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# Generalized Bivariate Kummer-Beta Distribution

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## Abstract

A new bivariate beta distribution based on the Humbert's confluent hypergeometric function of the second kind is introduced. Various representations are derived for its product moments, marginal densities, marginal moments, conditional densities and entropies.

**Keywords:** Beta function; beta distribution; entropy; bivariate distribution; gamma function; Kummer-beta distribution.

## Distribución Kummer-beta bivariada generalizada

## Resumen

En este artículo se propone una nueva distribución beta bivariada basada en distribuciones hipergeométricas Humbert de segundo tipo. También se derivan las representaciones de las densidades marginales, momentos marginales y productos, densidades condicionales y entropía.

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**Palabras clave:** Función beta; distribución beta; entropía; distribución bivariada; función gama; distribución Kummer-beta.

## 1 Introduction

Bivariate beta distributions have attracted useful applications in several areas; for example, in the modeling of the proportions of substances in a mixture, brand shares, i.e., the proportions of brands of some consumer product that are bought by customers, proportions of the electorate voting for the candidate in a two-candidate election and the dependence between two soil strength parameters. They have also been used extensively as priors in Bayesian statistics. Bivariate beta distributions have also been applied to drought data. In this article, we introduce a new bivariate beta distribution and study its properties. The joint pdf of this new distribution is taken to be

$$f(x,y;a,b,c,\lambda_1,\lambda_2) = K(a,b,c,\lambda_1,\lambda_2)\times x^{a-1}y^{b-1}(1-x-y)^{c-1}\exp[-(\lambda_1 x+\lambda_2 y)], \quad (1)$$

where $x>0$, $y>0$, $x+y<1$, $a>0$, $b>0$, $c>0$, $-\infty<\lambda_i<\infty$, $i=1,2$ and $K(a,b,c,\lambda_1,\lambda_2)$ is the normalized constant given by

$$K(a,b,c,\lambda_1,\lambda_2) = \left\{B(a,b,c)\ \mathbf{\Phi}_2[a,b;a+b+c;-\lambda_1,-\lambda_2]\right\}^{-1}, \quad (2)$$

where $B(a,b,c)$ is the beta function of three arguments given in Definition A.4, and $\mathbf{\Phi}_2$ is the Humbert's confluent hypergeometric function of two variables defined in the Appendix. For $\lambda_1=\lambda_2=\lambda$, the density (1) reduces to a bivariate Kummer-beta density (Bran-Cardona, Orozco-Castañeda and Nagar [1]) and further, if $\lambda_1=\lambda_2=0$, it slides to a Dirichlet density with parameters $a,b$ and $c$. We will, therefore, call the distribution given by the density (1) the *Generalized Bivariate Kummer-beta distribution.* Throughout this work we will denote this distribution by $GBKB(a,b;c;\lambda_1,\lambda_2)$.

The Dirichiet distribution is of interest to those studying proportions, spacings, or the random division of an interval. In Bayesian analysis, the Dirichlet distribution is used as a conjugate prior distribution for the parameters of the multinomial distribution. However, the Dirichlet family is not sufficiently rich in scope to represent many important distributional assumptions, because the Dirichlet distribution has fewer number of parameters. The generalized bivariate Kummer-beta distribution is a generalization of the Dirichlet distribution (a bivariate beta distribution) with the added number of parameters and will enrich the existing class of bivariate beta distributions. Further, the proposed generalized bivariate Kummer-beta distribution which has an elementary pdf (except for the normalizing constant) is sufficiently flexible and can be used in place of other bivariate beta distributions. Needless to say that generalized bivariate Kummer-beta distribution is conjugate prior for the multinomial distribution.

For an in-depth review of known bivariate beta distributions and their applications, we refer our readers to excellent texts by Arnold, Castillo and Sarabia [2], Balakrishnan and Lai [3], Hutchinson and Lai [4, 5], Kotz, Balakrishnan and Johnson [6], and Mardia [7], and for some recent work the reader is refereed to Ghosh [8], Gupta, Orozco-Castañeda and Nagar [9], Nadarajah [10, 11], Nadarajah and Kotz [12, 13, 14], Nadarajah, Shih and Nagar [15], Bran-Cardona, Orozco-Castañeda and Nagar [1], Nagar, Nadarajah and Okorie [16], Orozco-Castañeda, Nagar and Gupta [17], and Sarabia and Castillo [18].

The matrix variate generalizations of beta and Dirichlet distributions have been defined and studied extensively. For example, see Gupta and Nagar [19], Gupta, Cardeño and Nagar [20], and Nagar and Gupta [21].

In this article we study several properties such as marginal and conditional distributions, joint moments, correlation, and mixture representation of the bivariate Kummer-beta distribution defined by the density (1). We also derive distributions of $X+Y$, $X/(X+Y)$ and $XY$, where $(X,Y) \sim GBKB(a,b;c;\lambda_1,\lambda_2)$.

## 2 Properties

In this section we study several properties of the generalized bivariate Kummer-beta distribution defined in Section 1.

Writing $-(\lambda_1 x+\lambda_2 y) = -\lambda_1+\lambda_1(1-x-y)+(\lambda_1-\lambda_2)y$, $\lambda_1 \geq \lambda_2$, the density given in (1) can be rewritten as

$$K(a,b,c,\lambda_1,\lambda_2)\; x^{a-1}y^{b-1}(1-x-y)^{c-1} \\ \times \exp[-\lambda_1+\lambda_1(1-x-y)+(\lambda_1-\lambda_2)y].$$

Expanding $\exp[\lambda_1(1-x-y)+(\lambda_1-\lambda_2)y]$ in power series and rearranging certain factors, the joint density of $X$ and $Y$ in (1) can also be expressed as

$$\left\{\mathbf{\Phi}_2[a,b;a+b+c;-\lambda_1,-\lambda_2]\exp(\lambda_1)\right\}^{-1}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{\Gamma(a+b+c)}{\Gamma(b)\Gamma(c)} \\ \times\frac{\Gamma(b+i)\Gamma(c+j)}{\Gamma(a+b+c+i+j)}\frac{(\lambda_1-\lambda_2)^i}{i!}\frac{\lambda_1^j}{j!}\frac{x^{a-1}y^{b+i-1}\,(1-x-y)^{c+j-1}}{B(a,b+i,c+j)}.$$

Similarly, for $\lambda_1 \leq \lambda_2$, the density given in (1) can be written as

$$\left\{\mathbf{\Phi}_2[a,b;a+b+c;-\lambda_1,-\lambda_2]\exp(\lambda_2)\right\}^{-1}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(c)} \\ \times\frac{\Gamma(a+i)\Gamma(c+j)}{\Gamma(a+b+c+i+j)}\frac{(\lambda_2-\lambda_1)^i}{i!}\frac{\lambda_2^j}{j!}\frac{x^{a+i-1}y^{b-1}\,(1-x-y)^{c+j-1}}{B(a+i,b,c+j)}.$$

Thus the generalized bivariate Kummer-beta distribution is an infinite mixture of Dirichlet distributions.

In Bayesian analysis, if the posterior distributions are in the same family as the prior probability distribution; the prior and posterior are then called conjugate distributions, and the prior is called a conjugate prior. In case of multinomial distribution, the usual conjugate prior is the Dirichlet distribution. If

$$P(r,s,f|x,y)=\binom{r+s+f}{r,s,f}x^r y^s(1-x-y)^f$$

and

$$p(x,y)=K(a,b,c,\lambda_1,\lambda_2)x^{a-1}y^{b-1}(1-x-y)^{c-1}\exp[-(\lambda_1 x+\lambda_2 y)],$$

where $x > 0$, $y > 0$, and $x + y < 1$, then

$$p(x, y \,|\, r, s, f) = K(a + r, b + s, c + f, \lambda_1, \lambda_2) \times x^{a+r-1} y^{b+s-1} (1 - x - y)^{c+f-1} \exp[-(\lambda_1 x + \lambda_2 y)].$$

Thus, the generalized bivariate family of distributions considered in this article is conjugate prior for the multinomial distribution.

A distribution is said to be negatively likelihood ratio dependent if the density $f(x, y)$ satisfies

$$f(x_1, y_1) f(x_2, y_2) \leq f(x_1, y_2) f(x_2, y_1)$$

for all $x_1 > x_2$ and $y_1 > y_2$ (Lehmann [22], Tong [23]). In the case of generalized bivariate Kummer-beta distribution for $c > 1$ the above inequality reduces to

$$(1 - x_1 - y_1)(1 - x_2 - y_2) < (1 - x_1 - y_2)(1 - x_2 - y_1)$$

which clearly holds. Hence, the bivariate distribution defined by the density (1) for $c > 1$ is negatively likelihood ratio dependent.

**Theorem 2.1.** Let $(X, Y) \sim GBKB(a, b; c; \lambda_1, \lambda_2)$, and define $S = X + Y$ and $W = X/(X + Y)$. Then, the density of $S$ is given by

$$K(a, b, c, \lambda_1, \lambda_2) \, \frac{\Gamma(a)\Gamma(b)}{\Gamma(a + b)} s^{a+b-1} (1 - s)^{c-1} \exp(-\lambda_2 s) \times {}_1F_1(a; a + b; -(\lambda_1 - \lambda_2) s), \quad 0 < s < 1,$$

and the density of $W$ is given by

$$K(a, b, c, \lambda_1, \lambda_2) \frac{\Gamma(a + b)\Gamma(c)}{\Gamma(a + b + c)} w^{a-1} (1 - w)^{b-1} \times {}_1F_1(a + b; a + b + c; -\lambda_2 - (\lambda_1 - \lambda_2) w), \quad 0 < w < 1.$$

*Proof.* Substituting $x = ws$ and $y = s(1 - w)$ with the Jacobian $J(x, y \rightarrow w, s) = s$, in the joint density of $X$ and $Y$, we obtain the joint density of $W$ and $S$ as

$$K(a, b, c, \lambda_1, \lambda_2) s^{a+b-1} (1 - s)^{c-1} w^{a-1} (1 - w)^{b-1}$$

$$\times \exp[-\lambda_2 s - (\lambda_1 - \lambda_2) w s], \qquad (3)$$

where $0 < s < 1$ and $0 < w < 1$. Now, integrating appropriately by using the integral representation of confluent hypergeometric function (A.1), we obtain marginal densities of $S$ ans $W$. □

By using the above theorem and (A.8), it is straightforward to show that

$$E[(1-S)^r] = \frac{\Gamma(a+b+c)\Gamma(c+r)}{\Gamma(a+b+c+r)\Gamma(c)} \exp(-\lambda_2) \times \frac{\mathbf{\Phi}_2[c+r, a; a+b+c+r; \lambda_2, -(\lambda_1-\lambda_2)]}{\mathbf{\Phi}_2[a, b; a+b+c; -\lambda_1, -\lambda_2]}.$$

Further, by using (A.5), we write

$$\begin{aligned} &{}_1F_1(a+b; a+b+c; -\lambda_2 - (\lambda_1-\lambda_2)w) \\ &= \exp[-\lambda_2 - (\lambda_1-\lambda_2)w]\, {}_1F_1(c; a+b+c; \lambda_2 + (\lambda_1-\lambda_2)w) \\ &= \exp[-\lambda_2 - (\lambda_1-\lambda_2)w] \sum_{i=0}^{\infty} \frac{(c)_i}{(a+b+c)_i\, i!} [\lambda_2 + (\lambda_1-\lambda_2)w]^i \\ &= \exp[-\lambda_2 - (\lambda_1-\lambda_2)w] \sum_{i=0}^{\infty} \frac{(c)_i}{(a+b+c)_i\, i!} \sum_{j=0}^{i} \binom{i}{j} \lambda_2^{i-j} [(\lambda_1-\lambda_2)w]^j \end{aligned}$$

in the density of $W$ given in Theorem 2.1 and derive $E(W^r)$ as

$$\begin{aligned} E(W^r) =& K(a,b,c,\lambda_1,\lambda_2) \exp(-\lambda_2) \sum_{i=0}^{\infty} \frac{\Gamma(a+b)\Gamma(c+i)}{\Gamma(a+b+c+i)\, i!} \sum_{j=0}^{i} \binom{i}{j} \\ & \times \lambda_2^{i-j} (\lambda_1-\lambda_2)^j \int_0^1 w^{a+r+j-1} (1-w)^{b-1} \exp[-(\lambda_1-\lambda_2)w]\, \mathrm{d}w \\ =& K(a,b,c,\lambda_1,\lambda_2) \exp(-\lambda_2) \sum_{i=0}^{\infty} \frac{\Gamma(a+b)\Gamma(c+i)}{\Gamma(a+b+c+i)\, i!} \\ & \times \sum_{j=0}^{i} \binom{i}{j} \lambda_2^{i-j} (\lambda_1-\lambda_2)^j \frac{\Gamma(a+r+j)\Gamma(b)}{\Gamma(a+b+r+j)} \\ & \times {}_1F_1(a+r+j; a+b+r+j; -(\lambda_1-\lambda_2)). \end{aligned}$$

In next two theorems, we derive marginal distributions of $X$ and $Y$. It is interesting to note that these marginal distributions do not belong to the Kummer-beta family and differs by an additional factor containing confluent hypergeometric function ${}_1F_1$.

**Theorem 2.2.** If $(X, Y) \sim GBKB(a, b; c; \lambda_1, \lambda_2)$, then the marginal density of $X$ is

$$K(a, b, c, \lambda_1, \lambda_2)\Gamma(b)\Gamma(c)\{\Gamma(b + c)\}^{-1} \exp(-\lambda_1 x) \\ \times x^{a-1}(1 - x)^{b+c-1} {}_1F_1(b; b + c; -\lambda_2(1 - x)),$$

where $0 < x < 1$.

*Proof.* To find the marginal p.d.f. of $X$, we integrate (1) with respect to $y$ to get

$$K(a, b, c, \lambda_1, \lambda_2)x^{a-1} \exp(-\lambda_1 x)\int_0^{1-x} y^{b-1}(1-x-y)^{c-1} \exp(-\lambda_2 y)\, \mathrm{d}y.$$

Substituting $z = y/(1 - x)$ with $\mathrm{d}y = (1 - x)\,\mathrm{d}z$ above, one obtains

$$K(a, b, c, \lambda_1, \lambda_2)x^{a-1} \exp(-\lambda_1 x)(1 - x)^{b+c-1} \\ \times \int_0^1 \exp[-\lambda_2(1 - x)z]z^{b-1}(1 - z)^{c-1}\, \mathrm{d}z.$$

Now, the desired result is obtained by using (A.1). □

**Theorem 2.3.** If $(X, Y) \sim GBKB(a, b; c; \lambda_1, \lambda_2)$, then the marginal density of $Y$ is

$$K(a, b, c, \lambda_1, \lambda_2)\Gamma(a)\Gamma(c)\{\Gamma(a + c)\}^{-1} \exp(-\lambda_2 y) \\ \times y^{b-1}(1 - y)^{a+c-1} {}_1F_1(a; a + c; -\lambda_1(1 - y)),$$

where $0 < x < 1$.

*Proof.* Similar to the proof of Theorem 2.2. □

Using the above theorem, the conditional density function of $X$ given $Y=y$, $0<y<1$, is obtained as

$$\frac{\Gamma(a+c)}{\Gamma(a)\Gamma(c)}\frac{\exp(-\lambda_1 x)x^{a-1}(1-x-y)^{c-1}}{(1-y)^{a+c-1}{}_1F_1(a;a+c;-\lambda_1(1-y))},\quad 0<x<1-y.$$

Similarly, using Theorem 2.2, the conditional density function of $Y$ given $X=x$, $0<x<1$, is derived as

$$\frac{\Gamma(b+c)}{\Gamma(a)\Gamma(b)}\frac{\exp(-\lambda_2 y)y^{b-1}(1-x-y)^{c-1}}{(1-x)^{b+c-1}{}_1F_1(b;b+c;-\lambda_2(1-x))},\quad 0<y<1-x.$$

Further, using conditional densities given above, we derive

$$E(X^r\,|\,y)=(1-y)^r\ \frac{B(a+r,c)}{B(a,c)}\ \frac{{}_1F_1(a+r;a+c+r;-\lambda_1(1-y))}{{}_1F_1(a;a+c;-\lambda_1(1-y))}$$

and

$$E(Y^r\,|\,x)=(1-x)^r\ \frac{B(b+r,c)}{B(b,c)}\ \frac{{}_1F_1(b+r;b+c+r;-\lambda_2(1-x))}{{}_1F_1(b;b+c;-\lambda_2(1-x))}.$$

Further, using (1), the joint $(r,s)$-th moment is obtained as

$$\begin{aligned}\mathrm{E}(X^rY^s) &= K(a,b,c,\lambda_1,\lambda_2)\\ &\quad\times\int_0^1\int_0^{1-x} x^{a+r-1}y^{b+s-1}\,(1-x-y)^{c-1}\exp[-(\lambda_1x+\lambda_2y)]\,\mathrm{d}y\,\mathrm{d}x\\ &= \frac{K(a,b,c,\lambda_1,\lambda_2)}{K(a+r,b+s,c,\lambda_1,\lambda_2)}\\ &= \frac{\Gamma(a+r)\Gamma(b+s)\Gamma(d)}{\Gamma(a)\Gamma(b)\Gamma(d+r+s)}\frac{\mathbf{\Phi}_2[a+r,b+s;d+r+s;-\lambda_1,-\lambda_2]}{\mathbf{\Phi}_2[a,b;d;-\lambda_1,-\lambda_2]},\end{aligned}$$

where $d=a+b+c$, $a+r>0$ and $b+s>0$. Now, substituting appropriately, we obtain

$$\mathrm{E}(X)=\frac{a}{d}\cdot\frac{\mathbf{\Phi}_2[a+1,b;d+1;-\lambda_1,-\lambda_2]}{\mathbf{\Phi}_2[a,b;d;-\lambda_1,-\lambda_2]},$$

$$\mathrm{E}(Y)=\frac{b}{d}\cdot\frac{\mathbf{\Phi}_2[a,b+1;d+1;-\lambda_1,-\lambda_2]}{\mathbf{\Phi}_2[a,b;d;-\lambda_1,-\lambda_2]},$$

$$\mathrm{E}(X^2) = \frac{a(a+1)}{d(d+1)} \cdot \frac{\mathbf{\Phi}_2[a+2, b; d+2; -\lambda_1, -\lambda_2]}{\mathbf{\Phi}_2[a, b; d; -\lambda_1, -\lambda_2]},$$

$$\mathrm{E}(Y^2) = \frac{b(b+1)}{d(d+1)} \cdot \frac{\mathbf{\Phi}_2[a, b+2; d+2; -\lambda_1, -\lambda_2]}{\mathbf{\Phi}_2[a, b; d; -\lambda_1, -\lambda_2]},$$

$$\mathrm{E}(XY) = \frac{ab}{d(d+1)} \cdot \frac{\mathbf{\Phi}_2[a+1, b+1; d+2; -\lambda_1, -\lambda_2]}{\mathbf{\Phi}_2[a, b; d; -\lambda_1, -\lambda_2]},$$

$$\mathrm{E}(X^2Y^2) = \frac{ab(a+1)(b+1)}{d(d+1)(d+2)(d+3)} \cdot \frac{\mathbf{\Phi}_2[a+2, b+2; d+4; -\lambda_1, -\lambda_2]}{\mathbf{\Phi}_2[a, b; d; -\lambda_1, -\lambda_2]},$$

$$\begin{aligned}\mathrm{Var}(X) = \frac{a}{d}\Bigg[&\frac{a+1}{d+1} \cdot \frac{\mathbf{\Phi}_2[a+2, b; d+2; -\lambda_1, -\lambda_2]}{\mathbf{\Phi}_2[a, b; d; -\lambda_1, -\lambda_2]} \\ &- \frac{a}{d}\left\{\frac{\mathbf{\Phi}_2[a+1, b; d+1; -\lambda_1, -\lambda_2]}{\mathbf{\Phi}_2[a, b; d; -\lambda_1, -\lambda_2]}\right\}^2\Bigg],\end{aligned}$$

$$\begin{aligned}\mathrm{Var}(Y) = \frac{b}{d}\Bigg[&\frac{b+1}{d+1} \frac{\mathbf{\Phi}_2[a, b+2; d+2; -\lambda_1, -\lambda_2]}{\mathbf{\Phi}_2[a, b; d; -\lambda_1, -\lambda_2]} \\ &- \frac{b}{d}\left\{\frac{\mathbf{\Phi}_2[a, b+1; d+1; -\lambda_1, -\lambda_2]}{\mathbf{\Phi}_2[a, b; d; -\lambda_1, -\lambda_2]}\right\}^2\Bigg],\end{aligned}$$

and

$$\begin{aligned}\mathrm{Cov}(X, Y) = \frac{ab}{d}\Bigg[&\frac{1}{d+1} \frac{\mathbf{\Phi}_2[a+1, b+1; d+2; -\lambda_1, -\lambda_2]}{\mathbf{\Phi}_2[a, b; d; -\lambda_1, -\lambda_2]} \\ &- \frac{1}{d} \frac{\mathbf{\Phi}_2[a+1, b; d+1; -\lambda_1, -\lambda_2]}{\mathbf{\Phi}_2[a, b; d; -\lambda_1, -\lambda_2]} \frac{\mathbf{\Phi}_2[a, b+1; d+1; -\lambda_1, -\lambda_2]}{\mathbf{\Phi}_2[a, b; d; -\lambda_1, -\lambda_2]}\Bigg].\end{aligned}$$

Notice that the expressions for $\mathrm{E}(XY)$, $\mathrm{E}(X^2)$, $\mathrm{E}(Y^2)$, $\mathrm{E}(X)$ and $\mathrm{E}(Y)$ involve $\mathbf{\Phi}_2[a, b; c; x, y]$ which can be computed by using a suitable software. Table 1 provides correlations between $X$ and $Y$ for different values of $a, b, c$, $\lambda_1$ and $\lambda_2$. All the values of the correlation coefficient are negative because of the condition $x + y < 1$. Further, for selected values of the parameters, it is possible to find correlations close to 0 or $-1$. As can be seen that for fixed $a, b, \lambda_1, \lambda_2$ the correlation increases as $c$ increases. Thus for small values of $c$ the correlation is close to $-1$ whereas for large $c$ the correlation is close 0. The correlation is very small when $a$, $b$ and $c$ are smaller than one. Further, for fixed values of $a, b, c$, the correlation increases as $\lambda_1$ or $\lambda_2$ increases. Furthermore, the choices of $a, b$ small and $c$ large yield correlations close to zero, whereas large values of $a$ or $b$ and small values of $c$ or $\lambda_1, \lambda_2$ give large correlations.

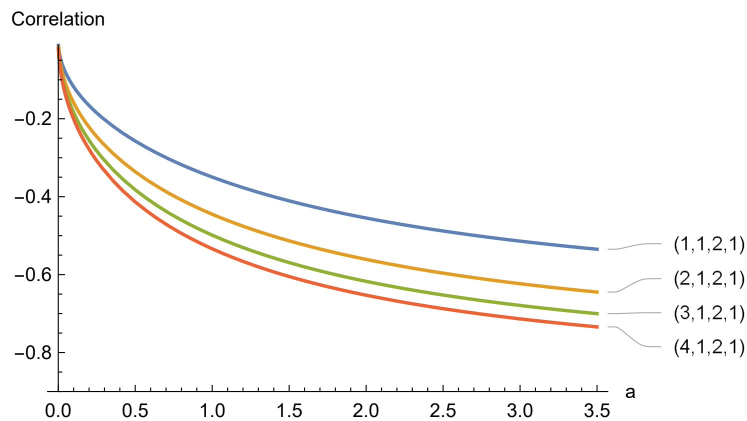


**Figure 1:** Plots of the correlation coefficient as a function of $a$, for $(b, c, \lambda_1, \lambda_2) = (1, 1, 2, 1), (2, 1, 2, 1), (3, 1, 2, 1), (4, 1, 2, 1)$.

## 3 Entropies

In this section, exact forms of Renyi and Shannon entropies are determined for the generalized bivariate Kummer-beta distribution defined in 1.

Let $(\mathcal{X}, \mathcal{B}, \mathcal{P})$ be a probability space. Consider a pdf $f$ associated with $\mathcal{P}$, dominated by $\sigma-$finite measure $\mu$ on $\mathcal{X}$. Denote by $H_{SH}(f)$ the well-known Shannon entropy introduced in Shannon [24]. It is define by

$$H_{SH}(f) = -\int_{\mathcal{X}} f(x) \ln f(x) \, \mathrm{d}\mu. \tag{4}$$

One of the main extensions of the Shannon entropy was defined by

**Table 1:** Correlations between $X$ and $Y$ for different values of $a, b, c$, $\lambda_1$ and $\lambda_2$.

| $a$ | $b$ | $\lambda_1$ | $\lambda_2$ | $c$ | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | | 0.1 | 0.5 | 1 | 2 | 4 | 6 | 10 |
| 6 | 6 | −5.5 | -5.5 | −0.992 | −0.959 | −0.92 | −0.848 | −0.726 | −0.628 | −0.485 |
| 6 | 6 | −2.5 | −2.5 | −0.988 | −0.943 | −0.892 | −0.802 | −0.663 | −0.561 | −0.425 |
| 6 | 6 | −0.5 | −0.5 | −0.985 | −0.928 | −0.865 | −0.761 | −0.613 | −0.512 | −0.385 |
| 3 | 3 | −5.5 | −5.5 | −0.989 | −0.946 | −0.895 | −0.799 | −0.639 | −0.518 | −0.358 |
| 3 | 3 | −2.5 | −2.5 | −0.981 | −0.911 | −0.834 | −0.707 | −0.53 | −0.416 | −0.284 |
| 3 | 3 | −0.5 | −0.5 | −0.971 | −0.871 | −0.77 | −0.624 | −0.449 | −0.349 | −0.241 |
| 2 | 2 | −5.5 | −5.5 | −0.988 | −0.941 | −0.882 | −0.772 | −0.589 | −0.454 | −0.29 |
| 2 | 2 | −2.5 | −2.5 | −0.977 | −0.89 | −0.795 | −0.645 | −0.45 | −0.336 | −0.216 |
| 2 | 2 | −0.5 | −0.5 | −0.959 | −0.823 | −0.697 | −0.531 | −0.356 | −0.266 | −0.176 |
| 1 | 1 | −5.5 | −5.5 | −0.987 | −0.934 | −0.865 | −0.728 | −0.498 | −0.345 | −0.189 |
| 1 | 1 | −2.5 | −2.5 | −0.967 | −0.846 | −0.718 | −0.529 | −0.32 | −0.218 | −0.127 |
| 1 | 1 | −1.5 | −2.5 | −0.952 | −0.79 | −0.64 | −0.45 | −0.267 | −0.184 | −0.111 |
| 1 | 1 | −0.5 | −0.5 | −0.927 | −0.713 | −0.549 | −0.371 | −0.221 | −0.156 | −0.097 |
| 0.5 | 0.5 | −5.5 | −5.5 | −0.987 | −0.929 | −0.848 | −0.675 | −0.395 | −0.24 | −0.113 |
| 0.5 | 0.5 | −2.5 | −2.5 | −0.955 | −0.79 | −0.622 | −0.403 | −0.207 | −0.13 | −0.069 |
| 0.5 | 0.5 | −1.5 | −1.5 | −0.925 | −0.694 | −0.509 | −0.313 | −0.162 | −0.105 | −0.059 |
| 0.5 | 0.5 | −0.5 | −0.5 | −0.873 | −0.57 | −0.39 | −0.234 | −0.126 | −0.085 | −0.051 |
| 0.5 | 0.5 | 0.5 | 0.5 | −0.783 | −0.428 | −0.28 | −0.17 | −0.098 | −0.069 | −0.044 |
| 0.5 | 0.5 | 1.5 | 1.5 | −0.642 | −0.29 | −0.188 | −0.121 | −0.076 | −0.057 | −0.038 |
| 0.5 | 0.5 | 2.5 | 2.5 | −0.458 | −0.175 | −0.119 | −0.085 | −0.059 | −0.047 | −0.033 |
| 0.5 | 0.5 | 5.5 | 5.5 | −0.008 | −0.010 | −0.022 | −0.029 | −0.029 | −0.027 | −0.023 |
| 1 | 1 | 0.5 | 0.5 | −0.886 | −0.615 | −0.45 | −0.297 | −0.181 | −0.131 | −0.085 |
| 1 | 1 | 1.5 | 1.5 | −0.82 | −0.498 | −0.349 | −0.232 | −0.147 | −0.11 | −0.075 |
| 1 | 1 | 2.5 | 2.5 | −0.719 | −0.372 | −0.257 | −0.176 | −0.119 | −0.093 | −0.066 |
| 1 | 1 | 5.5 | 5.5 | −0.223 | −0.081 | −0.075 | −0.072 | −0.064 | −0.057 | −0.046 |
| 2 | 2 | 0.5 | 0.5 | −0.944 | −0.774 | −0.634 | −0.469 | −0.311 | −0.235 | −0.158 |
| 2 | 2 | 2.5 | 2.5 | −0.889 | −0.632 | −0.481 | −0.343 | −0.232 | −0.181 | −0.128 |
| 2 | 2 | 5.5 | 5.5 | −0.664 | −0.332 | −0.243 | −0.185 | −0.143 | −0.121 | −0.095 |
| 3 | 3 | 0.5 | 0.5 | −0.964 | −0.842 | −0.728 | −0.576 | −0.408 | −0.318 | −0.221 |
| 3 | 3 | 1.5 | 1.5 | −0.953 | −0.805 | −0.679 | −0.524 | −0.368 | −0.287 | −0.203 |
| 3 | 3 | 2.5 | 2.5 | −0.938 | −0.759 | −0.623 | −0.47 | −0.329 | −0.259 | −0.186 |
| 3 | 3 | 5.5 | 5.5 | −0.845 | −0.555 | −0.418 | −0.309 | −0.226 | −0.186 | −0.142 |
| 6 | 6 | 0.5 | 0.5 | −0.982 | −0.918 | −0.849 | −0.738 | −0.587 | −0.487 | −0.365 |
| 6 | 6 | 1.5 | 1.5 | −0.98 | −0.906 | −0.83 | −0.712 | −0.559 | −0.462 | −0.346 |
| 6 | 6 | 2.5 | 2.5 | −0.976 | −0.892 | −0.808 | −0.684 | −0.53 | −0.437 | −0.327 |
| 6 | 6 | 5.5 | 5.5 | −0.96 | −0.833 | −0.722 | −0.583 | −0.439 | −0.361 | −0.274 |

Rényi [25]. This generalized entropy measure is given by

$$H_R(\eta, f) = \frac{\ln G(\eta)}{1-\eta} \qquad (\text{for } \eta > 0 \text{ and } \eta \neq 1), \tag{5}$$

where

$$G(\eta) = \int_{\mathcal{X}} f^{\eta} d\mu.$$

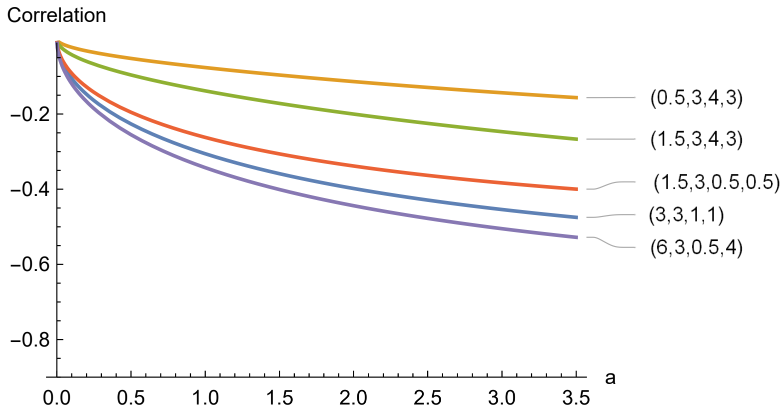


**Figure 2:** Plots of correlation coefficient as a function of $a$, for $(b, c, \lambda_1, \lambda_2) = (0.5, 3, 4, 3), (1.5, 3, 4, 3), (1.5, 3, 0.5, 0.5), (3, 3, 1, 1), (6, 3, 0.5, 4)$.

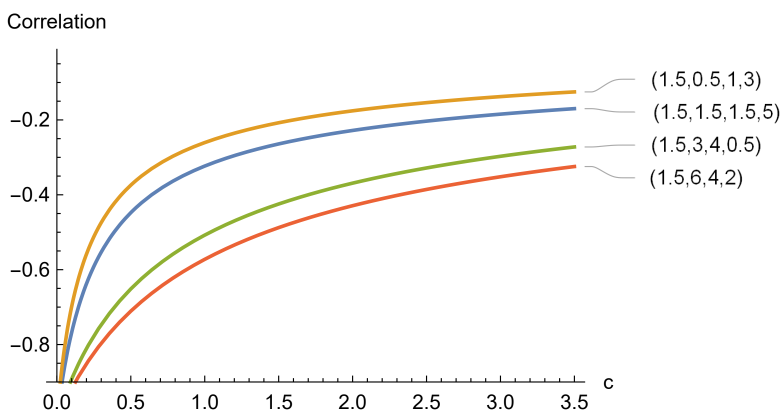


**Figure 3:** Plots of correlation coefficient as a function of $c$, for $(a, b, \lambda_1, \lambda_2) = (1.5, 0.5, 1, 3), (1.5, 1.5, 1.5, 5), (1.5, 3, 4, 0.5), (1.5, 6, 4, 2)$.

The additional parameter $\eta$ is used to describe complex behavior in probability models and the associated process under study. Rényi entropy is monotonically decreasing in $\eta$, while Shannon entropy (4) is obtained from (5) for $\eta \uparrow 1$. For details see Nadarajah and Zografos [26], Zografos and Nadarajah [27] and Zografos [28].

First, we give the following lemma useful in deriving these entropies.

**Lemma 3.1.** Let $g(a,b,c,\lambda_1,\lambda_2) = \lim_{\eta\to 1} h(\eta)$, where

$$h(\eta) = \frac{\mathrm{d}}{\mathrm{d}\eta}\mathbf{\Phi}_2\big[\eta(a-1)+1, \eta(b-1)+1; \eta(a+b+c-3)+3; -\lambda_1\eta, -\lambda_2\eta\big].$$

Then,

$$\begin{aligned} g(a,b,c,\lambda_1,\lambda_2) = & \sum_{r,s=1}^{\infty} \frac{\Gamma(a+r)\Gamma(b+s)\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(a+b+c+r+s)} \frac{(-\lambda_1)^r(-\lambda_2)^s}{r!\, s!} \\ & \times \Big[r+s+(a-1)\psi(a+r)+(b-1)\psi(b+r) \\ & + (a+b+c-3)\psi(a+b+c) - (a-1)\psi(a) \\ & - (b-1)\psi(b) - (a+b+c-3)\psi(a+b+c+r+s)\Big] \end{aligned} \qquad (6)$$

where $\psi(\alpha) = \Gamma'(\alpha)/\Gamma(\alpha)$ is the digamma function.

*Proof.* Expanding $\mathbf{\Phi}_2$ in series form by using A.6, we write

$$\begin{aligned} h(\eta) &= \frac{\mathrm{d}}{\mathrm{d}\eta} \sum_{r,s=0}^{\infty} \Delta_{r,s}(\eta) \frac{(-\lambda_1)^r(-\lambda_2)^s}{r!\, s!} \\ &= \sum_{r,s=0}^{\infty} \left[\frac{\mathrm{d}}{\mathrm{d}\eta}\Delta_{r,s}(\eta)\right] \frac{(-\lambda_1)^r(-\lambda_2)^s}{r!\, s!}, \end{aligned} \qquad (7)$$

where

$$\Delta_{r,s}(\eta) = \frac{\Gamma[\eta(a-1)+1+r]\Gamma[\eta(b-1)+1+s]\Gamma[\eta(a+b+c-3)+3]}{\Gamma[\eta(a-1)+1]\Gamma[\eta(b-1)+1]\Gamma[\eta(a+b+c-3)+3+r+s]}\eta^{r+s}.$$

Now, differentiating the logarithm of $\Delta_{r,s}(\eta)$ w.r.t. to $\eta$, one obtains

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\eta}\Delta_{r,s}(\eta) = \Delta_j(\eta) \Big[ & \frac{r+s}{\eta} + (a-1)\psi(\eta(a-1)+1+r) \\ & + (b-1)\psi(\eta(b-1)+1+s) \\ & + (a+b+c-3)\psi(\eta(a+b+c-3)+3) \\ & - (a-1)\psi(\eta(a-1)+1) - (b-1)\psi(\eta(b-1)+1) \\ & - (a+b+c-3)\psi(\eta(a+b+c-3)+3+r+s)\Big]. \end{aligned} \qquad (8)$$

Finally, substituting (8) in (7) and taking $\eta \to 1$, one obtains the desired result. □

**Theorem 3.1.** For the bivariate beta distribution defined by the pdf (1), the Rényi and the Shannon entropies are given by

$$H_R(\eta, f) = \frac{1}{1-\eta}\Big[\eta \ln K(a,b,c,\lambda_1,\lambda_2) + \ln\Gamma[\eta(a-1)+1] + \ln\Gamma[\eta(b-1)+1] + \ln\Gamma[\eta(c-1)+1] - \ln\Gamma[\eta(a+b+c-3)+3] + \ln\mathbf{\Phi}_2[\eta(a-1)+1, \eta(b-1)+1; \eta(a+b+c-3)+3; -\lambda_1\eta, -\lambda_2\eta]\Big]$$

and

$$H_{SH}(f) = -\ln K(a,b,c,\lambda_1,\lambda_2) - [(a-1)\psi(a) + (b-1)\psi(b) + (c-1)\psi(c) - (a+b+c-3)\psi(a+b+c)] - \frac{g(a,b,c,\lambda_1,\lambda_2)}{\mathbf{\Phi}_2[a,b;a+b+c;-\lambda_1,-\lambda_2]},$$

respectively, where $\psi(\alpha) = \Gamma'(\alpha)/\Gamma(\alpha)$ is the digamma function and $g(a,b,c,\lambda_1,\lambda_2)$ is given by (6).

*Proof.* For $\eta > 0$ and $\eta \neq 1$, using the joint density of $X$ and $Y$ given by (1), we have

$$\begin{aligned}
G(\eta) &= \int_0^1 \int_0^{1-x} f^\eta(x,y)\,\mathrm{d}y\,\mathrm{d}x \\
&= [K(a,b,c,\lambda_1,\lambda_2)]^\eta \int_0^1 \int_0^{1-x} x^{\eta(a-1)} y^{\eta(b-1)} \\
&\quad \times (1-x-y)^{\eta(c-1)} \exp[-\eta(\lambda_1 x + \lambda_2 y)]\,\mathrm{d}y\,\mathrm{d}x \\
&= \frac{[K(a,b,c,\lambda_1,\lambda_2)]^\eta}{K(\eta(a-1)+1, \eta(b-1)+1, \eta(c-1)+1, \eta\lambda_1, \eta\lambda_2)} \\
&= [K(a,b,c,\lambda_1,\lambda_2)]^\eta \frac{\Gamma[\eta(a-1)+1]\Gamma[\eta(b-1)+1]\Gamma[\eta(c-1)+1]}{\Gamma[\eta(a+b+c-3)+3]} \\
&\quad \times \mathbf{\Phi}_2\big[\eta(a-1)+1, \eta(b-1)+1; \eta(a+b+c-3)+3; -\lambda_1\eta, -\lambda_2\eta\big],
\end{aligned}$$

where the last line has been obtained by using (2). Now, taking logarithm of $G(\eta)$ and using (5) we get $H_R(\eta, f)$. The Shannon entropy is obtained from $H_R(\eta, f)$ by taking $\eta \uparrow 1$ and using L'Hopital's rule. □

## 4 Distribution of The Product

In Theorem 2.1, we have derived distributions of $X/(X+Y)$ and $X+Y$ where $(X,Y) \sim GBKB(a,b;c;\lambda_1,\lambda_2)$. In this section, we derive the density of $XY$, where $(X,Y) \sim GBKB(a,b;c;\lambda_1,\lambda_2)$. The distribution of $XY$, where $X$ and $Y$ are independent random variables, $X \sim KB(a_1,b_1,\lambda_1)$, $Y \sim KB(a_2,b_2,\lambda_2)$ has been derived in Nagar and Zarrazola [29].

**Theorem 4.1.** If$(X,Y) \sim GBKB(a,b;c;\lambda_1,\lambda_2)$, then the pdf of $W = XY$ is given by

$$\frac{\sqrt{\pi}K(a,b,c,\lambda_1,\lambda_2)\,\exp(-\lambda_1)}{2^{a+c-b-1}}\,\frac{w^{b-1}(1-4w)^{c-1/2}}{\left(1+\sqrt{1-4w}\right)^{b+c-a}}$$
$$\times\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{\Gamma(c+i)(\lambda_1-\lambda_2)^j\lambda_1^i}{\Gamma(c+1/2+i)\,2^{i+j}\,i!\,j!}\left(\frac{1-4w}{1+\sqrt{1-4w}}\right)^i\left(1-\sqrt{1-4w}\right)^j$$
$$\times\,{}_2F_1\left(c+i,c+b-a+i+j;2c+2i;\frac{2\sqrt{1-4w}}{1+\sqrt{1-4w}}\right),\qquad(9)$$

with $0<w<1/4$.

*Proof.* Making the transformation $W=XY$ with the Jacobian $J(x,y\to x,w)=x^{-1}$ in (1), we obtain the joint density of $X$ and $W$ as

$$K(a,b,c,\lambda_1,\lambda_2)\,\frac{w^{b-1}(-x^2+x-w)^{c-1}}{x^{b+c-a}}\exp\left[-\frac{\lambda_1x^2+\lambda_2w}{x}\right],\qquad(10)$$

where $p<x<q$ with

$$p=\frac{1-\sqrt{1-4w}}{2},\quad q=\frac{1+\sqrt{1-4w}}{2},\quad 0<w<\frac{1}{4}.$$

Now, integrating $x$ in (10), the marginal density of $W$ is obtained as

$$K(a,b,c,\lambda_1,\lambda_2)\,w^{b-1}\int_p^q\frac{(-x^2+x-w)^{c-1}}{x^{b+c-a}}\exp\left[-\frac{\lambda_1x^2+\lambda_2w}{x}\right]dx.\qquad(11)$$

Writing

$$-x^2+x-w=(q-x)(x-p)=(q-p)^2t(1-t),$$

$$\begin{aligned}\frac{-\lambda_1 x^2 - \lambda_2 w}{x} &= \frac{\lambda_1 (x-p)(q-x)}{x} + \frac{(\lambda_1 - \lambda_2) w}{x} - \lambda_1 \\ &= \frac{\lambda_1 (q-p)^2 t(1-t)}{q[1-(1-p/q)t]} + \frac{(\lambda_1 - \lambda_2) w}{q[1-(1-p/q)t]} - \lambda_1,\end{aligned}$$

$$\begin{aligned}&\exp\left[\frac{\lambda_1 (q-p)^2 t(1-t)}{q\,[1-(1-p/q)\,t]}\right] \exp\left[\frac{(\lambda_1 - \lambda_2) w}{q\,[1-(1-p/q)\,t]}\right] \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{1}{i!j!}\left[\frac{(q-p)^{2i}\lambda_1^i t^i (1-t)^i (\lambda_1-\lambda_2)^j w^j}{q^{i+j}\,[1-(1-p/q)\,t]^{i+j}}\right], \end{aligned} \tag{12}$$

where $t = (q-x)/(q-p)$ in (11) the marginal density of $W$ rewritten as

$$\begin{aligned}&K(a,b,c,\lambda_1,\lambda_2)\exp(-\lambda_1) w^{b-1} \\ &\quad\times \int_p^q \frac{[(x-p)(q-x)]^{c-1}}{x^{b+c-a}} \exp\left[\frac{\lambda_1 (x-p)(q-x)}{x} + \frac{(\lambda_1-\lambda_2) w}{x}\right] \mathrm{d}x \\ &= K(a,b,c,\lambda_1,\lambda_2)\ \exp(-\lambda_1) w^{b-1} \\ &\quad\times \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(q-p)^{2i+2c-1}\lambda_1^i (\lambda_1-\lambda_2)^j w^j}{q^{i+j+b+c-a}\ i!j!} \int_0^1 \frac{t^{c+i-1}(1-t)^{c+i-1}}{[1-(1-p/q)\,t]^{b+c-a+i+j}}\,\mathrm{d}t,\end{aligned}$$

where we have used the substitution $t = (q-x)/(q-p)$. Now, evaluating the above integral using (A.2) and simplifying the resulting expression, we get the desired result. □

**Theorem 4.2.** Let $(X,Y) \sim GBKB(a,b;c;c\lambda_1,c\lambda_2)$ and $U$ and $V$ be defined by $U = cX$ and $V = cY$. Then, $U$ and $V$ are asymptotically distributed as a product of independent gamma densities;

$$\lim_{c\to\infty} f_{U,V}(u,v) = \frac{\exp[-(1+\lambda_1)u]u^{a-1}}{(1+\lambda_1)^a\Gamma(a)} \frac{\exp[-(1+\lambda_2)v]v^{a-1}}{(1+\lambda_2)^b\Gamma(b)},$$

where $f_{U,V}(u,v)$ denotes the joint density of $U$ and $V$.

*Proof.* In the joint density of $X$ and $Y$ given by (1) transform $U = cX$ and $V = cY$ with the Jacobian $J(x,y \to u,v) = c^{-2}$ to get the joint density of $U$ and $V$ as

$$f_{U,V}(u,v) = c^{-(a+b)} K(a,b,c,c\lambda_1,c\lambda_2)$$

$$\times\, u^{a-1}v^{b-1}\left(1-\frac{u+v}{c}\right)^{c-1}\exp\left[-\left(\lambda_1 u+\lambda_2 v\right)\right].$$

Now, observing that

$$\lim_{c\to\infty}\frac{\Gamma(a+b+c)}{\Gamma(c)}c^{-(a+b)}=1,$$

$$\begin{aligned}\lim_{c\to\infty}\mathbf{\Phi}_2[a,b;a+b+c;-c\lambda_1,-c\lambda_2]&={}_1F_0(a;-\lambda_1){}_1F_0(b;-\lambda_2)\\&=(1+\lambda_1)^{-a}(1+\lambda_2)^{-b}\end{aligned}$$

and

$$\lim_{c\to\infty}\left(1-\frac{u+v}{c}\right)^{c-1}=\exp[-(u+v)],$$

we get the desired result. □

## 5 Multivariate Generalization

A multivariate generalization of (1) can be defined by the density

$$K(a_1,\ldots,a_n,c,\lambda_1,\ldots,\lambda_n)\prod_{i=1}^{n}x_i^{a_i-1}\left(1-\sum_{i=1}^{n}x_i\right)^{c-1}\exp\left(-\sum_{i=1}^{n}\lambda_i x_i\right),$$

where $x_i>0$, $i=1,\ldots,n$, $\sum_{i=1}^{n}x_i<1$, $a_i>0$, $i=1,\ldots,n$, $c>0$, $-\infty<\lambda_i<\infty$, $i=1,\ldots,n$ and $K(a_1,\ldots,a_n,c,\lambda_1,\ldots,\lambda_n)$ is the normalized constant given by

$$\begin{aligned}&[K(a_1,\ldots,a_n,c,\lambda_1,\ldots,\lambda_n)]^{-1}=\\&\qquad\frac{\Gamma(a_1)\cdots\Gamma(a_n)\Gamma(c)}{\Gamma(a_1+\cdots+a_n+c)}\mathbf{\Phi}_2^{(n)}\left[a_1,\ldots,a_n;\sum_{i=1}^{n}a_i+c;-\lambda_1,\ldots,-\lambda_n\right],\end{aligned}$$

where $\mathbf{\Phi}_2^{(n)}$ is the Humbert's confluent hypergeometric function of $n$ variables (Srivastava and Karlsson [30]).

## Appendix

The integral representations of the confluent hypergeometric function and the Gauss hypergeometric function are given as

$$ {}_1F_1(a;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}\exp(zt)\,\mathrm{d}t, \qquad \mathrm{Re}(c) > \mathrm{Re}(a) > 0, \tag{A.1} $$

and

$$ {}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}(1-zt)^{-b}\,\mathrm{d}t, \qquad \mathrm{Re}(c) > \mathrm{Re}(a) > 0,\ |\arg(1-z)| < \pi, \tag{A.2} $$

respectively. The series expansions for ${}_1F_1$ and ${}_2F_1$ can be obtained by expanding $\exp(zt)$ and $(1-zt)^{-b}$, $|zt| < 1$, in (A.1) and (A.2) and integrating $t$. Thus

$$ {}_1F_1(a;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} \frac{z^k}{k!}, \tag{A.3} $$

and

$$ {}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k} \frac{z^k}{k!}, \quad |z| < 1. \tag{A.4} $$

where the Pochammer symbol $(a)_n$ is defined by $(a)_n = a(a+1)\cdots(a+n-1) = (a)_{n-1}(a+n-1)$ for $n = 1, 2, \ldots$, and $(a)_0 = 1$.

The confluent hypergeometric function ${}_1F_1(a;c;z)$ satisfies Kummers relation

$$ {}_1F_1(a;c;-z) = \exp(-z){}_1F_1(c-a;c;z). \tag{A.5} $$

For properties and further results on these functions the reader is referred to Luke [31].

The Humbert's confluent hypergeometric function $\mathbf{\Phi}_2$ is defined by

$$ \mathbf{\Phi}_2[a,b;c;z_1,z_2] = \sum_{r,s=0}^{\infty} \frac{(a)_r(b)_s}{(c)_{r+s}} \frac{z_1^r z_2^s}{r!\,s!}, $$

$$= \sum_{r=0}^{\infty} \frac{(a)_r}{(c)_r} \frac{z_1^r}{r!} \, {}_1F_1(b; c + r; z_2)$$

$$= \sum_{s=0}^{\infty} \frac{(b)_s}{(c)_s} \frac{z_2^s}{s!} \, {}_1F_1(a; c + s; z_1). \tag{A.6}$$

The integral representations of $\mathbf{\Phi}_2$ is given by

$$\mathbf{\Phi}_2[a, b; c; z_1, z_2] = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c - a - b)} \int_0^1 \int_0^{1-u} u^{a-1} v^{b-1} \times (1-u-v)^{c-a-b-1} \exp(z_1 u + z_2 v) \, dv \, du, \tag{A.7}$$

where $\text{Re}(a) > 0$, $\text{Re}(b) > 0$ and $\text{Re}(c - a - b) > 0$. Substituting $t = (1 - u)^{-1} v$ and integrating $t$ in the above expression, the Humbert's confluent hypergeometric function $\mathbf{\Phi}_2$ can also be represented as

$$\mathbf{\Phi}_2[a, b; c; z_1, z_2] = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c - a)} \int_0^1 u^{a-1} (1 - u)^{c-a-1} \times \exp(z_1 u) {}_1F_1(b; c - a; z_2(1 - u)) \, du. \tag{A.8}$$

For properties and further results on these functions the reader is referred to Luke [31] and Srivastava and Karlsson [30]. Next, we define the Kummer-beta distribution due to Ng and Kotz [32].

**Definition A.1.** The random variable $X$ is said to have a Kummer-beta distribution, denoted by $X \sim KB(\alpha, \beta, \lambda)$, if its p.d.f. is given by

$$\frac{x^{\alpha-1}(1 - x)^{\beta-1} \exp[\lambda(1 - x)]}{B(\alpha, \beta) {}_1F_1(\beta; \alpha + \beta; \lambda)}, \quad 0 < x < 1,$$

where $\alpha > 0$, $\beta > 0$, $-\infty < \lambda < \infty$ and $B(a, b)$ is the beta function given by

$$B(a, b) = \Gamma(a)\Gamma(b)\{\Gamma(a + b)\}^{-1}.$$

Note that for $\lambda = 0$ the above density simplifies to a beta type I density with parameters $\alpha$ and $\beta$.

**Definition A.2.** The random variables $X$ and $Y$ are said to have a Dirichlet type 1 distribution of order 3 with parameters $(a, b, c)$, $a > 0$, $b > 0$, $c > 0$, denoted as $(X, Y) \sim \mathrm{D1}(a, b; c)$, if their joint p.d.f. is given by

$$\{B(a,b,c)\}^{-1}x^{a-1}y^{b-1}(1-x-y)^{c-1}, \quad x > 0, \quad y > 0, \quad x + y < 1,$$

where $B(a, b, c)$ is defined by

$$B(a,b,c) = \Gamma(a)\Gamma(b)\Gamma(c)\{\Gamma(a+b+c)\}^{-1}. \qquad \text{(A.9)}$$

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## Appendix

The integral representations of the confluent hypergeometric function and the Gauss hypergeometric function are given as

$$ {}_1F_1(a;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1} \exp(zt)\,\mathrm{d}t, \quad \mathrm{Re}(c) > \mathrm{Re}(a) > 0, \tag{A.1} $$

and

$$ {}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}(1-zt)^{-b}\,\mathrm{d}t, \quad \mathrm{Re}(c) > \mathrm{Re}(a) > 0,\ |\arg(1-z)| < \pi, \tag{A.2} $$

respectively. The series expansions for ${}_1F_1$ and ${}_2F_1$ can be obtained by expanding $\exp(zt)$ and $(1-zt)^{-b}$, $|zt| < 1$, in (A.1) and (A.2) and integrating $t$. Thus

$$ {}_1F_1(a;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} \frac{z^k}{k!}, \tag{A.3} $$

and

$$ {}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \quad |z| < 1. \tag{A.4} $$

where the Pochammer symbol $(a)_n$ is defined by $(a)_n = a(a+1)\cdots(a+n-1) = (a)_{n-1}(a+n-1)$ for $n = 1, 2, \ldots,$ and $(a)_0 = 1$.

The confluent hypergeometric function ${}_1F_1(a;c;z)$ satisfies Kummers relation

$$ {}_1F_1(a;c;-z) = \exp(-z)\,{}_1F_1(c-a;c;z). \tag{A.5} $$

For properties and further results on these functions the reader is referred to Luke [31].

The Humbert's confluent hypergeometric function $\mathbf{\Phi}_2$ is defined by

$$ \mathbf{\Phi}_2[a,b;c;z_1,z_2] = \sum_{r,s=0}^{\infty} \frac{(a)_r (b)_s}{(c)_{r+s}} \frac{z_1^r z_2^s}{r!\, s!}, $$

$$= \sum_{r=0}^{\infty} \frac{(a)_r}{(c)_r} \frac{z_1^r}{r!} \, {}_1F_1(b; c+r; z_2)$$
$$= \sum_{s=0}^{\infty} \frac{(b)_s}{(c)_s} \frac{z_2^s}{s!} \, {}_1F_1(a; c+s; z_1). \tag{A.6}$$

The integral representations of $\mathbf{\Phi}_2$ is given by

$$\mathbf{\Phi}_2[a, b; c; z_1, z_2] = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-a-b)} \int_0^1 \int_0^{1-u} u^{a-1} v^{b-1} \times (1-u-v)^{c-a-b-1} \exp(z_1 u + z_2 v) \, dv \, du, \tag{A.7}$$

where $\text{Re}(a) > 0$, $\text{Re}(b) > 0$ and $\text{Re}(c-a-b) > 0$. Substituting $t = (1-u)^{-1}v$ and integrating $t$ in the above expression, the Humbert's confluent hypergeometric function $\mathbf{\Phi}_2$ can also be represented as

$$\mathbf{\Phi}_2[a, b; c; z_1, z_2] = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 u^{a-1}(1-u)^{c-a-1} \times \exp(z_1 u) {}_1F_1(b; c-a; z_2(1-u)) \, du. \tag{A.8}$$

For properties and further results on these functions the reader is referred to Luke [31] and Srivastava and Karlsson [30]. Next, we define the Kummer-beta distribution due to Ng and Kotz [32].

**Definition A.3.** The random variable $X$ is said to have a Kummer-beta distribution, denoted by $X \sim KB(\alpha, \beta, \lambda)$, if its p.d.f. is given by

$$\frac{x^{\alpha-1}(1-x)^{\beta-1} \exp[\lambda(1-x)]}{B(\alpha, \beta) {}_1F_1(\beta; \alpha+\beta; \lambda)}, \quad 0 < x < 1,$$

where $\alpha > 0$, $\beta > 0$, $-\infty < \lambda < \infty$ and $B(a, b)$ is the beta function given by

$$B(a, b) = \Gamma(a)\Gamma(b)\{\Gamma(a+b)\}^{-1}.$$

Note that for $\lambda = 0$ the above density simplifies to a beta type I density with parameters $\alpha$ and $\beta$.

**Definition A.4.** The random variables $X$ and $Y$ are said to have a Dirichlet type 1 distribution of order 3 with parameters $(a, b, c)$, $a > 0$, $b > 0$, $c > 0$, denoted as $(X, Y) \sim \mathrm{D1}(a, b; c)$, if their joint p.d.f. is given by

$$\{B(a, b, c)\}^{-1} x^{a-1} y^{b-1} (1 - x - y)^{c-1}, \quad x > 0, \quad y > 0, \quad x + y < 1,$$

where $B(a, b, c)$ is defined by

$$B(a, b, c) = \Gamma(a)\Gamma(b)\Gamma(c)\{\Gamma(a + b + c)\}^{-1}. \tag{A.9}$$