The determination of the temperature by Conformal transformations at any inner point on a cuneiform surface

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Abstract
The conformal transformations are closely linked to various areas of physics. Thus, it is necessary for the undergraduates of the exact sciences to understand this branch of mathematics which is more refined and applicable. In this paper, we will use the conformal transformations to develop the problems of the temperature determination and heat transition across any cuneiform surface, where, through a succession of conforming transformations, we change the initial problem proposed to the problem in which the temperature determination is made in a stationary transition by two parallel surfaces.

Keywords: Thermodynamics; Textbooks for undergraduates.

Resumen
Las transformaciones conformales están estrechamente vinculadas a diversas áreas de la física. Por lo tanto, es necesario que los estudiantes universitarios de ciencias exactas comprendan esta rama de las matemáticas que es más refinada y aplicable. En este artículo, utilizaremos las transformaciones conformales para desarrollar los problemas de la determinación de la temperatura y la transición de calor a través de cualquier superficie cuneiforme, donde, a través de una sucesión de transformaciones conformes, cambiamos el problema inicial propuesto al problema en el que la determinación de la temperatura es realizado en una transición estacionaria por dos superficies paralelas.

Palabras clave: Termodinámica; Libros de texto para estudiantes universitarios.

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I. INTRODUCTION

The one-variable complex functions theory is one of the basic parts of mathematical analysis. Its influence can be noted in almost every studies involving mathematics and physics. In addition, to being prominent in mathematics, the theory proves to be a good tool applied to engineers and physicists [1, 2].

Harry Bateman and Ebenezer [3] were the first ones to study the conformal symmetry of Maxwell's equations. They called a generic expression symmetry as a spherical wave transformation.

It is notorious that the conformal symmetry has been showing as a central tool in the description of various physical models, ranging from condensed matter to high-energy physics, also going through cosmology. In condensed matter (and statistical physics), it can be seen that when a material approaches a phase transition, in second order, there will be no natural length scale in the sample (length scale tends to infinity [4]. That is the idea behind the renormalization group, the treatment of system information at the smallest length scales leads to the description of a system that takes into account only physics at large lengths, which allow physics to be studied near phase transitions.

In fundamental particle physics, it is hoped to explain the masses of various fundamental particles. Since the masses of these particles are "light", the length scale in the system becomes long and repeated. Hence, an invariance of scale develops. Therefore, this could be a motivation to analyze the conformal invariance [5].

In the context of quantum gravity, for example, the conformal invariance had proved promising as a gauge theory. In this sense, it is suggested that conformal gravity it best fits the galactic rotation curves, whose anomalies are generally attributed to the presence of non-baryonic dark matter. In quantum field theory the conformal invariance is a path towards renormalizable models, since conformal theories are unitary due to the simultaneous discrete symmetries of parity and temporal reversal.

Given the above, we see that conformal transformations are closely linked to various areas of physics. Thus, it is necessary for undergraduates in the various areas of the exact sciences to know this refined and highly applicable mathematics that has brought, with the development of science and technology, great advances in the understanding of the universe itself.

In this work to develop the knowledge of the conformal transformations in physics, we will work the problem of the temperature and the heat conduction through a cuneiform surface reducing it, through a succession of conformal transformations, to the problem of determination of the temperature and the heat flux in the one-dimensional stationary situation. [2, 6, 7, 8].

II. STATIONARY ONE-DIMENSIONAL HEAT CONDUCTION

Suppose a flat wall wide L, where its outer surfaces are kept at temperatures T_1 and T_2, as shown in Fig. 1.

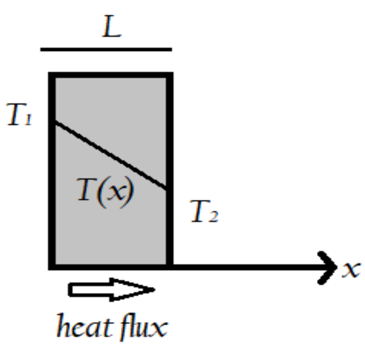

FIGURE 1. Heat flows through the surface maintained at T_1 to the surface maintained at T_2. $T(x)$ is the temperature any measurement at a point inside the wall. Note that $T_1 > T_2$.

Assuming that conduction heat transfer is permanent, without constant internal heat generation and transport properties (thermal conductivity), the temperature distribution $T(x)$ within the wall is linear [7].

To solve this problem we may taking account the general differential heat equation [9]. Where, we assume that there no exists internal heat generation. And, on the other hand, we also consider it does not change over time and the process takes place along the x direction. So, the equation is given by

$$\frac{d^2T}{dx^2} = 0 \cdot \tag{1}$$

The general solution for the Eq. (1) is a first degree function with constant coefficients, it means,

$$T(x) = ax + b \ , \tag{2}$$

where we determine the constants a and b through the boundary conditions [10] of the problem.

It is seen that if we take the zero point in the x-axis in the wall maintained at the T_1 temperature we have $T(x=0)=T_1$. Note that, if we take the L-point in the x-axis in the wall maintained at the T_2 we have $T(x=L)=T_2$. So, with these two boundary conditions we get:

$$b = T_1 \Rightarrow a = \frac{T_2 - T_1}{L} \ . \tag{3}$$

This way we can show that in the stationary regime of heat flow by conduction [8], the $T(x)$ temperature may be found by the equation bellow

$$T(x) = T_1 + \frac{(T_2 - T_1)x}{L} \ . \tag{4}$$

The Eq. (4) shows that, in fact, the temperature in the inner points of the wall is linear and it depends only on the position at which the point is taken relative to the surface of the highest temperature. In addition, heat flux can be obtained through Fourier's law, given by:

$$\varphi = -k\frac{dT(x)}{dx} \ . \tag{5}$$

where ϕ is the heat flux, Q is the heat, A is the surface area, k is the material's thermal conductivity, $T(x)$ is the temperature at the internal point and x is the position taking into account.

III. ABOUT CONFORMAL TRANSFORMATIONS

When we dealing with functions defined in the plane of the Reals, it should be noted that in this case a graphical representation is natural. These graphical representation present a fundamental importance when we need to understanding many concepts, such as derivatives and integrals. Regarding the functions of complex variables, we do not have such a graphical representation.

Thus, we need a plane for the representation of each variables. In this way, to display the corresponding set of points z and w defined in the complex domain C, where they may correlated each other by the transformation $w=f(z)$, two planes are drawn separately. For each point (x,y) of the z-plane, in the f definition domain, there is a point (u,v) in the w-plane, where $z = x + yi$ and $w = u + vi$.

The correspondence between the two planes is called transformation of the points in the z-plane into the points of the w-plane by the f function. The w points are called *Images* of the z points. This idea also applies between specific sets, such as image curves, of a particular region etc. [1]

In the general situation we may say that a conformal transformation is a kind of transformation that preserves local angles. An analytic function is conformal to any point that has a nonzero derivative. On the other hand, when we are dealing with complex variables, any conformal mapping of a complex variable that has continuous partial derivatives is analytical [2, 6].

To understand the above concepts let us take the following supposition. Let us take into account that a certain transformation takes the point (x_0, y_0) from the *z*-plane, in the point (u_0,v_0) on the *w*-plane, as shown in the Fig. 2, and the curves C_1 and C_1, that intersect each other at (x_0, y_0), on the curves C'_1 and C'_2, that also intersect each other at (u_0,v_0). So, if the transformation is such that the angle between C_1 and C_2 in (x_0, y_0) is equal to the angle between C'_1 and C'_2 in (u_0,v_0), the transformation is called conformal to (x_0, y_0).

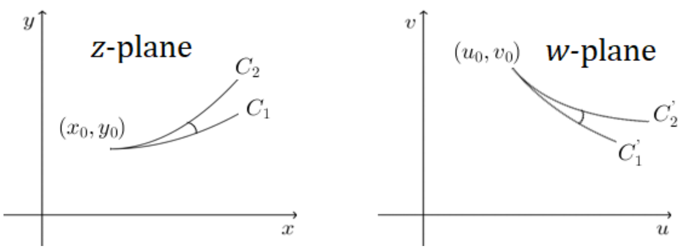

FIGURE 2. A conformal transformation takes the C_1 and C_2 curves, defined in the *z*-plane, in the C'_1 and C'_2 curves, defined in the *w*-plane, preserving the angle [6].

The Conformal transformations are very important because they may transform complex regions to work into regions knowed that are easily to be worked. The angle between the curves C_1 and C_2, shown in Fig. 2, is defined as the angle between their tangents at the intersection point of the curves.

IV. THE TEMPERATURE INSIDE A CUNEIFORM REGION

Let us start this section defined what is a cuneiform region. A cuneiform region is one that is shaped like a wedge. A wedge, on the other hand, is a piece that ends in very sharp dihedral angle. A dihedral angle is an expansion of the concept of angle to a three-dimensional space. It is defined as the space between two semi planes not contained on the same plane from a common edge, as showed in Fig. 3.

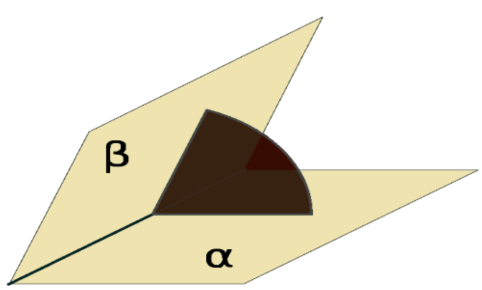

FIGURE 3. The α and β planes intersects each other forming between them an angle less than 90 degrees. The region among these planes is called the cuneiform region.

After the understanding what is a cuneiform region we may present the problem situation. Let us suppose the following situation. A infinite cuneiform region as we show in Fig. 4 where the *BA* line represents the wall with the temperature T_1. The *BC* line with width *L* is isolated, it means, there is no heat flux by conduction through it. The *CD* line represents the wall with the temperature T_2. The angle between these both surfaces with different temperatures is $\pi/4$. Note that, we are considering $T_1 > T_2$ during all steps of the problem, in this way, the heat flux will be through the *BA* line to the *CD* line.

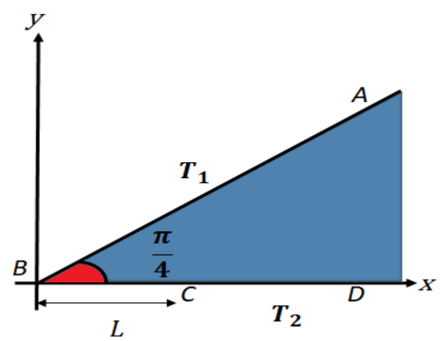

FIGURE 4. A cuneiform surface where the *AB* line is with temperature T_1 and the *CD* line with T_2 less than T_1. The *BC* line is isolated with width equals to *L*.

The problem consists in to temperature determination at any point inside this surface. We will develop this problem by the using of the conformal transformations. The mathematical steeps may be seen in Appendix. Mapping the points inside the surface, seen in Fig. 4, in the complex domain, defined in the *z*-plane, it means, any inner point at the cuneiform surface can be found by

$$z = x + iy\,, \tag{6}$$

we may use the correlation $w=f(z)=z^2$ to transform the inner points in the cuneiform region in the inner points in the *w*-plane. As we may see in Fig. 5.

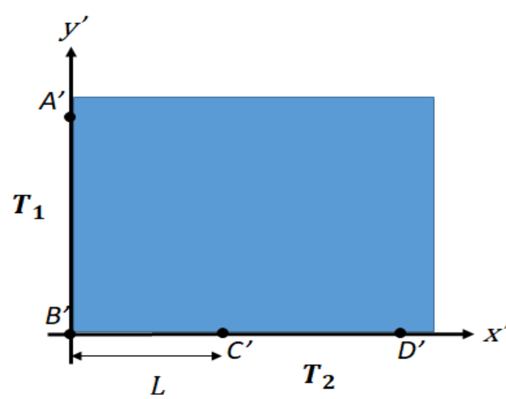

FIGURE 5. The representation of the w-plane.

On the other hand, by the transformation

$$\phi = \sin\frac{\pi w}{2L} \ , \tag{7}$$

takes the Fig. 5 into the Fig. 6(a). In this way we have the following representation.

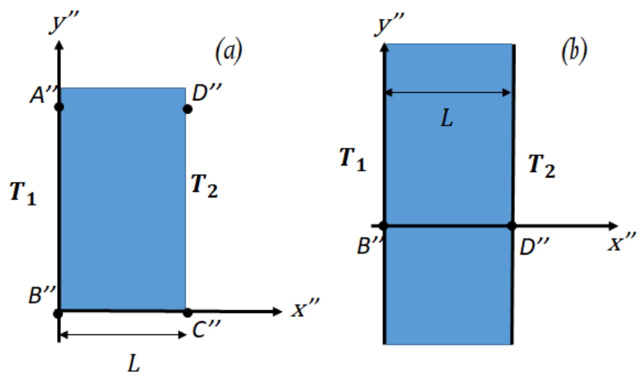

FIGURE 6. The representation of the semi-infinity *w*-plane. Any inner point in this plane can be found by the mapping *w=x''+iy''*. In the left side (*a*) we presents the transformation of the *w*-plane into the *w*-plane. In the right side (*a*) we presents the *w*-plane extends to infinity by symmetry.

Note that, if we consider there is no heat flux by through the *B''C''* surface, by symmetry we can represents the ϕ-plane as follow in the Fig. 6(b). Now, taking into account the inverse function in the Eq. (7) we have

$$w = \frac{2L}{\pi}\sin^{-1} z^2 \ . \tag{7}$$

But, in the other side we have that:

$$w = x'' + iy'' \ , \tag{8}$$

in this way we get:

$$x'' = \mathrm{Re}(w) = \frac{2L}{\pi}\mathrm{Re}(\sin^{-1} z^2) \ . \tag{9}$$

So, if we compare the Fig. 6(b) with the Fig. 1, we can see that the problem by a succession of transformations became in that problem of heat flux by conduction at the permanently condition. Hence, in any inner point of this cuneiform surface the temperature can be given by

$$T(x'') = T_1 + \frac{T_2 - T_1}{L} x'' \ , \tag{10}$$

Where *x''* is given by Eq. (10). Beyond this we can see that the heat flux through this surface is the same that one given by Eq. (5).

V. CONCLUSIONS

In order to promote a diversification in the literature on math applications in the determination of problems contained in physics, this work comes to contribute in the formation of academics of the exact and natural sciences. It was proposed to approach the conformal transformations to solve the thermodynamic problem of temperature determination at any inner point of a cuneiform surface.

In a one-dimensional case it can be seen that the temperature and the stationary conduction heat transfer, between two flat surfaces kept in a different temperature occurs linearly, may be determined as a fist degree function of the position *x*, as was shown in Eq. (5), and the heat flux between the surface can be determined by Fourier's Law.

A conformal transformation is understood to mean any transformation that moves from one reference system to another while preserving angles locally. Given the above, such transformations have a great applicability regarding the approach of complex variables, which in fact is also widely used in physics.

From there, for a given cuneiform region, defined by Fig. 4, by mapping the inner points in this surface, through the using the complex variables, we determined the temperature inside this surface. The temperature *T* is shown in Eq. (10), where *x*'', given by Eq. (9), is the mapping of points within this initial cuneiform surface.

We see that conforming transformations are indeed very useful in physics when dealing with more complex problems, such as the one addressed in this paper, and indeed a review of schematized literature and an application should be useful to students of the exact sciences and of nature during its academic formation process.

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APPENDIX

Here we develop the mathematical steeps used during this work. In fact the transformation that leaves the inner points in *z*-plane to that ones in the ϕ-plane is given by:

$$\phi = z^2 \ . \tag{A.1}$$

It follow that

$$\mathrm{Re}(\phi) = x' = x^2 - y^2 \geq 0 \, ; \ \mathrm{Im}(\phi) = 2xy \geq 0 \tag{A.3}$$

Showing that, in fact, $x' \geq 0$ and $y' \geq 0$ is true.

On the other hand, the inner region in the Fig. 5 leaves the in the inner region of the Fig. 6(a) given by the *w*-plane by the following transformation.

$$\varphi = \sin\frac{\pi w}{2L}. \tag{A.4}$$

As we know, any *w*-point in the inner region of the Fig. 6(a) is also given by

$$w = x'' + iy''. \tag{A.5}$$

Using Eq. (A.5) in Eq. (A.4) we get

$$\begin{aligned} \varphi &= \sin\frac{\pi(x''+iy'')}{2L}, \\ \varphi &= \sin\left(\frac{\pi x''}{2L} + \frac{i\pi y''}{2L}\right), \\ \varphi &= \frac{e^{i\left(\frac{\pi x''}{2L}+\frac{i\pi y''}{2L}\right)} - e^{-i\left(\frac{\pi x''}{2L}+\frac{i\pi y''}{2L}\right)}}{2i}, \\ \varphi &= \sin\frac{\pi x''}{2L}\cosh\frac{\pi y''}{2L} + i\cos\frac{\pi x''}{2L}\sinh\frac{\pi y''}{2L}. \end{aligned} \tag{A.6}$$

It means that the real part of Eq. (A.6) is given by

$$\mathrm{Re}(\phi) = x' = \sin\frac{\pi x''}{2L}\cosh\frac{\pi y''}{2L} \ . \tag{A.7}$$

and

$$\mathrm{Im}(\phi) = y' = \cos\frac{\pi x''}{2L}\sinh\frac{\pi y''}{2L} \ . \tag{A.8}$$

Taking into account that $0 \leq x'' \leq L$ and $y'' \geq 0$ it follows that the equations Eq. (A.7) and Eq. (A.8) are satisfied.