos(\omega)$ we get

$$\cos(\omega) = -\frac{12\, B_1 a_5 + 9\, \rho^2 b_3 a_5 - 4\, b_5 A_2}{4\rho\, b_4 \sqrt{-3\, a_5 A_2}}.$$

Substituting the expressions of $\cos(\omega)$ and s in $\xi_3 = 0$, we obtain an equation of the form $\rho(\mathbf{A} + \mathbf{B}\rho^2) = 0$. Hence, as in previous subcases, system (3) has at most two periodic solutions.

Subcase 2.2.2 Assume $b_4 = 0$. Then, the second equation $\xi_2 = 0$ is of the form $\mathbf{A} + \mathbf{B}\rho^2 = 0$, so, there is at most one positive solution for ρ. Hence, by substituting the value of s and ρ in $\xi_3 = 0$, we obtain an equation of the form $\mathbf{A} + \mathbf{B}\cos(\omega) = 0$. Therefore, we get at most one solution for ρ and $\cos(\omega)$. In short, putting aside those subcases where we obtain a continuum of solutions, there is at most one solution for s, ρ and $\cos(\omega)$ and so, there are at most two periodic solutions for system (3).

In the previous case, we gave a particular solution obtained from the subcase 1.1, that is the most general one. Now we are going to see the general solution of subcase 2.2.2, characterized by $a_3 = a_4 = b_4 = 0$.

Corollary 1. *If $a_3 = a_4 = b_4 = 0$ then*

a) If $b_5 = 0$ the system (10) has no solution or it has a continuum of solutions.

b) If $b_5 \neq 0$, the solutions of system (10) are given by:

$$\begin{aligned} \rho^2 &= -\frac{4B_1 + b_5 s^2}{3b_3} \\ s^2 &= -\frac{4A_2}{3a_5} \\ \cos(\omega) &= \frac{2(A_1 - B_2)}{b_5 \rho s} \end{aligned}.$$

Demonstração. a) If $b_5 = 0$, the system (10) reduces to

$$\begin{aligned} 4A_2 + 3a_5 s^2 &= 0, \\ 4B_1 + 3b_3\rho^2 &= 0, \\ 4\rho s(B_2 - A_1) + \cos(\omega)\Big(s^2(-4A_2 - 3a_5 s^2) + \rho^2(4B_1 + 3b_3\rho^2)\Big) &= 0 \end{aligned}$$

that is

$$\begin{aligned} 4A_2 + 3a_5 s^2 &= 0, \\ 4B_1 + 3b_3\rho^2 &= 0, \\ 4\rho s(B_2 - A_1) &= 0. \end{aligned}$$

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This system of equations does not depend on ω, hence either has no solution or has a continuum of solutions.

b) If $b_5 \neq 0$, we have the system:

$$\begin{aligned} 4A_2 + 3a_5 s^2 &= 0 \\ 4B_1 + 3b_3\rho^2 + b_5 s^2 &= 0 \\ 4\rho s(B_2 - A_1) + \cos(\omega)\Big(s^2(-4A_2 - 3a_5 s^2) + \rho^2\left(4B_1 + 3b_3\rho^2 + 3b_5 s^2\right)\Big) &= 0. \end{aligned}$$

From the first equation, $\xi_1 = 0$, we obtain that $s^2 = -\frac{4A_2}{3a_5}$, and, by substituting this value in the second equation, $\xi_2 = 0$, it follows that $\rho^2 = -\frac{4B_1 + b_5 s^2}{3b_3}$.

Finally, the equation $\xi_3 = 0$ reduces to

$$\rho s\Big(2(B_2 - A_1) + b_5\rho s\cos(\omega)\Big) = 0,$$

hence,

$$\cos(\omega) = \frac{2(A_1 - B_2)}{b_5\rho s}.$$

This three equalities provides all the possible solutions of this subcase. □

Example 2. *In the previous corollary, if we take* $A_1 = B_2$, *we have that* $\omega = \frac{\pi}{2}$ *or* $\omega = \frac{3\pi}{2}$, *and if we take the values* $a_5 = -1, b_3 = -1, b_5 = 1, A_2 = 9, B_1 = 1$, *we obtain the following two solutions:*

$$(\rho, s, \omega) = \left(\frac{4}{\sqrt{3}}, 2\sqrt{3}, \frac{\pi}{2}\right),$$

$$(\rho, s, \omega) = \left(\frac{4}{\sqrt{3}}, 2\sqrt{3}, \frac{3\pi}{2}\right).$$

Observe that this subcase does not depend on the constants not listed in this example, so, we can choose any value for them.

4 CONCLUSIONS

This paper shows that the application of averaging method of first-order it is useful for study the existence of limit cycles of perturbated system of second-order differential equations.

We have proved that, using Theorem 2, we can obtain at most four periodic solutions of system (3) when $f_1(x, y)$ and $g_1(x, y)$ are real cubic polynomials, and $f_2(x, y)$ and $g_2(x, y)$ are real quadratic polynomials. Moreover, if $a_4 = 0$ and $a_3 = 0$, the system (3) has at most two periodic solutions. We have also obtained the general solution in the case $a_3 = a_4 = b_4 = 0$.

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DATA STATEMENT

This paper is not related with any data.

REFERENCES

[1] TR Blows and Lawrence M Perko. Bifurcation of limit cycles from centers and separatrix cycles of planar analytic systems. *Siam Review*, 36(3):341–376, 1994.

[2] Adriana Buică and Jaume Llibre. Averaging methods for finding periodic orbits via brouwer degree. *Bulletin des sciences mathematiques*, 128(1):7–22, 2004.

[3] Lan Sun Chen and Ming Shu Wang. The relative position, and the number, of limit cycles of a quadratic differential system. *Acta Math. Sinica*, 22(6):751–758, 1979.

[4] Colin Christopher and Chengzhi Li. *Limit cycles of differential equations.* Springer Science & Business Media, 2007.

[5] Zouhair Diab, Juan LG Guirao, and Juan A Vera. On the limit cycles for a class of generalized liénard differential systems. *Dynamical Systems*, 37(1):1–8, 2022.

[6] Zouhair Diab and Amar Makhlouf. Limit cycles for the class of-dimensional polynomial differential systems. *Journal of Applied Mathematics*, 2016, 2016.

[7] Amina Feddaoui, Jaume Llibre, Chemseddine Berhail, and Amar Makhlouf. Periodic solutions for differential systems in $\mathbb{R}^3$ and $\mathbb{R}^4$. *Applied Mathematics and Nonlinear Sciences*, 6(1):373–380, 2021.

[8] Hector Giacomini, Jaume Llibre, and Mireille Viano. On the shape of limit cycles that bifurcate from hamiltonian centers. *Nonlinear Analysis, Theory, Methods and Applications*, 41:523–537, 2000.

[9] Maoan Han and Jibin Li. Lower bounds for the Hilbert number of polynomial systems. *Journal of Differential Equations*, 252(4):3278–3304, 2012.

[10] Maoan Han, Yuhai Wu, and Ping Bi. A new cubic system having eleven limit cycles. *Discrete & Continuous Dynamical Systems*, 12(4):675, 2005.

[11] Maoan Han, Junmin Yang, and Pei Yu. Hopf bifurcations for near-hamiltonian systems. *International Journal of Bifurcation and Chaos*, 19(12):4117–4130, 2009.

[12] Maoan Han, Tonghua Zhang, and Hong Zang. On the number and distribution of limit cycles in a cubic system. *International Journal of Bifurcation and Chaos*, 14(12):4285–4292, 2004.

[13] David Hilbert. Mathematische probleme, lecture, second internat. congr. math. (paris, 1900), nachr. ges. wiss. göttingen math. phys. kl. (1900), 253-297; english transl., bull. amer. math. soc. 8 (1902), 437-479.

[14] Yu Ilyashenko. Centennial history of Hilbert's 16th problem. *Bulletin of the American Mathematical Society*, 39(3):301–354, 2002.

[15] Chengzhi Li, Changjian Liu, and Jiazhong Yang. A cubic system with thirteen limit cycles. *Journal of Differential Equations*, 246(9):3609–3619, 2009.

[16] JB Li and CF Li. Distribution of limit cycles for planar cubic hamiltonian systems. *Acta Math Sinica*, 28:509–521, 1985.

[17] Jibin Li. Hilbert's 16th problem and bifurcations of planar polynomial vector fields. *International Journal of Bifurcation and Chaos*, 13(01):47–106, 2003.

[18] Jaume Llibre. Centers: their integrability and relations with the divergence. *Applied Mathematics and Nonlinear Sciences*, 1(1):79–86, 2016.

https://doi.org/10.17993/3ctecno.2022.v11n2e42.25-36

[19] Jaume Llibre and Marco Antonio Teixeira. Limit cycles for a mechanical system coming from the perturbation of a four–dimensional linear center. *Journal of Dynamics and Differential Equations*, 18(4):931–941, 2006.

[20] Jaume Llibre and Xiang Zhang. On the integrability of the hamiltonian systems with homogeneous polynomial potentials. *Applied Mathematics and Nonlinear Sciences*, 3(2):527–536, 2018.

[21] Jan A Sanders and Ferdinand Verhulst. *Averaging Methods in Nonlinear Dynamical Systems*, volume 59. Springer. Berlin, 1985.

[22] Songling Shi. A concrete example of the existence of four limit cycles for plane quadratic systems. *Scientia Sinica*, 23(2):153–158, 1980.

[23] Ferdinand Verhulst. *Nonlinear differential equations and dynamical systems.* Universitext, Springer. Berlin, 1991.