Exact solutions to cubic duffing equation for a nonlinear electrical circuit

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Abstract:

This work provides an exact solution to a cubic Duffing oscillator equation with initial conditions and bounded periodic solutions. This solution is expressed in terms of the Jacobi elliptic function (cn). This exact solution is used as a seed to give a good analytic approximate solution to a nonlinear equation that describes a nonlinear electrical circuit. This last equation is solved numerically and compared with the analytic solution obtained from solving the cubic Duffing equation. It is suggested that the methodology used herein may be useful in the study of other nonlinear problems described by differential equations of the form $z'' = F(z)$, $F(z)$ being an odd function in the sense that (z) may be approximated by an appropriate solution to a cubic Duffing oscillator equation. In particular, the exact solution may be applied in the study of the cubic nonlinear Schrodinger equation, which is reduced to a cubic Duffing oscillator equation by means of a travelling wave transformation.

Key words

Duffing equation, cubic Duffing oscillator equation, third-order Duffing model, analytic solution, Jacobi elliptic functions.

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Introduction

The cubic Duffing equation is a differential equation with third-power nonlinearity. There are many problems in physics and engineering that lead to the nonlinear differential cubic Duffing equation: from the oscillations of a simple pendulum, including nonlinear electrical circuits, to various applications in image processing [1,2,3]. The search for new methods that lead to the solution of the Duffing equation of different orders: third, fifth, seventh, and higher orders, is of vital importance since these solutions may be applied to a nonlinear cubic Schrodinger equation which has various applications in nonlinear optics, plasma physics, fluid mechanics and Bose-Einstein condensates [4,5]. By using the gauge transformation, the nonlinear cubic Schodinger equation may be reduced to a nonlinear ordinary differential cubic Duffing oscillator equation.

This work is aimed at finding, by means of the Jacobi elliptic functions, the exact periodic solution of the differential cubic Duffing equation with initial conditions that describe the behaviour of a nonlinear electrical circuit.

1. Nonlinear electrical circuit

Let us consider a capacitor of two terminals as a dipole in which a functional relationship between the electric charge , the voltage and the time has the following form:
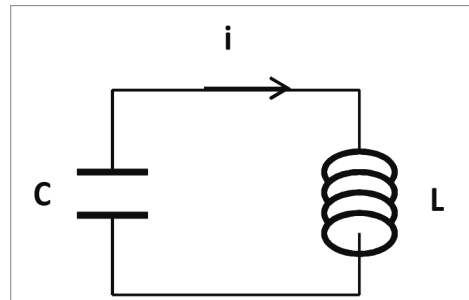
$$f(q, u, t) = 0 \qquad (1)$$

A nonlinear capacitor is said to be controlled by charge when it is possible to express the tension as a function of charge:

$$u \quad u(q) \qquad (2)$$

As an example of nonlinear electrical circuit, let us consider the circuit shown in Figure 1.

Figure1. LC circuit

Source: own elaboration

This circuit consists of a linear inductor in series with a nonlinear capacitor. The relationship between the charge of the nonlinear capacitor and the voltage drop across it may be approximated by the following cubic equations [6,7,8] :

$$u_c = sq + aq^3, \qquad (3)$$

where u_c is the potential across the plates of the nonlinear capacitor, q is the charge , and s and a are constants. The equation of the circuit may be written as

$$L\frac{di}{dt} + sq + aq^3 = 0, \qquad (4)$$

where L is the inductance of the coil. Dividing by L and considering that $i = \frac{dq}{dt}$, we obtain the cubic Duffing equation in the form

$$\frac{d^2q}{dt^2} + \alpha q + \beta q^3 = 0, \qquad (5)$$

being $\alpha = s/L = 1/LC$ and $\beta = a/L = 1/LCq_0^2$ constants.

2. A brief introduction to Jacobi elliptic functions cn, sn and dn

It is worth remembering that this work provides an exact solution to equation (5) in terms of the Jacobi elliptic function cn. This function is defined as follows:

$$\operatorname{cn}(t,m) = \cos\varphi, \text{ where } t = \int_0^\varphi \frac{d\theta}{\sqrt{1 - m^2 \sin^2 \theta}}.$$

$$(6)$$

There are other two important elliptic Jacobi functions, namely sn and dn, which are defined by:

$$\operatorname{sn}(t,m) = \sin\varphi \text{ and } \operatorname{dn}(t,k) =$$

$$\sqrt{1 - m^2 \sin^2 \varphi}. \tag{7}$$

The number m $(0 < m < 1)$ is called elliptic modulus and the number φ is called Jacobi amplitude, and it is denoted by am (t, m). Thus,

$$\varphi = \operatorname{am}(t,m), \ \sin(\varphi) = \sin(\operatorname{am}(t,m)) =$$

$$\operatorname{sn}(t,m)$$

For these equations, the following identities hold:

$$\operatorname{sn}^2(t,m) + \operatorname{cn}^2(t,m) = 1, \ \operatorname{dn}^2(t,m) =$$

$$1 - m^2 \operatorname{sn}^2(t,m) \tag{8}$$

$$\lim_{m\to 0} \operatorname{sn}(t,m) = \sin t, \ \lim_{m\to 0} \operatorname{cn}(t,m) = \cos t,$$

$$\lim_{m\to 0} \operatorname{dn}(t,m) = 1. \tag{9}$$

$$\lim_{m\to 1} \operatorname{sn}(t,m) = \tanh t, \ \lim_{m\to 1} \operatorname{cn}(t,m) = \operatorname{sech} t,$$

$$\lim_{m\to 1} \operatorname{dn}(t,m) = \operatorname{sech} t. \tag{10}$$

These functions are derivable and so:

$$\frac{d}{dt} \operatorname{sn}(t,m) = \operatorname{cn}(t,m)\operatorname{dn}(t,m),$$

$$\frac{d}{dt} \operatorname{cn}(t,m) = -\operatorname{sn}(t,m)\operatorname{dn}(t,m), \tag{11}$$

$$\frac{d}{dt} \operatorname{dn}(t,m) = -m^2 \operatorname{sn}(t,m)\operatorname{cn}(t,m).$$

The graph of functions sn and cn are shown in Figure 2 for $m = \frac{1}{4}$.

Figure 2. Graph of sn=sn $(t,1/4)$ and cn=cn $(t,1/4)$ on the interval $0 \le t \le 8$.
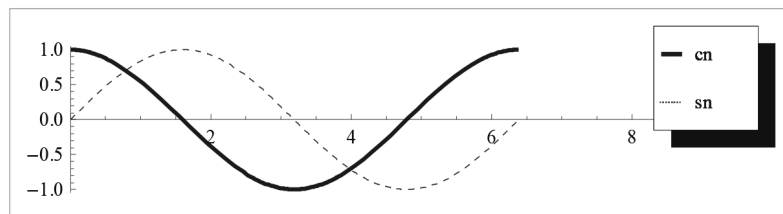
Source: own elaboration

From Figure 2, it can be observed that functions sn and cn are periodic. They have a common period equal to $4K(1/4) = 4K(m)$, where $K = K(m)$ is called the elliptic K function for modulus m. In our case, $K(1/4) \approx 1.5962422$.

Equations (9) indicate that functions sn and cn generalize the sine and cosine functions, respectively.

Jacobi elliptic functions $cn(t,m)$, $sn(t,m)$ and $dn(t,m)$ may also be calculated for complex values of $t = u + \sqrt{-1}\, v$ as follows :

$$sn(u + \sqrt{-1}\, v, m) = \frac{sn(u,m)\, dn(v,m')}{1 - dn^2(u,m)\, sn^2(v,m')} +$$

$$\sqrt{-1}\, \frac{cn(u,m)\, dn(u,m)\, sn(v,m')\, cn(v,m')}{1 - dn^2(u,m)\, sn^2(v,m')}$$

$$sn(\sqrt{-1}\, v, m) = \sqrt{-1}\, \frac{sn(v,m')}{cn(v,m')} =$$

$$\sqrt{-1}\, sn(v,m'),\ m' = \sqrt{1 - m^2},\ 0 < m < 1. \quad (12)$$

$$cn(u + \sqrt{-1}\, v, m) = \frac{cn(u,m)\, cn(v,m')}{1 - dn^2(u,m)\, sn^2(v,m')} -$$

$$\sqrt{-1}\, \frac{sn(u,m)\, dn(u,m)\, sn(v,m')\, dn(v,m')}{1 - dn^2(u,m)\, sn^2(v,m')}$$

$$cn(\sqrt{-1}\, v, m) = \frac{1}{cn(v,m')} =$$

$$nc(v,m'),\ m' = \sqrt{1 - m^2},\ 0 < m < 1. \quad (13)$$

$$dn(u + \sqrt{-1}\, v, m) = \frac{dn(u,m)\, cn(v,m')\, dn(v,m')}{1 - dn^2(u,m)\, sn^2(v,m')}$$

$$-\sqrt{-1}\, m^2\, \frac{sn(u,m)\, cn(u,m)\, sn(v,m')}{1 - dn^2(u,m)\, sn^2(v,m')}$$

$$dn(\sqrt{-1}\, v, m) = \frac{dn(v,m')}{cn(v,m')} =$$

$$dc(v,m'),\ m' = \sqrt{1 - m^2},\ 0 < m < 1. \quad (14)$$

When $m > 1$, these functions are defined by the following equations :

$$cn(t,m) = dn(m\, t, \frac{1}{m}),\ sn(t,m) = \frac{1}{m} sn(m\, t, \frac{1}{m}),$$

$$dn(t,m) = cn(t\, m, \frac{1}{m}),\ m > 1 \quad (15)$$

The following is also defined:

$$cn(t,m) = \frac{1}{dn(\sqrt{m^2 - 1}\, t, \frac{-m}{\sqrt{m^2 - 1}})} =$$

$$nd(\sqrt{m^2 - 1}\, t, \frac{-m}{\sqrt{m^2 - 1}})\ \text{for}\ m < -1 \quad (16)$$

In some cases, it is necessary to deal with a function of the form $cn(wt, k)$, where w is any number (real or complex) and k is a pure imaginary number, say $k = m\sqrt{-1}$, where m is a real number. We may define the expression $cn(wt, k) = cn(wt, m\sqrt{-1}))$ in a consistent way by using the formula:

$$cn(wt, m\sqrt{-1}) = \frac{cn(\sqrt{m^2+1}\, wt, m_0)}{dn(\sqrt{m^2+1}\, wt, m_0)} =$$

$$cd(\sqrt{m^2+1}\, wt, m_0), \text{ where } m_0 = \frac{|m|}{\sqrt{m^2+1}}$$

(17)

In the special case when both w and k are pure imaginary numbers, say $w = \omega\sqrt{-1}$ and $k = m\sqrt{-1}$ (ω and m are real numbers), it is possible to define $cn(wt, k) = cn(\omega\sqrt{-1}\, t, m\sqrt{-1})$ as follows:

$$cn(\omega\sqrt{-1}\, t, m\sqrt{-1}) = nd(\sqrt{m^2+1}\, \omega t,$$

$$\frac{1}{\sqrt{m^2+1}}) = \frac{1}{dn(\sqrt{m^2+1}\, \omega t, \frac{1}{\sqrt{m^2+1}})}$$

(18)

4. Exact solutions for the nonlinear model

In the linear case, the general solution to equation $y''(t) + \alpha y(t) = 0$ is $y(t) = c_1 \cos(\sqrt{\alpha}t + c_2)$, where c_1 and c_2 are the constants of integration which are determined from the initial conditions $y(0) = y_0$ and $y'(0) = y_0'$. When a cubic term is added we obtain the nonlinear equation $y''(t) + \alpha y(t) + \beta y^3(t) = 0$ and the solution cannot be expressed in terms of the cosine function. In this case, the Jacobi cn funtion solves this nonlinear equation. Indeed, direct calculations using equations (11) show that function $y = c_1 \, cn \, (\omega t + c_2, m)$ satisfies the fo-llowing differential equation

$$y''(t) + \omega^2 \left(1 - 2m^2\right) y(t) + \frac{2m^2\omega^2}{c_1^2} y^3(t) = 0$$

(19)

for any constants c_1 and c_2. Therefore the ge-neral solution to equation (5) is obtained by solving the system

$$\begin{cases} \alpha = \omega^2 \left(1 - 2m^2\right). \\ \beta = \dfrac{2m^2\omega^2}{c_1^2}. \end{cases}$$

(20)

which yields:

$$\omega = \sqrt{\alpha + c_1^2\beta} \text{ and } m = \sqrt{\frac{c_1^2\beta}{2(\alpha + c_1^2\beta)}}.$$

(21)

The values of c_1 and c_2 are determined from the initial conditions $q(0) = q_0$ and $i = \frac{dq}{dt}\big|_{t=0} = i_0$. If $q'(0) = 0$, then $c_1 = q_0$ and $c_2 = 0$. Thus, the solution to problem:
$$\frac{d^2q}{dt^2} + \alpha q + \beta q^3 = 0, q(0) = q_0, q'(0) = 0.$$

(22)

Is

$$q(t) = q_0 cn(\sqrt{\alpha + q_0^2\beta}\, t, \sqrt{\frac{q_0^2\beta}{2(\alpha + q_0^2\beta)}}), \ \alpha + q_0^2\beta \neq 0.$$

(23)

If $m = \frac{\beta q_0^2}{2(\alpha + \beta q_0^2)} \to 1$, then $\beta \to -\frac{2\alpha}{q_0^2}$ and since $\lim_{m\to 1} cn(\omega t, m) = sech(\omega t)$ the solution to (23) results in:

$$q(t) = q_0 \, sech(\sqrt{\alpha}t)$$

(24)

It is worth noting that solution (19) is boun-ded and non-periodic. This situation is typical for solitons. Some important non-linear diffe-rential equations admit soliton solutions that can be expressed in terms of sech.

On the other hand, if $m = \frac{\beta q_0^2}{2(\alpha + \beta q_0^2)} \to 0$ then $\beta \to 0$ and because $\lim_{m \to 0} \text{cn}(\omega t, m) = \cos(\omega t)$ (23) reduces to:

$$q(t) = q_0 \cos(\sqrt{\alpha} t) \qquad (25)$$

Which is the solution to linear equation $y''(t) + \alpha y(t) = 0$. This result is consistent with the theory already known for the linear case.

The behaviour of solution (16) depends on parameters α and β as well as on the initial condition q_0. In this paper we will consider five cases:

First Case: $\alpha + q_0^2 \beta > 0$ and $\beta > 0$. Let $\alpha = 1$, $\beta = 1$ and $q_0 = 10^{-3}$. Formula (23) yields $q(t) = 0.001 \text{cn}(1.000000499 t, 0.0009999995)$. This function is periodic with period $T \approx 6.2832$. The corresponding graph in the interval $[0, T]$ is shown in Figure 3.

Figure 3. Graph of
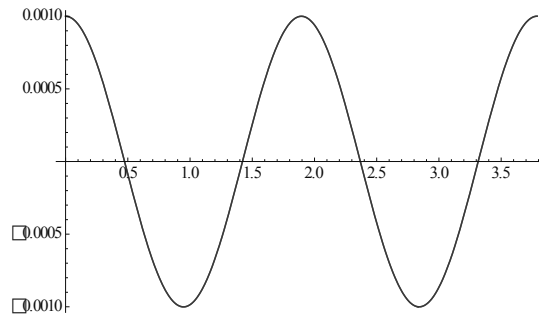$q(t) = 0.001 \text{cn}(1.000000499 t, 0.0009999995)$
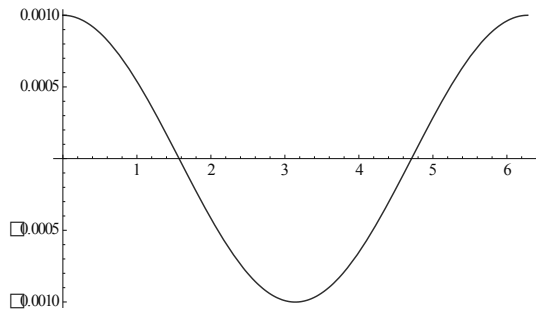
Source: own elaboration

Second Case: $\alpha + q_0^2 \beta > 0$ and $\beta < 0$. Let $\alpha = 11$, $\beta = -33$ and $q_0 = 10^{-3}$. By computing formulas (17) and (23),

$q(t) = 0.001 \text{cd}(3.31662479 t, 1.732 \times 10^{-3})$

This function is periodic with period $T \approx 1.894$. The corresponding graph in the interval $[0, 2T]$ is shown in Figure 4.

Figure 4. Graph of
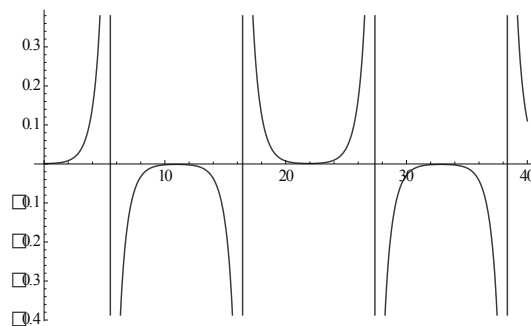$q(t) = 0.001 \text{cd}(3.31662479 t, 1.732 \times 10^{-3})$

Source: own elaboration

Third Case: $\alpha + q_0^2 \beta < 0$ and $\beta < 0$. Let $\alpha = -2$, $\beta = -6$ and $q_0 = 10^{-3}$. By computing formulas (18) and (23),

$q(t) = 0.001 \text{nd}(1.4142135 t, 1.732 \times 10^{-3})$. This function is unbounded and periodic with period $T \approx 16.429$. The corresponding graph in the interval $[0, 40]$ is shown in Figure 5.

Figure 5. Graph of

$q(t) = 0.001 \text{nd}(1.4142135 t, 1.732 \times 10^{-3})$
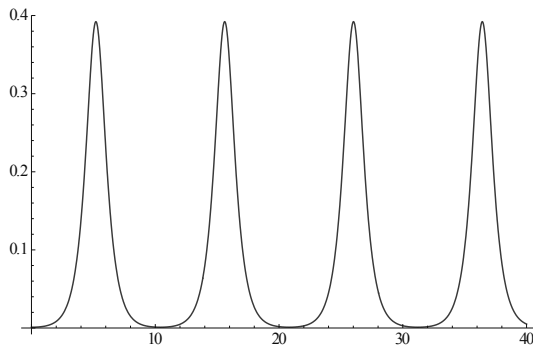

Source: own elaboration

Fourth Case: $\alpha + q_0^2\beta < 0$ and $\beta > 0$. Let $\alpha = -2$, $\beta = 13$ and $q_0 = 10^{-3}$. By computing formulas (18) and (23),
$q(t) = 0.001\,\mathrm{nd}(1.4142135t, 2.5495\times10^{-3})$. This function is bounded and periodic with period $T \approx 10.406$. The corresponding graph in the interval $[0, 40]$ is shown in Figure 6.

Figure 6. Graph of
$$q(t) = 0.001\,\mathrm{nd}(1.4142135t, 2.5495\times10^{-3})$$

Source: own elaboration

Fifth Case : $\alpha + q_0^2\beta = 0$. In this case it is not possible to apply formula (23). Equation (22) then reduces to

$$\frac{d^2q}{dt^2} - q_0^2\beta q + \beta q^3 = 0,\, q(0) = q_0,\, q'(0) = 0$$

$$(26)$$

Direct calculations show that function

$$q = q(t) = \sqrt{2}q_0\,\mathrm{sech}(q_0\sqrt{\beta}\,(t+C)),$$

$$\text{where } C = \frac{1}{q_0\sqrt{\beta}}\,\mathrm{arcsech}(\frac{1}{\sqrt{2}}) \qquad (27)$$

Finally, let us consider a nonlinear circuit consisting of a linear inductor in series with a nonlinear capacitor, as shown in Figure 1.

For $L = 2,81mH$, $C = 9pF$, $q_o = 10^{-10}C$, $i_0 = 0$, we have that $\alpha = 1/LC = 4\times10^{13}$ and $\beta = 1/LCq_0^2 = 4\times10^{33}$. The numerical solution for the Runge - Kutta method and the analytical solution we obtain from (23), i.e.
$$q = q(t) = \mathrm{cn}(8.94427\times10^6, 0.7071067811865475)$$
are compared (graphically) in Figure 7.

Figure 7. Numerical solution (dashed) vs analytical solution (solid)
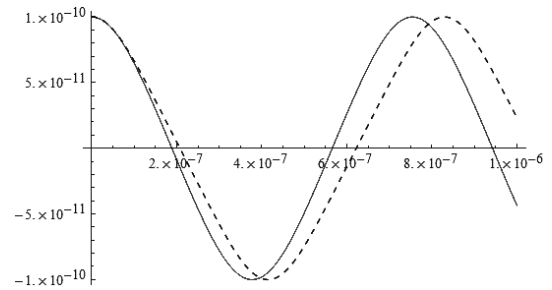
Source: own elaboration

5. Conclusions

A way to study non-linear circuits by means of the cubic Duffing differential equation has been shown, obtaining an exact solution in terms of a Jacobi elliptic function cn. It was possible to establish conditions under which the solution is periodic and harmonic. This is very important in the study of electric reso-nators and their applications in engineering. On the other hand, the results (already know) for the linear case were generalized for the nonlinear case in a consistent and natural way. It may be stated that the Jacobi elliptic functions are an essential way to solve many nonlinear problems.

Solutions (24) and (27) turned out to be very interesting because they represent a type of soliton solution (for a more thorough study of these solutions see [9]).

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