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New exact solutions for the combined sinh-cosh-Gordon equation

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Abstract. We present the general projective Riccati equations method to obtain exact solutions for the combined sinh-cosh-Gordon equation. The Painlevé property $v = e^u$ will be used to back up the method to derive travelling wave solutions of distinct physical structures. In addition we showed the behavior of the solutions with the graph of some of them. The method can also be applied to other nonlinear partial differential equation (NLPDE's) or systems in mathematical physics.

Key words and phrases. Nonlinear differential equation, travelling wave solution, Mathematica, projective Riccati equation method.

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Resumen. Presentamos el método proyectivo de ecuaciones de Riccati general, para obtener soluciones exactas para la ecuación sinh-cosh-Gordon combinada. La propiedad de Painvelé $v = e^u$ se usará para alcanzar el método, y derivar soluciones por ondas viajeras de distintas estructuras físicas. Ademas mostraremos el comportamiento de las soluciones con el gráfico de algunas de ellas. El método puede además ser aplicado a otras ecuaciones diferenciales parciales no lineales (NLPDEs) o sistemas en física matematica.

1. **Introduction**

In the study of nonlinear wave phenomena, the *travelling wave solutions* of partial differential equation (PDEs) have physical relevance. The knowledge of closed form solutions of nonlinear PDEs and ODEs facilitates the testing of numerical solvers, and aids in the stability analysis of solutions. It is well-known that searching the exact solutions for nonlinear partial differential equations is the great importance for many researches. A variety of powerful methods such that *tanh method*, *generalized tanh method*, *general projective Riccati equation method*, *Bäcklund transformation*, *Hirota bilinear forms*, and many other methods have been developed in this direction. Practically, there is not a unified method that can be used to handle all types of nonlinear problems. In this paper, we will use the general projective Riccati equation method, to construct exact solutions for the *combined sinh-cosh-Gordon equation*.

2. The general projective Riccati equations method

For a given nonlinear equation that does not explicitly involve independent variables

$$P(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \ldots) = 0, \tag{2.1}$$

when we look for its *travelling wave solutions*, the first step is to introduce the wave transformation, which have by definition the form

$$u(x,t) = v(\xi), \quad \xi = x + \lambda t, \tag{2.2}$$

where λ is a constant and change (1.1) to an ordinary differential equation (ODE) for the function $v(\xi)$

$$P(v, v', v'', \ldots) = 0. \tag{2.3}$$

The next crucial step is to introduce new variables $\sigma(\xi)$, $\tau(\xi)$ which are solutions of the system

$$\begin{cases} \sigma'(\xi) = e\sigma(\xi)\tau(\xi) \\ \tau'(\xi) = e\tau^2(\xi) - \mu\sigma(\xi) + r. \end{cases} \tag{2.4}$$

It is easy to see that the first integral of this system is given by

$$\tau^2 = -e[r - 2\mu\sigma(\xi) + \frac{\mu^2+\rho}{r}\sigma^2(\xi)], \tag{2.5}$$

where $\rho = \pm 1$. From this integral we obtain the following particular solutions:

1. Case I:
   If $r = \mu = 0$ then

$$\tau_1(\xi) = -\frac{1}{e\xi}, \quad \sigma_1(\xi) = \frac{C}{\xi}. \tag{2.6}$$

2. Case II:
   If $e = 1$ and $\rho = -1$

$$\begin{cases} \tau_2 = \dfrac{\sqrt{r}\tan(\sqrt{r}\xi)}{\mu\sec(\sqrt{r}\xi)+1} \quad (r>0) \\ \sigma_2 = \dfrac{r\sec(\sqrt{r}\xi)}{\mu\sec(\sqrt{r}\xi)+1} \quad (r>0). \end{cases} \tag{2.7}$$

3. Case III:
   If $e = -1$ and $\rho = -1$

$$\begin{cases} \tau_3 = \dfrac{\sqrt{r}\tanh(\sqrt{r}\xi)}{\mu \operatorname{sech}(\sqrt{r}\xi)+1} & (r>0) \\ \sigma_3 = \dfrac{r \operatorname{sech}(\sqrt{r}\xi)}{\mu \operatorname{sech}(\sqrt{r}\xi)+1} & (r>0). \end{cases} \tag{2.8}$$

4. Case IV:
   If $e = -1$ and $\rho = 1$

$$\begin{cases} \tau_4 = \dfrac{\sqrt{r}\coth(\sqrt{r}\xi)}{\mu \operatorname{csch}(\sqrt{r}\xi)+1} & (r>0) \\ \sigma_4 = \dfrac{r \operatorname{csch}(\sqrt{r}\xi)}{\mu \operatorname{csch}(\sqrt{r}\xi)+1} & (r>0). \end{cases} \tag{2.9}$$

5. Case V:
   If $e = 1$ and $\rho = 1$

$$\begin{cases} \tau_5 = \dfrac{-\sqrt{-r}\coth(\sqrt{-r}\xi)}{\mu \operatorname{csch}(\sqrt{-r}\xi)+1} & (r<0) \\ \sigma_5 = \dfrac{r \operatorname{csch}(\sqrt{-r}\xi)}{\mu \operatorname{csch}(\sqrt{-r}\xi)+1} & (r<0). \end{cases} \tag{2.10}$$

We seek a solution of (1.1) in the form

$$u(x,t) = v(\xi) = a_0 + \sum_{i=1}^{M} \sigma^{i-1}(\xi)(a_i\sigma(\xi) + b_i\tau(\xi)), \tag{2.11}$$

where $\sigma(\xi)$, $\tau(\xi)$ satisfy the system (2.4). The integer M can be determined by balancing the highest derivative term with nonlinear terms in (2.3), before the a_i and b_i can be computed. Substituting (2.11), along with (2.4) and (2.5) into (2.3) and collecting all terms with the same power in $\sigma^i(\xi)\tau^j(\xi)$, we get a polynomial in the variables $\sigma(\xi)$ and $\tau(\xi)$. Equaling the coefficients of this polynomial to zero, we obtain a system of algebraic equations, from which the constants μ, r, λ, a_i, b_i $(i = 1, 2, \ldots, M)$ are obtained explicitly. Using the solutions of the system (2.14) along with (2.11), we obtain the explicit solutions for (2.1) in the original variables.

3. The combined sinh-cosh-Gordon equation

This is the equation

$$u_{tt} - ku_{xx} + \alpha\sinh(u) + \beta\cosh(u) = 0, \tag{3.1}$$

where subscripts indicate partial derivatives, u is a real scalar function of the two independent variables x and t, while α and β are all model parameters and they are arbitrary, nonzero constants. This equation has been discussed in [8] by mean the variable separated ODE and the tanh methods. In this paper, we obtain new exact solution for many values of k, α and β. First introduce the transformations

$$\sinh u = \frac{V - V^{-1}}{2}, \quad \cosh u = \frac{V^1 + V^{-1}}{2}, \quad V = e^u, \tag{3.2}$$

after which we obtain the equation

$$2V(V_{tt} - kV_{xx}) + 2(kV_x^2 - V_t^2) + (\beta+\alpha)V^3 + (\beta-\alpha)V = 0. \tag{3.3}$$

The substitution $V = v(\xi) = v(x + \lambda\, t)$ in (2.15) gives us the equation

$$(\alpha+\beta)v^3 - (\alpha-\beta)v + 2(\lambda^2 - k)vv'' - 2(\lambda^2 - k)(v')^2 = 0. \tag{3.4}$$

According to the method described above, we seek solutions of (2.13) in the form

$$u(x,t) = v(\xi) = a_0 + a_1\sigma(\xi) + b_1\tau(\xi), \tag{3.5}$$

where $\sigma(\xi)$ and $\tau(\xi)$ satisfy the system (1.5). Substituting (2.17), along with (1.5) and (1.6) into (2.16) and collecting all terms with the same power in $\sigma^i(\xi)\tau^j(\xi)$ we get a polynomial in the variables $\sigma(\xi)$ and $\tau(\xi)$. Equaling the coefficients of these polynomial to zero and after simplifications (using $e = \pm 1$, $r \neq 0$) we get the following algebraic system:

- $4ea_1b_1(k-\lambda^2)(\mu^2+\rho) = 0$,
- $2e(k-\lambda^2)(\mu^2+\rho)(-ra_1^2 + eb_1^2(\mu^2+\rho)) = 0$,
- $a_0^3(\alpha+\beta) - a_0(\alpha-\beta+3erb_1^2(\alpha+\beta)) = 0$,
- $2\mu b_1^2(r(k-\lambda^2) + 3ea_0(\alpha+\beta)) + a_1(-\alpha+\beta+2era_0(k-\lambda^2) + 3a_0^2(\alpha+\beta) - 3erb_1^2(\alpha+\beta)) = 0$,
- $-a_0(6er\mu a_1(k-\lambda^2) - 3ra_1^2(\alpha+\beta) + 3eb_1^2(\alpha+\beta)(\mu^2+\rho)) + 2r(-b_1^2(k-\lambda^2)(3\mu^2+2\rho) + 3e\mu a_1 b_1^2(\alpha+\beta)) = 0$,
- $4ea_0a_1(k-\lambda^2)(\mu^2+\rho) - 2er\mu a_1^2(k-\lambda^2) + ra_1^3(\alpha+\beta) + 6\mu b_1^2(k-\lambda^2)(\mu^2+\rho) - 3ea_1b_1^2(\alpha+\beta)(\mu^2+\rho) = 0$,
- $b_1(\alpha-\beta-3a_0^2(\alpha+\beta) + erb_1^2(\alpha+\beta)) = 0$,
- $2b_1(a_0(-e\mu(k-\lambda^2) + 3a_1(\alpha+\beta)) + e(ra_1(k-\lambda^2) + \mu b_1^2(\alpha+\beta))) = 0$,
- $b_1(-4ea_0(k-\lambda^2)(\mu^2+\rho) + 4er\mu a_1(k-\lambda^2) - 3ra_1^2(\alpha+\beta) + eb_1^2(\alpha+\beta)(\mu^2+\rho)) = 0$.

Solving the previous system respect to the unknown variables r, a_0, a_1, b_1 we obtain the solutions

$$b_1 = 0,\ a_0 = \pm\frac{\alpha-\beta}{\sqrt{\alpha^2-\beta^2}},\ a_1 = \frac{2\mu e(k-\lambda^2)}{\alpha+\beta},\ r = -\frac{\sqrt{\alpha^2-\beta^2}}{e(k-\lambda^2)},$$

where $\mu^2 + \rho = 0$, $\rho = \pm 1$, $e = \pm 1$.

Therefore, according (2.17) and using (1.6) to (1.10), and after simplifications we obtain the following classification of some exact solutions for the equation (2.13): (in all cases $u(x,t) = v(\xi) = a_0 + a_1\sigma(\xi)$, $b_1 = 0$, $m = \sqrt{\alpha^2-\beta^2} \neq 0$, $n = (\lambda^2 - k) \neq 0$ and $\xi = x + \lambda t$):

For $e = 1$ and $\rho = 1$:

N°	r	μ	a_0	a_1	u
1	$-\frac{m}{n}$	$\imath$	$-\frac{\alpha-\beta}{m}$	$\frac{2\imath n}{\alpha+\beta}$	$\log(\frac{(\alpha-\beta)\left(\csc(\sqrt{-\frac{m}{n}}\,\xi)+1\right)}{m\left(\csc(\sqrt{-\frac{m}{n}}\,\xi)-1\right)})\,(\,n<0\,\text{and}\,\alpha>\beta\,)$
2	$\frac{m}{n}$	$-\imath$	$\frac{\alpha-\beta}{m}$	$\frac{2m}{\alpha+\beta}$	$\log(\frac{(\beta-\alpha)\left(\csc(\sqrt{\frac{m}{n}}\,\xi)+1\right)}{m\left(\csc(\sqrt{\frac{m}{n}}\,\xi)-1\right)})\,(\,n>0\,\text{and}\,\alpha<\beta\,)$

For $e=-1$ and $\rho=1$:

N°	r	μ	a_0	a_1	u
3	$\frac{m}{n}$	$\imath$	$-\frac{\alpha-\beta}{m}$	$\frac{2\imath n}{\alpha+\beta}$	$\log(\frac{(\alpha-\beta)\left(\csc(\sqrt{-\frac{m}{n}}\,\xi)-1\right)}{m\left(\csc(\sqrt{-\frac{m}{n}}\,\xi)+1\right)})\,(\,n<0\,\text{and}\,\alpha>\beta\,)$
4	$-\frac{m}{n}$	$\imath$	$\frac{\alpha-\beta}{m}$	$\frac{2\imath n}{\alpha+\beta}$	$\log(\frac{(\beta-\alpha)\left(\csc(\sqrt{\frac{m}{n}}\,\xi)-1\right)}{m\left(\csc(\sqrt{\frac{m}{n}}\,\xi)+1\right)})\,(\,n>0\,\text{and}\,\alpha<\beta\,)$

For $e=1$ and $\rho=-1$:

N°	r	μ	a_0	a_1	u
5	$-\frac{m}{n}$	-1	$-\frac{\alpha-\beta}{m}$	$\frac{2n}{\alpha+\beta}$	$\log(\frac{(\alpha-\beta)\,\cot^2(\frac{1}{2}\sqrt{-\frac{m}{n}}\,\xi)}{m})\,(\,n<0\,\text{and}\,\alpha>\beta\,)$
6	$\frac{m}{n}$	-1	$\frac{\alpha-\beta}{m}$	$\frac{2n}{\alpha+\beta}$	$\log(\frac{(\beta-\alpha)\,\cot^2(\frac{1}{2}\sqrt{\frac{m}{n}}\,\xi)}{m})\,(\,n>0\,\text{and}\,\alpha<\beta\,)$
7	$-\frac{m}{n}$	-1	$-\frac{\alpha-\beta}{m}$	$\frac{2n}{\alpha+\beta}$	$\log(\frac{(\beta-\alpha)\,\coth^2(\frac{1}{2}\sqrt{\frac{m}{n}}\,\xi)}{m})\,(\,n>0\,\text{and}\,\alpha<\beta\,)$
8	$-\frac{m}{n}$	1	$\frac{\alpha-\beta}{m}$	$\frac{2n}{\alpha+\beta}$	$\log(\frac{(\beta-\alpha)\,\tan^2(\frac{1}{2}\sqrt{\frac{m}{n}}\,\xi)}{m})\,(\,n>0\,\text{and}\,\alpha<\beta\,)$
9	$-\frac{m}{n}$	1	$\frac{\alpha-\beta}{m}$	$\frac{2n}{\alpha+\beta}$	$\log(\frac{(\alpha-\beta)\,\tanh^2(\frac{1}{2}\sqrt{-\frac{m}{n}}\,\xi)}{m})\,(\,n<0\,\text{and}\,\alpha>\beta\,)$
10	$\frac{m}{n}$	1	$-\frac{\alpha-\beta}{m}$	$\frac{2n}{\alpha+\beta}$	$\log(\frac{(\beta-\alpha)\,\tanh^2(\frac{1}{2}\sqrt{\frac{m}{n}}\,\xi)}{m})\,(\,n>0\,\text{and}\,\alpha<\beta\,)$
11	$-\frac{m}{n}$	-1	$-\frac{m}{\lvert\alpha+\beta\rvert}$	$\frac{2n}{\alpha+\beta}$	$\log(\frac{(\alpha-\beta)\,\coth^2(\frac{1}{2}\sqrt{-\frac{m}{n}}\,\xi)}{m})\,(\,n<0\,\text{and}\,\alpha>\beta\,)$
12	$-\frac{m}{n}$	1	$-\frac{\alpha-\beta}{m}$	$\frac{2n}{\alpha+\beta}$	$\log(\frac{(\alpha-\beta)\,\tan^2(\frac{1}{2}\sqrt{-\frac{m}{n}}\,\xi)}{m})\,(\,n<0\,\text{and}\,\alpha>\beta\,)$

The surface in Figure 1 corresponds to solution (1) with $\xi = x + \lambda t$, $k = 2$, $\lambda = 1$, $\alpha = 2$ and $\beta = 1$, for $x = -14$ to $x = 14$ and $t = -1$ to $t = 1$.

The surface in Figure 2 corresponds to solution (8) with $\xi = x + \lambda t$, $k = 1$, $\lambda = 2$, $\alpha = -192$ and $\beta = 1$, for $x = -115$ to $x = 115$ and $t = -1$ to $t = 1$.

The surface in Figure 3 corresponds to solution (7) with $\xi = x + \lambda t$, $k = 1$, $\lambda = 2$, $\alpha = -192$ and $\beta = 1$, for $x = -0{,}5$ to $x = 0{,}5$ and $t = -1$ to $t = 1$.

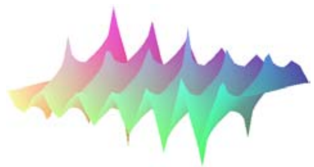

Figure 1

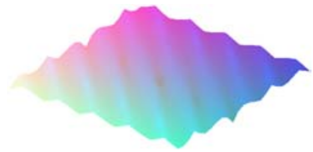

Figure 2

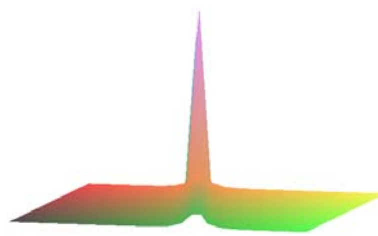

Figure 3

4. Conclusions

The projective Riccati equation method is a powerful method to search exact solutions for NLPDE´s . The projective method is more complicated than other methods, in the sense that demands more computer resources since the algebraic system may require a lot of time to be solved. In some cases, this system is so complicated that no computer algorithm may solve it, specially if the value of M is greater than four. In this paper, this method has been applied to the combined sinh-cosh-Gordon equation with $M = 1$.

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