Structure semicoelevator for semitopological categories

Semicoelevadores de estructura para categorías semitopológicas

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Abstract. To search the formation of new semitopological categories the semicoelevators of structure are introduced, whose fixed points form new semitopological categories. Additionally, if we add the notion of full functor, we can prove the equivalence between reflective and semitopological subcategories, on the other hand, the properties of the semitopological functors are studied.

Keywords: structure semicoelevator, semitopological category.

Resumen. En la busqueda de formación de nuevas categorías semitopológicas se introducen los funtores semicolevadores de estructura, cuyos puntos fijos forman nuevas categorías semitopológicas, este hecho permite junto a la noción de plenitud probar la equivalencia entre subcategorias reflexivas y semitopológicas, además, se estudian las propiedades de los funtores semitopológicos.

Palabras claves: semicoelevador de estructura, categoría semitopológica.

Mathematics Subject Classification: 55U40, 18B30.

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1. Introduction

The topological functors [1] arise from the study of the forgetful functor of the category of topological spaces to category of sets, these are defined with the lifting of sources and sinks. On the other hand, not every subcategory of a topological category is topological, for example the category of Haussdorf spaces (see Example 2.2). Similarly, categories of algebraic type like the category of groups and vector spaces aren't topological respect the category of sets.

A candidate to solve that inconvenient is the topologically algebraic functors, however the composition of topological functors is topological, but in case of topologically algebraic functors that is not true [2]. For that reason we don't

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work with these functors, the best candidate is the semitopological functors, these functors are the central axis of this work.

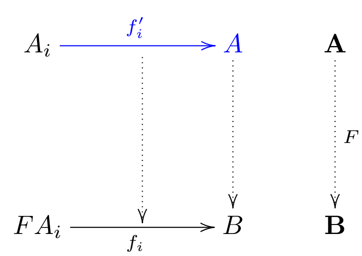
On the other hand, Montañez in [8] presents a method to build topological subcategories correspond to fixed points of endofunctors (these are called coelevators). Moreover, these are reflective of the initial category, though this method doesn't capture important categories, example the Haussdorf spaces. So we weaken this concept to semitopological categories, this creates again a source of generation of semitopological categories.

2. Background

To study the semitopological functors we need some terminology, we assume that the reader knows the basic notions of the category theory, references around this are [1] and [4]. The main reference for categorical notions of this paper is [1].

The first notion to define a topological category is a **sink**. Let $\mathbf{A}$ be a category, a pair $\{(f_i, A) : i \in I\}$ will be called a sink, where A is an object of $\mathbf{A}$ and $f_i : A_i \to A$ is a morphism with codomain A. If $F : \mathbf{A} \to \mathbf{B}$ be a functor, a pair (f_i, B) is a **structural sink** if B is an object of $\mathbf{B}$ and $f_i : FA_i \to B$ is a morphism in $\mathbf{B}$ where A_i is an object of $\mathbf{A}$. The dual notions of sink and structural sink are **source** and **structural source**.

Let (f_i, B) be a structural sink, if exist a sink (f_i', A) in $\mathbf{A}$ such that $Ff_i' = f_i$ and $FA = B$, we say that (f_i, B) has a **lifting**. Lifting property for a structural sink is illustrated in the following diagram:

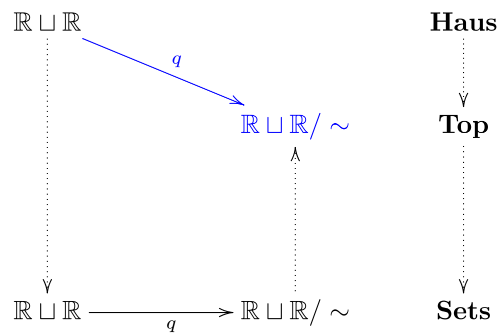


Definition 2.1. Let $F : \mathbf{A} \to \mathbf{B}$ be a functor. F is a topological functor if each structural sink has a unique lifting.

The main example is the forgetful functor from the category of topological spaces to the category of sets. The lifting property in this case has the name of the final topology for a sink. On the contrary, the category of Hausdorff spaces isn't topological with respect the forgetful functor to sets.

Example 2.2. Let $\mathcal{O} : \mathbf{Haus} \to \mathbf{Set}$ be the forgetful functor on the category of Hausdorff spaces to the category of sets. Let $\mathbb{R} \sqcup \mathbb{R}$ be the space formed with two copies of the real line $\mathbb{R}$ with the metric topology. If we take the quotient function $q : \mathbb{R} \sqcup \mathbb{R} \to \mathbb{R} \sqcup \mathbb{R}/\sim$ where $(a, 0) \sim (b, 1)$ iff $a \neq 0$ and $b \neq 0$, this equivalence relation can be thought as identified all the points except when is zero.

The final topology is the quotient topology over $\mathbb{R} \sqcup \mathbb{R}/\sim$, this space is called the line with two origins and isn't a Hausdorff space. This happens because if we take the two origins and build two open neighborhoods of their, the intersection of these is always non-empty.



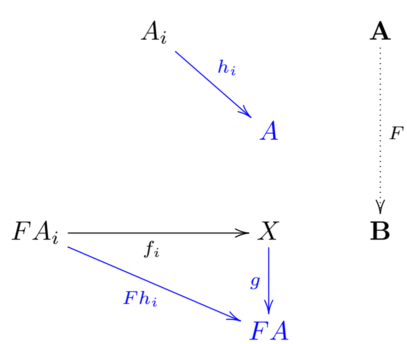
3. Semitopological categories

Lifting a structural sink can be some restrictive. To weak this notion, Tholen [9] works with factorizations of a structural sink.

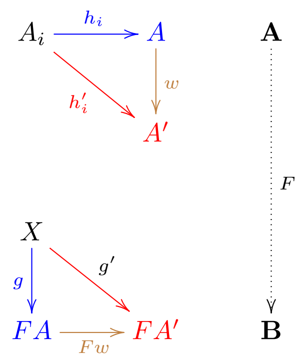
A **factorization** for a sink (f_i', A) is a pair (g, h_i) such that $h_i \circ g = f_i'$ for each morphism f_i in the sink. In case of working with structural sink (f_i, B) the notion of factorization is a pair (g, h_i) where $h_i : A_i \to A$ and $g : B \to FA$ such that $Fh_i = g \circ f_i$. To generalize the notion of lifting we use some special factorization of a structural sink.

Definition 3.1. [9] Let $F : \mathbf{A} \to \mathbf{B}$ be a functor and (f_i, X) a structural sink, with $f_i : FA_i \to X$; the pair (A, g), where $g : X \to FA$, is a **semifinal solution** if satisfies:

1. For each f_i, exist $h_i : A_i \to A$ such that $g \circ f_i = Fh_i$.



2. If exist (A', g') and $h'_i : A_i \to A'$ such that $g' \circ f_i = Fh'_i$, then exist a unique $w : A \to A'$ such that $Fw \circ g = g'$ and $w \circ h_i = h'_i$.



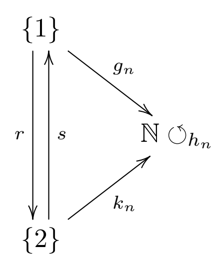
Definition 3.2. A functor $F : \mathbf{A} \to \mathbf{B}$ is semitopological if each structural sink has a unique semifinal solution.

If $F : \mathbf{A} \to \mathbf{B}$ is a semitopological functor, then $\mathbf{A}$ will be called a semitopological category.

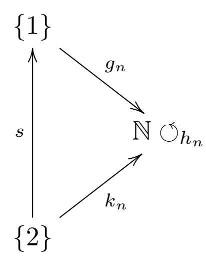
Theorem 3.3. *[1] Every topological functor is semitopological.*

The forgetful functor $O : \mathbf{Grp} \to \mathbf{Set}$ is semitopological but not topological. The following example of a semitopological functor which is not topological was proposed as an exercise in [1] and here we solved for understanding better the behavior of the semifinal solution.

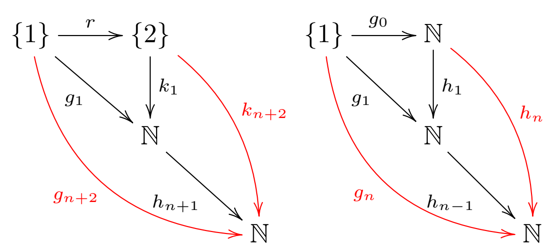
Example 3.4. Let $\mathbf{B}$ be the subcategory of $\mathbf{Set}$ generated by the following objects $\{1\}, \{2\}$ and $\mathbb{N}$, and morphisms: $r(1) = 2$, $s(2) = 1$, $g_n(1) = k_n(2) = n$ and $h_n(m) = n + m$.



and let $\mathbf{A}$ be a subcategory of $\mathbf{B}$, generated by the sets: $\{1\}, \{2\}$ and $\mathbb{N}$, and morphisms: $s(2) = 1$, $g_n(1) = k_n(2) = n$ and $h_n(m) = n+m$ with $g_0 \notin mor(\mathbf{A})$, the diagram of $\mathbf{A}$ is,



The inclusion functor $\imath : \mathbf{A} \to \mathbf{B}$ isn't topological because the structural sink $(g_0, \mathbb{N})$ doesn't have a lifting; but $\imath$ is semitopological, since **A** and **B** differ only in g_0 and r, we just need that these morphisms have semifinal solutions.



So the semifinal solution of r is $(k_1, \mathbb{N})$ and of g_0 is $(h_1, \mathbb{N})$.

4. Some properties of semitopological functors

To prove the theorem 4.1, we need some preliminary facts, no relevant with the object of the paper. In [1] can be found as proposition 25.12 and theorem 25.14.

Theorem 4.1. *[1] Let $F : \boldsymbol{A} \to \boldsymbol{B}$ be a semitopological functor then:*

- *F is faithful*
- *F has a left adjoint.*
- *F preserve and detect limits*
- *F detect colimits*

Where we say that a functor $F : \mathbf{A} \to \mathbf{B}$ detect limits if a diagram $C : \mathbf{I} \to \mathbf{A}$ has limit in **A** whenever $F \circ C$ has one in **B**.

The notion of source is dual to the notion of sink. The next theorem was proposed in [9] and here the proof is presented completely.

Theorem 4.2. *[9] F is a semitopological functor if and only if every structural source has a semi initial solution.*

Proof. Let $F : \mathbf{A} \to \mathbf{B}$ be a topological functor and (f_i, B) be a structural source where $f_i : B \to FA_i$. We define the sink (h_j, B) formed with the morphisms h_j such that $h_j : \overline{A}_i \to B$ satisfies that $f_i \circ h_j$ is a morphism in $\mathbf{A}$. Since F is semitopological, the sink (h_j, B) has a semifinal solution, which in turn is the semi initial solution for the source (f_i, B).

The previous construction doesn't consider the case the sink (h_j, B) is empty. In this case, since F is semitopological has a left adjoint (see Theorem 4.1) and therefore $\mathbf{B}$ can be associated with its free object A_B and a morphism m_B, this couple will be the semi initial solution of the source.

Conversely, if every source has semi initial solution and you take a structural sink (f_i, B) such that $f_i : FA_i \to B$, form the initial source of objects C_j such that exist a morphism $h_i : B \to C_j$ complying $h_j \circ f_i$ is a morphism in $\mathbf{A}$ and this source has a semi initial solution (g, A) then the sink (f_i, B) has semifinal solution. □

Additionally, theorem 4.1 implies the following.

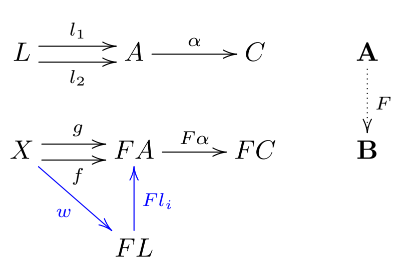
Corollary 4.3. *[1] Let $F : \boldsymbol{A} \to \boldsymbol{B}$ be a semitopological functor and $\boldsymbol{B}$ a complete category. Then $\boldsymbol{A}$ is complete.*

To prove theorem 4.4 we use an epi-structural morphism. Let $F : \mathbf{A} \to \mathbf{B}$ be a functor and g be a morphism in $\mathbf{B}$, g will be called **epi-structural** if for each two morphisms $f, h : A \to A'$ in $\mathbf{A}$ such that $Ff \circ g = Fh \circ g$ then $f = h$.

In case of work with a semitopological functor, thanks to its faithful, it is possible to prove that the morphism g of a semifinal solution(g, A) of a sink (X, f_i) is an epi-structural morphism [1].

Theorem 4.4. *Let $F : \boldsymbol{A} \to \boldsymbol{B}$ be a semitopological functor and α a monomorphism in $\boldsymbol{A}$ then $F\alpha$ is a monomorphism in $\boldsymbol{B}$.*

Proof. If $f, g : X \to FA$ are morphisms in $\mathbf{B}$ and $\alpha : A \to C$ a monomorphism in $\mathbf{A}$ such that $F\alpha \circ f = F\alpha \circ g$. The source $(X, \{f, g\})$ has semi initial solution given by $w : X \to FL$ and $l_i : FL \to FA$ with $i = 1, 2$, as w is epi-structural and $F\alpha \circ Fl_2 \circ w = F\alpha \circ Fl_1 \circ w$ then $\alpha \circ l_2 = \alpha \circ l_1$, since α is monomorphism $l_1 = l_2$ thus $f = Fl_1 \circ w = Fl_2 \circ w = g$, that is $f = g$ when $F\alpha \circ f = F\alpha \circ g$, as f and g were arbitrary then $F\alpha$ is a monomorphism.



□

5. Structures semicoelevators

The structure elevator and coelevator proposed by Montañez in [8] constitutes an alternative in the formation of new topological categories. In this section, we translate this notion for semitopological categories and prove that the category formed by the fixed points of these functors is a semitopological category.

Let $F : \mathbf{A} \to \mathbf{B}$ be a topological functor, an endofunctor $C : \mathbf{A} \to \mathbf{A}$ will be called a coelevator if $F \circ C = F$ and for every object X in $\mathbf{A}$ exist a morphism $c_x : X \to C(X)$ such that $Fc_x = id_{FX}$.

Example 5.1. [6] Let X be a topological space. X is completely regular iff for each closed subset A of X and for each $x \in X$ with $x \notin A$, exist a continuous function $f : X \to I$ such that $f(x) = 0$ and $f(A) = 1$, where $I = [0, 1]$ with the usual topology. Additionally a topological space is completely regular iff has the initial topology with respect the bounded and continuous functions of real value, then we can define the coelevator $C_I : \mathbf{Top} \to \mathbf{Top}$ such that for each topological space X, sends to the topological space $C_I(X)$, this has the initial topology of the set of all arrows of X to I.

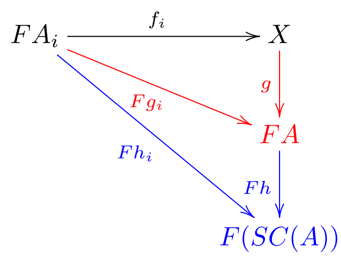
Definition 5.2. If $F : \mathbf{A} \to \mathbf{B}$ is a semitopological functor and $SC : \mathbf{A} \to \mathbf{A}$ a functor, it will be called **structure semicoelevator** if for every object X in $\mathbf{A}$ exist an epimorphism $g_x : X \to SC(X)$ in $\mathbf{A}$ such that the g_x form a natural transformation of the identity functor of $\mathbf{A}$ to the functor SC.

An example of the previous concept is the structure coelevators, since $Fg_x = id_{FX}$. The dual notion of semicoelevator is structure semielevator, which is a functor $SE : \mathbf{A} \to \mathbf{A}$ such that for every X in $\mathbf{A}$ exist a monomorphism $f_x : SE(X) \to X$ in $\mathbf{A}$ such that the f_x form a natural transformation of the functor SE to the identity functor of $\mathbf{A}$.

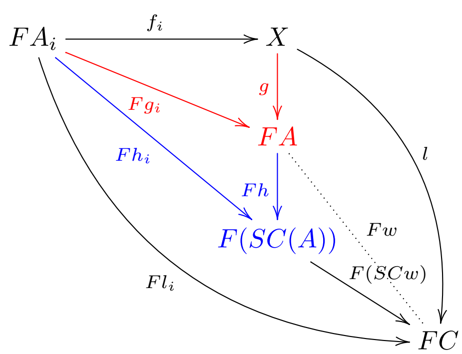
In case to work with an idempotent functor $S : \mathbf{A} \to \mathbf{A}$, the fixed points of S coincides with its image, since if we take a fixed point $A \in Obj(S(\mathbf{A}))$ and applying S then $S(A) \in Im(S)$ where $S(A) = A$, therefore $A \in Im(S)$. Now, let C in $Im(S)$, then exist X such that $S(X) = C$ applying S is met that $S(S(X)) = S(C) = S(X) = C$ i.e. $C \in Obj(S(\mathbf{A}))$. The structure semicoelevators generalize the concept of coelevation for semitopological functor, it is pertinent to ask under what conditions the category $SC(\mathbf{A})$ formed by the fixed points of SC is semitopological, this motivates the theorem 5.3.

Theorem 5.3. *Let $F : \mathbf{A} \to \mathbf{B}$ be a semitopological functor and $SC : \mathbf{A} \to \mathbf{A}$ an idempotent structural semicoelevator. Then $SC(\mathbf{A})$ is a semitopological category relative to $\mathbf{B}$.*

Proof. Let $A_i \in SC(\mathbf{A})$ and $f_i : FA_i \to X$ as F is semitopological functor the sink (f_i, X) has semifinal solution (g, A) in $\mathbf{A}$. With this in mind, we build the semifinal solution in $SC(\mathbf{A})$, as SC is a semicoelevator, it will be exist morphism $h : A \to SC(A)$, as SC is idempotent $SC(A) \in Obj(\mathbf{A})$, whereby applying F, we have the following commutative diagram in $\mathbf{B}$:



With $h_i = h \circ g_i$, we will prove that $(h \circ g, SC(A))$ is the semifinal solution in SC($\mathbf{A}$) of the sink (f_i, X); for this, let C be an object of $(SC(\mathbf{A}))$ such that exist $l : X \to FC$, $l_i : A_i \to C$ and $l \circ f_i = Fl_i$. Since A is the semifinal solution of (f_i, X) there is a morphism of $w : A \to C$ satisfying $l = F(w) \circ g$ and $l_i = w \circ g_i$. Applying SC to w, we have $SC(w) : SC(A) \to SC(C)$, since C is a fixed point, $SC(w)$ can be rewritten as $SC(w) : SC(A) \to C$. Applying F, the following diagram is commutative:



Therefore $(F(h) \circ g, SC(A))$ is the semifinal solution in SC($\mathbf{A}$) of the sink (f_i, X), where the morphism $F(h) \circ g$ is an epi-structural because g is an epi-structural and $F(h)$ is an epimorphism from the definition 5.2. So $F : SC(\mathbf{A}) \to \mathbf{B}$ is semitopological and $SC(\mathbf{A})$ is a semitopological category. □

Remark 5.4. In case g_x isn't an epimorphism in the definition 5.2, the constant functors are semicoelevators and therefore the previous theorem would be false. A counterexample will be to consider the category of topological spaces and build a constant functor to a point of a topological space.

An example of the theorem 5.3 is the category of Hausdorff spaces, let (X, τ) be a topological space, if we consider the source formed by the $f : (X, \tau) \to (Y, \eta)$ such that (Y, η) is a Hausdorff space and the relation of equivalence $\sim$ in $X \times X$ given by $a \sim b$ if and only if for all $f : (X, \tau) \to (Y, \eta)$ such that (Y, τ) is a Hausdorff space and $f(a) = f(b)$, thus $(X/\sim, \tau^*)$ is a Hausdorff space where τ^* is the quotient topology.

The previous construction induces an endofunctor in **Top**, designated by $H : \mathbf{Top} \to \mathbf{Top}$, which acts on objects by $H(X) = (X/\sim, \tau^*)$, where $\sim$ is the $\tau-$closure of $\overline{\{(x,x) : x \in X)\}}$ and τ^* is the quotient topology. In morphisms $H(f) = \overline{f}$; where $\overline{f}(\overline{a}) := f(a)$, whose fixed points correspond to the category of Hausdorff spaces **Haus**.

If we remember that **Top** is a topological category, especially semitopological and applying theorem 5.3 to H then **Haus** is semitopological category relative to category of sets. Note that in this case, the functor H is left adjoint to the inclusion functor of $\imath$ in **Top**, in other words the category **Haus** is a reflective subcategory of **Top**.

The scheme used in the proof that **Haus** is a semitopological category when it view as the fixed points of an endofunctor of **Top** motivates the corollary 5.5.

Let **C** be a reflective subcategory of **A**. The reflection induces an endofunctor $R_{\mathbf{C}} : \mathbf{A} \to \mathbf{A}$ such that every object A in **A** goes to its reflection in **C**, which consists of an object $R_{\mathbf{C}}(A)$ and a morphism $h_A : A \to R_C(A)$ in **A**. Additionally, let $f : A \to B$ be a morphism in **A**, exist a morphism $h_B : B \to R_{\mathbf{C}}(B)$ and the composition $f \circ h_B$ is a morphism of A to $R_C(B)$. Since $R_{\mathbf{C}}(A)$ is the reflection of A in **C** exist a morphism $R_{\mathbf{C}}(f) : R_{\mathbf{C}}(A) \to R_{\mathbf{C}}(B)$, this morphism is the image of f under the functor $R_{\mathbf{C}}$. Note that the fixed points of the functor R_C correspond to the category **C** and where the image of an object of **C** is the same object, therefore R_C is an idempotent functor.

Corollary 5.5. *Let $F : \boldsymbol{A} \to \boldsymbol{B}$ be a semitopological functor and $\boldsymbol{C}$ a reflective subcategory of $\boldsymbol{A}$ then $\boldsymbol{C}$ is a semitopological category.*

Proof. Let **C** be a reflective subcategory, so exist a morphism $r_X : X \to C_X$ where C_X is the reflection of X in **C**. If $f : X \to Y$ is a morphism in **A**, there is a morphism $r_Y \circ f : X \to C_Y$ with $r_Y : Y \to C_Y$, therefore exist a morphism $g : C_X \to C_Y$ in **C** such that $r_Y \circ f = g \circ r_X$. Applying F to f we have the following commutative diagram:

$$\begin{array}{ccc} FX & \xrightarrow{F(r_X)} & F(C_X) \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Fg} \\ FY & \xrightarrow[F(r_Y)]{} & F(C_Y) \end{array}$$

If $\tau_x = Fr_X$ for each $X \in Obj(\mathbf{A})$ then $R_{\mathbf{C}}$ is a semicoelevator, it is idempotent since $R_{\mathbf{C}} \circ R_{\mathbf{C}}(X) = C_X = R_{\mathbf{C}}(X)$, the fixed points are exactly the objects of **C** and applying theorem 5.3, **C** is a semitopological category. □

If we think in a reflective subcategory of a topological category, it couldn't be topological. For example the forgetful functor of Hausdorff spaces to Sets. The corollary 5.5 only generates semitopological categories despite the fact that category **A** is topological; some examples of the corollary 5.5 are the following:

Example 5.6. • The abelian group category **Ab** is a reflective subcategory of the category of groups **Grp**.

Let $(G, *)$ be a group and the normal subgroup $G' = \langle aba^{-1}b^{-1} | \ a, b \in G \rangle$ of G, then $(G/G', *)$ is an abelian group [3]. The canonical homomorphism $q_G : (G, *) \to (G/G', *)$ is the reflector morphism. In case of existing $(H, \circ)$ in **Ab** and a homomorphism $h : (G, *) \to (H, \circ)$, there is a natural homomorphism $\overline{h} : (G/G', *) \to (H, \circ)$ such that $\overline{h} \circ q_G = h$. Therefore **Ab** is a semitopological category.

- The category of torsion free abelian groups **LTAB** is a reflective subcategory of **Ab**. Let $(H, *)$ be an abelian group, using the torsion subgroup $T = \{a \in H \mid \exists n, a^n = e\}$, the group $(H/T, *)$ is torsion free abelian group and the canonical homomorphism $q_H : (H, *) \to (H/T, *)$ is the reflector morphism. Therefore **LTAB** is a semitopological category.

- The category $\mathbf{Top_{T_0}}$ whose objects are topological spaces that meet the first axiom of separability $\mathbf{T}_0$, is a reflective subcategory of **Top**. Let (X, τ) be a topological space, defining in X the relation $a \sim b$ if only if $\overline{\{a\}} = \overline{\{b\}}$ (the closure of a is equal to the closure of b), this relationship is an equivalence relation and determines the quotient space $(X/\sim, \tau*)$ that results T_0 and the continuous function $q_X : (X, \tau) \to (X/\sim, \tau*)$ is the reflector morphism [7]. Therefore, $\mathbf{Top_{T_0}}$ is a semitopological category.

From the corollary 5.5, reflective subcategories are semitopological. The reciprocal motives the following theorem:

Theorem 5.7. *Let $F : \boldsymbol{A} \to \boldsymbol{B}$ be a full and a semitopological functor with $\boldsymbol{C}$ a full subcategory of $\boldsymbol{A}$. Then $\boldsymbol{C}$ is a semitopological category relative to $\boldsymbol{B}$ if only if $\boldsymbol{C}$ is a reflective subcategory of $\boldsymbol{A}$.*

Proof. Let **C** be a semitopological category relative to **B** and $A \in Obj(\mathbf{A})$. Applying F, FA has a free object C_A in **C** and there is a morphism $m_A : FA \to FC_A$, since the functor F is full, there exist r_A such that $Fr_A = m_A$ applying this construction to each object A of **A**, the inclusion functor from **C** to **A** has a left adjoint, which implies that **C** is a reflective subcategory.

From corollary 5.5 it follows that **C** is a reflective subcategory of **A**, the category **C** is a semitopological category. □

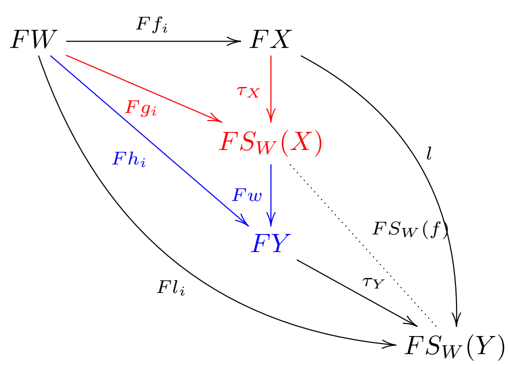
Example 5.8. The theorem 5.7 shows another way to prove that the category **Haus** is reflective in **Top**, which is happens because the functor i : **Haus** $\hookrightarrow$ **Top** is semitopological, a fact seen as a consequence of theorem 5.3.

Let **A** be a semitopological category and an object W of **A**, an endofunctor is determined in **A** using the semifinal solutions, we will denote this functor as S_W, where S_W sends an object X to the semifinal solution of the sink formed by all the morphisms from W to X.

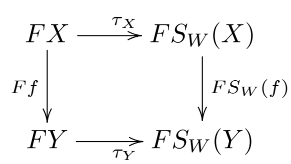
To establish the theorem 5.9, we need the notion of idempotent functor. Let $G : \mathbf{A} \to \mathbf{B}$ be a functor. G will be called an **idempotent** functor if $G^2(X)$ is isomorphic to $G(X)$.

Theorem 5.9. *Let $F : \boldsymbol{A} \to \boldsymbol{B}$ be a full semitopological functor, then given an object W in $\boldsymbol{A}$, the functor $S_W : \boldsymbol{A} \to \boldsymbol{A}$ is an idempotent semicoelevator.*

Proof. It will be seen that S_W is a semicoelevator. Let $f : X \to Y$ be a morphism in $\mathbf{A}$ and $f_i : W_i \to X$ all the possibles arrows to X in $\mathbf{A}$. Applying F to f_i then $Ff_i : FW \to FY$ is a morphism in $\mathbf{B}$. Since F is semitopological functor exist the semifinal solution of (Ff_i, FX) denote that object $S_W(X)$. Additionally, exist $w : S_W(X) \to Y$ such that $Fw \circ \tau_X = Ff$, therefore we have the morphism $\tau_y \circ (Ff \circ Ff_i) : FW \to FS_W(Y)$ then there will be $S_W(f) : S_W(X) \to S_W(Y)$ such that $S_W(f) \circ \tau_X = Ff \circ \tau_Y$, the above generates that commutes the following diagram:



i.e., S_W is a semicoelevator,



From the idempotent condition of S_W is clear the uniqueness of the semifinal solutions. □

Example 5.10. Let $\mathbf{C}$ be a small category, the functor $R : \mathbf{C} \to Set^{\mathbf{C}^{op}}$ is full, faithful and injective in objects [5]. An object A of $\mathbf{C}$ goes to the representable functor $R(A)$, that assigns to each object X of $\mathbf{C}$, the set $[X, A]$ of all morphisms of X to A and each morphism $f : X \to Y$ goes to the function $R(f) : [Y, A] \to [X, A]$ where $H(f)(g) = f \circ g$.

Let $\mathbf{C}$ be a small category and $F : \mathbf{C} \to \mathbf{D}$ a full semitopological functor, so the functor $R : \mathbf{C} \to Sets^{\mathbf{C}^{op}}$ is a full semitopological functor.

The previous example motivates the construction of representable subcategories.

Definition 5.11. Let $F : \mathbf{A} \to \mathbf{B}$ be a full semitopological functor, a subcategory $\mathbf{C}$ of $\mathbf{A}$ is representable if exist an object W in $\mathbf{A}$ such that $\mathbf{C} = S_W(\mathbf{A})$.

Theorem 5.5 and 5.9 help to connect the reflective subcategories and representables under conditions of fullness.

Corollary 5.12. *Let $F : \boldsymbol{A} \to \boldsymbol{B}$ be a full semitopological functor and $\boldsymbol{C}$ a representable subcategory of $\boldsymbol{A}$ then $\boldsymbol{C}$ is a reflective subcategory of $\boldsymbol{A}$.*

6. Conclusions

To generalize the concept of coelevator for semitopological categories, we need to use the notions of semifinal solutions and epi-structural morphisms, these functors received the name of semicoelevator and give new semitopological categories, the formed by the fixed points of the semicoelevator.

On the other hand the generalization of concept of elevator (dual of the coelevator) didn't give new semitopological categories. Other relevant thing was the study of the equivalence of semitopological functors and the notion of reflection under the fullness of the semitopological functor. Also interesting examples were drawn from that equivalence.

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