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Pricing Options under Telegraph Processes

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Abstract. In this paper we introduce a financial market model based on continuous time random motions with alternating constant velocities and jumps, which occur with velocity switches. Given that jump directions match velocity directions of the underlying random motion properly in relation to interest rates, in this setting will be free of arbitrage. Additionally, we suppose also the interest rate depending on the market state.
The replicating strategies for options are constructed in detail, and closed form formulas for option prices are obtained.

Keywords: jump telegraph process, European option pricing, perfect hedging, self-financing strategy, fundamental equation.
JEL Classification: G10, G12, D8.

Resumen. En este artículo introducimos un nuevo modelo del mercado financiero basado en movimientos aleatorios en tiempo continuo con velocidades constantes y alternantes. Este movimiento está complementado con saltos que ocurren cuando se presentan cambios de la tendencia. Este modelo está libre de arbitraje, si la dirección del salto es opuesta a la diferencia entre tendencia y tasa de interés. Suponemos que las tasas de interés dependen del estado del mercado. Las estrategias repicables son construidas en detalle. Las fórmulas completas para los precios de las opciones son obtenidas.

Palabras clave: proceso telegráfico con saltos, valoración de opciones europeas, estrategias autofinancieras, ecuación fundamental.
Clasificación JEL: G10, G12, D8.

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1. Introduction

Option pricing models based on the exponential Brownian motion have well known limitations. These models have infinite velocities of propagation, independent log-returns increments on separated time intervals, among others. Moreover it is widely accepted that financial time series are not Gaussian.

It seems rather natural to replace in the basic models a Brownian motion by a finite velocity random evolution (with statistically dependent increments). However, it is conventional that such substitution creates arbitrage opportunities (see e. g. Ratanov, 2004). The main cause of the arbitrage here is the persistent character of such random motions. To avoid arbitrage possibilities, we propose a model with jumps occurring every time a tendency changes.

Cox and Ross (1975, 1976) and Merton (1976) initiated the research of the option pricing models with jump diffusion processes, but jumps introduced in these models are usually motivated by empirical adequacy. In the present paper the use of jumps is motivated not only by the adequacy problems, but also by the possibility to avoid an arbitrage as well.

More specifically, we suppose the market can have two possible states, alternating at independent and exponentially distributed time intervals, which form a continuous time Markov chain. The interest rates $r_\pm$ and the velocities $c_\pm$ of log-returns of the risky asset are defined by the current market state. Moreover we suppose that log-returns of a risky asset follow the so-called telegraph process (see Kac, 1959), with jumps occurring each time of velocity changes. Thus we have a complete market model, and hedging is perfect. Unfortunately, the underlying process is not a Lévy process, and therefore the general theory does not work.

It is known (see Kac, 1959, 1974; Ratanov, 1997) that, at least in the homogeneous setting, the underlying process converges to Brownian motion under suitable rescaling. More precisely, we prove that this model converges to the Black-Scholes model if the size of jumps vanishes, but the velocities of the asset's return and the frequencies of jumps go to infinity in a particular manner.

The paper is organized as follows. Section 2 presents inhomogeneous telegraph processes and martingales related to the telegraph evolutions and to the driving inhomogeneous Poisson process. Here, also the Girsanov theorem for the telegraph processes with jumps is obtained. In Section 2.3 we introduce the main model: for that purpose, we consider a friction-free financial market, where a risk-free (bond) asset has two constant return rates $r_\pm$ depending on the market's state, and a risky asset price is given by the stochastic exponential $S_0\mathcal{E}_t(X+J)$. Here $X = X(t)$ is the integrated telegraph process and $J = J(t)$ is a pure jump process. The common inhomogeneous

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Poisson process drives both of them. The martingale measure for the asset price process is constructed. In Section 3 we derive the fundamental equation for the option price and the perfect hedging strategy formulas. The left continuity in time of the portfolio dynamics is proved as well. The closed formulas for the price of the standard call option are presented in Section 4. These formulas are analytic tractable and combine the outlines of the Black-Scholes and Merton formulas. Appendices contain the proof of the convergence to the Black-Scholes model and the exact formulas for the distributions of the underlying processes, which are necessary for the call option price formula.

This paper exploits the ideas presented by the author at the 2nd Nordic-Russian Symposium on Stochastic Analysis Ratanov (1997) and continues the author's previous papers devoted to the telegraph model, Ratanov (2004, 2005).

2. Inhomogeneous Telegraph Processes and Martingales. Dynamics of the Basic Assets and the Martingale Measure

2.1. *Telegraph and Poisson Martingales*

The state of the market is denoted by $\sigma = \sigma(t),\ \ t \geq 0$ with values ± 1 such that

$$\mathrm{P}(\sigma(t+\Delta t) = 1 \,|\, \sigma(t) = -1) = \lambda_- \Delta t + o(\Delta t),$$

$$\mathrm{P}(\sigma(t+\Delta t) = -1 \,|\, \sigma(t) = 1) = \lambda_+ \Delta t + o(\Delta t),\quad \Delta t \to 0.$$

Here $\lambda_-,\ \lambda_+ > 0$ and $\sigma(0) = \xi$, where ξ is a random variable with two values ± 1. Time intervals $\tau_j - \tau_{j-1},\ \ j = 1,\ 2,\ \dots$ ($\tau_0 = 0$), separated by instants $\tau_j,\ \ j = 1,\ 2,\ \dots$ of value changes of σ are independent and exponentially distributed random variables. Denote by $N(t)$ the number of value changes of σ in time t, i.e. $\sigma(t) = \xi(-1)^{N(t)}$. The process $N = N(t)$ is an inhomogeneous Poisson process with alternating parameters $\lambda_\pm$.

Let $c_- < c_+,\ \ h_-,\ \ h_+$ be real numbers. We denote

$$V(t) = c_{\sigma(t)},\quad X(t) = \int_0^t V(s)ds \tag{2.1}$$

and

$$J(t) = \sum_{j=1}^{N(t)} h_{\sigma(\tau_j -)},\quad t \geq 0. \tag{2.2}$$

The process $(X,\ V)$ is called a (inhomogeneous) telegraph process with states $(c_-,\ \lambda_-)$ and $(c_+,\ \lambda_+)$. The process $J = J(t),\ \ t \geq 0$ is a pure jump process with jumps at the Poisson times $\tau_j,\ \ j = 1,\ 2,\ \dots$ For $\lambda_- = \lambda_+$ and $-c_- = c_+ = c$, processes $V(t) = \xi c(-1)^{N(t)}$ and $X(t) = \xi c \int_0^t (-1)^{N(s)} ds,\ \ t \geq 0$ are well known, due to S. Goldstein (1951) and M. Kac (1959, 1974), and they are called telegraph and integrated telegraph processes, respectively. Also, it is known that if $\lambda,\ c \to \infty$ and $c^2/\lambda \to 1$, the process $X(t)$ converges to the standard Brownian motion.

The inhomogeneous process is less known (see for example Beghin et al, 2001), where the exact distributions of inhomogeneous $X(t)$ are calculated).

Remark 2.1. *Let* $X = X(t)$ *and* $\tilde{X} = \tilde{X}(t),\ t \ge 0$ *be telegraph processes with states* $(c_\pm,\ \lambda_\pm)$ *and* $(\tilde{c}_\pm,\ \lambda_\pm)$ *respectively, governed by the common Poisson process* $N = N(t)$. *Then*

$$\tilde{X}(t) = aX(t) + bt \tag{2.3}$$

with

$$a = a_{\tilde{c}} = \frac{\tilde{c}_+ - \tilde{c}_-}{c_+ - c_-},\quad b = b_{\tilde{c}} = \frac{c_+\tilde{c}_- - c_-\tilde{c}_+}{c_+ - c_-}. \tag{2.4}$$

Notice that $c_\sigma a + b \equiv \tilde{c}_\sigma,\ \ \sigma = \pm 1$.

To construct related martingales we have the following lemma.

Lemma 2.1. *The conditional expectations* $j_\sigma(t) = \mathrm{E}(J(t) \mid \xi = \sigma)$, $n_\sigma(t) = \mathrm{E}(N(t) \mid \xi = \sigma)$ *and* $v_\sigma(t) = \mathrm{E}(V(t) \mid \xi = \sigma)$, $\sigma = \pm 1,\ t \ge 0$ *can be calculated as follows*

$$j_\sigma(t) = \frac{\gamma H}{2} t + \lambda_\sigma \alpha_\sigma \frac{1 - \mathrm{e}^{-\Lambda t}}{\Lambda}, \tag{2.5}$$

$$n_\sigma(t) = \gamma t + \lambda_\sigma \beta_\sigma \frac{1 - \mathrm{e}^{-\Lambda t}}{\Lambda}, \tag{2.6}$$

$$v_\sigma(t) = g + \lambda_\sigma \delta_\sigma \mathrm{e}^{-\Lambda t}, \tag{2.7}$$

where $\Lambda = \lambda_- + \lambda_+$, $H = h_- + h_+$, $\gamma = \frac{2\lambda_+\lambda_-}{\Lambda}$, $g = \frac{c_+\lambda_- + c_-\lambda_+}{\Lambda}$, $\alpha_\sigma = \frac{\lambda_\sigma h_\sigma - \lambda_{-\sigma} h_{-\sigma}}{\Lambda}$, $\beta_\sigma = \frac{\lambda_\sigma - \lambda_{-\sigma}}{\Lambda}$, $\delta_\sigma = \frac{c_\sigma - c_{-\sigma}}{\Lambda},\ \ \sigma = \pm 1$.

Remark 2.2. *In the homogeneous case* $\lambda_- = \lambda_+ = \lambda$, $c_+ = a + c,\ \ c_- = a - c$ *formulas (2.6)-(2.7) are known:*

$n_\sigma(t) = \lambda t,\ \ v_\sigma(t) = a + \sigma c \mathrm{e}^{-2\lambda t},\ \ \sigma = \pm 1.$

Proof. Formulas (2.6) and (2.7) follow from (2.5). Indeed, the Poisson process $N(t),\ \ t \ge 0$ is a pure jump process with $h_\pm = 1$. Hence (2.6) coincides with (2.5), which has $H = h_- + h_+ = 2$ and $\alpha_\sigma \equiv \beta_\sigma$. Moreover, $V(t) - c_\sigma,\ \ t \ge 0$ is again a pure jump process with alternating jump values $h_\sigma = c_{-\sigma} - c_\sigma,\ \ \sigma = \pm 1$. Thus $H = 0$ and $\alpha_\sigma = -(c_\sigma - c_{-\sigma})$. Therefore, (2.7) follows from (2.5) and the identity $c_\sigma - \lambda_\sigma(c_\sigma - c_{-\sigma})/\Lambda = g$.

To prove (2.5), first notice that conditioning on a switch at the time interval $(0,\ \Delta t)$ we have

$$j_\sigma(t + \Delta t) = (1 - \lambda_\sigma \Delta t) j_\sigma(t) + \lambda_\sigma \Delta t (j_{-\sigma}(t) + h_\sigma) + o(\Delta t),\ \Delta t \to 0.$$

Hence, expectations $j_\sigma(t),\ \sigma = \pm 1$ fit the equations

$$\frac{\mathrm{d}j_\sigma}{\mathrm{d}t}(t) = -\lambda_\sigma(j_\sigma - j_{-\sigma}) + \lambda_\sigma h_\sigma,\ \ t > 0 \tag{2.8}$$

with initial data $j_\sigma\,|_{t=0} = 0,\ \ \sigma = \pm 1$.

Since $\lambda_\sigma\alpha_\sigma - \lambda_{-\sigma}\alpha_{-\sigma} = \lambda_\sigma h_\sigma - \lambda_{-\sigma}h_{-\sigma}$ and $\alpha_{-\sigma} = -\alpha_\sigma$, the unique solution of system (2.8) is given by (2.5). Thus, the lemma is proved.

The following formulas are the evident consequence of Lemma 2.1.

Corollary 2.1. *Let* $(X(t),\ V(t)),\ \ t \ge 0$ *be the telegraph process with states* $(c_-,\ \lambda_-)$ *and* $(c_+,\ \lambda_+)$. *Let* $J = J(t),\ \ t \ge 0$ *be the jump process with values* $h_\pm$ *driven by the same Poisson process. Then the conditional expectations are given by*

$$\mathrm{E}(J(t)\,|\,\mathcal{F}_s) = J(s) + \frac{\gamma H}{2}(t - s) + \lambda_\sigma\alpha_\sigma\frac{1 - \mathrm{e}^{-\Lambda(t-s)}}{\Lambda}, \tag{2.9}$$

$$\mathrm{E}(X(t)\,|\,\mathcal{F}_s) = X(s) + g(t - s) + \lambda_\sigma\delta_\sigma\frac{1 - \mathrm{e}^{-\Lambda(t-s)}}{\Lambda}, \tag{2.10}$$

with $\sigma = \sigma(s)$, $s \le t$.

From these formulas it is easy to obtain the following theorem:

Theorem 2.1. *Let* $(X(t),\ V(t))$ *be the telegraph process with states* $(c_-,\ \lambda_-)$ *and* $(c_+,\ \lambda_+)$. *Let* $J = J(t) = \sum_{j=1}^{N(t)} h_{\sigma(\tau_j -)}$. *Then* $X + J$ *is the martingale if and only if* $\lambda_\sigma h_\sigma = -c_\sigma$, $\sigma = \pm 1$.

Proof. From formulas (2.9) and (2.10), it follows that $X + J$ is the martingale if and only if

$$\begin{cases} g + \dfrac{\gamma H}{2} = 0. \\ \alpha_\sigma + \delta_\sigma = 0. \end{cases}$$

The unique solution to this system is $h_\sigma = -c_\sigma / \lambda_\sigma,\ \ \sigma = \pm 1$.

2.2. *Change of Measure*

Let $X = X(t),\ \ t \ge 0$ be the telegraph process with the states $(c_\pm,\ \lambda_\pm)$, $\lambda_\pm > 0$, $c_+ > c_-$, and $N = N(t),\ \ t \ge 0$ be the driving Poisson process.

Fix time horizon T. Let

$$Z(t) = \frac{\mathrm{dP}^*_t}{\mathrm{dP}_t} = \mathcal{E}_t(X^* + J^*),\ \ 0 \le t \le T \tag{2.11}$$

be the density of new measure P^* relative to P. Here X^* is the telegraph process with the states $(c_\pm^*,\ \lambda_\pm)$, $\kappa_n^{*,\,\sigma} = (1+h_\sigma^*)^k(1+h_{-\sigma}^*)^k$, is the pure jump process with the jump values

$$h_\sigma^* = -c_\sigma^*/\lambda_\sigma > -1,\ \ \sigma = \pm 1. \tag{2.12}$$

Both of these processes are driven by the same inhomogeneous Poisson process *N*. $\mathcal{E}_t(\cdot)$ denotes the stochastic exponential.

From (2.11) we obtain

$$Z(t) = \mathrm{e}^{X^*(t)}\kappa^*(t), \tag{2.13}$$

Where

$$\kappa^*(t) = \prod_{s\le t}\left(1+\Delta J^*(s)\right). \tag{2.14}$$

Here $\Delta J^*(s) = J^*(s) - J^*(s-)$.

Let us consider the sequence $\kappa_n^{*,\,\sigma}$, which is defined as follows

$$\kappa_n^{*,\,\sigma} = \kappa_{n-1}^{*,-\sigma}(1+h_\sigma^*),\ \ n\ge 1,\ \ \kappa_0^{*,\,\sigma} = 1,\ \ \sigma = \pm 1. \tag{2.15}$$

Thus, if $n = 2k$,

$$\kappa_n^{*,\,\sigma} = (1+h_\sigma^*)^k(1+h_{-\sigma}^*)^k, \tag{2.16}$$

And $n = 2k+1$,

$$\kappa_n^{*,\,\sigma} = (1+h_\sigma^*)^{k+1}(1+h_{-\sigma}^*)^k. \tag{2.17}$$

Therefore $\kappa^*(t) = \kappa_{N(t)}^{*,\,\sigma}$, where $\sigma = \pm 1$ indicates the initial direction.

The following theorem replaces the Girsanov theorem in this framework.

Theorem 2.2. *Under the probability* P^* *with density Z(t) relative to* P*, process* $N = N(t),\ \ t\ge 0$ *is again the Poisson process with intensities* $\lambda_-^* = \lambda_- - c_-^* = \lambda_-(1+h_-^*) > 0$ *and* $\lambda_+^* = \lambda_+ - c_+^* = \lambda_+(1+h_+^*) > 0$ *(see (2.12)).*

Proof. Let $\pi_n^{(\sigma)}(t) = \mathsf{P}(N(t) = n \mid \xi = \sigma)$ and $\pi_{*,n}^{(\sigma)}(t) = \mathsf{P}^*(N(t) = n \mid \xi = \sigma),\ n = 0,1,2\ldots$. Probabilities $\pi_n^{(\sigma)}(t),\ \ \sigma = \pm 1$ are completely defined as the solution of the following system, which can be obtained in the same manner as (2.8)

$$\begin{cases} \dfrac{\mathrm{d}\pi_n^{(\sigma)}}{\mathrm{d}t} = -\lambda_\sigma \pi_n^{(\sigma)}(t) + \lambda_\sigma \pi_{n-1}^{(-\sigma)}(t),\ t>0,\ n\ge 1 \\ \pi_n^{(\sigma)}|_{t=0} = 0,\ n\ge 1;\ \pi_0^{(\sigma)}|_{t=0} = 1 \end{cases}$$

Moreover, from (2.13)-(2.17) and (2.3)-(2.4) it follows

$$\pi_{*,n}^{(\sigma)}(t) = \mathrm{E}(Z(t)\mathbf{1}_{\{N(t)=n\}} \mid \xi = \sigma) = \kappa_{*,n}^{(\sigma)}\int_{-\infty}^{\infty}\mathrm{e}^{a^*x+b^*t}p_n^{(\sigma)}(x,t)dx \tag{2.18}$$

with $a^* = \frac{c_+^* - c_-^*}{c_+ - c_-}$ and $b^* = \frac{c_+c_-^* - c_-c_+^*}{c_+ - c_-}$. Here $p_n^{(\sigma)},\ \ n\ge 0$ are the probability densities with respect to measure P of the current position of the process $X(t),\ \ 0\le t\le T$, which has *n* turns, i. e. for any measurable set Δ

$$P(X(t)\in \Delta,\ N(t)=n \mid \xi=\sigma)=\int_{\Delta} p_n^{(\sigma)}(x,t)dx \tag{2.19}$$

Conditioning on the number of jumps at $(0,\ \Delta t)$ and passing to limit as $\Delta t \to 0$ we obtain (see Ratanov (2004) for details)

$$\frac{\partial p_n^{(\sigma)}}{\partial t}+c_\sigma \frac{\partial p_n^{(\sigma)}}{\partial x}=-\lambda_\sigma p_n^{(\sigma)}+\lambda_\sigma p_{n-1}^{(-\sigma)},\quad n\geq 1 \tag{2.20}$$

with zero initial conditions: $p_n^{(\sigma)}|_{t=0}=0,\ n\geq 1.$ Moreover $p_0^{(\sigma)}(x,\ t)=\mathrm{e}^{-\lambda_\sigma t}\delta(x-c_\sigma t)$.

From this equation and (2.18)

$$\frac{\mathrm{d}\pi_{*,\,n}^{(\sigma)}}{\mathrm{d}t}=(b^*-\lambda_\sigma+a^*c_\sigma)\pi_{*,\,n}^{(\sigma)}(t)+\lambda_\sigma(1-c_\sigma^*/\lambda_\sigma)\pi_{*,\,n-1}^{(-\sigma)}(t).$$

The following evident equalities complete the proof:

$$b^*-\lambda_\sigma+a^*c_\sigma=c_\sigma^*-\lambda_\sigma=-\lambda_\sigma^*,$$
$$\lambda_\sigma(1-c_\sigma^*/\lambda_\sigma)=\lambda_\sigma^*,$$
$$p_n^{(\sigma)}|_{t=0}=0,\quad n\geq 1,$$
$$p_0^{(\sigma)}|_{t=0}=\delta(x).$$

Corollary 2.2. *Under the probability* P^* *with density Z*(*t*) *relative to* P*, process* $X=X(t),\ 0\leq t\leq T$ *is the telegraph process with the states* $(c_-,\ \lambda_-^*)$ *and* $(c_+,\ \lambda_+^*)$ *with* $\lambda_\sigma^*=\lambda_\sigma-c_\sigma^*=\lambda_\sigma(1+h_\sigma^*)>0,\ \ \sigma=\pm 1$ *(see (2.12)).*

2.3. *Dynamics of the basic assets and the martingale measure*

We assume the bond price follows

$$B(t)=\mathrm{e}^{Y(t)},\quad Y(t)=\int_0^t r_{\sigma(s)}ds,\quad r_-,\ \ r_+>0. \tag{2.21}$$

To introduce the price process for the risky asset let $X=X(t),\ \ t\geq 0$ be the telegraph process with the states $(c_-,\ \lambda_-)$ and $(c_+,\ \lambda_+)$, $c_+>c_-$ and $J=J(t)=\sum_{j=1}^{N(t)} h_{\sigma(\tau_j-)},\ \ h_\pm>-1$ (see (2.1)-(2.2)).

We assume the price of the risky asset follows the equation

$$dS(t)=S(t-)d\left(X(t)+J(t)\right),\quad t>0. \tag{2.22}$$

Process $S(t),\ t\geq 0$ assumed to be right-continuous.

Integrating (2.22) we obtain

$$S(t)=S_0\mathcal{E}_t\left(X+J\right)=S_0\mathrm{e}^{X(t)}\kappa(t), \tag{2.23}$$

where

$$k(t)=\prod_{s\leq t}(1+\Delta J(s))=k_{N(t)}^{\sigma},\qquad S_0=S(0)$$

The sequence $\kappa_n^\sigma,\ n \geq 0$ is defined as in (2.15)-(2.17) (with $h_\pm$ instead of $h_\pm^*$).

We assume the following restrictions to the parameters of the model

$$\frac{r_\sigma - c_\sigma}{h_\sigma} > 0,\ \ \sigma = \pm 1. \tag{2.24}$$

Since the process N is the unique source of randomness, there is the only one equivalent martingale measure. To construct it we are looking for the respective martingale in the form $X^*(t) + J^*(t),\ \ t \geq 0$. By Theorem 2.1 we suppose that $\lambda_\sigma h_\sigma^* = -c_\sigma^*$.

Theorem 2.3. *Let* $Z(t) = \mathcal{E}_t(X^* + J^*)$, $t \geq 0$ *with* $h_\sigma^* = -c_\sigma^* / \lambda_\sigma$ *be the density of probability* P^* *relative to* P. *The process* $(B(t)^{-1}S(t))_{t\geq 0}$ *is the* P^* *- martingale if and only if*

$$c_\sigma^* = \lambda_\sigma + \frac{c_\sigma - r_\sigma}{h_\sigma},\ \ \sigma = \pm 1.$$

Under the probability P^* *the Poisson process N is driven by the parameters*

$$\lambda_\sigma^* = \frac{r_\sigma - c_\sigma}{h_\sigma} > 0,\ \sigma = \pm 1.$$

Proof. First notice that by Corollary 2.2 $X(t)$-$Y(t)$ is the telegraph process (with respect to P^*) with the states $(c_\sigma - r_\sigma,\ \ \lambda_\sigma - c_\sigma^*),\ \ \sigma = \pm 1$. From Theorem 2.1 it follows that $X(t) - Y(t) + J(t),\ \ t \geq 0$ is the P^*-martingale if and only if

$$(\lambda_\sigma - c_\sigma^*)h_\sigma = -(c_\sigma - r_\sigma).$$

Hence $c_\sigma^* = \lambda_\sigma + (c_\sigma - r_\sigma)/h_\sigma$ and $h_\sigma^* = -c_\sigma^*/\lambda_\sigma = -1 + (r_\sigma - c_\sigma)/(\lambda_\sigma h_\sigma)$. Thus, theorem is proved.

Remark 2.3. *From condition (2.24) it follows that* $h_\sigma^* > -1$ *and* $\lambda_\sigma^* = \lambda_\sigma - c_\sigma^* = (r_\sigma - c_\sigma)/h_\sigma > 0$. *Therefore* $Z = Z(t) = \mathcal{E}_t(X^* + J^*)$ *really defines the new probability measure.*

3. Pricing and Hedging Options

3.1. *Fundamental equation*

Fix time horizon T and consider the function

$$F(t, x, \sigma) = \mathsf{E}_{T-t}^*\left[\mathrm{e}^{-Y(T-t)} f(x\mathrm{e}^{X(T-t)}\kappa(T-t)) \mid \xi = \sigma\right]$$

$$\sigma = \pm 1,\ \ 0 \leq t \leq T,$$

where E^* denotes the expectation with respect to martingale measure P^*, which is defined in Theorem 2.3. The density $Z(t)$ of P^* relative to P is defined in (2.13)-(2.17). Function $F_t = F(t,\ S(t),\ \sigma(t)) = \varphi_t S(t) + \psi_t B(t)$ is the strategy value at time t of the option with the claim $f(S_T)$ at the maturity time T.

Notice that $Y(t) = a_r X(t) + b_r t$ with $a_r = \frac{r_+ - r_-}{c_+ - c_-},\ \ b_r = \frac{c_+ r_- - c_- r_+}{c_+ - c_-}$ (see Remark 2.1). Conditioning on the number of jumps we can write

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$$F(t,\ x,\ \sigma)=\mathrm{e}^{-b_r(T-t)}\sum_{n=0}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{-a_r y}f(x\mathrm{e}^{y}\kappa_n^{\sigma})p_{*,\ n}^{(\sigma)}(y,\ T-t)dy, \tag{3.1}$$

where $p_{*,\ n}^{(\sigma)},\ n\geq 0,\ \sigma=\pm 1$ are the probability densities of telegraph process $X(t),\ 0\leq t\leq T$, which commences n turns, with respect to martingale measure P^{*}. Densities $p_{*,\ n}^{(\sigma)}$ are defined as in (2.19).

Function F solves the following difference-differential equation, which plays the same role as the fundamental equation in the Black-Scholes model. Exploiting equation (2.20) (with $\lambda_{\sigma}^{*}=\lambda_{\sigma}-c_{\sigma}^{*}=(r_{\sigma}-c_{\sigma})/h_{\sigma}$ instead of λ_{σ}) and the identity $c_{\sigma}a_r+b_r=r_{\sigma},\ \sigma=\pm 1$ from (3.1) we obtain

$$\frac{\partial F}{\partial t}(t,\ x,\ \sigma)+c_{\sigma}x\frac{\partial F}{\partial x}(t,\ x,\ \sigma)$$

$$=(r_{\sigma}+\lambda_{\sigma}^{*})F(t,\ x,\ \sigma)-\lambda_{\sigma}^{*}\mathrm{e}^{-b_r(T-t)}\sum_{n=1}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{-a_r y}f(x\mathrm{e}^{y}\kappa_n^{\sigma})p_{*,\ n-1}^{(-\sigma)}(y,\ T-t)\mathbf{d}y.$$

By equalities (2.15) and $\lambda_{\sigma}^{*}=\frac{r_{\sigma}-c_{\sigma}}{h_{\sigma}}$ the latter equation takes the form

$$\frac{\partial F}{\partial t}(t,\ x,\ \sigma)+c_{\sigma}x\frac{\partial F}{\partial x}(t,\ x,\ \sigma)$$

$$=(r_{\sigma}+\frac{r_{\sigma}-c_{\sigma}}{h_{\sigma}})F(t,\ x,\ \sigma)-\frac{r_{\sigma}-c_{\sigma}}{h_{\sigma}}F(t,\ x(1+h_{\sigma}),-\sigma),\ \ \sigma=\pm 1 \tag{3.2}$$

with the terminal condition $F_{t\uparrow T}=f(x)$.

Remark 3.1. Note that the above equations do not depend on $\lambda_{\pm}$ as the respective equation in the Black-Scholes model does not depend on the drift parameter.

3.2. *Predictability of the Strategy*

To identify the self-financing trading strategy $\Pi_t=(\varphi_t,\ \psi_t),\ 0\leq t\leq T$ such that $F_t=\varphi_t S(t)+\psi_t B(t),\ 0\leq t\leq T$ we have $\mathrm{d}F_t=\mathrm{d}F(t,S(t),\sigma(t))=\varphi_t\mathrm{d}S(t)+\psi_t\mathrm{d}B(t)$.

The predictability of the strategy means the left continuity of φ_t.

To prove it notice that

$$F_t=F_0+\int_0^t\varphi_s S(s)V(s)ds+\int_0^t\psi_s dB(s)+\sum_{j=1}^{N(t)}\varphi_{\tau j}h_{\sigma(\tau j-)}S(\tau j-).$$

From the identity $\psi_t=B(t)^{-1}(F_t-\varphi_t S(t))$ we obtain

$$F_t=F_0+\int_0^t r_{\sigma(s)}F_s ds+\int_0^t\varphi_s S(s)(c_{\sigma(s)}-r_{\sigma(s)})+\sum_{j=1}^{N(t)}\varphi_{\tau j}h_{\sigma(\tau j-)}S(\tau j-).$$

On the other hand

$$F_t=F_0+\int_0^t\frac{\partial F}{\partial s}(s,\ S(s),\ \sigma(s))ds+\int_0^t\frac{\partial F}{\partial x}(s,\ S(s),\ \sigma(s))S(s)c_{\sigma(s)}ds+\sum_{j=1}^{N(t)}(F_{\tau_j}-F_{\tau_j-}).$$

Comparing the latter two equations we have between jumps

$$\varphi_t = \frac{S(t)V(t)\frac{\partial F}{\partial x} + \frac{\partial F}{\partial t} - rF}{S(t)(V(t) - r)}.$$

From the fundamental equation (3.2) it follows that between the jumps

$$\varphi_t = \frac{r_{\sigma(t)} - c_{\sigma(t)}}{h_{\sigma(t)}S(t)(c_{\sigma(t)} - r_{\sigma(t)})}\left[F(t,\ S(t),\ \sigma(t)) - F(t,\ S(t)(1+h_{\sigma(t)}), -\sigma(t))\right] \tag{3.3}$$

$$= \frac{F(t,\ S(t)(1+h_{\sigma(t)}), -\sigma(t)) - F(t,\ S(t),\ \sigma(t))}{S(t)h_{\sigma(t)}}.$$

The jump values of φ are

$$\varphi_{\tau_j} = \frac{F_{\tau_j} - F_{\tau_j -}}{S(\tau_j -)h_{\sigma(\tau_j -)}} \tag{3.4}$$

$$= \frac{F(\tau_j,\ S(\tau_j),\ \sigma(\tau_j)) - F(\tau_j,\ S(\tau_j -), -\sigma(\tau_j))}{S(\tau_j -)h_{\sigma(\tau_j -)}}.$$

Formulas (3.3)-(3.4) remind the CRR and BS-formulas for the amounts of risky asset held over the time.

Lemma 3.1. *The strategy* $\varphi_t,\ \ 0 \le t < T$ *is left-continuous.*

Proof. To prove $\varphi_{\tau_j -} = \varphi_{\tau_j}$ first notice that by (2.23)

$$S(\tau_j -)(1+h_{\sigma(\tau_j -)}) = S(\tau_j). \tag{3.5}$$

Applying (3.5) to (3.3)-(3.4) it is easy to finish the proof.

4. Pricing a Standard Call

In the framework of the market model (2.21), (2.22)-(2.23) the price of the option with contingent claim f can be expressed as follows

$$c = c^{\sigma} = E^*_{\sigma}(B(T)^{-1}f) = \sum_{n=0}^{\infty} E^*_{\sigma}(B(T)^{-1}f \mid N(T) = n)\pi^{(\sigma)}_{*,n}(T), \tag{4.1}$$

$\sigma = \pm 1$,

where σ indicates the initial state. If $\lambda^*_- = \lambda^*_+ := \lambda$, then $\pi^{(\sigma)}_{*,\,n}(T) = \frac{(\lambda T)^n}{n!}e^{-\lambda T}$. In general case $\lambda^*_- \ne \lambda^*_+$ probabilities $\pi^{(\sigma)}_{*,\,n}(T),\ \ \sigma = \pm 1,\ \ n \ge 0$ are calculated in Appendix B.

For the standard call option with contingent claim $f = (S(T) - K)^+$ we rewrite (4.1) in the form

$$c = \sum_{n=0}^{\infty} c_n(K,T) \tag{4.2}$$

with

$$c_n(K,T) = S_0 U^{(\sigma)}_n(y - b^{(\sigma)}_n, T) - K u^{(\sigma)}_n(y - b^{(\sigma)}_n, T), \tag{4.3}$$

where $y = \ln K / S_0$ and $b_n^{(\sigma)} = \ln \kappa_n^{*,\sigma}$. Here functions $u_n^{(\sigma)}$ and $U_n^{(\sigma)}$, $n \geq 0$ are defined as follows:

$$u_n^{(\sigma)}(y,t) = u_n^{(\sigma)}(y,t;\lambda_\pm^*, c_\pm, r_\pm) = \mathsf{E}_\sigma^*\left[B(t)^{-1}\mathbf{1}_{\{X(t)>y, N(t)=n\}}\right] \tag{4.4}$$

$$= \mathrm{e}^{-b_r t}\int_y^\infty \mathrm{e}^{-a_r x} p_{*,n}^{(\sigma)}(x, t)dx$$

with $a_r = \frac{r_+ - r_-}{c_+ - c_-}$ and $b_r = \frac{c_+ r_- - c_- r_+}{c_+ - c_-}$ (see (2.3)-(2.4) in Remark 2.1);

$$U_n^{(\sigma)}(y, t) = U_n^{(\sigma)}(y, t; \lambda_\pm^*, c_\pm, r_\pm)$$

$$= \mathrm{E}_\sigma^*(B(t)^{-1}\mathcal{E}_t(X+J)\mathbf{1}_{\{X(t)>y\}} \mid N(t)=n)\pi_{*,n}^{(\sigma)}(t) \tag{4.5}$$

$$= \kappa_n^{*,\sigma}\mathrm{e}^{-b_r t}\int_y^\infty \mathrm{e}^{-a_r x + x} p_{*,n}^{(\sigma)}(x, t)dx$$

Functions $u_n^{(\sigma)}(y,t)$, $n \geq 1$ satisfy the equation (see (2.20))

$$\frac{\partial u_n^{(\sigma)}}{\partial t}(y, t) + c_\sigma \frac{\partial u_n^{(\sigma)}}{\partial y}(y, t) = -(\lambda_\sigma^* + r_\sigma)u_n^{(\sigma)}(y, t) + \lambda_\sigma^* u_{n-1}^{(-\sigma)}(y, t) \tag{4.6}$$

with initial conditions $u_n^{(\sigma)}|_{t=0} = 0$, $n \geq 1$. Functions $u_n^{(\sigma)}$, $n \geq 1$ are assumed to be continuous and piece-wise continuously differentiable.

It is plain, that $u_0^{(\sigma)}(y, t) = \mathrm{e}^{-(\lambda_\sigma^* + r_\sigma)t}\theta(c_\sigma t - y)$, $\sigma = \pm 1$. Moreover $u_n^{(\sigma)} \equiv 0$, if $y > c_+ t$, and for $y < c\_t$,

$$u_n^{(\sigma)}(y, t) \equiv \rho_n^{(\sigma)}(t) = \mathrm{e}^{-b_r t}\int_{-\infty}^\infty \mathrm{e}^{-a_r x} p_{n,*}^{(\sigma)}(x, t)dx. \tag{4.7}$$

In the latter case system (4.6) has the form

$$\frac{\mathrm{d}\rho_n^{(\sigma)}}{\mathrm{d}t} = (-\lambda_\sigma^* + r_\sigma)\rho_n^{(\sigma)} + \lambda_\sigma^* \rho_{n-1}^{(-\sigma)}, \quad n \geq 1, \tag{4.8}$$

$\rho_0^{(\sigma)} = \mathrm{e}^{-(\lambda_\sigma^* + r_\sigma)t}$ and $\rho_n^{(\sigma)}|_{t=0} = 0$, $n \geq 1$, $\sigma = \pm 1$.

As it is demonstrated in Appendix B the solution of (4.8) can be written in the form

$$\rho_n^{(\sigma)}(t) = \mathrm{e}^{-(\lambda_-^* + r_-)t}\Lambda_n^{(\sigma)}P_n^{(\sigma)}(t), \quad \sigma = \pm 1, \quad n \geq 0,$$

where $\Lambda_n^{(\sigma)} = \lambda_\sigma^{[(n+1)/2]}\lambda_{-\sigma}^{[n/2]}$ and functions $P_n^{(\sigma)}$ are defined as follows:

$$P_0^{(+)} = \mathrm{e}^{-at}, \quad P_0^{(-)} \equiv 1,$$

$$P_n^{(\sigma)} = P_n^{(\sigma)}(t) = \frac{t^n}{n!}\left[1 + \sum_{k=1}^\infty \frac{(m_n^{(\sigma)}+1)_k}{(n+1)_k} \cdot \frac{(-at)^k}{k!}\right], \quad \sigma = \pm 1, \quad n \geq 1. \tag{4.9}$$

Here

$$m_n^{(+)} = [n/2], \quad m_n^{(-)} = [(n-1)/2],$$

$$(m)_k = m(m+1) \ \ldots \ (m+k-1), \quad a = \lambda_+^* - \lambda_-^* + r_+ - r_-.$$

Notice that $\Lambda_{2n}^{(+)} = \Lambda_{2n}^{(-)} \equiv \Lambda_{2n}$, $P_{2n+1}^{(+)} \equiv P_{2n+1}^{(-)} \equiv P_{2n+1}$.

To write down $u_n^{(\sigma)} = u_n^{(\sigma)}(y, t)$ for $c_- t < y < c_+ t$ let us define coefficients $\beta_{k,j}$, $j < k$: $\beta_{k,0} = \beta_{k,1} = \beta_{k,k-2} = \beta_{k,k-1} = 1$,

$$\beta_{k,j} = \frac{(k-j)_{[j/2]}}{[j/2]!}. \tag{4.10}$$

Let functions $\varphi_{k,n}$ be defined as follows: $\varphi_{0,n} = P_{2n+1}$ and

$$\varphi_{k,n} = \sum_{j=0}^{k-1} a^{k-j-1} \beta_{k,j} P_{2n-j}^{-}, \quad 1 \le k \le n. \tag{4.11}$$

For $p, q > 0$ we denote $v_0^{(-)} \equiv 0$, $v_0^{(+)} = e^{-ap}$, $v_1^{(\sigma)} = P_1(p), \sigma = \pm 1$ and for $n \ge 1$

$$\begin{aligned} v_{2n+1}^{(\pm)} &= v_{2n+1}^{(\pm)}(p, q) = P_{2n+1}(p) + \sum_{k=1}^{n} \tfrac{q^k}{k!} \varphi_{k,n}(p), \\ v_{2n}^{(-)} &= v_{2n}^{(-)}(p, q) = P_{2n}^{(-)}(p) + \sum_{k=1}^{n-1} \tfrac{q^k}{k!} \varphi_{k+1,n}(p), \\ v_{2n}^{(+)} &= v_{2n}^{(+)}(p, q) = P_{2n}^{(+)}(p) + \sum_{k=1}^{n} \tfrac{q^k}{k!} \varphi_{k-1,n-1}(p), \end{aligned} \tag{4.12}$$

Theorem 4.1. *Then the solution of system (4.6) has the form*

$$u_n^{(\sigma)} = \begin{cases} 0, & y > c_+ t, \\ w_n^{(\sigma)}(p,q), & c_- t \le y \le c_+ t, \ \sigma = \pm 1, \\ \rho_n^{(\sigma)}(t), & y < c_- t, \end{cases} \tag{4.13}$$

where $w_n^{(\sigma)} = e^{-(\lambda_+^* + r_+)q - (\lambda_-^* + r_-)p} \Lambda_n^{(\sigma)} v_n^{(\sigma)}(p, q)$, $p = \frac{c_+ t - y}{c_1 - c_-}$, $q = \frac{y - c_- t}{c_+ - c_-}$. *This solution is unique.*

See the proof in Appendix B.

Remark 4.1. *If* $\lambda_-^* = \lambda_+^* = \lambda$, $r_+ = r_- = r$, *then* $P_n^{(\sigma)} = \frac{t^n}{n!}$, $\pi_n^{(\sigma)} \equiv \pi_n = \frac{(\lambda t)^n}{n!} e^{-\lambda t}$, $\rho_n^{(\sigma)} = e^{-rt} \pi_n(t)$ *and* $\varphi_{k,n} = P_{2n-k+1}^{(\sigma)}$. *Moreover*

$$v_n^{(\sigma)} = \frac{1}{n!} \sum_{k=0}^{m_n^{(\sigma)}} \binom{n}{k} q^k p^{n-k}.$$

Remark 4.2. *By definition function* $u_0^{(-)}$ *is discontinuous at* $q = 0$ *and* $u_0^{(+)}$ *has the discontinuity at* $p = 0$. *It is easy to see that functions* $u_n^{(\sigma)}$, $n \ge 1$, *defined in (4.13), are continuous. The points of possible discontinuity of derivatives are concentrated on the lines* $p = 0$ *and* $q = 0$. *For example for* $u_1^{(\sigma)}$, $\sigma = \pm 1$ *we have*

$$\frac{\partial u_1^{(\sigma)}}{\partial p}\Big|_{q \to +0} - \frac{\partial u_1^{(\sigma)}}{\partial p}\Big|_{q \to -0} = \lambda_\sigma^* e^{-(\lambda_+^* + r_+)p}$$

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and

$$\frac{\partial u_1^{(\sigma)}}{\partial p}\Big|_{p=+0} - \frac{\partial u_1^{(\sigma)}}{\partial p}\Big|_{p=-0} = \lambda_\sigma^* e^{-(\lambda_+^*+r_+)q}.$$

Moreover, using (4.13) it is possible to proof that $u_n^{(\sigma)} \in C^{n-1}$.

Similarly, functions $U_n^{(\sigma)} = U_n^{(\sigma)}(y,t),\ n \ge 1$ fit the equation

$$\frac{\partial U_n^{(\sigma)}}{\partial t} + c_\sigma \frac{\partial U_n^{(\sigma)}}{\partial y} = -(\lambda_\sigma^* + r_\sigma - c_\sigma)U_n^{(\sigma)} + \lambda_\sigma^*(1+h_\sigma)U_{n-1}^{(-\sigma)}. \tag{4.14}$$

For $\lambda_\sigma^* = \frac{r_\sigma - c_\sigma}{h_\sigma}$ (see Theorem 2.3) it follows that $\lambda_\sigma^*(1+h_\sigma) = \lambda_\sigma^* + r_\sigma - c_\sigma := \bar{\lambda}_\sigma$. Therefore equation (4.14) has the same form as (4.6) with $\bar{\lambda}_\sigma$ instead of λ_σ^*, $r_\pm = 0$ and $U_0^{(\sigma)} = e^{-\bar{\lambda}_\sigma t}\theta(c_\sigma t - y)$.

$$U_n^{(\sigma)}(y,\ t;\ \lambda_\pm^*,\ c_\pm,\ r_\pm) \equiv u_n^{(\sigma)}(y,\ t;\ \bar{\lambda}_\pm,\ c_\pm,\ 0). \tag{4.15}$$

Exploiting (4.2)-(4.3) we can consider the following particular cases in detail.

1) Merton model.[1]

Assume that $r_- = r_+ = r,\ c_- = c_+ = c,\ h_- = h_+ = -h,\ \lambda_- = \lambda_+ = \lambda$. Then equation (2.22) has the form

$$dS(t) = S(t-)(cdt - hdN(t)),$$

where $N = N(t),\ t \ge 0$ is the (homogeneous) Poisson process with parameter $\lambda > 0$. From call option pricing formula (4.2)-(4.3) we obtain

$$c = S_0 U(\ln K/S_0, T) - Ku(\ln K/S_0, T). \tag{4.16}$$

If $0 < h < 1$ and $c < r$, then $b_n^{(\sigma)} \equiv b_n = n\ln(1-h) \downarrow -\infty$ and

$$u = u(\ln K/S_0,\ T) = e^{-rT}\sum_{n=0}^{n_0} u_n^{(\sigma)}(\ln(K/S_0) - b_n,\ T)$$

$$= e^{-rT}P_\sigma(N(T) \le n_0) = e^{-rT}\Psi_{n_0}(\lambda^* T)$$

Here $\lambda^* = (c-r)/h > 0$ and $\Psi_{n_0}(z) = e^{-z}\sum_{n=0}^{n_0}\frac{z^n}{n!}$. Function U has the form

$$U(y,\ T) = \Psi_{n_0}(\lambda^*(1-h)T).$$

For $h < 0$ and $c < 0$, i. e. $b_n^{(\sigma)} = n\ln(1-h) \uparrow +\infty$, we have

$$u(y,T) = e^{-rT}(1 - \psi_{n0}(\lambda^* T)),$$

$$U(y,T) = 1 - \psi_{n0}(\lambda^*(1-h)T).$$

---

[1] This model is called the Merton model (see Melnikov et al, 2002; Merton, 1976), but Merton (1976) contains the reference to Cox, Ross (1975). See also Cox, Ross (1976).

By n_0 we denote

$$n_0 = \inf\{n : S_0 e^{n\ln(1-h)+(c-r)T} > B(T)^{-1} K\} = \left[\frac{\ln(K/S_0) - cT}{\ln(1-h)}\right].$$

2) If $(1+h_-)(1+h_+) < 1$, then $\ln(1+h_-) + \ln(1+h_+) < 0$ and $b_n^{(\sigma)} \to -\infty$. The call option price is given by the same formula (4.16) with

$$u^{(\sigma)}(y,T) = \sum_{k=0}^{n_-^{(\sigma)}} \rho_k^{(\sigma)}(T) + \sum_{k=n_-^{(\sigma)}+1}^{n_+^{(\sigma)}} u_k^{(\sigma)}(y - b_k^{(\sigma)}), T; \lambda_\pm^*, c_\pm, r_\pm),$$

and from (4.15) it follows

$$U^{(\sigma)}(y,\ T) = u^{(\sigma)}(y,\ T;\ \bar{\lambda}_\pm,\ c_\pm,\ 0), \tag{4.17}$$

$$y = \ln K / S_0.$$

Here

$$n_-^{(\sigma)} = \min\left\{n\ :\ y - b_n^{(\sigma)} > c_- T\right\},$$

$$n_+^{(\sigma)} = \min\left\{n\ :\ y - b_n^{(\sigma)} > c_+ T\right\}.$$

3) If $(1+h_-)(1+h_+) > 1$, then $\ln(1+h_-) + \ln(1+h_+) > 0$ and $b_n^{(\sigma)} \to +\infty$. Denoting

$$m_-^{(\sigma)} = \max\left\{n\ :\ y - b_n^{(\sigma)} > c_- T\right\},$$

$$m_+^{(\sigma)} = \max\left\{n\ :\ y - b_n^{(\sigma)} > c_+ T\right\},$$

we obtain the call option price formula of the form (4.16) with

$$u^{(\sigma)}(y,\ T) = \sum_{k=m_+^{(\sigma)}}^{m_-^{(\sigma)}} u_k^{(\sigma)}(y - b_k^{(\sigma)},\ T;\ \lambda_\pm^*,\ c_\pm,\ r_\pm) + \sum_{k=m_-^{(\sigma)}+1}^{\infty} \rho_k^{(\sigma)}(T).$$

For $U^{(\sigma)}(y,\ T)$ we again apply (4.17).

References

Abramowitz, A., Stegun, I. A. (1972). Handbook of mathematical functions (Dover, New York)

Albanese, C., Lawi, S. (2004). Generating functions for stochastic integrals. Preprint University of Toronto.

Beghin, L., Nieddu, L., Orsingher, E. (2001). "Probabilistic analysis of the telegrapher's process with drift by mean of relativistic transformations". *Journal of Applied Mathematics and Stochastic Analysis* 14, 11-25.

Cox, J. C., Ross, S. (1975). The pricing of options for jump processes. Rodney L. White Center Working Paper no. 2-75 (University of Pennsylvania, Philadelphia, Penn.)

Cox, J. C., Ross, S. (1976). "The valuation of options for alternative stochastic processes". *Journal of Financial Economics* 3, 145-166.

3. Nikita.p65 144 27/09/05, 08:18 p.m.

Goldstein, S. (1951). “On diffusion by discontinuous movements and on the telegraph equation”. The Quarterly Journal of Mechanics and Applied Mathematics 4, 129-156.

Kac, M. (1959). Probability and related topics in physical sciences. Interscience, London.

Kac, M. (1974). “A Stochastic model related to the telegraph equation”. *Rocky Mountain Journal of Mathematics* 4, 497-509.

Melnikov, A., Volkov, S., Nechaev, M. (2002). Mathematics of financial obligations. (Translations of Mathematical Monographs. 212. Providence, RI: American Mathematical Society. ix, 194 p)

Merton, R. C. (1976). “Option pricing when underlying stock returns are discontinuous”. *Journal of Financial Economics* 3, 125-144.

Ratanov, N. (1997). “Telegraph processes with reflecting and absorbing barriers in inhomogeneous media”. *Theoretical and Mathematical Physics* 112, 857-865.

Ratanov, N. (1999). Telegraph processes and option pricing. 2nd Nordic-Russian Symposium on Stochastic Analysis, Beitostølen, Norway.

Ratanov, N. (2004). Option pricing model based on telegraph process with jumps. Borradores de investigación 44, Universidad del Rosario, Colombia.

Ratanov, N. (2004). A jump telegraph model for option pricing. Borradores de investigación 58, Universidad del Rosario, Colombia.

Ratanov, N. (2005). Quantile hedging for telegraph markets and its applications to a pricing of equity-linked life insurance contracts. Borradores de investigación 62, Universidad del Rosario, Colombia.

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Appendix A. Convergence to Black-Scholes Model

It is known from Kac (1959) (see also Ratanov, 1997) that (homogeneous) telegraph process $X = X(t),\ t \geq 0$ converges to the standard Brownian motion $w(t),\ t \geq 0$, if $c, \lambda \to \infty,\ c^2/\lambda \to 1$. Moreover, we have the following theorem (at least for the symmetric case $\lambda_- = \lambda_+,\ c_- = a - c,\ c_+ = a + c$).

Theorem A.1. *Let* $\lambda_- = \lambda_+ = \lambda \to \infty,\ c \to \infty,$

$$c^2/\lambda \to v_c^2 \qquad a^2/\lambda \to v_a^2 \tag{A.1}$$

Let $h_-, h_+ \to 0$ and

$$a + \lambda B/2 \to \mu \tag{A.2}$$

where $B = \ln\left[(1+h_-)(1+h_+)\right]$.

Then model (2.22) converges in distribution to the Black-Scholes model:

$$S(\cdot) \xrightarrow{\text{D}} S_0 \exp(vw(\cdot) + \mu t), \tag{A.3}$$

with $v = \sqrt{v_c^2 + v_a^2}$.

Proof. Let $f(z,t) = \mathrm{e}^{z\bar{X}(t)}$ be the moment generating function of $\bar{X}(t) = X(t) + \ln \kappa(t)$. We prove here the convergence

$$f(z,t) \to \exp(\mu z t + v^2 z^2 t/2) \tag{A.4}$$

which is sufficient for the convergence of one-point distributions in (A.3). From Remark 2.1 it follows that

$$f(z,t) = \mathrm{e}^{z\bar{X}(t)} = \mathrm{e}^{z(cX^{st}(t) + at + \ln k(t))}$$

$$\sim \mathrm{e}^{azt} \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \mathrm{e}^{z(xc + nB/2)} p_n^{st}(x,\ t)dx, \tag{A.5}$$

where X^{st} is the standard telegraph process with the states $(\pm 1,\ \lambda)$, and $p_n^{st},\ n \geq 0$ are the probability densities of $X^{st}(t)$ which are defined as in (2.19).

Changing variables in the integral in (A.5) we obtain

$$f(z,\ t) \sim \mathrm{e}^{azt} \int_{-\infty}^{\infty} \mathrm{e}^{czx} \sum_{n=0}^{\infty} \mathrm{e}^{znB/2} p_n^{st}(x,\ t)dx$$

$$= \mathrm{e}^{azt + (\bar{\lambda} - \lambda)t} \int_{-\infty}^{\infty} \mathrm{e}^{zx} \bar{p}(x,\ t)dx,$$

where $\bar{p}(x,\ t)$ is the density of telegraph process $\bar{X}(t)$ with the states $(\pm c,\ \bar{\lambda})$, $\bar{\lambda} = \lambda \mathrm{e}^{zB/2} \sim \lambda$.

Then notice that

$$\bar{\lambda} - \lambda + az = \lambda(\mathrm{e}^{zB/2} - 1) + az$$

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$$\sim \frac{\lambda z B}{2} + \frac{\lambda z^2 B^2}{8} + az.$$

From (A.1)-(A.2) it follows that $\sqrt{\lambda} B/2 \sim -a/\sqrt{\lambda}$ and

$$\bar{\lambda} - \lambda + az \to \mu z + v_a^2 z^2 / 2$$

The densities $\bar{p}(\cdot,\ t)$ converge to the probability density of $v_c w(t)$:

$$\bar{p}(x,t) \to \frac{1}{v_c\sqrt{2\pi t}} \mathrm{e}^{-x^2/2tv_c^2}.$$

Summarizing we obtain (A.4). The complete proof of (A.3) is a bit tricky and it is omitted here.

Remark A.1. *Condition (A.2) in this theorem means that the total drift* $a + \lambda B/2$ *is asymptotically finite. Here* $a = (c_- + c_+)/2$ *is generated by the velocities of telegraph process X and summand* $\lambda B/2$ *represents the drift component (possibly with infinite asymptotics), which is provoked by jumps. If in (A.2) the limit of* $\lambda B/2$ *is finite, then* $a \to \alpha \equiv const$ *and in (A.3) the drift volatility term* $v_a = 0$.

In general, by (A.1)-(A.2) $\sqrt{\lambda} B/2 \to v_a$, and so $-\sqrt{\lambda} B/2$ has the meaning of the jump component of volatility.

Appendix B. Proof of Theorem 4.1

As it follows from (2.20), functions

$$\rho_n^{(\sigma)} = \rho_n^{(\sigma)}(t) = \mathrm{E}_\sigma^*(B(t)^{-1}\mathbf{1}_{\{N(t)=n\}})$$

$$= \mathrm{e}^{-b_r t}\int_{-\infty}^{\infty}\mathrm{e}^{-a_r x}p_{n,*}^{(\sigma)}(x,\ t)dx,\ \ t\ge 0,\ \ \sigma=\pm 1,\ \ n\ge 1$$

satisfy the system

$$\begin{cases}\dot\rho_n^{(+)} = \lambda_+^*\rho_{n-1}^{(-)} - (\lambda_+^* + r_+)\rho_n^{(+)}\\ \dot\rho_n^{(-)} = \lambda_-^*\rho_{n-1}^{(+)} - (\lambda_-^* + r_-)\rho_n^{(-)}\end{cases} \tag{B.1}$$

with $\rho_0^{(\sigma)}(t) = \mathrm{e}^{-(\lambda_\sigma^* + r_\sigma)t}$, $t\ge 0$, $t\ge 0$, $\sigma = \pm 1$ and $\rho_n^{(\pm)}|_{t=0} = 0$, $n\ge 1$. Here $\dot\rho_n^{(\pm)} = \frac{\mathrm{d}\rho_n^{(\pm)}}{\mathrm{d}t}$. For $\lambda_+^* = \lambda_-^* = \lambda$ and $r_\pm = 0$ the solution is well known:

$$\rho_n^{(\pm)}(t) = \pi_n(t) = \mathsf{P}(N(t)=n) = \frac{(\lambda t)^n}{n!}\mathrm{e}^{-\lambda t}.$$

Generally, we imply the following change of variables

$$\rho_n^{(\sigma)}(t) = \mathrm{e}^{-(\lambda_-^* + r_-)t}\Lambda_n^{(\sigma)}P_n^{(\sigma)}(t)$$

with $\Lambda_n^{(\sigma)} = (\lambda_\sigma)^{[(n+1)/2]}(\lambda_{-\sigma})^{[n/2]}$. In these notations we have $P_0^{(+)}(t) = \mathrm{e}^{-at}$, $a = (\lambda_+^* + r_+) - (\lambda_-^* + r_-)$; $P_0^{(-)}(t) = 1$; $P_n^{(\pm)}|_{t=0} = 0$, $n\ge 1$ and the system

$$\begin{cases}\dot P_n^{(+)} + aP_n^{(+)} = P_{n-1}^{(-)}\\ \dot P_n^{(-)} = P_{n-1}^{(+)}\end{cases},n\ge 1. \tag{B.2}$$

$$\dot P_n^{(\pm)} = \frac{\mathrm{d}P_n^{(\pm)}}{\mathrm{d}t}.$$

The latter system has the following solution (see (4.9))

$$\begin{aligned} P_{2n+1} \equiv P_{2n+1}^{(\pm)} &= \frac{t^{2n+1}}{(2n+1)!}\left[1+\sum_{k=1}^{\infty}\frac{(n+1)...(n+k)}{(2n+2)...(2n+k+1)}\cdot\frac{(-at)^k}{k!}\right],\\ P_{2n}^{(-)} &= \frac{t^{2n}}{(2n)!}\left[1+\sum_{k=1}^{\infty}\frac{n(n+1)...(n+k-1)}{(2n+1)...(2n+k)}\cdot\frac{(-at)^k}{k!}\right],\\ P_{2n}^{(+)} &= \frac{t^{2n}}{(2n)!}\left[1+\sum_{k=1}^{\infty}\frac{(n+1)...(n+k)}{(2n+1)...(2n+k)}\cdot\frac{(-at)^k}{k!}\right].\end{aligned} \tag{B.3}$$

Remark 2.1. *Formulas (B.3) can be expressed by hypergeometric function (Abramowitz, Stegun (1972)):*

$$P_n^{(\sigma)}(t) = \frac{t^n}{n!}\cdot{}_1F_1(m_n^{(\sigma)}+1;\ \ n+1;\ \ -at),\ \ m_n^{(+)} = [n/2],\ \ m_n^{(-)} = [(n-1)/2].$$

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Hypergeometric function ${}_1F_1(\alpha;\ \beta;\ z)$ is defined as follows (see e. g. Albanese and Lawi, 2004, formula (1.6))

$${}_1F_1(\alpha;\ \beta;\ z) = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha+1)\dots(\alpha+n-1)}{n!\beta(\beta+1)\dots(\beta+n-1)} z^n = 1 + \sum_{n=1}^{\infty} \frac{(\alpha)_n}{n!(\beta)_n} z^n.$$

As well, using (B.3) it easy to check that $P_{2n}^{(-)} - P_{2n}^{(+)} = aP_{2n+1},\ \ n \ge 0$.

To obtain

$$u_n^{(\sigma)}(y,t) = \mathrm{E}_\sigma^*(B(t)^{-1}\mathbf{1}_{\{X(t)>y, N(t)=n\}})$$

we apply the change of variables $p = \frac{c_+t-y}{c_+-c_-}$, $q = \frac{y-c_-t}{c_+-c_-}$ and

$$u_n^{(\sigma)} = \mathrm{e}^{-(\lambda_+^*+r_+)q-(\lambda_-^*+r_-)p}\Lambda_n^{(\sigma)} v_n^{(\sigma)}(p,\ q)$$

to equation (4.6).

Evidently, $u_n^{(\sigma)}(y,\ t) \equiv 0$, if $p < 0$, and $u_n^{(\sigma)}(y,\ t) \equiv \rho_n^{(\sigma)}(t)$, if $q < 0$. For $p,\ q > 0$ we have the system

$$\begin{cases} \dfrac{\partial v_n^{(+)}}{\partial q} = v_{n-1}^{(-)} \\ \dfrac{\partial v_n^{(-)}}{\partial p} = v_{n-1}^{(+)} \end{cases},\ n \ge 1 \tag{B.4}$$

with

$$v_0^{(+)} = \mathrm{e}^{-ap}\theta(p),\ \ v_0^{(-)} = \mathrm{e}^{aq}\theta(-q),\ \ v_n^{(\pm)}|_{p<0} \equiv 0$$

and

$$v_n^{(\sigma)}|_{q<0} = \mathrm{e}^{aq} P_n^{(\sigma)}(p+q). \tag{B.5}$$

Here $a = (\lambda_+^* + r_+) - (\lambda_-^* + r_-)$ and $P_n^{(\sigma)},\ \ n \ge 0,\ \ \sigma = \pm 1$ are defined in (B.3).

It is plain to check that the exact representation of the solution of (B.4) for $p,\ q > 0$ has the form of (4.12)

$$v_{2n+1}^{(\pm)} = v_{2n+1} = P_{2n+1}(p) + \sum_{k=1}^{n} \frac{q^k}{k!}\varphi_{k,n}(p),$$

$$v_{2n}^{(+)} = P_{2n}^{+}(p) + \sum_{k=1}^{n} \frac{q^k}{k!}\varphi_{k-1,n-1}(p),$$

$$v_{2n}^{(-)} = P_{2n}^{-}(p) + \sum_{k=1}^{n-1} \frac{q^k}{k!}\varphi_{k+1,n}(p),$$

where $\varphi_{0,\ n} = P_{2n+1},\ \ \varphi_{1,\ n} = P_{2n}^{(-)}$ and

$$\varphi_{k,n}^{!} = \varphi_{k-1,n-1,} 1 \le k \le n. \tag{B.6}$$

Proposition B.1. *The solution of system (B.6) has the form (4.11):* $\varphi_{k,\,n} = \sum_{j=0}^{k-1} a^{k-j-1}\beta_{k,\,j} P^{-}_{2n-j}$.

Proof. Indeed, from (4.11) and (B.2) it follows

$$\varphi'_{k,\,n} = \sum_{j=0}^{k-1} a^{k-j-1}\beta_{k,\,j} P^{(+)}_{2n-j-1}.$$

By the identities $P^{(+)}_{2n+1} = P^{(-)}_{2n+1}$ and $P^{(-)}_{2n} - P^{(+)}_{2n} = aP_{2n+1},\ n \geq 0$ (see Remark B.1) we have

$$\varphi'_{k,\,n} = \sum_{j\geq 0,\ j \textbf{ is even}} a^{k-j-1}\beta_{k,\,j} P_{2n-j-1} + \sum_{j\geq 0,\ j \textbf{ is odd}} a^{k-j-1}\beta_{k,\,j} P^{(-)}_{2n-j-1} - \sum_{j\geq 0,\ j \textbf{ is odd}} a^{k-j}\beta_{k,\,j} P_{2n-j}.$$

To complete the proof it is sufficient to apply the following identities $\beta_{k,\,2m+1} = \beta_{k-1,\,2m}$, $\beta_{k,\,2m} - \beta_{k,\,2m+1} = \beta_{k-1,\,2m-1}$, which are evident from the definition of $\beta_{k,\,n}$ (see (4.10)).