Parabolic type equations on $p-$adic balls

Ecuaciones de tipo parabólico sobre bolas $p-$ádicas

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Abstract. In this article we solve the Cauchy problem associated to a radial symbol constant on a ball of radius p^r and we show that the fundamental solution, $Z_r(x,t)$, vanishes outside the ball of radius p^{-r}.

Keywords: $p-$adic numbers, pseudodifferential equations, Cauchy problem.

Resumen. En este artículo encontramos la solución al problema de Cauchy asociado a un símbolo radial que es constante en bolas de radio p^r, y mostramos que la solución fundamental $Z_r(x,t)$ se anula fuera de una bola de radio p^{-r}.

Palabras claves: Números $p-$ádicos, ecuaciones pseudodiferenciales, problema de Cauchy.

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1. Introduction

A $p-$adic pseudodifferential operator $f(\partial,\alpha)$, with symbol $f(\xi)$, is an operator of the form

$$f(\partial,\alpha)u := \mathcal{F}^{-1}_{\xi\to x}\left(|f(\xi)|_p^{\alpha}\mathcal{F}_{x\to\xi}u\right),\ \alpha>0, \tag{1}$$

where $\varphi\in\mathbf{S}$, $\mathbf{S}$ denotes the $\mathbb{C}$-vector space of Bruhat-Schwartz functions over $\mathbb{Q}_p^n$, $\mathcal{F}$ denotes the Fourier transform on $L_2(\mathbb{Q}_p^n)$, and $f(\xi)$ is a p-adic valued function on $\mathbb{Q}_p^n$.

If the symbol $|f(\xi)|_p^{\alpha}$ is continuous, then the operator $f(D)$ is continuous and has a self-adjoint extension with dense domain in $L^2(\mathbb{Q}_p^n)$. This operator is considered to be a p-adic analogue of a linear partial elliptic differential operator with constant coefficients.

In [4] Kochubei studies the Cauchy problem

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} + D^{\alpha}u(x,t) = g(x,t),\ x\in\mathbb{Q}_p,\ n\geq 1,\ t\in(0,T], \\ u(x,0)=\varphi(x), \qquad x\in\mathbb{Q}_p, \end{cases} \tag{2}$$

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where

$$D^{\alpha}u := \mathcal{F}^{-1}_{\xi\to x}\left(|\xi|^{\alpha}_{p}\mathcal{F}_{x\to\xi}u\right),\ \alpha>0, \tag{3}$$

is the Valdimirov operator. He finds a fundamental solution in certain space $\mathfrak{M}_{\lambda}$, and gives some properties of it.

In [5] Rodríguez-Vega and Zúñiga-Galindo study the same problem, but in dimension n. They consider the Taibleson operator, $D^{\alpha}u := \mathcal{F}^{-1}_{\xi\to x}(||\xi||^{\alpha}_{p}\mathcal{F}_{x\to\xi}u)$, for $\alpha>0$, and $x\in\mathbb{Q}_p^n$, and by using the same method and the same space $\mathfrak{M}_{\lambda}$, they find a fundamental solution for the Cauchy problem and some of its properties.

In [7] Zúñiga-Galindo considers a more general class of pseudodifferential operadors, defined as

$$f(\partial,\alpha)u := \mathcal{F}^{-1}_{\xi\to x}\left(|f(\xi)|^{\alpha}_{p}\mathcal{F}_{x\to\xi}u\right),\ \alpha>0, \tag{4}$$

where f is an elliptic polynomial and $x\in\mathbb{Q}_p^n$. He build a fundamental solution for the homogeneous Cauchy problem

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} + f(\partial,\alpha)u(x,t) = 0,\ x\in\mathbb{Q}_p^n,\ n\geq 1,\ t\in(0,T], \\ u(x,0)=\varphi(x), \qquad x\in\mathbb{Q}_p^n, \end{cases} \tag{5}$$

with $\alpha>0$ and φ a bounded and continuous function. This fundamental solution is called the heat kernel and it is given by

$$Z(x,t) = \int_{\mathbb{Q}_p^n}\chi(-x\cdot\xi)e^{-t|f(\xi)|^{\alpha}_{p}}d^n\xi, \tag{6}$$

where χ is a standard additive character on $\mathbb{Q}_p^n$.

In [3] Casas-Sánchez and Zúñiga-Galindo study the Cauchy problem with variable coefficients associated to an elliptic quadratic form in dimension two and four, more recently Casas-Sánchez, Galeano-Peñaloza and Rodríguez-Vega [2], studied the Cauchy problem associated with an elliptic quadratic form of dimension three.

The main goal of this article is to solve the Cauchy problem associated to a radial symbol constant on a ball of radius p^r, see Definition 3.1, and to see that the fundamental solution, $Z_r(x,t)$, vanishes outside a ball, Proposition 3.4.

The article is organized as follows. In Section 2, we give the definitions and some preliminars results. In Section 3, we define the heat kernel $Z(x,t)$ and we show that the heat kernel is a fundamental solution of the Cauchy problem, see Theorem 3.14.

2. Preliminary Results

Along this article p will denote a prime number different from 2. Let $\mathbb{Q}_p$ be the field of the p-adic numbers, and let $\mathbb{Z}_p$ be the ring of p-adic integers. For $x\in\mathbb{Q}_p$, let $v(x)$ denote the valuation of x normalized by the condition $v(p)=1$, and let $|x|_p = p^{-v(x)}$, $|0|_p = 0$ be the normalized absolute value. We extend the p-adic absolute value to $\mathbb{Q}_p^n$ as follows:

$$||x||_p := \max\{|x_1|_p, \ldots, |x_n|_p\}, \text{ for } x = (x_1, \ldots, x_n) \in \mathbb{Q}_p^n.$$

The Bruhat-Schwartz space $\mathbf{S}(\mathbb{Q}_p^n)$ consists of all complex-valued and locally-constant functions with compact support, such functions are called *test functions*, where the *exponent of local constancy* of $\varphi(x) \in S(\mathbb{Q}_p^n)$ is the smallest integer, $l \geq 0$, with the property:

$$\varphi(x + x') = \varphi(x) \text{ if } ||x'||_p \leq p^{-l}, \quad \text{for any } x \in \mathbb{Q}_p^n.$$

A linear functional f defined over $\mathbf{S}(\mathbb{Q}_p^n)$, $f : \mathbf{S}(\mathbb{Q}_p^n) \longrightarrow \mathbb{C}$, $\phi \mapsto \langle f, \phi\rangle$ is called a *generalized function* or a *distribution*. Denote by $\mathbf{S}'(\mathbb{Q}_p^n)$ the set of all distributions over $\mathbb{Q}_p^n$. We note that all distributions over $\mathbb{Q}_p^n$ are continuous.

The Fourier transform of $\varphi \in \mathbf{S}(\mathbb{Q}_p^n)$ is defined as

$$(\mathcal{F}\varphi)(\xi) = \int_{\mathbb{Q}_p^n} \chi(\xi \cdot x)\varphi(x)dx \quad \text{ for } \xi \in \mathbb{Q}_p^n,$$

where $\xi \cdot x := \sum_{j=1}^{n} \xi_j x_j$, dx is the Haar measure on $\mathbb{Q}_p^n$ normalized by the condition $vol(B_0^n) = 1$ and χ denote an additive character of $\mathbb{Q}_p$, trivial on $\mathbb{Z}_p$, but not on $p^{-1}\mathbb{Z}_p$. The Fourier transform is a linear isomorphism from $\mathbf{S}(\mathbb{Q}_p^n)$ onto itself satisfying $(\mathcal{F}(\mathcal{F}\varphi))(\xi) = \varphi(-\xi)$.

For a more detailed explanation, the reader may consult [1] and [6].

3. A General Type of Equations

In this section we consider the following integral.

Definition 3.1. We define the heat kernel as

$$Z_r(x,t) := \int_{\mathbb{Q}_p^n} \chi(-x \cdot \xi)\, e^{-t\left(f(||\xi||_p) - \lambda\right)}\, d\xi, \tag{7}$$

where $f(||\xi||_p)$ satisfies the following conditions

(i) $f(||\xi||_p) = \lambda$ if $||\xi||_p \leq p^r$.

(ii) $f(||\xi||_p)$ is a non-decreasing function.

(iii) There are positive constants A_1, A_2, γ_1 and γ_2, such that

$$A_1||\xi||_p^{\gamma_1} \leq f(||\xi||_p) \leq A_2||\xi||_p^{\gamma_2}, \quad \text{ for } ||\xi||_p \gg 1.$$

We first note that the function $e^{-tf(||\xi||_p)}$ belongs to $L^1(\mathbb{Q}_p^n)$.

Lemma 3.2. *For $\rho \geq 1$, and fixed $t > 0$, we have $e^{-tf(||\xi||_p)} \in L^\rho(\mathbb{Q}_p^n)$.*

Proof. It is a consequence of the fact that

$$\int_{\mathbb{Q}_p^n} e^{-\rho t||\xi||_p^{\alpha}}\, d\xi < +\infty$$

for any $\rho, t, \alpha > 0$. Indeed

$$\begin{aligned}||\exp\big(-tf(||\xi||_p)\big)||_{L^\rho}^{\rho} &= \int_{||\xi||_p>p^r} e^{-t\rho f(||\xi||_p)}\, d\xi + \int_{||\xi||_p\le p^r} e^{-\lambda\rho t}\, d\xi \\ &\le \int_{\mathbb{Q}_p^n} e^{-\rho t A_1||\xi||_p^{\gamma_1}}\, d\xi + e^{-\lambda\rho t} < +\infty.\end{aligned}$$

□

Lemma 3.3. *For all $x \in \mathbb{Q}_p^n$ and $t > 0$,*

$$Z_r(x,t) \ge 0.$$

Proof.

$$\begin{aligned}Z_r(x,t) &= e^{\lambda t}\sum_{k=-\infty}^{\infty} e^{-tf(p^k)}\int_{||\xi||_p=p^k}\chi(-x\cdot\xi)\, d\xi \\ &= e^{\lambda t}\sum_{k=-\infty}^{\infty} e^{-tf(p^k)}\Big(p^{nk}\Omega_{-k}(x) - p^{n(k-1)}\Omega_{-k+1}(x)\Big) \\ &= e^{\lambda t}\sum_{k=-\infty}^{\infty} p^{nk}\Big(e^{-tf(p^k)} - e^{-tf(p^{k+1})}\Big)\Omega_{-k}(x),\end{aligned}$$

where Ω_k is the indicator function of the ball $\{x \in \mathbb{Q}_p^n \mid ||x||_p \le p^k\}$. □

Proposition 3.4. *If $x \neq 0$ and $t > 0$, then we have that*

$$Z_r(x,t) = \begin{cases} p^{nr} - ||x||_p^{-n} e^{-t(f(p||x||_p^{-1})-\lambda)} & \\ +(1-p^{-n})||x||_p^{-n}\displaystyle\sum_{l=0}^{v(x)-r-1} p^{-nl} e^{-t(f(p^{-l}||x||_p^{-1})-\lambda)} & \text{if } ||x||_p < p^{-r}, \\ p^{nr} - ||x||_p^{-n} e^{-t(f(p||x||_p^{-1})-\lambda)} & \text{if } ||x||_p = p^{-r}, \\ 0 & \text{if } ||x||_p > p^{-r}. \end{cases}$$

Proof. As $f(||\xi||_p) = \lambda$ for $||\xi||_p \le p^r$, then

$$Z_r(x,t) = \int_{||\xi||_p\le p^r}\chi(-x\cdot\xi)\, d\xi + e^{\lambda t}\int_{||\xi||_p>p^r}\chi(-x\cdot\xi)\, e^{-tf(||\xi||_p)}\, d\xi, \qquad (8)$$

the first integral is equal to the Fourier transform of $\Omega_r(\xi)$, i.e., $p^{nr}\Omega_{-r}(x)$. For the second integral note that if $||x||_p > p^{-r}$, then for $||\xi||_p = p^{\upsilon}$, $\upsilon > r$, we have

$$\int_{||\xi||_p=p^{\upsilon}}\chi(-x\cdot\xi)\, d\xi = 0,$$

therefore $Z_r(x,t)=0$ for $||x||_p > p^{-r}$.

Now

$$\int_{||\xi||_p>p^r} \chi(-x\cdot\xi)\, e^{-tf(||\xi||_p)}\, d\xi = \sum_{\nu=r+1}^{\infty} e^{-tf(p^\nu)} \int_{||\xi||_p=p^\nu} \chi(-x\cdot\xi)\, d\xi,$$

the last sum has only a finitely number of terms not equal to zero. In fact if $||x||_p = p^{-v(x)}$ $(v(x)>r)$, then $||\xi||_p$ can be equal to $p^{r+1}, p^{r+2}, \ldots, p^{v(x)+1}$. In the last case if $||\xi||_p = ||x||_p^{-1} p$ we have that

$$e^{-tf(||x||_p^{-1}p)} \int_{||\xi||_p=||x||_p^{-1}p} \chi(-x\cdot\xi)\, d\xi = -e^{-tf(||x||_p^{-1}p)} ||x||_p^{-n},$$

and that $||\xi||_p$ varies between $p^{r+1},\ldots,p^{v(x)}$ is equivalent to $||\xi||_p$ varies between

$$||x||_p^{-1}p^{r+1-v(x)},\ldots,||x||_p^{-1}p^{-1},||x||_p^{-1},$$

and therefore

$$\begin{aligned}
&\int_{p^r<||\xi||_p<||x||_p^{-1}} \chi(-x\cdot\xi)\, e^{-tf(||\xi||_p)}\, d\xi \\
&\qquad = \sum_{l=0}^{v(x)-r-1} \int_{||\xi||_p=p^{-l}||x||_p^{-1}} \chi(-x\cdot\xi)\, e^{-tf(||\xi||_p)}\, d\xi \\
&\qquad = \sum_{l=0}^{v(x)-r-1} e^{-tf(p^{-l}||x||_p^{-1})} p^{-ln} ||x||_p^{-n} (1-p^{-n}) \\
&\qquad = (1-p^{-n})||x||_p^{-n} \sum_{l=0}^{v(x)-r-1} e^{-tf(p^{-l}||x||_p^{-1})} p^{-ln}. \qquad (9)
\end{aligned}$$

□

Corollary 3.5.

$$Z_r(x,t) \le Ct||x||_p^{-n}\left(f(p||x||_p^{-1})-\lambda\right). \qquad (10)$$

Proof. (i) If $||x||_p < p^{-r}$, we have that $p^{nr} < ||x||^{-n}$, and

$$\begin{aligned}
Z_r(x,t) &\le ||x||^{-n} - ||x||^{-n} e^{-t(f(p||x||^{-1})-\lambda)} \left(1-(1-p^{-1})\sum_{l=0}^{v(x)-r-1} p^{-nl}\right) \\
&\le ||x||^{-n} - ||x||^{-n} e^{-t(f(p||x||^{-1})-\lambda)} \left(1-(1-p^{-1})\frac{1-p^{-n(v(x)-r)}}{1-p^{-n}}\right) \\
&\le ||x||^{-n}\left(1-\frac{p^{n-1}-1}{p^n-1} e^{-t(f(p||x||^{-1})-\lambda)}\right) \\
&\le Ct||x||_p^{-n}\left(f(p||x||_p^{-1})-\lambda\right).
\end{aligned}$$

(ii) If $||x||_p = p^{-r}$, then

$$
\begin{aligned}
Z_r(x,t) &= ||x||^{-n} - ||x||^{-n} e^{-t(f(p||x||^{-1})-\lambda)} \\
&\le ||x||^{-n}\left(1 - e^{-t(f(p||x||^{-1})-\lambda)}\right) \\
&\le Ct||x||_p^{-n}\left(f(p||x||_p^{-1}) - \lambda\right).
\end{aligned}
$$

□

Lemma 3.6. *For $\alpha > 0$, $t > 0$, the following assertions hold*

(i) $Z_r(x,t) \in C(\mathbb{Q}_p^n, \mathbb{R}) \cap L^1(\mathbb{Q}_p^n) \cap L^2(\mathbb{Q}_p^n)$, *for* $t > 0$.

(ii) $\displaystyle\int_{\mathbb{Q}_p^n} Z_r(x,t)\, dx = 1$.

(iii) $\displaystyle\lim_{t\to 0^+} Z_r(x,t) * \varphi(x) = \varphi(x)$, *for* $\varphi \in \mathbf{S}(\mathbb{Q}_p^n)$.

(iv) $Z_r(x,t) * Z(x,t') = Z(x,t+t')$, *for* $t, t' > 0$.

Proof. (i) It is a direct consequence of Lemma 3.2.

(ii) It follows from (i) by the inversion formula for the Fourier transform.

(iii) By using (ii), we have that

$$
\int_{\mathbb{Q}_p^n} Z_r(x-\xi,t)\varphi(\xi)\, d\xi - \varphi(x) = \int_{\mathbb{Q}_p^n} Z_r(x-\xi,t)(\varphi(\xi)-\varphi(x))\, d\xi.
$$

Since $\varphi(x)$ is locally constant, there exists $m \in \mathbb{Z}$ such that $\varphi(\xi) - \varphi(x) = 0$ if $||\xi - x||_p \le p^m$. Then

$$
\begin{aligned}
I :&= \left| \int_{\mathbb{Q}_p^n} Z_r(x-\xi,t)(\varphi(\xi)-\varphi(x))\, d\xi \right| \\
&= \left| \int_{||\xi-x||_p > p^m} Z_r(x-\xi,t)(\varphi(\xi)-\varphi(x))\, d\xi \right| \\
&\le Ct \int_{||\xi-x||_p > p^m} ||\xi-x||_p^{-n} f(p||\xi-x||_p^{-1})\, |\varphi(\xi)-\varphi(x)|\, d\xi \\
&\le A_2 Ctp \int_{||\xi-x||_p > p^m} ||\xi-x||_p^{-n-\gamma_2}\, |\varphi(\xi)-\varphi(x)|\, d\xi \\
&= C_2 t \int_{||z||_p > p^m} ||z||_p^{-n-\gamma_2}\, |\varphi(x-z)-\varphi(x)|\, dz
\end{aligned}
$$

$$\leq C_2 t \int\limits_{||z||_p > p^m} ||z||_p^{-n-\gamma_2} \left|\varphi(x-z) - \varphi(x)\right| dz$$
$$\leq C_2 t (C_3 + C_4 |\varphi(x)|) \to 0, \text{ as } t \to 0^+.$$

(iv) By (i) the formula is equivalent to

$$\mathcal{F}_{x\to\xi}\big(Z_r(x,t)\big) \cdot \mathcal{F}_{x\to\xi}\big(Z_r(x,t')\big) = \mathcal{F}_{x\to\xi}\big(Z_r(x,t+t')\big)$$

but

$$\mathcal{F}_{x\to\xi}\big(Z_r(x,\tau)\big) = e^{-\tau\left(f(||\xi||_p)-\lambda\right)}.$$

□

Definition 3.7. We define the following pseudodifferential operators

$$J\varphi = \mathcal{F}^{-1}_{\xi\to x}\big(f(||\xi||_p)\mathcal{F}_{y\to\xi}\,\varphi(y)\big), \quad \varphi \in \mathbf{S}(\mathbb{Q}_p^n) \tag{11}$$

and

$$H\varphi = J\varphi - \lambda\varphi, \quad \varphi \in \mathbf{S}(\mathbb{Q}_p^n). \tag{12}$$

Lemma 3.8. *We have the following*

(i) The map $J : \mathbf{S}(\mathbb{Q}_p^n) \to \mathbf{S}(\mathbb{Q}_p^n)$, sending φ to $J\varphi$ is a homeomorphism.

(ii) The map $H : \mathbf{S}(\mathbb{Q}_p^n) \to \mathbf{S}(\mathbb{Q}_p^n)$, sending φ to $H\varphi$ is continuous.

Proof. For the first claim recall that $\varphi \mapsto \mathcal{F}\varphi$ is a homeomorphism of $\mathbf{S}(\mathbb{Q}_p^n)$, the results follows from this observation since $\varphi \mapsto f(||\xi||_p)\mathcal{F}\varphi$ is a homeomorphism.

The second statement follows from (i). □

Definition 3.9. We set for $\varphi \in \mathbf{S}(\mathbb{Q}_p^n)$

$$u(x,t) = \begin{cases} Z_r(x,t) * \varphi(x), & \text{if } t > 0, \\ \varphi(x), & \text{if } t = 0. \end{cases} \tag{13}$$

Lemma 3.10. *For $t \geq 0$, $u(x,t) \in \mathbf{S}(\mathbb{Q}_p^n)$.*

Proof. Since $Z_r(x,t) * \varphi(x) \in L^1(\mathbb{Q}_p^n) \cap L^2(\mathbb{Q}_p^n)$ for $t > 0$

$$\mathcal{F}_{x\to\xi}(u(x,t)) = e^{-t(f(||\xi||_p)-\lambda)}\widehat{\varphi}(\xi) \in \mathbf{S}(\mathbb{Q}_p^n),$$

since the Fourier transform is a homomorphism on $\mathbf{S}$, we conclude that $u(x,t) \in \mathbf{S}$ for $t > 0$. □

Remark 3.11. For $t \geq 0$, $\alpha > 0$

$$H(u(x,t)) = \mathcal{F}^{-1}_{\xi\to x}\left[(f(||\xi||_p) - \lambda)e^{-t(f(||\xi||_p)-\lambda)}\ \widehat{\varphi}(\xi)\right].$$

Lemma 3.12. *Let $\varphi \in \mathbf{S}(\mathbb{Q}_p^n)$ and $u(x,t)$, $t \geq 0$ defined by (13). Then $u(x,t)$ is continuously differentiable in time for $t \geq 0$, in the pointwise sense, and the derivative is given by*

$$\frac{\partial u}{\partial t}(x,t) = -H(u(x,t)).$$

Proof. The results follow from the following claim

Claim 3.13. (i) For $t_0 \geq 0$

$$\lim_{t\to t_0} \frac{e^{-t(f(||\xi||_p)-\lambda)} - e^{-t_0(f(||\xi||_p)-\lambda)}}{t-t_0} - (f(||\xi||_p) - \lambda)e^{-t_0(f(||\xi||_p)-\lambda)} = 0.$$

(ii) For $t, t_0 \in [0,T]$

$$\Big|\frac{e^{-t(f(||\xi||_p)-\lambda)} - e^{-t_0(f(||\xi||_p)-\lambda)}}{t-t_0} + (f(||\xi||_p) - \lambda)e^{-t_0(f(||\xi||_p)-\lambda)}\Big| \leq C(t)||\xi||_p^{\gamma_2}.$$

Indeed, note that

$$\lim_{t\to t_0} \frac{u(x,t)-u(x,t_0)}{t-t_0} + \mathcal{F}^{-1}\big((f(||\xi||_p) - \lambda)e^{-t_0(f(||\xi||_p)-\lambda)}\widehat{\varphi}(\xi)\big)$$
$$= \lim_{t\to t_0} \mathcal{F}^{-1}\left[\widehat{\varphi}(\xi)\Big(\frac{e^{-t(f(||\xi||_p)-\lambda)} - e^{-t_0(f(||\xi||_p)-\lambda)}}{t-t_0} + (f(||\xi||_p) - \lambda)e^{-t_0(f(||\xi||_p)-\lambda)}\Big)\right]$$

The first part follows from the Claim $(i)-(ii)$ by using the Dominated Convergence Theorem. The second part follows from Claim (i) by the Parseval-Steklov Theorem. □

Proof of the Claim. The Claim is a consequence of the following calculation. By applying the Mean Value Theorem twice we obtain

$$\begin{aligned}
&\frac{e^{-t(f(||\xi||_p)-\lambda)} - e^{-t_0(f(||\xi||_p)-\lambda)}}{t-t_0} + (f(||\xi||_p) - \lambda)e^{-t_0(f(||\xi||_p)-\lambda)} \\
&\quad = -(f(||\xi||_p) - \lambda)e^{-t'(f(||\xi||_p)-\lambda)} + (f(||\xi||_p) - \lambda)e^{-t_0(f(||\xi||_p)-\lambda)} \\
&\quad = (f(||\xi||_p) - \lambda)^2(t'-t_0)e^{-t''(f(||\xi||_p)-\lambda)},
\end{aligned}$$

where $t' = t'(||\xi||_p)$ is a point between t_0 and t and $t'' = t''(||\xi||_p)$ is a point between t_0 and t' (and thus between t_0 and t). □

Theorem 3.14. *Consider the following Cauchy problem*

$$\begin{cases} u \in C\big([0,\infty], \mathbf{S}(\mathbb{Q}_p^n)\big) \cap C^1\big([0,\infty], L^2(\mathbb{Q}_p^n)\big), \\ \dfrac{\partial u}{\partial t}(x,t) + \big(H\,u\big)(x,t) = 0, \quad x \in \mathbb{Q}_p^n, \quad t \in (0,T], \quad \alpha > 0, \\ u(x,0) = \varphi(x), \quad \varphi \in \mathbf{S}(\mathbb{Q}_p^n). \end{cases} \tag{14}$$

Then the function $u(x,t)$ defined in (13) is a solution to Problem (14).

Proof. This is a consequence of Lemmas 3.6 and 3.12. □

4. Heat equation on a ball

Consider the following pseudodifferential operator

$$f(\partial,\alpha,r)\varphi := \mathcal{F}^{-1}_{\xi\to x}\big(\max(||\xi||_p, p^r)^\alpha \mathcal{F}_{y\to\xi}\,\varphi(y)\big), \quad \varphi\in \mathbf{S}(\mathbb{Q}_p^n), \tag{15}$$

and the following equation

$$\frac{\partial u}{\partial t}(x,t) + \big(f(\partial,\alpha,r)\,u\big)(x,t) = \lambda u(x,t), \tag{16}$$

then the fundamental solution to this equation is given by

$$Z_r(x,t) = \int_{\mathbb{Q}_p^n} \chi(-x\cdot\xi)\, e^{-t\left(\max(||\xi||_p,p^r)^\alpha-\lambda\right)}\, d\xi, \quad \lambda = p^{r\alpha} \tag{17}$$

and

$$Z_r(x,t) = e^{\lambda t} Z(x,t) + p^n r - e^{\lambda t}(1-p^{-n})p^{nr}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}t^n\frac{p^{r\alpha n}}{1-p^{-\alpha n-1}},$$

where $Z(x,t)$ is given by (6) in the case $|f(\xi)|_p = ||\xi||_p$.

We note that our fundamental solution (17) agrees with the solution obtained by Kochubei in the case $\lambda = p^{r\alpha}$, see [4], Theorem 4.9.

References

[1] S. Albeverio, A. Yu. Khrennikov, and V. M. Shelkovich, *Theory of p-adic distributions: linear and nonlinear models*, Cambridge University Press, 2010.

[2] O. F. Casas-Sánchez, J. Galeano-Peñaloza, and J. J. Rodríguez-Vega, *Parabolic-type pseudodifferential equations with elliptic symbols in dimension 3 over $p-$adics*, $p-$Adic Numbers, Ultrametric Analysis and Applications **7(1)** (2015), 1–16.

[3] O. F. Casas-Sánchez and W. A. Zúñiga-Galindo, *p-adic elliptic quadratic forms, parabolic-type pseudodifferential equations with variable coefficients, and Markov processes*, p-Adic Numbers Ultrametric Analysis and Applications **6(1)** (2014), 1–20.

[4] Anatoly N. Kochubei, *Pseudo-differential equations and stochastics over non-archimedean fields*, Marcel Dekker, Inc., New York, 2001.

[5] J. J. Rodríguez-Vega and W. A. Zúñiga-Galindo, *Elliptic pseudodifferential equations and Sobolev spaces over p-adic fields*, Pacific J. Math. **246** (2010), 407–420.

[6] V. S. Vladimirov, I. V. Volovich, and E. I. Zelenov, *p-adic analysis and mathematical physics*, World Scientific, 1994.

[7] W. A. Zúñiga-Galindo, *Parabolic equations and Markov processes over p-adic fields*, Potential Anal. 28 **2** (2008), 185–200.