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ON VERTICAL RIVULETS

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Abstract. A rivulet is a uni-directional liquid stream due to gravity. A steady stream of a viscous fluid flowing down a vertical plane has been considered in this paper. A conformal transformation and the Raylegh-Ritz method have been used to solve the Poisson's equations with Dirichlet-Neumann boundary conditions and in computing the cord-length 2L. The computation determines completely the cross-section size and shape of the rivulet in the form $\lambda = \lambda(\alpha)$ where α is the contact angle.

Key words and phrases. Laplace, Poisson and Navier-Stokes equations; Dirichlet-Newmann boundary conditions; conformal mappings; Schwartz-Christoffel transformations; Raliegh-Ritz method.

1991 Mathematics Subject Classification. Primary 76D05. Secondary 76D20, 76D25.

1. Introduction

A rivulet is a liquid stream due to gravity, along a solid surface, the liquid sharing a surface with a surrounding gaseous medium. A simple example of a rivulet is the stream of water seen on the windshield of a car after a rainfall. Rivulets occur in a wide variety of engineering applications. Drops rolling on a surface used for condensation may coalesce forming a rivulet. Rivulets arise in the melting and casting of metals. In processes of heat exchange and gas absorption rivulets play a major role, since they have a large surface area to cross-sectional area ratio. The theory of rivulets was introduced in [6] and most subsequent work (see, for instance, [2], [3], [7]) has been related to their mechanical stability. Here we are only concerned with parameters which describe the basic steady state.

Towell and Rothfeld [6] developed a theoretical analysis of the hydrodynamics of liquid rivulets with straight parallel contact lines. They obtained for thin liquid films (i.e. for a small contact angle of the surface of flow with a solid

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plane) a relation between the rivulet width and the flow rate, and this relation contains the plane inclination and the contact angle of the liquid on the plane. They also showed that surface tension ensures that the horizontal cross-section of the stream is a lenticular region D bounded partly by a free surface S in the form of a circular arc, and partly by the plane boundary B. Such a region is sketched in Figure 1, which also shows the appropriate system of dimensional coordinates, the non-dimensional (x, y) cartesian coordinates, based on the half width L of the rivulet.

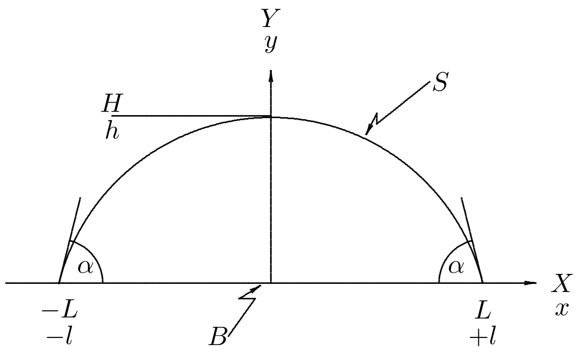

Figure 1. Horizontal section of a vertical rivulet.
Cartesian coordinates are denoted by upper and lower case symbols, respectively. The contact angle α determines the shape of the section in a non-dimensional plane.

The objective of this paper is the accurate computation of the cross-sectional shape and length-scale which determine the surface area. The external gas phase is assumed to have no effect. The point of view and the results differ from those of TOWELL and ROTHELD in that we study thick films of flow (flows of contact angle $\alpha \gg 20.0$), and in that the method we use to find the volume flux Q and the velocity distribution of the flow is quite different.

2. Flow description

(a) The flow is assumed to be a steady incompressible viscous flow in the direction of the negative Z–axis with a free surface. Thus, if U, V and W are the velocity components of the the liquid, then $U = V = 0$ and $W = W(X, Y)$.
(b) The flow is fully developed, that is, the stream is of uniform width and the derivatives with respect to Z are zero. This amounts to assume that the stream has traveled far enough as for the shape of the cross section no longer to change along the stream path.
(c) The flow has constant density ρ and viscosity μ.
(d) No shear stress is present at the gas-liquid interface.

Under these assumptions the equations of motion (Navier-Stokes equations) simplify to a balance of viscous and gravitational forces and reduce to the following

$$\frac{\partial P}{\partial X} = \frac{\partial P}{\partial Y} = 0 \tag{2.1}$$

and

$$\frac{\partial^2 W(X,Y)}{\partial X^2} + \frac{\partial^2 W(X,Y)}{\partial Y^2} - \frac{\partial P}{\partial Z} = -\frac{\rho g}{\mu}, \tag{2.2}$$

where P is pressure and g is gravity. Equation (2.2) describes how the upward viscous force is equal to the downward gravitational force.

Assuming the ambient gas has no significant effect, the appropiate boundary conditions for the liquid gas interface S are

(a) the shear stress condition:

$$(\partial W/\partial n)_S = 0, \tag{2.3}$$

where n is a coordinate normal to the free surface;

(b) the normal stress from the Laplace condition

$$P_S = \delta/R, \tag{2.4}$$

where R is the local radius of curvature of S and δ is the surface tension of the liquid;

(c) the no-slip condition on B

$$W(X,Y) = 0. \tag{2.5}$$

From equation (2.4) it follows that P_S is independent of Z, that is

$$\frac{\partial P_S}{\partial Z} = 0. \tag{2.6}$$

On the other hand, from equation (2.1) we get that P is a function of Z only. Thus, using (2.6) we get

$$\frac{\partial P}{\partial Z} = 0 \tag{2.7}$$

everywhere across the rivulet, and equation (2.2) becomes:

$$\frac{\partial^2 W}{\partial X^2} + \frac{\partial^2 W}{\partial Y^2} = -\frac{g}{\nu}, \tag{2.8}$$

where $\nu = \mu/\rho$ is the kinematic viscosity.

Under these conditions, equations (2.1) and (2.7) show that the pressure P is uniform throughout the entire rivulet, and (2.4) then shows that the curvature of the free surface S is also uniform. The domain of flow D is therefore a longitudinal slice of a circular cylinder, as shown in Figure 1.

In the complete dynamical problem, the most natural specification of the entire rivulet and configuration is the volume flux

$$Q = \int_D\int W(X,Y)\; dX\; dY, \tag{2.9}$$

where D is the domain of the liquid.

3. Dimensional analysis

It is convenient at this stage to scale the problem. The basic set of prescribed dimensional constants are ν/g and Q. If the dimensions of a quantity x are denoted by $[x]$, we therefore have

$$\left[\frac{\nu}{g}\right] = L^*\ T^*, \qquad [Q] = \frac{L^{*\ 3}}{T^*}$$

where L^* and T^* denote the primary dimension of length and time, so that

$$L^* = \left[Q\frac{\nu}{g}\right]^{1/4}, \qquad \frac{L^*}{T^*} = \left[Q\frac{g}{\nu}\right]^{1/2}.$$

Now, since the shape of the domain D is known to be a circular segment, we may expret the half-width L of the rivulet to be of the form

$$L = \lambda\left[Q\frac{\nu}{g}\right]^{1/4}, \tag{3.1}$$

where λ is a non-dimensional parameter to be determided and from which α follows. Mechanically speaking, an alternate and pernhaps more significant measure of α is the aspect ratio h (see Figure 1),

$$h = \lambda\frac{H}{L} \tag{3.2}$$

where H is the maximum thickness of the rivulet. The h, λ play the role of dimensionless H, L. The ratio h is given in terms of α by

$$h = \frac{1-\cos\alpha}{\sin\alpha}.$$

To determine $\lambda = \lambda(h) = \lambda(\alpha)$ we introduce non dimensional variables

$$x = \frac{X}{L}, \quad y = \frac{Y}{L} \tag{3.3}$$

as noted in Figure 1, and a non-dimensional velocity in the form

$$w(x,y) = \frac{W(X,Y)}{\lambda^2}\left[\frac{\nu}{Q\ g}\right]^{1/2}. \tag{3.4}$$

The factor λ^2 appears in (3.4) in order to yield a universal form to the governing equation. With this transformation the equation of vertical motion (2.8) becomes

$$\nabla^2 w \equiv \frac{\partial^2 w(x,y)}{\partial x^2} + \frac{\partial^2 w(x,y)}{\partial y^2} = -1\ . \tag{3.5}$$

Since the usual non-slip and non-stress boundary conditions are homogeneous:

$$w(x,y) = 0, \tag{3.6}$$

and

$$(\partial w(x,y)/\partial n)_s = 0 \ , \tag{3.7}$$

the unique solution of Poisson's problem (3.5) with Dirichlet-Neumann boundary conditions (3.6), (3.7) yields a velocity distribution $w(x,y)$ which only depends on α or h via their effect on the shape of the domain and not via any further prescribed dimensional scales. The definition of Q in (2.9) now provides the fundamental equation for $\lambda(h)$:

$$\lambda^4 \int_D \int w(x,y)\ dx\ dy = 1 \ . \tag{3.8}$$

When h is given, the shape of D is immediately known and the velocity distribution $w(x,y)$ then follows from the solution of the boundary-value problem. Thus, $\lambda(h)$ is deduced from (3.8) and the dimensional lengths and velocities are finally determined in terms of ν/g and Q by (3.1) and (3.4).

4. Numerical solution

4.1 Conformal Transformation. Most methods in two dimensional partial differential equations theory are designed for rectangular domains (which are natural or artificial coordinates). The domain of the present problem is complicated but the boundary which is in part a straight line and in part a circular arc, and these domains are ideally suited for conformal transformation onto rectangular domains. A conformal transformation $\zeta = f(z)$ can be employed to map a circular segment in the z-plane onto a square in the ζ–plane.

Using the Schwarz-Christoffel transformation (see Figure 2),

$$\zeta = 1 + \ \mathrm{i} \ - \frac{2\sqrt{2\pi}\ \mathrm{i}}{\Gamma^2(1/4)} \int \frac{(1+z/1-z)^{\pi/\alpha}}{\sqrt{\varphi(\varphi^2-1)}}\, d\varphi, \tag{4.1}$$

where Γ is the gama function.

Using (4.1), the Poisson equation (3.5), with Dirichlet-Neumann boundary conditions (3.6), (3.7), will take the form

$$\nabla^2 x(\xi,\eta) = -G(\xi,\eta), \tag{4.2}$$

with

$$w(1,\eta) = w(\xi,1) = 0, \tag{4.3a}$$

$$\frac{\partial w}{\partial \xi}(0,\eta) = 0 \tag{4.3b}$$

and

$$\frac{\partial w}{\partial \eta}(\xi,0) = 0, \tag{4.3c}$$

and where

$$G(\xi,\eta) = \left|\frac{d\zeta}{dz}\right|^{-2}$$

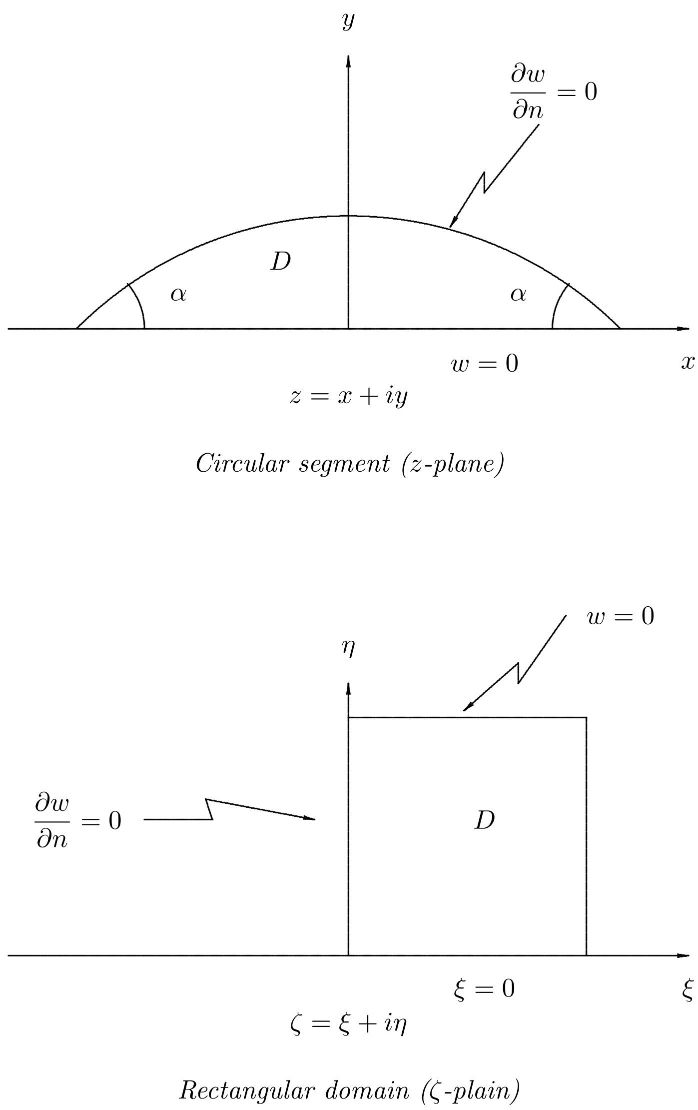

Figure 2. The conformal transformation of the circular segment onto the rectangular domain.

is the transformation modulus, the ratio of lenght in the ζ–plane to lenght in the z–plane, which is a position function.

4.2 Optimization method. Of all the numerical methods available we choose the Rayleigh-Ritz optimization method as the most suitable, mainly because our primary objective is to calculate λ from the flux integral

$$\int_D \int w(x, y)\, dxdy, \tag{4.4}$$

and this integral is directly obtained from this method. Thus, we minimize the integral (see for instance [4])

$$J = \int_0^1 \int_0^1 \left\{ \left(\nabla w(\xi,\eta)\right)^2 + 2w(\xi,\eta)G(\xi,\eta) \right\} \, d\xi d\eta \tag{4.5}$$

over the class of functions w which satisfy the boundary conditions (4.3). As a minimizing sequence for J we try a sequence $\{w_n(\xi,\eta)\}$ of functions of the form

$$w_n(\xi,\eta) = \sum_{j=1}^{n} A_{nj} f_j(\xi,\eta) \tag{4.6}$$

where

$$f_j(\xi,\eta) = \sum_{i=1}^{j} \beta_i \; T_i(\xi) T_i(\eta) \tag{4.7}$$

and

$$T_i(x) = 1 - x^{2i}, \qquad i \geq 1. \tag{4.8}$$

The $\beta_{i,s}$ in (4.7) are chosen in such a way that

$$\int_0^1 \int_0^1 \nabla f_j(\xi,\eta) \; \nabla f_k(\xi,\eta) d\xi \; d\eta = \begin{cases} 0, & \text{if } j \neq k \\ 1, & \text{if } j = k \end{cases} \tag{4.9}$$

and the A_{nj}, in (4.6), so that $w_n(\xi,n)$ satisfies the boundary conditions (4.3).

For our minimization we choose

$$w_4(\xi,\eta) = \sum_{j=1}^{4} A_j f_j(\xi,\eta). \tag{4.10}$$

The minimization condition is

$$\frac{\partial J}{\partial A_j} = 0, \qquad j = 1,2,3,4.$$

We get

$$A_j = -\int_0^1 \int_0^1 f_j(\xi,\eta) G(\xi,\eta) \, d\xi \; d\eta. \tag{4.11}$$

Using (4.10) in (4.5), we can see that the approximate value of λ is 1.29099 for the case $\alpha = 90.0$. In comparing this value of λ with the exact value obtained in [1], we find that they are in total agreement to four significant figures.

5. Conclusions

The results for λ coresponding to successive values of h are shown in Table 1 and Figure 3. From Figure 3 we conclude that for values of h in the interval $0 \leq h \leq 1$ the Rayleigh-Ritz technique yields a very accurate evaluation of the function $\lambda(h)$. In this range, therefore, the objective of this paper has been achieved. For larger values of h, however, the position is less clear. It was originally hoped that a further contribution to this aspect of the problem could be made by finding an analytic expansion for the asymptotic behaviour of $\lambda(h)$ as $h \to \infty$ (physically speaking this limit would be approached by a rivulet of mercury on a glass). But although this boundary value problem for Poisson's equation can be very simply stated, its solutions is much more awkward than expected, and further progress has not yet been made.

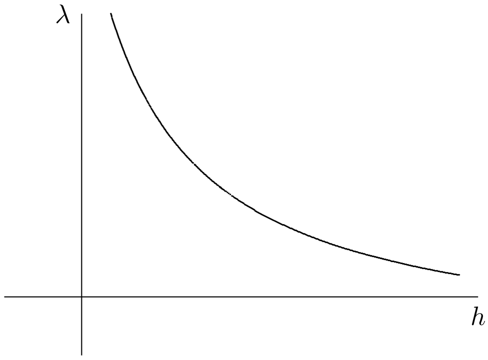

Figure 3. The results for λ coresponding to successive values of h.

h	α	flux	λ
0.2	22.62	0.00243	4.50546
0.4	43.60	0.01912	2.68923
0.6	61.93	0.06278	1.99775
0.8	77.32	0.14317	1.62570
1.0	90.0	0.35995	1.29099
1.2	100.39	0.44509	1.22430
1.4	108.92	0.68965	1.09734
1.6	115.99	1.02124	0.99476
1.8	121.89	1.46432	0.90906
2.0	126.87	2.04684	0.83604
2.2	131.11	2.79990	0.77306
2.4	134.76	3.75759	0.71824

Table 1. Scale Parameters of Vertical Flow

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