On Prime and Weakly Prime $\mathcal{LA}$-submodules of $\mathcal{LA}$-modules

Sobre $\mathcal{LA}$-submódulos de $\mathcal{LA}$-módulos primos y débilmente primos

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Abstract. In this paper, we introduce the concept of prime and weakly prime $\mathcal{LA}$-submodules and give some basic results about prime and weakly prime $\mathcal{LA}$-submodules of $\mathcal{LA}$-modules. Moreover, we investigated relationships between prime and weakly prime $\mathcal{LA}$-submodules in $\mathcal{LA}$-modules. Finally, we obtain sufficient conditions of a weakly prime $\mathcal{LA}$-submodule in order to be a prime $\mathcal{LA}$-submodule.

Keywords: $\mathcal{LA}$-module, prime $\mathcal{LA}$-submodule, weakly prime $\mathcal{LA}$-submodule, $\mathcal{LA}$-ring, prime ideal.

Resumen. En este artículo, introducimos el concepto de $\mathcal{LA}$-submódulos primos y débilmente primos y damos algunos resultados básicos acerca de los conceptos de $\mathcal{LA}$-submódulos de $\mathcal{LA}$-módulos primos y débilmente primos. Más aún, investigamos relaciones entre $\mathcal{LA}$-submódulos primos y débilmente primos en $\mathcal{LA}$-módulos. Finalmente, obtenemos condiciones suficientes para que un $\mathcal{LA}$-submódulo débilmente primo sea un $\mathcal{LA}$-submódulo primo.

Palabras claves: $\mathcal{LA}$-módulos, $\mathcal{LA}$-submódulo primo, $\mathcal{LA}$-submódulo débilmente primo, $\mathcal{LA}$-anillo, ideal primo.

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1. Introduction

Throughout this paper, we assume that all rings are $\mathcal{LA}$-rings. Let R be an $\mathcal{LA}$-ring and let M be an $\mathcal{LA}$-module. In 2003, Anderson and Smith [1] introduced the concepts of a weakly prime ideal in commutative rings. According to their definition, a proper ideal I of a ring R is called a weakly prime ideal if whenever $0 \neq ab \in I$ for $a, b \in R$, then $a \in I$ or $b \in I$. In [3], Dauns introduced the concepts of a prime submodule. A proper submodule N of an R-module M to be a prime submodule [3] of M if whenever $am \in N$ for $a \in R, m \in M$, then $m \in N$ or $a \in (N : M)$. Atani and Farzalipour [2] introduced the concepts of

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a weakly prime submodule. According to their definition, a proper submodule N of an R-module M to be a weakly prime submodule of M if whenever $0 \neq am \in N$ for $a \in R, m \in M$, then $m \in N$ or $a \in (N : M)$. Mostafanasab et al. [6] introduced the concepts of a weakly classical prime submodule. According to their definition, a proper submodule N of M to be a almost 2-absorbing submodule of M if whenever $0 \neq abm \in N$ for $a, b \in R, m \in M$, then $am \in N$ or $bm \in N$.

In 1996, Mushtaq and Kamran [7] introduced the notion of left almost group ($\mathcal{LA}$-group). Yusuf in [10] introduces the concept of a left almost ring ($\mathcal{LA}$-ring). That is, a non empty set R with two binary operations " $+$ " and " $\cdot$ " is called a left almost ring, if $(R, +)$ is an $\mathcal{LA}$-group, $(R, \cdot)$ is an $\mathcal{LA}$-semigroup and distributive laws of " $\cdot$ " over " $+$ " holds. Further in [9] Shah and Rehman generalize the notions of commutative semigroup rings into $\mathcal{LA}$-rings.

In 2010, Shah and Rehman [9] define the notion of an $\mathcal{LA}$-module over an $\mathcal{LA}$-ring, a non abelian non associative structure but closer to abelian group. Now we shall use the paper [8] which deals with the notion of $\mathcal{LA}$-modules. In this study we followed lines as adopted in [9, 8] and established the notion of prime and weakly prime $\mathcal{LA}$-submodules of an $\mathcal{LA}$-module. Specifically we characterize the prime and weakly prime $\mathcal{LA}$-submodules in $\mathcal{LA}$-modules. Moreover, we investigated relationships between prime and weakly prime $\mathcal{LA}$-submodules in $\mathcal{LA}$-modules.

2. Preliminaries

In this section, we refer to [4, 5, 7, 9, 8, 10] for some elementary aspects and quote few definitions, and essential examples to step up this study. For more details, we refer to the papers in the references.

Recall that a groupoid $(S, \cdot)$ is called a **left almost-semigroup ($\mathcal{LA}$-semigroup)** if it satisfies the left invertive law; $(xy)z = (zy)x$ for all $x, y, z \in S$ (see [4]). An $\mathcal{LA}$-semigroup $(G, \cdot)$ is called a **left almost group ($\mathcal{LA}$-group)** if there exists left identity $e \in G$ (that is $ex = x$ for all $x \in G$), for all $x \in G$ there exists $x^{-1} \in G$ such that $xx^{-1} = e = x^{-1}x$ (see [7]).

Definition 2.1. [10] A **left almost ring ($\mathcal{LA}$-ring)** is a non empty set R together with two binary operations, addition (denoted by $+$) and multiplication (denoted by $\cdot$), such that for all x, y, z in R:

1. $(R, +)$ is an $\mathcal{LA}$-group.
2. $(R, \cdot)$ is an $\mathcal{LA}$-semigroup.
3. $x(y + z) = xy + xz$ and $(x + y)z = xz + yz$.

Definition 2.2. [9] Let R be an $\mathcal{LA}$-ring. A **left almost module ($\mathcal{LA}$-module)** is an $\mathcal{LA}$-group M together with a function $R \times M \mapsto M$ (denote the image of (r, m) by rm) such that for all $r, s \in R$ and $n, m \in M$:

1. $r(n+m) = rn + rm$,
2. $(r+s)m = rm + sm$,
3. $s(rm) = r(sm)$,
4. If R has a multiplicative left identity element e, then $em = m$.

Example 2.3. [8] Every locally associative $\mathcal{LA}$-group is an $\mathcal{LA}$-module over the $\mathcal{LA}$-ring of integers.

Lemma 2.4. *[8] Let M be an $\mathcal{LA}$-module over an $\mathcal{LA}$-ring R. Then the following properties hold.*

1. $r0 = 0$,
2. $0m = 0$,
3. $(-r)m = -(rm) = r(-m)$,
4. $(-r)(-m) = rm$ *for all* $r \in R$ *and* $m \in M$.

Proof. See [8]. □

Definition 2.5. [8] An $\mathcal{LA}$-subgroup N of an $\mathcal{LA}$-module M over an $\mathcal{LA}$-ring R is called an **$\mathcal{LA}$-submodule** over R, if $RN \subseteq N$, i.e., $rn \in N$ for all $r \in R$ and $n \in N$.

Definition 2.6. [8] Let M be an $\mathcal{LA}$-module over an $\mathcal{LA}$-ring R and let A be an $\mathcal{LA}$-submodule of M. We define the **quotient module or factor module** M/A by $M/A = \{m + A : m \in M\}$.

Lemma 2.7. *[5, 8] Let A and B be two $\mathcal{LA}$-submodules of an $\mathcal{LA}$-module M over an $\mathcal{LA}$-ring R. Then $(A+B)/A \cong B/(A \cap B)$.*

Proof. See [5, 8]. □

Recall that an ideal P of an $\mathcal{LA}$-ring R is a **completely prime ideal** if for each elements a, b of R, $ab \in P$ implies that either $a \in P$ or $b \in P$.

Example 2.8. [5] Let $R = \{0, 1, 2\}$ be a set under the binary operations defined as follows,

$+$	0	1	2
0	0	1	2
1	2	0	1
2	1	2	0

$\cdot$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Then R is an $\mathcal{LA}$-ring (See [5]). It is easy to see that $\{0\}$ is a completely prime ideal of R.

Recall that an ideal P of an $\mathcal{LA}$-ring R is a **weakly completely prime ideal** if for each elements a, b of R, $0 \neq ab \in P$ implies that either $a \in P$ or $b \in P$. Clearly, every completely prime ideal of an $\mathcal{LA}$-ring R is weakly completely prime.

Example 2.9. Let $R = \{0, 1, 2, 3, 4, 5, 6, 7\}$ be an $\mathcal{LA}$-ring (See [5]) under the binary operations defined as follows,

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	0	3	2	6	7	4	5
2	2	3	0	1	5	4	7	6
3	3	2	1	0	7	6	5	4
4	4	6	5	7	0	2	1	3
5	5	7	4	6	2	0	3	1
6	6	4	7	5	1	3	0	2
7	7	5	6	4	3	1	2	0

+	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	2	0	2	0	0	2	2
2	0	4	0	4	0	0	4	4
3	0	5	0	5	0	0	5	5
4	0	0	0	0	0	0	0	0
5	0	4	0	4	0	0	4	4
6	0	2	0	2	0	0	2	2
7	0	5	0	5	0	0	5	5

It is easy to see that $\{0\}$ is a weakly completely prime ideal of R. But $\{0\}$ is not a completely prime ideal of R, since $6 \cdot 4 = 0 \in \{0\}$, while $6 \notin \{0\}$ and $4 \notin \{0\}$.

3. Prime and Weakly Prime $\mathcal{LA}$-Submodules

In this section, we define and study the prime and weakly prime $\mathcal{LA}$-submodules in an $\mathcal{LA}$-module. Moreover, we investigated relationships between prime and weakly prime $\mathcal{LA}$-submodules in $\mathcal{LA}$-modules.

Let M be an $\mathcal{LA}$-module over an $\mathcal{LA}$-ring R and $\emptyset \neq A, B \subseteq M$. The colon ideal of R is considered to be $(A : B)$ such that $a \in (A : B) \Leftrightarrow aB \subseteq A$ and $ab \in (A : B) \Leftrightarrow a(bB) \subseteq A$, where $a, b \in R$. If $A = \{m\}$, then we write $(\{m\} : B)$ as $(m : A)$ and similarly if $B = \{m\}$, we write $(A : m)$.

Remark 3.1. Let A be an $\mathcal{LA}$-submodule of an $\mathcal{LA}$-module M over an $\mathcal{LA}$-ring R and $\emptyset \neq B \subseteq M$. It is easy to see that $0 \in (A : B) \neq \emptyset$, since $0B = \{0\} \subseteq A$.

Lemma 3.2. *Let M be an $\mathcal{LA}$-module over an $\mathcal{LA}$-ring R and $m \in M$. If A is an $\mathcal{LA}$-submodule of M, then $(A : m)$ is an ideal of R.*

Proof. Let a, b and r be any elements of R such that $a, b \in (A : m)$. Then we have $am, bm \in A$. Since

$$(a - b)m \quad = \quad am - bm \in A + A \subseteq A$$

and $a(rm) = r(am) \in rA \subseteq A$, we have $a - b, ar, ra \in (A : m)$. Therefore, $(A : m)$ is an ideal of R. □

By Lemma 3.2, we immediately obtain the following corollary:

Corollary 3.3. *Let A and B be two $\mathcal{LA}$-submodules of an $\mathcal{LA}$-module M over an $\mathcal{LA}$-ring R. Then $(A : B)$ is an ideal of R.*

Remark 3.4. Let M be an $\mathcal{LA}$-module over an $\mathcal{LA}$-ring R. Then the following properties hold.

1. For every $\mathcal{LA}$-submodule A of M, $(A : m) = R$ for all $m \in A$.
2. If A and B are any $\mathcal{LA}$-submodules of M such that $B \subseteq A$, then $(A : B) = R$.
3. Let $C, D \subseteq M$ such that $D \subseteq C$. If A is an $\mathcal{LA}$-submodule of M, then $(A : C) \subseteq (A : D)$.

In the following we shall introduce the notion of prime $\mathcal{LA}$-submodules of an $\mathcal{LA}$-module M over an $\mathcal{LA}$-ring R.

Definition 3.5. A proper $\mathcal{LA}$-submodule N of an $\mathcal{LA}$-module M over an $\mathcal{LA}$-ring R is called **prime** if for $r \in R$ and $m \in M, rm \in N$ implies that $m \in N$ or $r \in (N : M)$.

Example 3.6. Let $M = \{0, 3, 8\}$ the binary operation "$+$" be defined on M as follows:

$+$	0	3	8
0	0	3	8
3	8	0	3
8	3	8	0

Then M is an $\mathcal{LA}$-group. Let $R = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ be an $\mathcal{LA}$-ring under the binary operations defined as follows,

$\oplus$	0	1	2	3	4	5	6	7	8
0	3	4	6	8	7	2	5	1	0
1	2	3	7	6	8	4	1	0	5
2	1	5	3	4	2	0	8	6	7
3	0	1	2	3	4	5	6	7	8
4	5	0	4	2	3	1	7	8	6
5	0	4	0	4	0	0	4	4	8
6	7	6	0	1	5	8	3	2	4
7	6	8	1	5	0	7	4	3	2
8	8	7	5	0	1	6	2	4	3

$\cdot$	0	1	2	3	4	5	6	7	8
0	3	1	6	3	1	6	6	1	3
1	0	3	0	3	8	8	3	0	8
2	8	1	5	3	7	2	6	4	0
3	3	3	3	3	3	3	3	3	3
4	0	6	7	3	5	4	1	2	8
5	8	6	4	3	2	7	1	5	0
6	8	3	8	3	0	0	3	8	0
7	0	1	2	3	4	5	6	7	8
8	3	6	1	3	6	1	1	6	3

Define a map $R \times M \mapsto M$ by $(r, m) \mapsto r \cdot m$. Then M is an $\mathcal{LA}$-module over an $\mathcal{LA}$-ring R. It is easy to see that $\{3\}$ is an $\mathcal{LA}$-submodule of an $\mathcal{LA}$-module M over an $\mathcal{LA}$-ring R. But $\{0\}$ is not a prime $\mathcal{LA}$-submodule of M over an $\mathcal{LA}$-ring R, since $1 \cdot 1 = 3 \in \{3\}$, while $1 \notin \{3\}$ and $1M = \{0, 3, 8\} \nsubseteq \{3\}$.

Let M_1 and M_2 be two $\mathcal{LA}$-modules. Then

$$M_1 \times M_2 := \{(x, y) \in M_1 \times M_2 : x \in M_1, y \in M_2\}.$$

For any $(a, b), (c, d) \in M_1 \times M_2$ and $r \in R$ we define $(a, b) + (c, d) := (a + c, b + d)$ and $r(a, b) := (ra, rb)$, then $M_1 \times M_2$ is an $\mathcal{LA}$-module as well.

Example 3.7. Let $R = \{0, 1, 2\}$ be an $\mathcal{LA}$-ring under the binary operations defined as follows,

$+$	0	1	2
0	0	1	2
1	2	0	1
2	1	2	0

$\cdot$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

It is easy to see that R is an $\mathcal{LA}$-module over an $\mathcal{LA}$-ring R. Next, let $A = \{0\} \times \{0\}$. Then, A is a prime $\mathcal{LA}$-submodule of an $\mathcal{LA}$-module $R \times R$ over an $\mathcal{LA}$-ring R.

Theorem 3.8. *Let M be an $\mathcal{LA}$-module over an $\mathcal{LA}$-ring R and $m \in M$. If A is a prime $\mathcal{LA}$-submodule of M, then $(A : m)$ is a completely prime ideal of R.*

Proof. Then by Lemma 3.2, we have $(A : m)$ is a proper ideal of R. Let a and b be any elements of R such that $ab \in (A : m)$. Thus $a(bm) \in A$. Since A is a prime $\mathcal{LA}$-submodule of M, we have $bm \in A$ or $a \in (A : M)$, that is $b \in (A : m)$ or $a \in (A : m)$.Therefore, $(A : m)$ is a completely prime ideal of R. □

By Theorem 3.8, we immediately obtain the following corollary:

Corollary 3.9. *Let A be a prime $\mathcal{LA}$-submodule of an $\mathcal{LA}$-module M over an $\mathcal{LA}$-ring R. Then $(A : M)$ is a completely prime ideal of R.*

In the following we shall introduce the notion of weakly prime $\mathcal{LA}$-submodules of an $\mathcal{LA}$-module M over an $\mathcal{LA}$-ring R.

Definition 3.10. A proper $\mathcal{LA}$-submodule N of an $\mathcal{LA}$-module M over an $\mathcal{LA}$-ring R is called **weakly prime** if for $r \in R$ and $m \in M, 0 \neq rm \in N$ implies that $m \in N$ or $r \in (N : M)$.

As is easily seen, every weakly prime $\mathcal{LA}$-submodule of an $\mathcal{LA}$-module M over an $\mathcal{LA}$-ring R is a weakly prime $\mathcal{LA}$-submodule.The following example shows that the converse of this property does not hold in general.

Example 3.11. Let $R = \{0, 1, 2, 3, 4, 5, 6, 7\}$ be an $\mathcal{LA}$-ring under the binary operations defined as follows,

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	2	0	3	1	6	4	7	5
2	1	3	0	2	5	7	4	6
3	3	2	1	0	7	6	5	4
4	4	5	6	7	0	1	2	3
5	6	4	7	5	2	0	3	1
6	5	7	4	6	1	3	0	2
7	7	6	5	4	3	2	1	0

$\cdot$	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	4	4	0	0	4	4	0
2	0	4	4	0	0	4	4	0
3	0	0	0	0	0	0	0	0
4	0	3	3	0	0	3	3	0
5	0	7	7	0	0	7	7	0
6	0	7	7	0	0	7	7	0
7	0	3	3	0	0	3	3	0

It is easy to see that $R \times R$ is an $\mathcal{LA}$-module over an $\mathcal{LA}$-ring R. Then, $\{0\} \times \{0\}$ is a weakly prime $\mathcal{LA}$-submodule of $R \times R$. But $\{0\} \times \{0\}$ is not a prime $\mathcal{LA}$-submodule of $R \times R$, since $4(0,3) = (0,0) \in \{0\} \times \{0\}$, while $(0,3) \notin \{0\} \times \{0\}$ and $4(R \times R) = \{0,3\} \times \{0,3\} \not\subseteq \{0\} \times \{0\}$.

Theorem 3.12. *Let M be an $\mathcal{LA}$-module over an $\mathcal{LA}$-ring R. Then the following conditions are equivalent.*

1. *A is a weakly prime $\mathcal{LA}$-submodule of M.*
2. *$(A : m) = (A : M) \cup (0 : m)$ for any $m \in M - A$.*
3. *$(A : m) = (A : M)$ or $(A : m) = (0 : m)$ for any $m \in M - A$.*

Proof. First assume that A is a weakly prime $\mathcal{LA}$-submodule of M. Let m be any element of $M - A$ such that $r \in (A : m)$. Then $rm \in A$. If $rm = 0$, then $r \in (0 : m) \subseteq (A : M) \cup (0 : m)$. Net, let $rm \neq 0$. Since A is a weakly prime $\mathcal{LA}$-submodule of M, we have $m \in A$ or $r \in (A : M)$. Thus it is clear that $r \in (A : M) \subseteq (A : M) \cup (0 : m)$. Next, we prove that $(A : M) \cup (0 : m) \subseteq (A : m)$. Clearly, $(A : M) \subseteq (A : m)$ and $(0 : m) \subseteq (A : m)$. Hence $(A : M) \cup (0 : m) \subseteq (A : m)$. Therefore $(A : m) = (A : M) \cup (0 : m)$ and so (1) implies (2).

Assume that (2) holds. It is well-known that the union of two ideals I, J of an $\mathcal{LA}$-ring R is an ideal if $I \subseteq J$ or $J \subseteq I$. By condition, the ideals $(A : M)$ is the union of the ideals $(A : M)$ and $(0 : m)$, so either $(A : m) \subseteq (A : M)$ or $(0 : m) \subseteq (A : m)$. Thus either $(A : m) = (A : M)$ or $(A : m) = (0 : m)$ and so (2) implies (3).

Finally, assume that (3) holds. Let r be any element of R and let m be any element of M such that $0 \neq rm \in A$. Then $r \in (A : m)$. By condition 3, we have $r \in A : M)$. Therefore, A is a weakly prime $\mathcal{LA}$-submodule of M and so (3) implies (1). □

Theorem 3.13. *Let A be a weakly prime $\mathcal{LA}$-submodule of an $\mathcal{LA}$-module M over an $\mathcal{LA}$-ring R with left identity. For every $m \in M$ if I is an ideal of R such that $\{0\} \neq Im \subseteq A$, then either $m \in A$ or $I \subseteq (A : M)$.*

Proof. Let m be any element of M and let I be an ideal of R such that $\{0\} \neq Im \subseteq A$. Clearly, $I \subseteq (A : m)$ and $I \not\subseteq (0 : m)$. If $m \in A$, then there is nothing to prove. Next, let $m \notin A$. Then by Theorem 3.12, we have $I \subseteq (A : m) = (A : M)$ i.e., $I \subseteq (A : M)$. □

By Theorem 3.8 and Corollary 3.9, we immediately obtain the following corollary:

Corollary 3.14. *Let A be a weakly prime $\mathcal{LA}$-submodule of an $\mathcal{LA}$-module M over an $\mathcal{LA}$-ring R. Then the following properties hold.*

1. *A is a weakly prime $\mathcal{LA}$-submodule of M.*
2. *$(A : m) = (A : M) \cup (0 : m)$ for any $m \in M - A$.*

Theorem 3.15. *Let A be a weakly prime $\mathcal{LA}$-submodule of an $\mathcal{LA}$-module M over an $\mathcal{LA}$-ring R. If A is not prime, then $(A : M)A = \{0\}$.*

Proof. Suppose that $(A : M)A \neq \{0\}$. We will show that A is a prime $\mathcal{LA}$-submodule of an $\mathcal{LA}$-module M over an $\mathcal{LA}$-ring R. Let m be any element of M and let r be any element of R such that $rm \in A$. If $rm \neq 0$, then either $m \in A$ or $r \in (A : M)$, since A is weakly prime $\mathcal{LA}$-submodule. Now, assume that $rm = 0$. Next, let $rA \neq \{0\}$. Then there exists element n of A such that $0 \neq rn \in A$. Thus $0 \neq rn + 0 = rn + rm = r(n + m) \in A$, which implies that either $n + m \in A$ or $r \in (A : M)$. Therefore either $n \in A$ or $r \in (A : M)$. Now we can assume that $rA = \{0\}$ and $(A : M)m = \{0\}$. Since $(A : M)A \neq \{0\}$, then there exists $s \in (A : M)$ and $n \in A$ such that $0 \neq sn \in A$. Clearly,

$$\begin{aligned}
(s + r)(n + m) &= (s + r)n + (s + r)m \\
&= (sn + rn) + (sm + rm) \\
&= (sn + 0) + (0 + 0) \\
&= (0 + 0) + (0 + sn) \\
&= sn.
\end{aligned}$$

It is clear that $0 \neq (s + r)(n + m) \in A$. Thus, since A is a weakly prime $\mathcal{LA}$-submodule of M, we have $n + m \in A$ or $s + r \in (A : M)$. Therefore $m \in A$ or $r \in (A : M)$ and hence A is a prime $\mathcal{LA}$-submodule of M. □

By Theorem 3.15, we immediately obtain the following theorem:

Theorem 3.16. *Let A be an $\mathcal{LA}$-submodule of an $\mathcal{LA}$-module M over an $\mathcal{LA}$-ring R such that $(A : M)A = \{0\}$. Then the following conditions are equivalent.*

1. *A is a prime $\mathcal{LA}$-submodule of M.*
2. *A is a weakly prime $\mathcal{LA}$-submodule of M.*

Lemma 3.17. *Let A_i and B_i be two $\mathcal{LA}$-submodules of an $\mathcal{LA}$-module M_i over an $\mathcal{LA}$-ring R_i. Then the following properties hold.*

1. *For any $m_i \in M_i$, $\left(\prod_{i=1}^{n} A_i : (m_1, m_2, \ldots, m_n)\right) = \prod_{i=1}^{n} (A_i : m_i)$.*

2. $\left(\prod_{i=1}^{n} A_i : \prod_{i=1}^{n} B_i\right) = \prod_{i=1}^{n} (A_i : B_i)$.

Proof. Straightforward. □

Theorem 3.18. *Let M_1 and M_2 be two $\mathcal{LA}$-modules over $\mathcal{LA}$-rings R_1 and R_2, respectively. If $A \times M_2$ is a weakly prime $\mathcal{LA}$-submodule of an $\mathcal{LA}$-module $M_1 \times M_2$ over an $\mathcal{LA}$-rings $R_1 \times R_2$, then A is a weakly prime $\mathcal{LA}$-submodule of M_1.*

Proof. Let m be any element of M_1 and let r be any element of R such that $0 \neq rm \in A$. Clearly, $(0,0) \neq (a,a)(m,0) = (am,0) \in A \times M_2$. Since $A \times M_2$ is a weakly prime $\mathcal{LA}$-submodule of $M_1 \times M_2$, we have $(m,0) \in A \times M_2$ or $(a,a) \in (A \times M_2 : M_1 \times M_2)$. By Lemma 3.17, it follows that $m \in A$ or $a \in (A : M_1)$. Hence, A is a weakly prime $\mathcal{LA}$-submodule of M_1. □

By Theorem 3.18, we immediately obtain the following corollary:

Corollary 3.19. *Let M_1 and M_2 be two $\mathcal{LA}$-modules over $\mathcal{LA}$-rings R_1 and R_2, respectively. If $M_1 \times A$ is a weakly prime $\mathcal{LA}$-submodule of an $\mathcal{LA}$-module $M_1 \times M_2$ over an $\mathcal{LA}$-rings $R_1 \times R_2$, then A is a weakly prime $\mathcal{LA}$-submodule of M_2.*

Theorem 3.20. *Let M_1 and M_2 be two $\mathcal{LA}$-modules over $\mathcal{LA}$-rings R_1 and R_2, respectively. Then the following conditions are equivalent.*

1. *A is a prime $\mathcal{LA}$-submodule of M_1.*
2. *$A \times M_2$ is a prime $\mathcal{LA}$-submodule of an $\mathcal{LA}$-module $M_1 \times M_2$ over an $\mathcal{LA}$-rings $R_1 \times R_2$.*

Proof. First assume that A is a prime $\mathcal{LA}$-submodule of M_1. Let (m_1, m_2) be any element of $M_1 \times M_2$ and let (a_1, a_2) be any element of $R_1 \times R_2$ such that $(a_1, a_2)(m_1, m_2) = (a_1 m_1, a_2 m_2) \in A \times M_2$. Clearly, $a_1 m_2 \in A$. Then $m_1 \in A$ or $a_1 \in (A : M)$, since A is a prime $\mathcal{LA}$-submodule of M_1. By Lemma 3.17, it follows that $(m_1, m_2) \in A \times M_2$ or $(a_1, a_2) \in (A : M_1) \times R_2 = (A : M_1) \times (M_2 : M_2) = (A \times M_2 : M_1 \times M_2)$. Therefore, $A \times M_2$ is a prime $\mathcal{LA}$-submodule of $M_1 \times M_2$ and so (2) implies (1).

It is clear that $2 \Rightarrow 1$. □

By Theorem 3.20, we immediately obtain the following corollary:

Corollary 3.21. *Let M_1 and M_2 be two $\mathcal{LA}$-modules over $\mathcal{LA}$-rings R_1 and R_2, respectively. Then the following conditions are equivalent.*

1. *A is a prime $\mathcal{LA}$-submodule of M_2.*
2. *$M_1 \times A$ is a prime $\mathcal{LA}$-submodule of an $\mathcal{LA}$-module $M_1 \times M_2$ over an $\mathcal{LA}$-rings $R_1 \times R_2$.*

By Theorem 3.20 and Corollary 3.21, we immediately obtain the following theorem:

Theorem 3.22. *Let M_i be an $\mathcal{LA}$-module over $\mathcal{LA}$-rings R_i. Then the following conditions are equivalent.*

1. *A_j is a prime $\mathcal{LA}$-submodule of M_j.*
2. *$M_1 \times M_2 \times \ldots \times M_{j-1} \times A_j \times M_{j+1} \times \ldots \times M_n$ is a prime $\mathcal{LA}$-submodule of an $\mathcal{LA}$-module $\prod_{i=1}^{n} M_i$ over an $\mathcal{LA}$-ring $\prod_{i=1}^{n} R_i$.*

Theorem 3.23. *Let A and B be two proper $\mathcal{LA}$-submodules of an $\mathcal{LA}$-module M over an $\mathcal{LA}$-ring R such that $B \subseteq A$. Then the following properties hold.*

1. *If A is a weakly prime (prime) $\mathcal{LA}$-submodule of M, then A/B is a weakly prime (prime) $\mathcal{LA}$-submodule of M/B.*
2. *Let B be a weakly prime $\mathcal{LA}$-submodule of M. If A/B is a weakly prime (prime) $\mathcal{LA}$-submodule of M/B, then A is a weakly prime (prime) $\mathcal{LA}$-submodule of M.*

Proof. 1. Let m be any element of M and let r be any element of R such that $0 \neq r(m+B) \in A/B$. Then we have $rm \in A$. If $rm = 0 \in A$, then $r(m+B) = rm + B = 0 + B = B$, a contradiction. Since A is a weakly prime $\mathcal{LA}$-submodule of M, we have $m \in A$ or $r \in (A : M)$. Therefore $m + A \in A/B$ or $r \in (A/B : M/B)$ and hence A/B is a weakly prime $\mathcal{LA}$-submodule of M/B.

2. Let m be any element of M and let r be any element of R such that $0 \neq rm \in A$. Then we have $r(m+B) = rm + B \in A/B$. For all $rm \in B$ since B is a weakly prime $\mathcal{LA}$-submodule of M, we have $m \in B \subseteq A$ or $r \in (B : M) \subseteq (A : M)$. So we may assume that $rm \notin B$. This implies that $m + B \in A/B$ or $r \in (A/B : M/B)$. Therefore $m \in A$ or $r \in (A : M)$ and hence A is a weakly prime $\mathcal{LA}$-submodule of M. □

Theorem 3.24. *Let A and B be two weakly prime $\mathcal{LA}$-submodules of an $\mathcal{LA}$-module M over an $\mathcal{LA}$-ring R that are not prime $\mathcal{LA}$-submodule. Then $A + B$ is a weakly prime $\mathcal{LA}$-submodule of M.*

Proof. Since $(A+B)/B \cong B/(A+B)$, we have $(A+B)/B$ is weakly prime $\mathcal{LA}$-submodule by Theorem 3.23 (1). Now the assertion follows from Theorem 3.23(2). □

4. Conclusion

In this paper, we have studied some characteristics of prime and weakly prime $\mathcal{LA}$-submodules, and give some basic results about prime and weakly prime submodules of $\mathcal{LA}$-modules. First, we demonstrated that the notions of prime and weakly prime $\mathcal{LA}$-submodules in an $\mathcal{LA}$-module. Then, we proved that an $\mathcal{LA}$-submodule A_j is a prime $\mathcal{LA}$-submodule of an $\mathcal{LA}$-module M_j over an $\mathcal{LA}$-ring R_j if and only if $M_1 \times M_2 \times \ldots \times M_{j-1} \times A_j \times M_{j+1} \times \ldots \times M_n$ is a prime $\mathcal{LA}$-submodule of an $\mathcal{LA}$-module $\prod_{i=1}^{n} M_i$ over an $\mathcal{LA}$-rings $\prod_{i=1}^{n} R_i$. At last, we discussed the relations between prime and weakly prime $\mathcal{LA}$-modules in an $\mathcal{LA}$-module.

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