Investigación

Hyperbolic classification of natural numbers and Goldbach Conjecture

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Abstract

By means of a theoretical development of lecture [3], we provide a characterization of the Goldbach Conjecture in an infinite set of even numbers in terms of gradients of deformed hyperbolas.

Keywords: Hyperbolic Classification, Natural Numbers, Goldbach Conjecture, Characterization.

1 Hyperbolic classification of natural numbers

1.1 Introduction

For a natural number $n > 1$ the fact of being a prime is equivalent to stating that the hyperbola $xy = n$ does not contain non-trivial natural number coordinate points that is, the only natural number coordinate points in the hyperbola are $(1, n)$ and $(n, 1)$.

We establish a family of bijective functions between non-negative real numbers and a half-open interval of real numbers. Bijectivity allows us to transport usual real number operations, sum and product, to the interval. It also allows us to deform the $xy = k$ hyperbolas with k as a real positive number in such a way that we can distinguish whether a natural number n is a prime or not by its behaviour in terms of gradients of the deformed hyperbolas near the deformed of $xy = n$ (Hyperbolic Classification of Natural Numbers).

In this section we define a function ψ which ranges from non-negative real numbers to a half-open interval, strictly increasing, continuous in $\mathbb{R}^+ = \mathbb{R}_{\geq 0}$ and class 1 in each interval $[m, m+1]$ $(m \in \mathbb{N} = \{0, 1, 2, 3, ...\})$. The bijectivity of ψ allows to transport the usual sum and product of $\mathbb{R}^+$ to the set $\widehat{\mathbb{R}}^+ := \psi(\mathbb{R}^+)$ in the usual manner. That is, calling $\hat{x} := \psi(x)$, we define $\hat{s} \oplus \hat{t} := \psi(s+t)$, $\hat{s} \otimes \hat{t} := \psi(st)$. Therefore, $(\widehat{\mathbb{R}}^+, \oplus, \otimes)$ is an algebraic structure isomorphic to the usual one $(\mathbb{R}^+, +, \cdot)$ and as a result, we obtain an algebraic structure $(\widehat{\mathbb{N}} := \psi(\mathbb{N}), \oplus, \otimes)$ isomorphic to the usual one $(\mathbb{N}, +, \cdot)$. The function ψ also preserves the usual orderings. Thus we transport the notation from $\mathbb{R}^+$ to $\widehat{\mathbb{R}}^+$, that is $\hat{n}$ is natural iff n is natural, $\hat{p}$ is prime iff p is prime, $\hat{x}$ is rational iff x is rational, etc. Assume that, for example $\widehat{0}=0,\ \widehat{1}=0'72,\ \widehat{2}=1'3,\ \widehat{3}=3'0001,\ \widehat{4}=\pi,\ \widehat{5}=6'3,\ \widehat{7}=7'21,...,\widehat{12}=9'03,...$ then, the following situation would arise: the even number $9'03$ is the sum of the prime numbers $6'3$ and $7'21$ and the number π is the product of the numbers $0'72$ and π.

Obviously, until now, we have only actually changed the symbolism by means of the function ψ. If we call $\hat{x}\hat{y}$ plane the set $(\psi(\mathbb{R}^+))^2$, the hyperbolas $xy=k$ $(k>0)$ of the xy plane with $x>0$ and $y>0$ are transformed by means of the function $\psi\times\psi$ at the $\hat{x}\otimes\hat{y}=\hat{k}$ "hyperbolas" of the $\hat{x}\hat{y}$ plane. We will restrict our attention to the points in the $\hat{x}\hat{y}$ plane that satisfy $\hat{x}>\hat{0}$ and $\hat{y}\geq\hat{x}$. Then, with these restrictions for the ψ function, it is possible to choose right-hand and left-hand derivatives of ψ at $m\in\mathbb{N}^*=\{1,2,3,...\}$ such that we can characterize the natural number coordinate points in the $\hat{x}\hat{y}$ plane in terms of differentiability of the functions which determine the transformed hyperbolas. As a result, we can distinguish prime numbers from composite numbers in the aforementioned terms.

Definition 1.1. Let $\psi:\mathbb{R}^+\to\mathbb{R}$ be a map and let ψ_m be the restriction of ψ to each closed interval $[m.m+1]$ $(m\in\mathbb{N})$. We say that ψ is an $\mathbb{R}^+$ *coding function* if:
(i) $\psi(0)=0$.
(ii) $\psi\in\mathcal{C}(\mathbb{R}^+)$.
(iii) $\forall m\in\mathbb{N},\ \psi_m\in\mathcal{C}^1([m,m+1])$ with positive derivative in $[m,m+1]$.

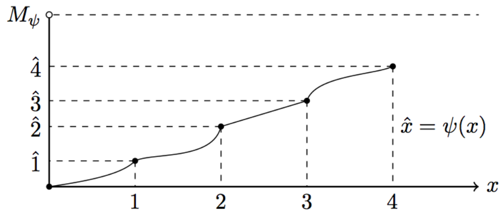


Figure 1. $\mathbb{R}^+$ coding function.

Remarks 1.2. (1) Easily proved, if ψ is an $\mathbb{R}^+$ coding function then it is strictly increasing and consequently, injective.
(2) If $M_\psi:=\sup\{\psi(x):x\in\mathbb{R}^+\}$ then, $M_\psi\in(0,+\infty]$ (being $M_\psi=+\infty$ iff ψ is not bounded), and so $\psi(\mathbb{R}^+)=[0,M_\psi)$. Therefore $\psi:\mathbb{R}^+\to\psi(\mathbb{R}^+)=[0,M_\psi)$ is bijective, and here onwards we will refer to the ψ function as a bijective function.
(3) We will frequently use the notation $\hat{x}:=\psi(x)$. Due to the ψ bijection, we transport the sum and the product from $\mathbb{R}^+$ to $[0,M_\psi)$ in the usual manner ([2]), that is we define in $[0,M_\psi)$ the operations ψ-sum as $\hat{x}\oplus\hat{y}:=\psi(x+y)$ and ψ-product as $\hat{x}\otimes\hat{y}:=\psi(x\cdot y)$. Thus, $\psi:(\mathbb{R}^+,+,\cdot)\to([0,M_\psi),\oplus,\otimes)$ is an isomorphism.
(4) The ψ function preserves the usual orderings, that is, $\hat{s}\leq\hat{t}\Leftrightarrow s\leq t,\ \hat{s}=\hat{t}\Leftrightarrow s=t$.
(5) For $\hat{x}\in[0,M_\psi)$ we say that $\hat{x}$ is a ψ*-natural number* iff x is a natural number, $\hat{x}$ is ψ*-prime* iff x is prime, $\hat{x}$ is ψ*-rational* iff x rational, etc.
(6) When we work on the set $[0,M_\psi)^2$, we say that we are on the $\hat{x}\hat{y}$ plane.
(7) For $x\geq y$ we denote $\hat{x}\sim\hat{y}:=\psi(x-y)=$ (ψ*-subtraction*) and for $y\neq0$, $\hat{x}\div\hat{y}:=\psi(x/y)$ (ψ*-quotient*).

1.2 ψ-hyperbolas in the $\hat{x}\hat{y}$ plane

The aim here is to study the transformed curves of the $y=k/x$ hyperbolas $(k\in\mathbb{R}^+-\{0\})$ by means of an $\mathbb{R}^+$ coding function in terms of differentiability. Consider the function

$$h_k:(0,+\infty)\to(0,+\infty)\,,\ h_k(x)=\frac{k}{x}.$$

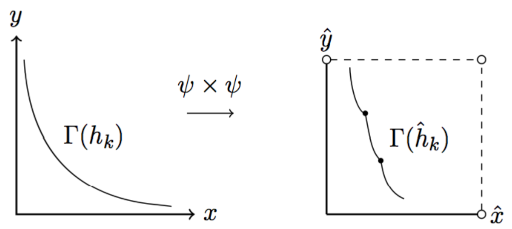


Figure 2. ψ-hyperbolas in the $\hat{x}\hat{y}$ plane.

We call *ψ-hyperbola* any transformed curve graph of $\Gamma(h_k)$ by means of $\psi \times \psi$. Notice that the function which defines the ψ-hyperbola is:

$$\hat{h}_k : (0, M_\psi) \to (0, M_\psi), \ \hat{h}_k(u) = \psi\left(\frac{k}{\psi^{-1}(u)}\right).$$

Theorem 1.3. *Let $\psi : \mathbb{R}^+ \to [0, M_\psi)$ be an $\mathbb{R}^+$ coding function. Then, $\hat{h}_k : (0, M_\psi) \to (0, M_\psi)$ is continuous and strictly decreasing.*

Proof. $(0, M_\psi) \overset{\psi^{-1}}{\to} (0, +\infty) \overset{h_k}{\to} (0, +\infty) \overset{\psi}{\to} (0, M_\psi)$, thus $\hat{h}_k = \psi \circ h_k \circ \psi^{-1}$ is a composition of continuous functions, and is consequently continuous. In addition

$$0 < s < t < M_\psi \Rightarrow \psi^{-1}(s) < \psi^{-1}(t) \Rightarrow \frac{k}{\psi^{-1}(s)} > \frac{k}{\psi^{-1}(t)}$$

$$\Rightarrow \psi\left(\frac{k}{\psi^{-1}(s)}\right) > \psi\left(\frac{k}{\psi^{-1}(t)}\right) \Rightarrow \hat{h}_k(s) > \hat{h}_k(t),$$

that is, $\hat{h}_k$ is strictly decreasing. $\square$

Let us call $(\psi_{i-1})'_-(i) = a_i$ and $(\psi_i)'_+(i) = b_i$ $(i = 1, 2, 3, \ldots)$ We will now analyse the differentiability of $\hat{h}_k$ distinguishing, for this, the cases in which the dependent and/or independent variable takes ψ-natural number values or not.

Theorem 1.4. *Where $x, y \in \mathbb{R}^+ - \mathbb{N}$, $\lfloor x \rfloor = n$, $\lfloor y \rfloor = m$.*
1.- If $(\hat{x}, \hat{y}) \in \Gamma(\hat{h}_k)$, then

$$(\hat{h}_k)'(\hat{x}) = \frac{-k}{x^2} \cdot \frac{(\psi_m)'(y)}{(\psi_n)'(y)}.$$

2.- If $(\hat{x}, \hat{m}) \in \Gamma(\hat{h}_k)$, then

$$(\hat{h}_k)'_+(\hat{x}) = \frac{-k}{x^2} \cdot \frac{a_m}{(\psi_n)'(x)}, \quad (\hat{h}_k)'_-(\hat{x}) = \frac{-k}{x^2} \cdot \frac{b_m}{(\psi_n)'(x)}.$$

3.- If $(\hat{n}, \hat{y}) \in \Gamma(\hat{h}_k)$, then

$$(\hat{h}_k)'_+(\hat{n}) = \frac{-k}{n^2} \cdot \frac{(\psi_m)'(y)}{b_n}, \quad (\hat{h}_k)'_-(\hat{n}) = \frac{-k}{n^2} \cdot \frac{(\psi_m)'(y)}{a_n}.$$

4.- If $(\hat{n}, \hat{m}) \in \Gamma(\hat{h}_k)$, then

$$(\hat{h}_k)'_+(\hat{n}) = \frac{-k}{n^2} \cdot \frac{a_m}{b_n}, \quad (\hat{h}_k)'_-(\hat{n}) = \frac{-k}{n^2} \cdot \frac{b_m}{a_n}.$$

Proof. <u>Case 1</u> $u \in (\hat{n}, \hat{n} \oplus \hat{1})$ $(n \in \mathbb{N})$ that is, u is not a ψ-natural number. We obtain

$$(\hat{n}, \hat{n} \oplus \hat{1}) \overset{\psi^{-1}}{\longrightarrow} (n, n+1) \overset{h_k}{\longrightarrow} (k/(n+1), k/n) \overset{\psi}{\longrightarrow} (\hat{k} \div (\hat{n} \oplus \hat{1}), \hat{k} \div \hat{n})$$

so, $\hat{h}_k$ maps $\hat{h}_k : (\hat{n}, \hat{n} \oplus \hat{1}) \to (\hat{k} \div (\hat{n} \oplus \hat{1}), \hat{k} \div \hat{n})$.
1.a) Suppose $\hat{h}_k(u)$ is not a ψ-natural number (Fig. 3). Since $k/\psi^{-1}(u)$ is not a natural number, in a neighbourhood of u, the expression of the $\hat{h}_k$ function is:

$$\hat{h}_k(t) = \psi_{\left\lfloor \frac{k}{\psi^{-1}(u)} \right\rfloor} \left(\frac{k}{\psi^{-1}(t)} \right).$$

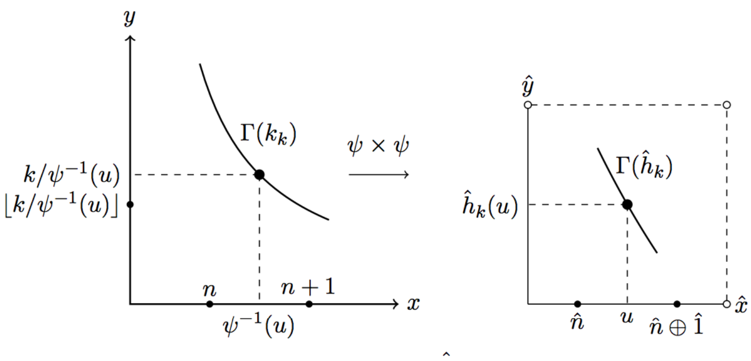


Figure 3. Finding $(\hat{h}_k)'(u)$.

$$(\hat{h}_k)'(u) = \left(\psi_{\left\lfloor \frac{k}{\psi^{-1}(u)} \right\rfloor} \right)' \left(\frac{k}{\psi^{-1}(u)} \right) \cdot \frac{-k}{(\psi^{-1}(u))^2} \cdot \frac{1}{(\psi_n)'(\psi^{-1}(u))}.$$

Consequently $\hat{h}_k$ is differentiable at u.
1.b) Suppose $\hat{h}_k(u)$ is a ψ-natural number (Fig. 4). This is equivalent to say that $k/\psi^{-1}(u)$ is a natural number. For a sufficiently small $\epsilon > 0$ we obtain

$$(u \sim \epsilon, u] \overset{\psi^{-1}}{\to} \left(\psi^{-1}(u \sim \epsilon), \psi^{-1}(u)\right] \overset{h_k}{\to}$$

$$\left[\frac{k}{\psi^{-1}(u)}, \frac{k}{\psi^{-1}(u \sim \epsilon)} \right) \overset{\psi}{\to} \left[\hat{k} \div \widehat{\psi^{-1}(u)}, \hat{k} \div \widehat{\psi^{-1}(u \sim \epsilon)} \right).$$

We can choose $\epsilon > 0$ such that $n < \psi^{-1}(u \sim \epsilon) < \psi^{-1}(u) < n+1$ and as a consequence for every $t \in (u \sim \epsilon, u]$ we verify $k/\psi^{-1}(u) \leq k/\psi^{-1}(t)$. That is, we can choose $\epsilon > 0$ such that $\forall t \in (u \sim \epsilon, u]$, $\hat{h}_k(t) = \psi_{\frac{k}{\psi^{-1}(u)}} \left(k/\psi^{-1}(t)\right)$. Thus:

$$(\hat{h}_k)'_-(u) = \left(\psi_{\frac{k}{\psi^{-1}(u)}} \right)'_+ \left(\frac{k}{\psi^{-1}(u)} \right) \cdot \frac{-k}{(\psi^{-1}(u))^2} \cdot \frac{1}{(\psi_n)'(\psi^{-1}(u))}.$$

Let us now examine the value of $(\hat{h}_k)'_+(u)$. For a sufficiently small $\epsilon > 0$ we obtain (Fig. 5).

$$[u, u \oplus \varepsilon) \overset{\psi^{-1}}{\to} \left[\psi^{-1}(u), \psi^{-1}(u \oplus \varepsilon)\right) \overset{h_k}{\to}$$

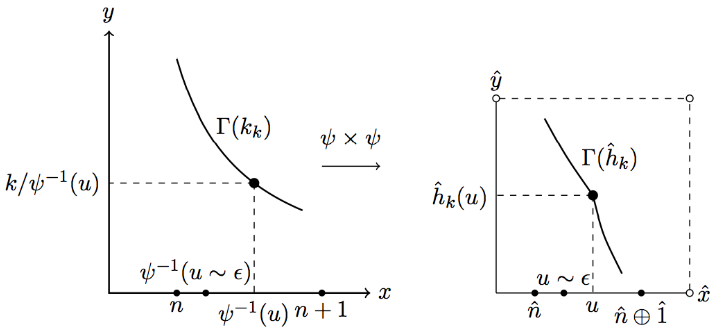


Figure 4. Finding $(\hat{h}_k)'_-(u)$.

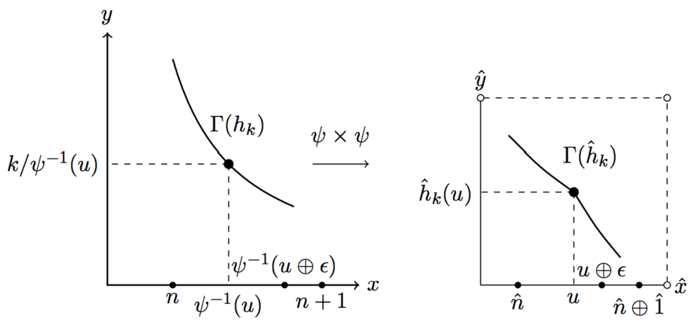


Figure 5. Finding $(\hat{h}_k)'_+(u)$.

$$\left(\frac{k}{\psi^{-1}(u \oplus \epsilon)}, \frac{k}{\psi^{-1}(u)}\right] \xrightarrow{\psi} \left(\hat{k} \div \widehat{\psi^{-1}(u \oplus \epsilon)}, \hat{k} \div \widehat{\psi^{-1}(u)}\right].$$

We can choose $\epsilon > 0$ such that $n < \psi^{-1}(u) < \psi^{-1}(u \oplus \epsilon) < n+1$ and as a consequence for every $t \in [u, u \oplus \epsilon)$ we verify $k/\psi^{-1}(t) \leq k/\psi^{-1}(u)$. That is, we can choose $\epsilon > 0$ such that $\forall t \in [u, u \oplus \epsilon)$,

$$\hat{h}_k(t) = \psi_{\frac{k}{\psi^{-1}(u)}-1}\left(\frac{k}{\psi^{-1}(t)}\right).$$

Would result:

$$(\widehat{h}_k)'_+(u) = \left(\psi_{\frac{k}{\psi^{-1}(u)}}-1\right)'_-\left(\frac{k}{\psi^{-1}(u)}\right) \cdot \frac{-k}{(\psi^{-1}(u))^2} \cdot \frac{1}{(\psi_n)'(\psi^{-1}(u))}.$$

<u>Case 2</u> $u = \hat{n}$ $(n \in \mathbb{N}^*)$ that is, u is a ψ-natural number ($u > 0$). For a sufficiently small $\epsilon > 0$ and $\psi(n+\delta) = \hat{n} \oplus \epsilon$ we obtain (Fig. 6)

$$[\hat{n}, \hat{n} \oplus \epsilon) \xrightarrow{\psi^{-1}} [n, n+\delta) \xrightarrow{h_k} \left(\frac{k}{n+\delta}, \frac{k}{n}\right] \xrightarrow{\psi} \left(\hat{k} \div (\hat{n} \oplus \hat{\delta}), \hat{k} \div \hat{n}\right].$$

For every $t \in [\hat{n}, \hat{n} \oplus \epsilon)$, we verify $\hat{h}_k(t) = \psi_{\lfloor \frac{k}{n} \rfloor}(k/\psi^{-1}(t))$ if $k/n \notin \mathbb{N}^*$ and $\hat{h}_k(t) = \psi_{\frac{k}{n}-1}(k/\psi^{-1}(t))$ if $k/n \in \mathbb{N}^*$. As a consequence

$$(\hat{h}_k)'_+(\hat{n}) = \left(\psi_{\lfloor \frac{k}{n} \rfloor}\right)'\left(\frac{k}{n}\right) \cdot \frac{-k}{n^2} \cdot \frac{1}{(\psi_n)'_+(n)} \text{ (if } k/n \notin \mathbb{N}^*),$$

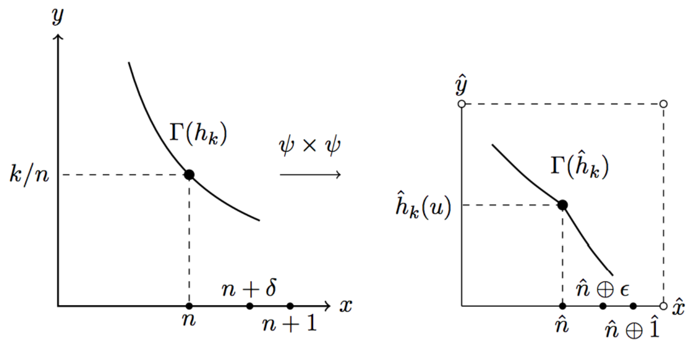


Figure 6. Finding $(\hat{h}_k)'_+(\hat{n})$.

$$(\hat{h}_k)'_+(\hat{n}) = \left(\psi_{\frac{k}{n}-1}\right)'_- \left(\frac{k}{n}\right) \cdot \frac{-k}{n^2} \cdot \frac{1}{(\psi_n)'_+(n)} \text{ (if } k/n \in \mathbb{N}^*).$$

Finally we have to study the differentiability of $\hat{h}_k$ at $u = \hat{n}$ from the left side. For a sufficiently small $\epsilon > 0$ and $\psi(n-\delta) = \hat{n} \sim \epsilon$, we obtain (fig. 7)

$$(\hat{n} \sim \epsilon, \hat{n}] \xrightarrow{\psi^{-1}} (n-\delta, n] \xrightarrow{h_k} \left[\frac{k}{n}, \frac{k}{n-\delta}\right) \xrightarrow{\psi} \left[\hat{k} \div \hat{n}, \hat{k} \div (\hat{n} \sim \hat{\delta})\right).$$

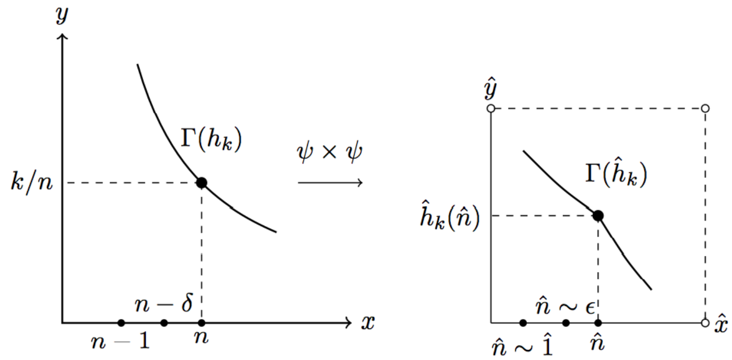


Figure 7. Finding $(\hat{h}_k)'_-(\hat{n})$.

We can choose $\epsilon > 0$ such that $\forall t \in (\hat{n} \sim \epsilon, \hat{n}]$ we verify

$$\hat{h}_k(t) = \psi_{\lfloor \frac{k}{n} \rfloor}\left(\frac{k}{\psi^{-1}(t)}\right)$$

regardless of whether k/n is a natural number or not. This therefore would result

$$(\hat{h}_k)'_-(\hat{n}) = \left(\psi_{\lfloor \frac{k}{n} \rfloor}\right)'_+ \left(\frac{k}{n}\right) \cdot \frac{-k}{n^2} \cdot \frac{1}{(\psi_{n-1})'_-(n)}.$$

We have completed our examination of the differentiability of $\hat{h}_k$ when dependent and/or independent variables take ψ-natural number values or not. Since $(\psi_{i-1})'_-(i) = a_i$ and $(\psi_i)'_+(i) = b_i$ $(i = 1, 2, 3, \ldots)$, the proposition is proven. $\square$

Corollary 1.5. If we want the $\hat{h}_k$ functions to be only differentiable at the points where both the ordinate and the abscissa are not ψ-natural numbers, we must select ψ in such a way that

$$(a_n \neq b_n) \wedge (a_m \neq b_m) \wedge (a_n a_m \neq b_n b_m) \quad \forall n \in \mathbb{N}^*, \forall m \in \mathbb{N}^*. \tag{1}$$

Definition 1.6. We say that an $\mathbb{R}^+$ coding function *identifies primes* if the $\hat{h}_k$ functions are only differentiable at the non-ψ-natural number abscissa and ordinate points

1.3 Classification of points in the $\hat{x}\hat{y}$ plane

Let $\psi : \mathbb{R}^+ \to [0, M_\psi)$ be an $\mathbb{R}^+$ coding function that identifies primes. The class of sets $\mathcal{H} = \{\Gamma(h_k) : k \in \mathbb{R}^+ - \{0\}\}$ is a partition of $(0, +\infty)^2$ and being ψ a bijective function, the class $\widehat{\mathcal{H}} = \{\Gamma(\hat{h}_k) : k \in \mathbb{R}^+ - \{0\}\}$ of all ψ-hyperbolas is a partition of $\left(0, M_\psi\right)^2$. Every subset of $\mathbb{R}^2$ will be considered as a topological subspace of $\mathbb{R}^2$ with the usual topology. We have the following cases:

1.- $(\hat{x}, \hat{y}) \in \left(0, M_\psi\right)^2$ $(x \notin \mathbb{N} \wedge y \notin \mathbb{N})$. Then, in a neighbourhood V of $(\hat{x}, \hat{y})$ we verify: $\forall (\hat{s}, \hat{t}) \in V$, the ψ-hyperbola which contains $(\hat{s}, \hat{t})$ is differentiable at $\hat{s}$. Of course, we mean to say the function which represents the graph of the ψ-hyperbola (Fig. 8).

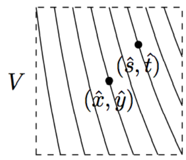


Figure 8. $x \notin \mathbb{N}, y \notin \mathbb{N}$.

2.- $(\hat{x}, \hat{m}) \in \left(0, M_\psi\right)^2$ $(x \notin \mathbb{N} \wedge m \in \mathbb{N}^*)$. Then, in a neighbourhood V of $(\hat{x}, \hat{m})$ we verify: $\forall (\hat{s}, \hat{t}) \in V$, the ψ-hyperbola which contains $(\hat{s}, \hat{t})$ is differentiable at $\hat{s}$ iff $\hat{t} \neq \hat{m}$ (Fig 9).

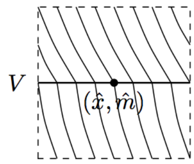


Figure 9. $x \notin \mathbb{N}, m \in \mathbb{N}^*$.

3.- $(\hat{n}, \hat{y}) \in \left(0, M_\psi\right)^2$ $(n \in \mathbb{N}^* \wedge y \notin \mathbb{N})$. Then, in a neighbourhood V of $(\hat{n}, \hat{y})$ we verify: $\forall (\hat{s}, \hat{t}) \in V$, the ψ-hyperbola which contains $(\hat{s}, \hat{t})$ is differentiable at $\hat{s}$ iff $\hat{s} \neq \hat{n}$ (Fig. 10).

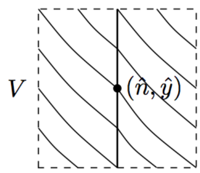


Figure 10. $n \in \mathbb{N}^*, y \notin \mathbb{N}$.

4.- $(\hat{n}, \hat{m}) \in \left(0, M_\psi\right)^2$ $(n \in \mathbb{N}^* \wedge m \in \mathbb{N}^*)$. Then, in a neighbourhood V of $(\hat{n}, \hat{m})$ we verify: $\forall (\hat{s}, \hat{t}) \in V$, the ψ-hyperbola which contains $(\hat{s}, \hat{t})$ is differentiable at $\hat{s}$ iff $\hat{s} \neq \hat{n}$ and $\hat{t} \neq \hat{m}$ (Fig. 11).

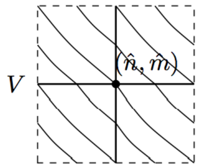


Figure 11. $n \in \mathbb{N}^*, m \in \mathbb{N}^*$.

Given the symmetry of the ψ-hyperbolas with respect to the line $\hat{x} = \hat{y}$, let us consider the triangular region of the $\hat{x}\hat{y}$ plane $\mathcal{T}_\psi = \left\{(\hat{x}, \hat{y}) : \hat{y} \geq \hat{x}, \hat{x} > \hat{0}\right\}$.

Definition 1.7. Let ψ be an $\mathbb{R}^+$ coding function that identifies primes and assume that $(\hat{x}, \hat{y}) \in \mathcal{T}_\psi$. If $(\hat{x}, \hat{y}) = (\hat{n}, \hat{m})$ with $n \in \mathbb{N}^*$, $m \in \mathbb{N}^*$ we say that it is a *vortex point* with respect to ψ (Fig. 12).

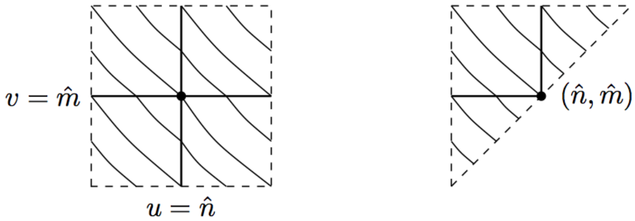


Figure 12. Vortex points.

The existence of vortex points in a ψ-hyperbola allows us to identify ψ-natural numbers (ψ-prime iff we have only one vortex point). We call this *Hyperbolic Classification of Natural Numbers.*

Corollary 1.8. Let $\hat{k} \in \left(\hat{0}, M_\psi\right)$. According to the statements made above, we may classify $\hat{k}$ in terms of the behaviour of ψ-hyperbolas in $\mathcal{T}_\psi$ that are near the ψ-hyperbola $\hat{x} \otimes \hat{y} = \hat{k}$. We obtain the following classification:
1) $\hat{k}$ is a ψ-natural number iff the ψ-hyperbola $\hat{x} \otimes \hat{y} = \hat{k}$ in $\mathcal{T}_\psi$ contains at least a vortex point.
2) $\hat{k}$ is a ψ-prime number iff $\hat{k} \neq \hat{1}$ and the ψ-hyperbola $\hat{x} \otimes \hat{y} = \hat{k}$ in $\mathcal{T}_\psi$ contains one and only one vortex point.
3) $\hat{k}$ is a ψ-composite number iff the ψ-hyperbola $\hat{x} \otimes \hat{y} = \hat{k}$ in $\mathcal{T}_\psi$ contains at least two vortex points.
4) $\hat{k}$ is not a ψ-natural number iff the ψ-hyperbola $\hat{x} \otimes \hat{y} = \hat{k}$ in $\mathcal{T}_\psi$ does not contain vortex points.

So, vortex points are characterized in terms of differentiability of the ψ-hyperbolas in $\mathcal{T}_\psi$ near these points. For every $k > 0$, denote $\bar{k} := \Gamma(\hat{h}_k) \cap \mathcal{T}_\psi$ and let $\bar{0}$ be one element different from $\bar{k}$ $(k > 0)$. Define $\mathfrak{R} = \left\{\bar{k} : k \geq 0\right\}$ and consider the operations on $\mathfrak{R}$:

$$(a) \;\; \bar{k} + \bar{s} = \overline{k+s}\,,\; \bar{k} \cdot \bar{s} = \overline{k \cdot s} \quad (k > 0, s > 0).$$
$$(b) \;\; \bar{t} + \bar{0} = \bar{0} + \bar{t} = \bar{t}\,,\; \bar{t} \cdot \bar{0} = \bar{0} \cdot \bar{t} = \bar{0} \quad (t \geq 0).$$

Then, $(\mathfrak{R}, +, \cdot)$ is an isomorphic structure to the usual one $(\mathbb{R}^+, +, \cdot)$ and prime numbers $p \in \mathbb{N}$ are characterized by the fact that $\bar{p} \neq \bar{1}$ and $\bar{p}$ contains one and only one vortex point.

Amongst the $\mathbb{R}^+$ coding functions that identifies primes, it will be interesting to select those given by $\psi_m : [m, m+1] \to \mathbb{R}^+ \;\; (m = 0, 1, 2, ...)$ functions that are affine (Fig. 13).

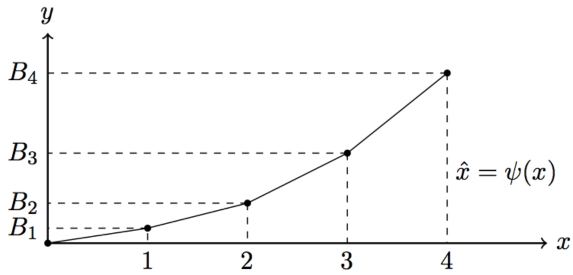


Figure 13. $\mathbb{R}^+$ *prime coding.*

$$\psi_m(x) = \xi_m(x - m) + B_m \ (\xi_m > 0 \ \forall m \in \mathbb{N}, B_0 = 0, B_m = \sum_{j=0}^{m-1} \xi_j \text{ if } m \geq 1). \tag{2}$$

We can easily prove that the ψ functions defined by means of the sequence $(\psi_m)_{m\geq 0}$ are $\mathbb{R}^+$ coding functions. Now, we have $\xi_0 = a_1$, $\xi_1 = a_2 = b_1$, $\xi_2 = a_3 = b_2$, $\xi_3 = a_4 = b_3 \ldots$, that is $a_n = \xi_{n-1}$, $a_m = \xi_{m-1}$, $b_n = \xi_n$, $b_m = \xi_m$. The conditions (1) for ψ to identify primes are clearly guaranteed by choosing ξ_i such that

$$0 < \xi_0 < \xi_1 < \xi_2 < \xi_3 < \ldots$$

though this is not the only way of choosing it.

Definition 1.9. Any $\mathbb{R}^+$ coding function ψ that is defined by means of ψ_m affine functions that also satisfies $0 < \xi_i < \xi_{i+1}$ $(\forall i \in \mathbb{N})$ it is said to be an $\mathbb{R}^+$ *prime coding*. We call the numbers $\xi_0, \xi_1, \xi_2, \xi_3, \ldots$ coefficients of the $\mathbb{R}^+$ prime coding.

2 Essential regions and Goldbach Conjecture

Goldbach's Conjecture is one of the oldest unsolved problems in number theory and in all of mathematics. It states: *Every even integer greater than 2 can be written as the sum of two primes* ($\mathcal{S}$). Furthermore, in his famous speech at the mathematical society of Copenhagen in 1921 G.H. Hardy pronounced that $\mathcal{S}$ is probably as difficult as any of the unsolved problems in mathematics and therefore Goldbach problem is not only one of the most famous and difficult problems in number theory, but also in the whole of mathematics ([5]). In this section, and using the Hyperbolic Classification of Natural Numbers we provide a characterization of $\mathcal{S}$.

In the $\hat{x}\hat{y}$ plane determined by any $\mathbb{R}^+$ prime coding function ψ and for any given ψ-even number $\hat{\alpha} \geq \widehat{16}$ we will consider the function in which any number $\hat{k}$ of the closed interval $[\hat{4}, \hat{\alpha} \div \hat{2}]$ corresponds to the area of the region of $\hat{x}\hat{y}$: $\hat{x} \geq \hat{2}, \hat{y} \geq \hat{x}, \hat{x} \otimes \hat{y} \leq \hat{k}$ (called lower area) and also the function that associates each to the area of the region of $\hat{x}\hat{y}$: $\hat{x} \geq \hat{2}, \hat{y} \geq \hat{x}, \hat{\alpha} \sim \hat{k} \leq \hat{x} \otimes \hat{y} \leq \hat{\alpha} \sim \hat{4}$ (called upper area). The $\hat{x}\hat{y}$ plane is considered imbedded in the xy plane with the Lebesgue Measure ([4]). This means that for any given ψ-even number $\hat{\alpha} \geq \widehat{16}$ we have $\hat{\alpha} = \hat{k} \oplus (\hat{\alpha} \sim \hat{k})$ and, associated to this decomposition, two data pieces, lower and upper areas. We will study if $\hat{\alpha}$ is the ψ-sum of the two ψ-prime numbers $\hat{k}_0$ and $\hat{\alpha} \sim \hat{k}_0$ taking into account the restrictions $\hat{\alpha} \sim \hat{3}$ and $\hat{\alpha} \div \hat{2}$ both ψ-composite. The upper and lower area functions will not yet yield any characterizations to the Goldbach Conjecture. We will need the second derivative of the total area function (the sum of the lower and upper areas).

To this end, we define the concept of essential regions associated to a hyperbola which, simply put, is any region in the xy plane with the shape $[n, n+1] \times [m, m+1]$ where n and m

are natural numbers, $m > n > 1$ and the hyperbola intersects it in more than one point or else the shape $[n, n+1]^2$ where $n > 1$ and $x \leq y$ and the hyperbola intersects in more than one point.

These essential regions are then transported to the $\hat{x}\hat{y}$ plane by means of the $\psi \times \psi$ function, and we will find the total area function adding the areas determined by each hyperbola in the respective essential regions, and the second derivative of this area function in each essential region. After this process we obtain the formula which determines the second derivative function of the total area $\widehat{A}_T$ in each sub-interval $[\hat{k}_0, \hat{k}_0 \oplus \hat{1}]$, $k_0 = 4, 5, ..., \alpha/2 - 1$ a derivative which is continuous.

$$(\widehat{A}_T)''(\hat{k}) = \frac{x_{k_0}}{\xi_{k_0}^2} \cdot \frac{1}{k} + \frac{y_{k_0}}{\xi_{\alpha-k_0-1}^2} \cdot \frac{1}{\alpha - k} \quad (\hat{k} \in [\hat{k}_0, \hat{k}_0 \oplus \hat{1}]).$$

Both x_{k_0} and y_{k_0} are numeric values in homogeneous polynomials of degree two obtained from substituting in their variables the ξ_i coefficients of the ψ $\mathbb{R}^+$ prime coding function . We call $P_{k_0} = \left(x_{k_0}, y_{k_0}\right)$ an essential point. The study of the behaviour of the second derivative in these intervals allows the following characterization of the Goldbach Conjecture for any even number $\alpha \geq 16$ with the restrictions $\alpha - 3$ and $\alpha/2$ composite.

Claim 2.1. Let $\alpha \geq 16$ be an even number. Then, α is the sum of two prime numbers k_0 and $\alpha - k_0$ $(5 \leq k_0 < \alpha/2)$ iff the consecutive essential points P_{k_0-1} and P_{k_0} are repeated, that is, if $P_{k_0-1} = P_{k_0}$.

Remark 2.2. Consider the infinite sets:

$$\begin{aligned} P &= \{\alpha \in \mathbb{N} : \alpha \text{ even}, \alpha \geq 16\}, \\ A &= \{\alpha \in \mathbb{N} : \alpha \text{ even}, \ \alpha \geq 16, \text{with } \alpha/2 \text{ and } \alpha - 3 \text{ composite}\}, \\ B &= \{\alpha \in \mathbb{N} : \alpha \text{ even}, \ \alpha \geq 16, \text{with } \alpha/2 \text{ prime or } \alpha - 3 \text{ prime}\}. \end{aligned}$$

Then, the Goldbach Conjecture is trivially satisfied in B, and P is a disjoint union of A and B so, Claim 2.1 provides a characterization of this conjecture in the infinite set A.

Definition 2.3. Consider the family of functions

$$\mathcal{H} = \{h_k : [\,2,\ \sqrt{k}\,] \to \mathbb{R},\ h_k(x) = k/x,\ k \geq 4\}$$

whose graphs represent the pieces of the hyperbolas $xy = k$ $(k \geq 4)$ included in the subset of $\mathbb{R}^2$, $S \equiv (x \geq 2) \wedge (x \leq y)$. For n, m natural numbers consider the subsets of $\mathbb{R}^2$:
a) $R_{(n,m)} = [n, n+1] \times [m, m+1] \quad (2 \leq n < m)$
b) $R_{(n,n)} = ([n, n+1] \times [n, n+1]) \cap \left\{(x, y) \in \mathbb{R}^2 : y \geq x\right\}$
Let h_k be an element of $\mathcal{H}$. We say that $R_{(n,m)}$ is a *square essential region* of h_k if $R_{(n,m)} \cap \Gamma(h_k)$ contains more than one point. We say that $R_{(n,n)}$ is a *triangular essential region* of h_k if $R_{(n,n)} \cap \Gamma(h_k)$ contains more than one point.

Example 2.4. The essential regions of the $xy = 17$ hyperbola are $R_{(2,8)}$, $R_{(2,7)}$, $R_{(2,6)}$, $R_{(2,5)}$, $R_{(3,5)}$, $R_{(3,4)}$ and $R_{(4,4)}$ (Fig. 14).

Analyse the different types of essential regions depending on the way the hyperbola $xy = k$ intersects with $R_{(n,m)}$ $(m > n)$. If the hyperbola passes through point $P(n, m+1)$ (Fig.15), then the equation for the hyperbola is $xy = n(m+1)$.

The abscissa of the Q point is $x = n(m+1)/m$. We verify that $n < n(m+1)/m < n+1$. This is equivalent to say $nm < nm + n$ and $nm + n < mn + m$ or equivalently $(0 < n) \wedge (n < m)$, which are trivially true.

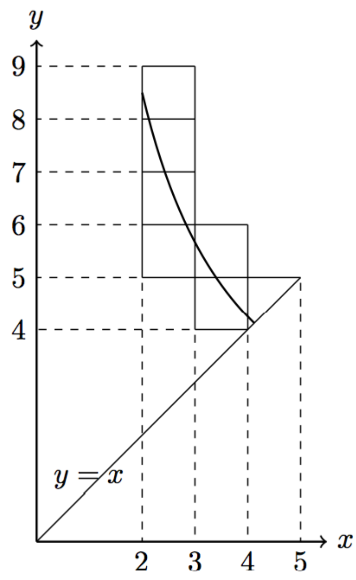


Figure 14. Essential regions of $xy = 17$.

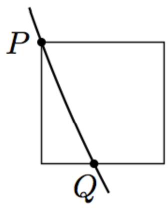


Figure 15. Intersection between hyperbolas and essential regions.

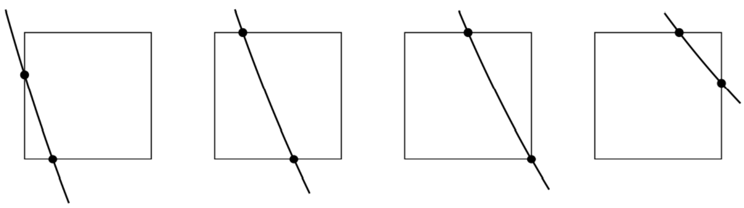


Figure 16. Types of square essential regions.

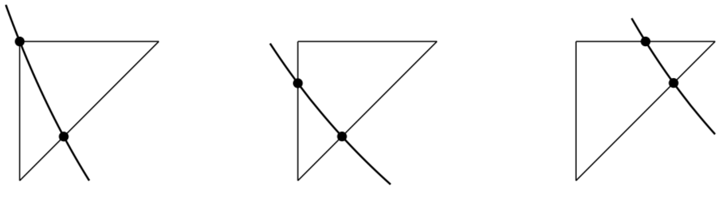


Figure 17. Types of triangular essential regions.

The remaining types are reasoned in a similar way (Fig. 16).

We use the same considerations for the triangular essential regions $R_{(n,n)}$ (Fig. 17).

Let $k_0 \in \mathbb{N}, k_0 \geq 4$. We will examine which are the types of essential regions for the hyperbolas $xy = k$ $(y \geq x)$ where $k_0 < k < k_0 + 1$. The passage through essential regions of points P_0, Q_0 of the $xy = k_0$ hyperbola with relation to P, Q points of the $xy = k$ hyperbola corresponds to the following diagrams (Fig. 18).

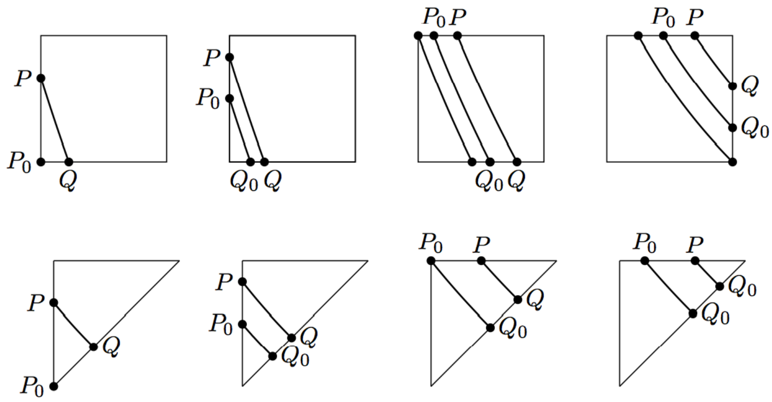


Figure 18. Square and triangular essential regions ($k_0 < k < k_0 + 1$).

As a consequence, the essential regions for the hyperbola $xy = k$ $(k > 4)$ are of the following types

a) Square essential regions $R_{(n,m)}$.

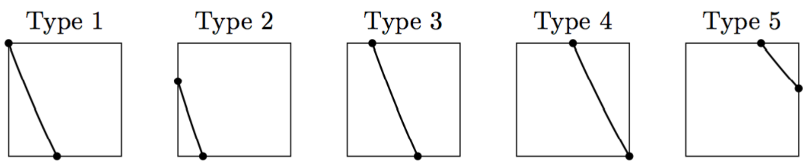


Figure 19. Square essential regions ($k > 4$).

b) Triangular essential regions $R_{(n,n)}$.

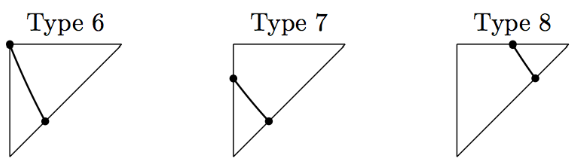


Figure 20. Triangular essential regions ($k > 4$).

We will find the essential regions of the $xy = k$ hyperbolas with the conditions $k_0 \in \mathbb{N},\ k_0 \geq 4,\ k_0 < k < k_0 + 1$. The abscissa of $xy = k_0$ varies in the interval $[2, \sqrt{k_0}]$ (Fig 21).

a) For $n \in \{2, 3, \ldots, \lfloor \sqrt{k_0} \rfloor - 1\}$ the $R_{(n,m)}$ essential regions of the $xy = k$ hyperbolas are obtained when m varies in the set (Fig. 22):

$$\{\lfloor k_0/(n+1) \rfloor, \lfloor k_0/(n+1) \rfloor + 1, \ldots, \lfloor k_0/n \rfloor\}.$$

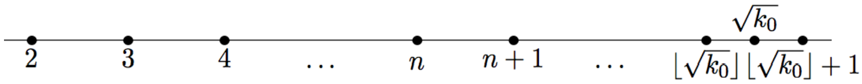


Figure 21. Finding all essential regions (1).

We can easily verify that if $m = \lfloor k_0/n \rfloor$ then $R_{(n,m)}$ is a square essential region of Type 2, if $m = \lfloor k_0/(n+1) \rfloor$, $R_{(n,m)}$ is a square essential region of Type 5 and the remaining $R_{(n,m)}$ are of Type 3.

b) For $n = \lfloor \sqrt{k_0} \rfloor$, the $R_{(\lfloor \sqrt{k_0} \rfloor, m)}$ essential regions are obtained when m varies in the set:

$$\{\lfloor \sqrt{k_0} \rfloor, \lfloor \sqrt{k_0} \rfloor + 1, \ldots, \lfloor k_0 / \lfloor \sqrt{k_0} \rfloor \rfloor\}.$$

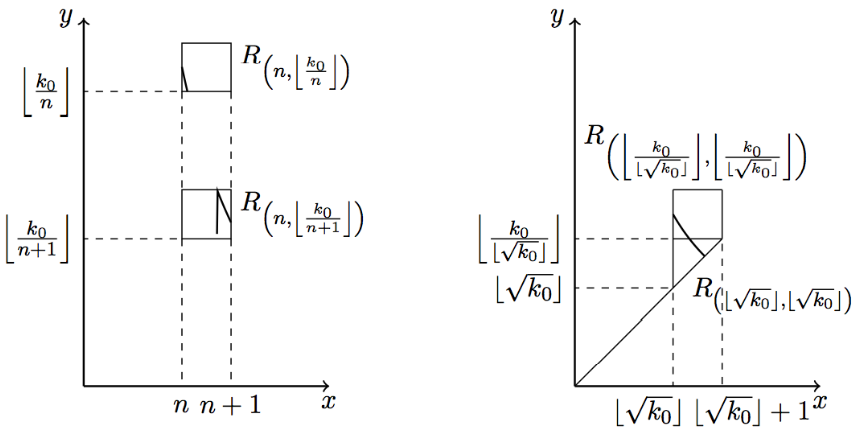


Figure 22. Finding all essential regions (2).

If $m = \lfloor \sqrt{k_0} \rfloor$ we obtain a triangular essential region and could eventually exist a square essential region (Fig. 22). Consider the set of indexes $\{(n, i_n)\}$ such that

(1) For $n = 2, 3, \ldots, \lfloor \sqrt{k_0} \rfloor - 1$ then

$$i_n = \lfloor k_0/(n+1) \rfloor, \lfloor k_0/(n+1) \rfloor + 1, \ldots, \lfloor k_0/n \rfloor.$$

(2) For $n = \lfloor \sqrt{k_0} \rfloor$ then

$$i_n = \lfloor \sqrt{k_0} \rfloor, \lfloor \sqrt{k_0} \rfloor + 1, \ldots, \lfloor k_0 / \lfloor \sqrt{k_0} \rfloor \rfloor.$$

Let $E_s(k_0)$ be the set $\{(n, i_n)\}$, where (n, i_n) are pairs of type (1) or of type (2). We obtain the following theorem:

Theorem 2.5. *Let $k_0 \in \mathbb{N}^*$ $(k_0 \geq 4)$. Then,*
i) All the $xy = k$ $(k_0 < k < k_0 + 1)$ hyperbolas have the same essential regions, each of the same type.
ii) The $xy = k$ essential regions are the elements of the set

$$\{R_{(n,i_n)} : (n, i_n) \in E_s(k_0)\}.$$

Example 2.6. For $k_0 = 18$ the essential regions of the $xy = k$ $(18 < k < 19)$ hyperbolas are (Fig. 23) $R_{(2,9)}$, $R_{(3,6)}$ (type 2), $R_{(2,8)}$, $R_{(2,7)}$, $R_{(3,5)}$ (type 3), $R_{(2,6)}$, $R_{(3,4)}$ (type 5) and $R_{(4,4)}$ (type 7). The essential regions of the $xy = k$ $(19 < k < 20)$ hyperbolas are exactly the same, due to the fact that 19 is a prime number.

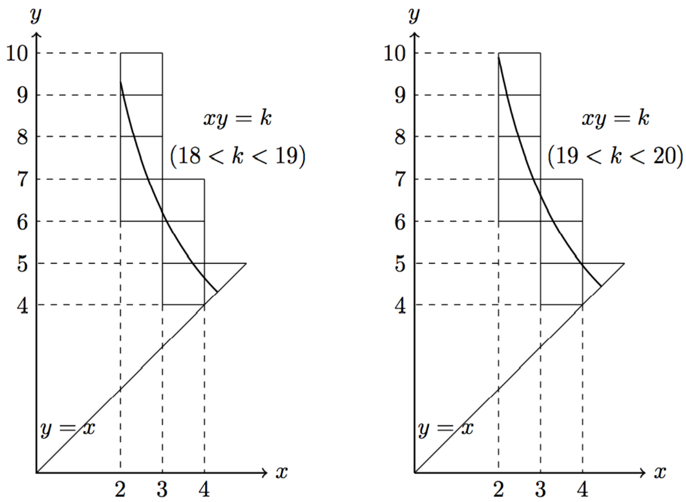


Figure 23. Essential regions ($18 < k < 19$ and $19 < k < 20$).

2.1 Areas in essential regions associated with a hyperbola

To every $R_{(n,m)}$ $(n \leq m)$ essential region of the $xy = k$ $(k \notin \mathbb{N}^*, k > 4)$ hyperbola, we will associate the region of the xy plane below the hyperbola (we call it $D_{(n,m)}(k)$). Denote $A_{(n,m)}(k)$ the area of $D_{(n,m)}(k)$. We have the following cases (Fig. 24).

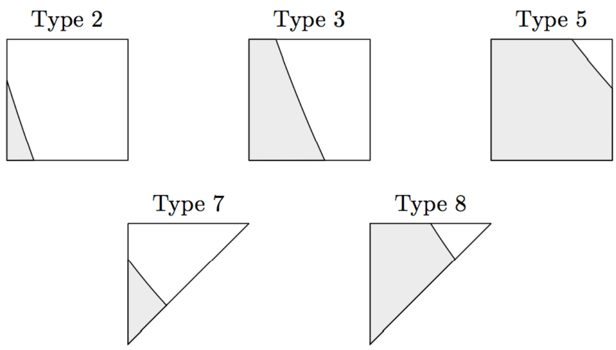


Figure 24. Areas in essential regions.

(i) Type 2 essential region

$A_{(n,m)}(k) = \iint_{D_{(n,m)}(k)} dxdy$ with $D_{(n,m)}(k) \equiv n \leq x \leq k/m$, $m \leq y \leq k/x$. Then,

$$A_{(n,m)}(k) = \int_n^{\frac{k}{m}} dx \int_m^{\frac{k}{x}} dy = \int_n^{\frac{k}{m}} \left(\frac{k}{x} - m \right) dx = k \log \frac{k}{nm} + nm - k.$$

If $k \in [k_0, k_0 + 1]$ ($k_0 \geq 4$ natural number), then $A'_{(n,m)}(k) = \log k/(nm)$ and the second derivative is $A''_{(n,m)}(k) = 1/k$. Note that we have used the closed interval $[k_0, k_0 + 1]$ so we may extend the definition of the essential region for $k \in \mathbb{N}$ $(k \geq 4)$ in a natural manner. In some cases the "essential region" would consist of a single point (null area).

(ii) Type 3 essential region

In this case $D_{(n,m)}(k) = D' \cup D''$ where $D' = [n, k/(m+1)] \times [m, m+1]$ and $D'' \equiv k/(m+1) < x \leq k/m$, $m \leq y \leq k/x$. Besides, $D' \cap D'' = \emptyset$.

$$A_{(n,m)}(k) = \iint_{D_{(n,m)}(k)} dxdy = \frac{k}{m+1} - n + \iint_{D''} dxdy$$

$$= \frac{k}{m+1} - n + k \log \frac{m+1}{m} + mk \left(\frac{1}{m+1} - \frac{1}{m} \right).$$

If $k_0 \leq k \leq k_0 + 1$ then, $A''_{(n,m)}(k) = 0$.

<u>(*iii*) Type 5 essential region</u>

In this case $D_{(n,m)}(k) = D' \cup D''$ where $D' = [n, k/(m+1)] \times [m, m+1]$ and $D'' \equiv k/(m+1) < x \leq n+1$, $m \leq y \leq k/x$. Besides, $D' \cap D'' = \emptyset$.

$$A_{(n,m)}(k) = \iint_{D_{(n,m)}(k)} dxdy = \frac{k}{m+1} - n + \iint_{D''} dxdy$$

$$= \frac{k}{m+1} - n + k \log \frac{(n+1)(m+1)}{k} - m \left(n + 1 - \frac{k}{m+1} \right).$$

In the interval $[k_0, k_0 + 1]$ we obtain $A'_{(n,m)}(k) = \log((n+1)(m+1)/k)$ and $A''_{(n,m)}(k) = -1/k$.

<u>(*iv*) Type 7 essential region</u>

$$D_{(n,n)}(k) \equiv n \leq x \leq \sqrt{k} \, , \; x \leq y \leq k/x.$$

$$A_{(n,n)}(k) = \int_n^{\sqrt{k}} dx \int_x^{\frac{k}{x}} dy = \int_n^{\sqrt{k}} \left(\frac{k}{x} - x \right) dx$$

$$= \left[k \log x - \frac{x^2}{2} \right]_n^{\sqrt{k}} = \frac{k}{2} \log k - \frac{k}{2} - k \log n + \frac{n^2}{2}.$$

If $k_0 \leq k \leq k_0 + 1$, $A'_{(n,n)}(k) = (1/2) \log k - \log n$ and $A''_{(n,n)}(k) = 1/(2k)$.

<u>(*v*) Type 8 essential region</u>

In this case $D_{(n,n)}(k) = D' \cup D''$ where $D' \equiv n \leq x \leq k/(n+1)$, $x \leq y \leq n+1$ and $D'' \equiv k/(n+1) < x \leq \sqrt{k}$, $x \leq y \leq k/x$. Besides, $D' \cap D'' = \emptyset$.

$$A_{(n,n)}(k) = \iint_{D'} dxdy + \iint_{D''} dxdy \int_n^{\frac{k}{n+1}} dx \int_x^{n+1} dy + \int_{\frac{k}{n+1}}^{\sqrt{k}} dx \int_x^{\frac{k}{x}} dy$$

$$= \int_n^{\frac{k}{n+1}} (n+1-x) \, dx + \int_{\frac{k}{n+1}}^{\sqrt{k}} \left(\frac{k}{x} - x \right) dx = \frac{k}{2} - n(n+1) + \frac{n^2}{2} + k \log \frac{n+1}{\sqrt{k}}.$$

If $k_0 \leq k \leq k_0 + 1$, $A'_{(n,n)}(k) = \log((n+1)/\sqrt{k})$ and $A''_{(n,n)}(k) = -1/(2k)$.

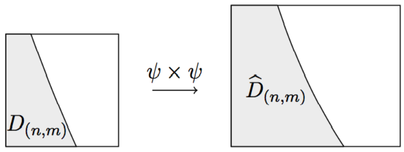


Figure 25. Relationship between $\widehat{A}_{(n,m)}$ and $A_{(n,m)}$.

2.2 Areas of essential regions in the $\hat{x}\hat{y}$ plane

Consider in the xy plane, an essential region $R_{(n,m)}(n \leq m)$ of the $xy = k$ $(k \geq 4)$ hyperbola and ψ an $\mathbb{R}^+$ prime coding function with ξ_i coefficients. Let $\widehat{R}_{(n,m)}$ be the corresponding region in the $\hat{x}\hat{y}$ plane that is, $\widehat{R}_{(n,m)} = (\psi \times \psi)(R_{(n,m)})$. We call $\widehat{A}_{(n,m)}$ the area of $\widehat{D}_{(n,m)} = (\psi \times \psi)(D_{(n,m)})$ supposing the $\hat{x}\hat{y}$ plane embedded in the xy plane.

Theorem 2.7. *In accordance with the aforementioned conditions*

$$\widehat{A}_{(n,m)} = \xi_n \xi_m A_{(n,m)}.$$

Proof. The transformation that maps $D_{(n,m)}$ in $\widehat{D}_{(n,m)}$ is $\hat{x} = \psi_n(x),\ \hat{y} = \psi_m(y)$. The Jacobian for this transformation is

$$J = \det \begin{bmatrix} \dfrac{\partial \hat{x}}{\partial x} & \dfrac{\partial \hat{x}}{\partial y} \\ \dfrac{\partial \hat{y}}{\partial x} & \dfrac{\partial \hat{y}}{\partial y} \end{bmatrix} = \det \begin{bmatrix} \psi_n'(x) & 0 \\ 0 & \psi_m'(y) \end{bmatrix} = \psi_n'(x)\psi_m'(y) \neq 0.$$

Thus, ([1]) $\widehat{A}_{(n,m)} = \iint_{\widehat{D}_{(n,m)}} d\hat{x}d\hat{y} = \iint_{D_{(n,m)}} |\psi_n'(x)\psi_m'(y)|\ dxdy$. Since ψ is an $\mathbb{R}^+$ prime coding function, then $|J| = \xi_n\xi_m$ and as a result the relationship between the areas of the essential regions in xy and in $\hat{x}\hat{y}$ is $\widehat{A}_{(n,m)} = \displaystyle\iint_{D_{(n,m)}} \xi_n\xi_m\ dxdy = \xi_n\xi_m \iint_{D_{(n,m)}} dxdy = \xi_n\xi_m A_{(n,m)}$. □

Let α be an even number. We will assume for technical reasons that $\alpha \geq 16$. Let $k \in [4, \alpha/2]$ and consider the subsets of $\mathbb{R}^2$ (FIg, 26):

$$D_I(k) = \{(x,y) \in \mathbb{R}^2 : x \geq 2,\ y \geq x,\ xy \leq k\},$$
$$D_S(k) = \{(x,y) \in \mathbb{R}^2 : x \geq 2,\ y \geq x,\ \alpha - k \leq xy \leq \alpha - 4\}.$$

Let ψ be an $\mathbb{R}^+$ prime coding function and consider the subsets of $[0, M_\psi)^2$

$$\widehat{D}_I(\hat{k}) = (\psi \times \psi)(D_I(k)) \ ,\ \widehat{D}_S(\hat{k}) = (\psi \times \psi)(D_S(k)).$$

We now define the functions

1) $\widehat{A}_I : [\hat{4}, \hat{\alpha} \div \hat{2}] \to \mathbb{R}^+$, $\hat{k} \to \widehat{A}_I(\hat{k})$ (area of $\widehat{D}_I(\hat{k})$).
2) $\widehat{A}_S : [\hat{4}, \hat{\alpha} \div \hat{2}] \to \mathbb{R}^+$, $\hat{k} \to \widehat{A}_S(\hat{k})$ (area of $\widehat{D}_S(\hat{k})$).
3) $\widehat{A}_T : [\hat{4}, \hat{\alpha} \div \hat{2}] \to \mathbb{R}^+$, $\widehat{A}_T = \widehat{A}_I + \widehat{A}_S$.

Let α be an even number $(\alpha \geq 16)$ and ψ an $\mathbb{R}^+$ prime coding function with coefficients ξ_i. We take $k_0 = 4, 5, \ldots, \alpha/2 - 1$ and we study the second derivative of $\widehat{A}_I$ at each closed interval $[\hat{k}_0, \hat{k}_0 \oplus \hat{1}]$. For this, we consider the corresponding function $A_I(k)$. Then $\forall k \in [k_0, k_0 + 1]$ we

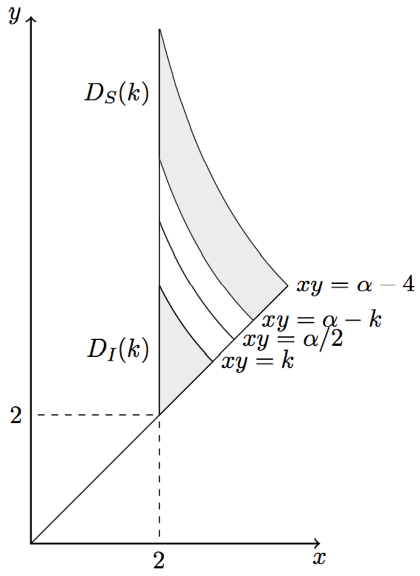


Figure 26. $D_I(k)$ and $D_S(k)$.

verify $A_I(k) = A_I(k_0) + A_I(k) - A_I(k_0)$. Additionally, $A_I(k) - A_I(k_0)$ is the sum of the areas in the essential regions associated with the $xy = k$ hyperbola, minus the area in the essential regions associated with the $xy = k_0$ hyperbola so,

$$A_I(k) - A_I(k_0) = \sum_{(n,i_n)\in E_S(k_0)} \left[A_{(n,i_n)}(k) - A_{(n,i_n)}(k_0) \right].$$

We know that functions $A_{(n,i_n)}(k)$ have a second derivative in $[k_0, k_0+1]$, therefore

$$A_I''(k) = \sum_{(n,i_n)\in E_S(k_0)} A''_{(n,i_n)}(k) \quad (\forall k \in [k_0, k_0+1]).$$

We now want to find the expression of $(\widehat{A}_I)''$ as a function of the variable $\hat{k}$, where $\hat{k} \in [\hat{k}_0, \hat{k}_0 \oplus \hat{1}]$. By proposition 2.7, $\widehat{A}_{(n,m)}(\hat{k}) = \xi_n \xi_m A_{(n,m)}(k)$. If we derive with respect to $\hat{k}$, we obtain

$$(\widehat{A}_{(n,m)})'(\hat{k}) = \xi_n \xi_m A'_{(n,m)}(k) \frac{dk}{d\hat{k}}.$$

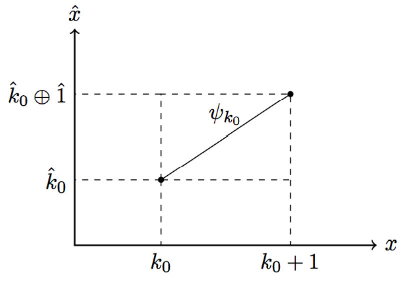


Figure 27. Finding $(\widehat{A}_{(n,m)})''(\hat{k})$).

At $k \in [k_0, k_0+1]$, the expression of $\hat{k}$ is $\hat{k} = \xi_{k_0}(k - k_0) + B_{k_0}$ (2). Then $dk/d\hat{k} = 1/\xi_{k_0}$, therefore $(\widehat{A}_{(n,m)})'(\hat{k}) = (\xi_n \xi_m / \xi_{k_0}) A'_{(n,m)}(k)$. Deriving once again:

$$(\widehat{A}_{(n,m)})''(\hat{k}) = \frac{\xi_n \xi_m}{\xi_{k_0}^2} A''_{(n,m)}(k).$$

We get the following theorem:

Theorem 2.8. *Let α be an even number ($\alpha \geq 16$). Then for every $\hat{k}_0 = \hat{4}, \hat{5}, \ldots, (\hat{\alpha} \div \hat{2}) \sim \hat{1}$*

a) $(\widehat{A}_I)''(\hat{k}) = \displaystyle\sum_{(n,i_n) \in E_S(k_0)} (\widehat{A}_{(n,i_n)})''(\hat{k}) \quad (\forall \hat{k} \in [\hat{k}_0, \hat{k}_0 \oplus \hat{1}])$.

b) For $\hat{k} \in [\hat{k}_0, \hat{k}_0 \oplus \hat{1}]$ and bearing in mind the different types of essential regions, we obtain

(i) Type 2 essential region: $(\widehat{A}_{(n,m)})''(\hat{k}) = \dfrac{\xi_n \xi_m}{\xi_{k_0}^2} \cdot \dfrac{1}{k}$.

(ii) Type 3 essential region: $(\widehat{A}_{(n,m)})''(\hat{k}) = 0$.

(iii) Type 5 essential region: $(\widehat{A}_{(n,m)})''(\hat{k}) = -\dfrac{\xi_n \xi_m}{\xi_{k_0}^2} \cdot \dfrac{1}{k}$.

(iv) Type 7 essential region: $(\widehat{A}_{(n,n)})''(\hat{k}) = \dfrac{\xi_n^2}{\xi_{k_0}^2} \cdot \dfrac{1}{2k}$.

(v) Type 8 essential region: $(\widehat{A}_{(n,n)})''(\hat{k}) = -\dfrac{\xi_n^2}{\xi_{k_0}^2} \cdot \dfrac{1}{2k}$.

Example 2.9. We will find $(\widehat{A}_I)''(\hat{k})$ in $[\widehat{12}, \widehat{13}]$ with $\hat{\alpha} \geq \widehat{26}$ (Fig. 28).

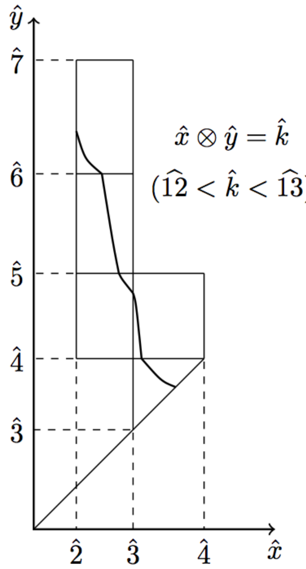


Figure 28. Finding $(\widehat{A}_I)''(\hat{k})$ in $[\widehat{12}, \widehat{13}]$.

$$\begin{aligned}(\widehat{A}_I)''(\hat{k}) &= \frac{\xi_2\xi_6}{\xi_{12}^2}\cdot\frac{1}{k} - \frac{\xi_2\xi_4}{\xi_{12}^2}\cdot\frac{1}{k} + \frac{\xi_3\xi_4}{\xi_{12}^2}\cdot\frac{1}{k} - \frac{1}{2}\cdot\frac{\xi_3^2}{\xi_{12}^2}\cdot\frac{1}{k}\\ &= \frac{1}{k\xi_{12}^2}(\xi_2\xi_6 - \xi_2\xi_4 + \xi_3\xi_4 - \xi_3^2/2)\end{aligned}$$

Now, consider the polynomial $p(t_2, t_3, t_4, t_6)) = t_2t_6 - t_2t_4 + t_3t_4 - t_3^2/2$. We call this polynomial a *lower essential polynomial* of $k_0 = 12$ and we write it as P_{I,k_0}. Let us generalize this definition.

Definition 2.10. Let α be an even number ($\alpha \geq 16$). The polynomial obtained naturally by removing the common factor function $1/(k\xi_0^2)$ in $(\widehat{A}_I)''(\hat{k})$ in the interval $[\hat{k}_0, \hat{k}_0 \oplus \hat{1}]$ ($k_0 = 4, 5, \dots, \alpha/2 - 1$) is called a *lower essential polynomial* of k_0. It is written as P_{I,k_0}.

Remarks 2.11. (*i*) Lower essential polynomials are homogeneous polynomials of degree 2. (*ii*) The variables that intervene in P_{I,k_0} are at most t_n and t_{i_n} where $(n, i_n) \in E_s(k_0)$, some of which may be missing (those which correspond to essential regions in which the second derivative is 0). (*iii*) We will also use P_{I,k_0} as the coefficient of $1/(k\xi_{k_0}^2)$ in $(\widehat{A}_I)''(\hat{k})$.

Corollary 2.12. Let α be an even number ($\alpha \geq 16$). Then, $\forall \hat{k} \in [\hat{k}_0, \hat{k}_0 \oplus \hat{1}]$ with $\hat{k}_0 \in \{\hat{4}, \hat{5}, \dots, \hat{\alpha} \div \hat{2} \sim \hat{1}\}$ we verify $(\widehat{A}_I)''(\hat{k}) = P_{I,k_0}/(k\xi_{k_0}^2)$.

2.3 $(\widehat{A}_S)''$ and $(\widehat{A}_T)''$ functions

Let α be an even number ($\alpha \geq 16$). We take $k_0 \in \{4, 5, \dots, \alpha/2 - 1\}$ and we examine the second derivative of $\widehat{A}_S$ at each closed interval $[\hat{k}_0, \hat{k}_0 \oplus \hat{1}]$. Then, $\forall k \in [k_0, k_0 + 1]$ we verify $A_S(k) = A_S(k_0) + A_S(k) - A_S(k_0)$. Additionally, $A_S(k) - A_S(k_0)$ is the area included between the curves

$$xy = \alpha - k_0,\ xy = \alpha - k,\ x = 2,\ y = x.$$

As a result, it is the sum of the areas in the essential regions of the $xy = \alpha - k_0$ hyperbola minus the area in the essential regions of $xy = \alpha - k$. We obtain:

$$A_S\,(k) - A_S\,(k_0) = \sum_{(n,i_n)\in E_S(\alpha-k_0-1)} \Big[A_{(n,i_n)}\,(\alpha - k_0) - A_{(n,i_n)}\,(\alpha - k)\Big],$$

$$A_S''(k) = -\sum_{(n.i_n)\in E_S(\alpha-k_0-1)} A_{(n,i_n)}''(\alpha - k).$$

Of course, the same relationships as in the lower areas are maintained with the expression $(\widehat{A}_S)''$ as a function of $\hat{k}$. We are left with:

$$(\widehat{A}_{(n,i_n)})''(\hat{k}) = -\frac{\xi_n\xi_{i_n}}{\xi_{\alpha-k_0-1}^2}\cdot A_{(n,i_n)}''(\alpha - k).$$

We define upper essential polynomial in a similar way we defined lower essential polynomial and we write them as P_{S,k_0}. The same remarks are maintained.

Remarks 2.13. (*i*) Upper essential polynomials are homogeneous polynomials of degree 2. (*ii*) The variables that intervene in P_{S,k_0} are at most $t_n,\ t_{i_n}$ where $(n, i_n) \in E_S(\alpha - k_0 - 1)$, some of which may be missing (those which correspond to essential regions in which the second derivative is 0). (*iii*) We will also use P_{S,k_0} as the coefficient of $1/(\alpha - k)\xi_{\alpha-k_0-1}^2$ in $(\widehat{A}_S)''(\hat{k})$.

2.4 Signs of the essential point coordinates

Definition 2.14. Let ψ be an $\mathbb{R}^+$ prime coding function and α an even number ($\alpha \geq 16$). For $k_0 \in \{4, 5, \ldots, \alpha/2 - 1\}$ we write $P_{k_0} = (x_{k_0}, y_{k_0}) = (P_{I,k_0}, P_{S,k_0})$. We call any P_{k_0} an *essential point* associated with ψ.

Hence, we can express

$$(\widehat{A}_T)''(\hat{k}) = \frac{x_{k_0}}{\xi_{k_0}^2} \cdot \frac{1}{k} + \frac{y_{k_0}}{\xi_{\alpha-k_0-1}^2} \cdot \frac{1}{\alpha - k} \quad (\hat{k} \in [\hat{k}_0, \hat{k}_0 \oplus \hat{1}]). \tag{3}$$

The formula from proposition 2.8 is

$$(\widehat{A}_I)''(\hat{k}) = \sum_{(n.i_n) \in E_S(k_0)} (\widehat{A}_{(n,i_n)})''(\hat{k}) \quad (\forall \hat{k} \in [\hat{k}_0, \hat{k}_0 \oplus \hat{1}]).$$

where the $E_S\,(k_0)$ sub-indexes are:
For $n = 2, 3, \ldots\,, \lfloor\sqrt{k_0}\rfloor - 1$,

$$i_n = \lfloor k_0/(n+1)\rfloor, \lfloor k_0/(n+1)\rfloor + 1, \ldots\,, \lfloor k_0/n\rfloor. \tag{4}$$

For $n = \lfloor\sqrt{k_0}\rfloor$,

$$i_n = \lfloor\sqrt{k_0}\rfloor, \lfloor\sqrt{k_0}\rfloor + 1, \ldots\,, \lfloor\, k_0/\lfloor\sqrt{k_0}\rfloor\,\rfloor. \tag{5}$$

Thus, for sub-index n in (1), in $(\widehat{A}_I)''$ only intervene $i_n = \lfloor k_0/(n+1)\rfloor$ and $i_n = \lfloor k_0/n\rfloor$, since we have already seen that all the sub-indexes included between them two, $(\widehat{A}_{(n,i_n)})''(\hat{k}) = 0$, as the essential regions are of type 3. In the lower essential polynomial we obtain $\xi_n(\xi_{\lfloor k_0/n\rfloor} - \xi_{[k_0/(n+1)]}) > 0$ (for any $\mathbb{R}^+$ prime coding function). For $n = \lfloor\sqrt{k_0}\;\rfloor$ we obtain the cases:

$$(i)\ \lfloor\sqrt{k_0}\rfloor = \lfloor k_0/\lfloor\sqrt{k_0}\rfloor\rfloor \quad (ii)\ \lfloor\sqrt{k_0}\rfloor < \lfloor k_0/\lfloor\sqrt{k_0}\rfloor\rfloor. \tag{6}$$

In case (i) we would obtain the addend $(1/2)\xi^2_{\lfloor\, k_0\,\rfloor}$, in case (ii) we would obtain (Fig. 29):

$$\xi_{\lfloor\sqrt{k_0}\rfloor}\xi_{\lfloor k_0/\lfloor\sqrt{k_0}\rfloor\rfloor} - (1/2)\xi^2_{\lfloor\sqrt{k_0}\rfloor} = \xi_{\lfloor\sqrt{k_0}\rfloor}(\xi_{\lfloor k_0/\lfloor\sqrt{k_0}\rfloor\rfloor} - (1/2)\xi_{\lfloor\sqrt{k_0}\rfloor}) > 0.$$

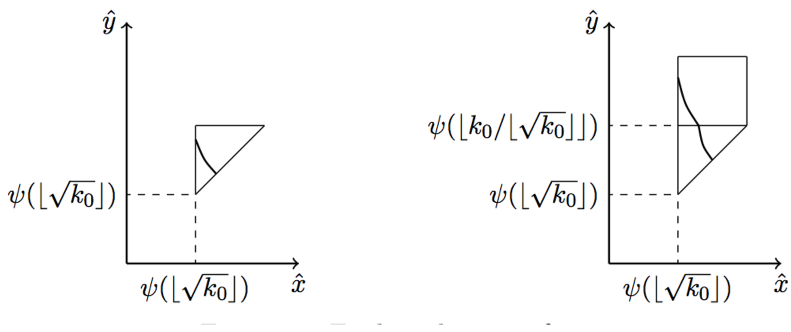


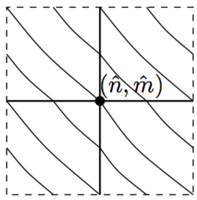
Figure 29. Finding the sign of x_{k_0}.

As a result, for an $\mathbb{R}^+$ prime coding function we obtain $x_4 > 0$, $x_5 > 0, \ldots,\ x_{\alpha/2-1} > 0$. The reasoning is entirely analogous for the upper essential polynomials that is, $y_4 < 0,\ y_5 < 0,\ \ldots,\ y_{\alpha/2-1} < 0$. We will now arrange the coordinates for the essential points.

1. Lower essential polynomials If $k_0 \in \mathbb{N},\ (k_0 > 4)$ is composite, there is at least one ψ- natural number coordinates point $(\hat{n}, \hat{m})$ such that $\hat{2} \leq \hat{n} \leq \hat{m}$ which the ψ-hyperbola $\hat{x} \otimes \hat{y} = \hat{k}_0$

goes through.

(a) If $2 < n < m$ we obtain the changes

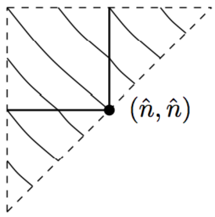


Changes

P_{I,k_0-1}	P_{I,k_0}
0	$-\xi_{n-1}\xi_m$
$-\xi_{n-1}\xi_{m-1}$	$\xi_n\xi_m$
$\xi_n\xi_{m-1}$	0

Figure 30. Arranging x_{k_0} in order. Case (a).

(b) If $2 < n = m$

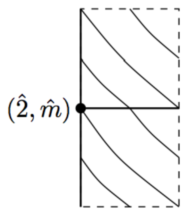


Changes

P_{I,k_0-1}	P_{I,k_0}
0	$-\xi_{n-1}\xi_n$
$-\xi_{n-1}^2/2$	$\xi_n^2/2$

Figure 31. Arranging x_{k_0} in order. Case (b).

(c) If $2 = n < m$



Changes

P_{I,k_0-1}	P_{I,k_0}
0	$\xi_2\xi_m$
$\xi_2\xi_{m-1}$	0

Figure 32. Arranging x_{k_0} in order. Case (c).

Then $P_{I,k_0} - P_{I,k_0-1} > 0$, since where there are transformations we obtain, for any prime coding function, either (a) or (b) or (c)

$$
\begin{aligned}
(a) \quad & \xi_n\xi_m - \xi_{n-1}\xi_m + \xi_{n-1}\xi_{m-1} - \xi_n\xi_{m-1} = \\
& \xi_m\left(\xi_n - \xi_{n-1}\right) - \xi_{m-1}\left(\xi_n - \xi_{n-1}\right) = \\
& \left(\xi_n - \xi_{n-1}\right)\left(\xi_m - \xi_{m-1}\right) > 0.
\end{aligned}
$$

$$
(b) \quad \frac{\xi_n^2}{2} - \xi_{n-1}\xi_n + \frac{\xi_{n-1}^2}{2} = \frac{\xi_n^2 - 2\xi_{n-1}\xi_n + \xi_{n-1}^2}{2} = \frac{(\xi_n - \xi_{n-1})^2}{2} > 0.
$$

$$
(c) \quad \xi_2\xi_m - \xi_2\xi_{m-1} = \xi_2\left(\xi_m - \xi_{m-1}\right) > 0.
$$

If k_0 is prime then $x_{k_0-1} = x_{k_0}$ since the same essential regions exist for the hyperbolas $\hat{x} \otimes \hat{y} = \hat{k}$ in $(\hat{k}_0 \sim \hat{1}, \hat{k}_0) \cup (\hat{k}_0, \hat{k}_0 \oplus \hat{1})$. For $\alpha - k_0$ prime we obtain $y_{k_0} = y_{k_0-1}$ since the same essential regions exist for the hyperbolas $\hat{x} \otimes \hat{y} = \hat{\alpha} \sim \hat{k}$ if $\hat{k} \in (\hat{k}_0 \sim \hat{1}, \hat{k}_0) \cup (\hat{k}_0, \hat{k}_0 \oplus \hat{1})$.

2. Upper essential polynomials If $\alpha - k_0$ is composite, and reasoning in the same way, we obtain $P_{S,k_0} - P_{S,k_0-1} > 0$. For $\alpha - k_0$ prime we obtain $P_{S,k_0} = P_{S,k_0-1}$ since the same essential regions exist for the hyperbolas $\hat{x} \otimes \hat{y} = \hat{\alpha} \sim \hat{k}$ if $\hat{k} \in (\hat{k}_0 \sim \hat{1}, \hat{k}_0) \cup (\hat{k}_0, \hat{k}_0 \oplus \hat{1})$. We obtain the theorem:

Theorem 2.15. *Let α be an even number ($\alpha \geq 16$), and ψ an $\mathbb{R}^+$ prime coding function. Let $P_{k_0} = (x_{k_0}, y_{k_0})$ be the essential points. Then,*
(i) $0 < x_4 \leq x_5 \leq \ldots \leq x_{\alpha/2-1}$. Additionally, $x_{k_0-1} = x_{k_0} \Leftrightarrow k_0$ is prime.
(ii) $y_4 \leq y_5 \leq \ldots \leq y_{\alpha/2-1} < 0$. Additionally, $y_{k_0-1} = y_{k_0} \Leftrightarrow \alpha - k_0$ is prime.

Corollary 2.16. In the hypotheses from the above theorem: The even number α is the sum of two primes k_0 and $\alpha - k_0$, $k_0 \in \{5, 6, \ldots, \alpha/2 - 1\}$ iff the consecutive essential points P_{k_0-1} and P_{k_0} are repeated, that is $P_{k_0-1} = P_{k_0}$.

Remark 2.17. Corollary 2.16 proves Claim 2.1 and according to Remark 2.2 we have provided a characterization of de Goldbach Conjecture in the infinite set

$$A = \{\alpha \in \mathbb{N} : \alpha \text{ even}, \ \alpha \geq 16, \text{with } \alpha/2 \text{ and } \alpha - 3 \text{ composite}\}.$$

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