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generalization of rakotch's fixed point theorem*

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Abstract

In this paper we get some generalizations of Rakotch's results [10] using the notion of ω-distance on a metric space.

Keywords: fixed point, completeness, ω-Rakotch contraction.

Resumen

En este trabajo usando la nocion de $\omega - distancia$ sobre un espacio mtrico obtenemos alugunas generalizaciones del teorema de Rakotch [10].

Palabras clave: punto fijo, completitud, contracción ω-Rakotch.

Mathematics Subject Classification: 47H10, 54E50

1 Introduction

In 1996, O. Kada, T. Suzuki & W. Takahashi [6] introduced the concept of ω-distance on a metric space, gave some examples, properties of ω-distance and they improved Caristi's fixed point [1], Ekeland's ε-variational principle [5] and the non-convex minimization theorem according to W. Takahashi [17]. Finally, by the use of the concept of ω-distance they proved a fixed point theorem in a complete metric space. This theorem generalized the fixed theorems of Subrahmanyan [14], Kannan [7] and Ciric [3]. In the same year T. Suzuki & W. Takahashi [15] gave another property of the ω-distance and using this notion they proved a fixed point theorem for set-valued mappings on complete metric spaces

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which are related with Nadler's fixed point theorem [9] and Edelstein theorem [4]. Moreover, they gave a characterization of completeness metric spaces. In 1997, T. Suzuki [16], proved several fixed point theorems which are generalizations of the Banach contraction principle and Kannan's fixed point theorems, and moreover, they discuss a characterization of metric completeness. In this paper we prove some fixed point theorems which are generalizations of Rakotch's theorem.

2 Preliminaries

Throughout this paper we denote by $\mathbb{N}$ the set of positive integers, by $\mathbb{R}$ the set of real numbers and $\mathbb{R}^+ = [0, +\infty]$.

Definition 2.1. *Let (M, d) be a metric space. A function $p : M \times M \to [0, +\infty]$ is called a ω-distance on M if the following conditions are satisfied:*

w_1.- *$p(x, z) \leq p(x, y) + p(y, z)$ for any $x, y, z \in M$.*

w_2.- *For any $x \in M$, $p(x, \cdot) : M \to [0, +\infty]$ is lower semi continuous.*

w_3.- *For any $\varepsilon > 0$ exists $\delta = \delta(\varepsilon) > 0$ such that,*
$p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ imply $d(x, y) \leq \varepsilon$.

The metric d is a ω-distance on M. Some other examples of ω-distances are given in [6] and [15]. The following results are crucial in the proof of our theorems. The next lemma was proved in [6].

Lemma 2.2. *Let (M, d) be a metric space and let p be a ω-distance on M. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0, +\infty)$ converging to 0, and let $x, y, z \in M$. Then the following hold:*

a.- If $p(x_n, y) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for any $n \in \mathbb{N}$ then $y = z$. In particular, if $p(x, y) = 0$ and $p(x, z) = 0$ then $y = z$.

b.- If $p(x_n, y_n) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for any $n \in \mathbb{N}$ then $\{y_n\}$ converges to z.

c.- If $p(x_n, x_m) \leq \alpha_n$ for any $n, m \in \mathbb{N}$ with $m > n$ then $\{x_n\}$ is a Cauchy sequence.

d.- If $p(y, x_n) \leq \alpha_n$ for any $n \in \mathbb{N}$ then $\{x_n\}$ is a Cauchy sequence. ■

Definition 2.3. *Let (M, d) be a metric space. A finite sequence $\{x_0, x_1, \ldots, x_n\}$ of points of M is called an ε-chain joining x_0 and x_n if $d(x_{i-1}, x_i) < \varepsilon$ for each $\varepsilon > 0$, $i = 1, 2, \ldots, n$.*

Definition 2.4. *A metric space (M, d) is said to be ε-chainable if for each pair (x, y) of its points there exists an ε-chain joining x and y.*

Every connected metric space is ε-chainable but the converse in not always true. However, for compact spaces both are equivalent. The following result was proved in [15].

Lemma 2.5. *Let $\varepsilon \in (0, +\infty)$ and let (M, d) be an ε-chainable metric space. Then the function $p : M \times M \to [0, +\infty)$ defined by*

$$p(x, y) = \inf\{\sum_{i=1}^{n} d(x_{i-1}, x_i)/\{x_0, x_1, \dots, x_n\} \text{is an } \varepsilon\text{-chain joining } x \text{ and } y\}$$

is a ω-distance on M. ∎

We extend the class of functions introduced by Rakotch [10] in the following definition.

Definition 2.6. *Let (M, d) be a metric space and let p be a ω-distance on M. We denote by $\mathcal{F}$ the family of functions $\lambda(x, y)$ satisfying the following conditions:*

a.- $\lambda(x, y) = \lambda(p(x, y))$, i.e., λ is dependent on the ω-distance p on M.

b.- $0 \leq \lambda(p) < 1$ for every $p > 0$.

c.- $\lambda(p)$ is monotonically decreasing function of p.

Now we introduce the following definition.

Definition 2.7. *Let (M, d) be a metric space and let p be a ω-distance on M. A mapping $T : M \to M$ is called a ω-Rakotch contraction if there exists a function $\lambda(x, y) \in \mathcal{F}$ such that*

$$p(Tx, Ty) \leq \lambda(x, y)p(x, y)$$

for all $x, y \in M$.

Remarks:

a.- If $p = d$ then T is called a Rakotch contraction.

b.- If $\lambda(x, y) = k,\ 0 \leq k < 1$ then we get for all $x, y \in M$

$$p(Tx, Ty) \leq kp(x, y).$$

T is called an ω-contraction [6] and [15], and if $p = d$ then T is a Banach contraction.

c.- If $\lambda(x, y) = k\ 0 \leq k < 1$ then for all $x \neq y$ implies

$$p(Tx, Ty) < p(x, y)$$

and we call T a ω-contractive mapping. It is clear that if $p = d$ then $x \neq y$ implies $d(Tx, Ty) < d(x, y)$ and T is called a contractive mapping.

3 Fixed point theorems

The next result generalizes Rakotch's theorem [10].

Theorem 3.1. *Let (M, d) be a complete metric space and let p be an ω-distance on M. Let $T : M \to M$ be an ω-Rakotch contraction. Then there exists a unique $z \in M$ such that $Tz = z$. Further, the z satisfies $p(z, z) = 0$*

PROOF: Since T is a ω-Rakotch contraction there exists a mapping $\lambda(x, y) \in \mathcal{F}$ such that

$$p(Tx, Ty) \leq \lambda(x, y)p(x, y)$$

for all $x, y \in M$.
Let $x_0 \in M$ and define $x_n = T^n x_0, n \in \mathbb{N}$

$$p(x_n, x_{n+1}) = p(Tx_{n-1}, Tx_n) \leq \lambda(x_{n-1}, x_n)p(x_{n-1}, x_n) \leq \ldots \leq$$

$$\leq \prod_{k=0}^{n-1} \lambda(p(x_k, x_{k+1}))p(x_0, Tx_0)$$

and

$$p(x_{n+1}, x_n) = p(Tx_n, Tx_{n-1}) \leq \lambda(x_n, x_{n-1})p(x_n, x_{n-1}) \leq \ldots \leq$$

$$\leq \prod_{k=0}^{n-1} \lambda(p(x_k, x_{k+1}))p(x_0, Tx_0).$$

It follows that

$$p(x_n, x_{n+1}) < p(x_0, Tx_0)$$

and

$$p(x_{n+1}, x_n) < p(Tx_0, x_0)$$

for all $n = 1, 2, \ldots$.
Now we prove that

$$p(x_0, x_n) \leq C$$

for some $C > 0$ and $n = 1, 2, 3, \ldots$
In fact,

$$p(x_1, x_{n+1}) \leq \lambda(p(x_0, x_n))p(x_0, x_n)$$

and by the triangle inequality

$$p(x_0, x_n)\lambda p(x_0, x_1) + p(x_1, x_{n+1}) + p(x_{n+1}, x_n)$$

and

$$p(x_0, x_n) \leq p(x_0, Tx_0) + \lambda(p(x_0, x_n))p(x_0, x_n) + p(x_{n+1}, x_n)$$

hence

$$p(x_0, x_n) < \frac{p(x_0, Tx_0) + p(Tx_0, x_0)}{1 - \lambda(p(x_0, Tx_n))}.$$

Now if $p(x_0, Tx_n) \geq \alpha_0$ for a given $\alpha_0 > 0$, then by the monotonicity of $\lambda(p)$ it follows that

$$\lambda(p(x_0, Tx_n)) \leq \lambda(\alpha_0)$$

and therefore

$$p(x_0, x_n) < \frac{p(x_0, Tx_0) + p(Tx_0, x_0)}{1 - \lambda(\alpha_0)} = C.$$

On the other hand if $p(x_k, x_{k+1}) \geq \varepsilon_0, k = 0, 1, \dots, n-1$ for any $\varepsilon_0 > 0$ then by monotonicity of λ it follows that

$$\lambda(p(x_k, x_{k+1})) \leq \lambda(\varepsilon_0)$$

and hence

$$p(x_n, x_{n+1}) \leq [\lambda(\varepsilon_0)]^n p(x_0, Tx_0).$$

But $0 \leq \lambda(\varepsilon_0) < 1$ by lemma 2.1 we have $\lim_{n\to\infty} p(x_n, x_{n+1}) = 0$.
We shall show that $\{x_n\}$ is a Cauchy sequence in (M, d). For $m > 0$, $p(x_n, x_{n+m}) \leq \prod_{k=0}^{n-1} \lambda[p(x_k, x_{k+m})]p(x_0, Tx_0)$.
If $p(x_k, x_{k+m}) \geq \varepsilon_0$ for any given $\varepsilon_0 > 0$ and $k = 0, 1, \dots, n-1$ then

$$p(x_n, x_{n+m}) \leq [\lambda(\varepsilon_0]^n)p(x_0, Tx_0) \to 0$$

as $n \to \infty$ and by lemma 2.1 we have that $\{x_n\}$ is a Cauchy sequence. Since (M, d) is complete, $\{x_n\}$ converges to some $z \in M$. Since $x_m \to z$ and $p(x_n, .)$ is lower semicontinuous,

$$p(x_n, z) \leq \lim_{m\to\infty} p(x_n, x_m) \leq \lambda^n(\varepsilon_0)p(x_0, Tx_0)$$

so $\lim_{n\to\infty} p(x_n, z) = 0$.
On the other hand,

$$p(x_n, Tz) = p(Tx_{n-1}, Tz) \leq \lambda(p(x_{n-1,z}))p(x_{n-1}, z) < p(x_{n-1}, z)$$

so $\lim_{n\to\infty} p(x_n, Tz) = 0$ and by lemma 2.2 we have $Tz = z$.
Now,

$$p(z, z) = p(Tz, Tz) \leq \lambda(z, z)p(z, z) < p(z, z)$$

so $p(z, z) = 0$.
If $y = Ty$ then

$$p(z, y) = p(Tz, Ty) \leq \lambda(z, y)p(z, y) < p(z, y)$$

and $p(z, y) = 0$ so by lemma 2.1 we have $z = y$. ■

Remarks:

a.- In case $p = d$, (M, d) is a complete metric space and $T : M \to M$ is a Rakotch contraction then we get the Rakotch's theorem [10].

b.- If (M, d) a complete metric space and $\lambda(x, y) = k$, $0 \leq k < 1$ we get a generalization of the Banach Contraction Principle [8] and [15].

Theorem 3.2. *Let (M,d) be a complete metric space, let p be a ω-distance on M and $T : M \to M$ is a mapping such that for some integer $m \in \mathbb{N}$ T^m is an ω-Rakotch contraction. Then T has a unique fixed point, i.e., there exists $z \in M$ such that $Tz = z$ and moreover holds $p(z,z) = 0$.*

Proof: Since for some $m \in \mathbb{N}$ T^m is a ω-Rakotch contraction, then there exists a function $\lambda(x,y) \in \mathcal{F}$ such that

$$p(T^m x, T^m y) \le \lambda(x,y) p(x,y)$$

for every $x, y \in M$.
Hence by theorem 3.1 there exists a unique $z \in M$ such that $z = T^m z$ for $m \in \mathbb{N}$ and $Tz = T(T^m z) = T^m(Tz)$ it follows that $z = Tz$.
Let us remark that in case $\lambda(x,y) = k,\ 0 \le k < 1,\ p = d$ and (M,d) complete metric space we get the Chu-Diaz's Theorem [2].
Now we get another generalization of Rakotch's Theorem [10] using Maia's Theorem [11].
■

Theorem 3.3. *Let M be a non empty set, d, and ρ two metrics on M, p and q their respective ω-distances on M and $T : M \to M$ a mapping. Suppose that:*

a.- $p(x,y) \le q(x,y)$ for all $x, y \in M$.

b.- (M,d) is a complete metric space.

c.- $T : (M,\rho) \to (M,\rho)$ is a ω-Rakotch contraction, i.e., there exists $\lambda(x,y) \in \mathcal{F}$ such that

$$q(Tx,Ty) \le \lambda(x,y) q(x,y)$$

for every $x, y \in M$.
Then there exists $z \in M$ such that $Tz = z$ and moreover $p(z,z) = 0$.

Proof: Let $x_0 \in M$ and define $x_n = T^n x_0, n \in \mathbb{N}$. from (c), $\{x_n\}$ is a Cauchy sequence in (M,ρ). By (a) and lemma 2.2, $\{x_n\}$ is a Cauchy sequence in (M,d) and by (b) it converges. The rest of the proof is similar to Theorem 3.1. ■

Now we generalize a result given by Singh-Deb-Gardner in [13].

Theorem 3.4. *Let $\varepsilon \in (0,+\infty)$ be and let (M,d) be a complete ε-chainable metric space. If T is a mapping from M into itself satisfying, $0 < d(x,y) < \varepsilon$ implies $d(Tx,Ty) \le \lambda(x,y) d(x,y)$ for all $x, y \in M$ and $\lambda(x,y) \in \mathcal{F}$. Then T has a unique $z \in M$ such that $z = Tz$.*

Proof: Since (M,d) is ε-chainable for every $x, y \in M$ we define the function $p : M \times M \to [0,+\infty)$ as follows:

$$p(x,y) = \inf\{\sum_{i=1}^{n} d(x_{i-1},x_i) / \{x_0,\ldots,x_n\} \text{ is an } \varepsilon\text{-chain joining } x \text{ and } y\}.$$

From lemma 2.2, p is a ω-distance on M satisfying $d(x,y) \leq p(x,y)$. Given $x,y \in M$ and any ε-chain $\{x_0, \ldots, x_n\}$ with $x_0 = x$ and $x_n = y$ we have for $i = 1, \ldots, n$,

$$d(Tx_{i-1}, Tx_i) \leq \lambda[d(x_{i-1}, x_i)]d(x_{i-1}, x_i) < \lambda(\varepsilon)\varepsilon < \varepsilon$$

. Hence $Tx_0, \ldots, Tx_n$ is an ε-chain joining Tx and Ty, and

$$p(Tx, Ty) \leq \sum_{i=1}^{n} d(Tx_{i-1}, Tx_i) \leq \sum_{i=1}^{n} \lambda(d(x_{i-1}, x_i)d(x_{i-1}, x_i))$$

. Since $\{x_0, \ldots, x_n\}$ is an arbitrary ε-chain we have

$$p(Tx, Ty) \leq \lambda(x,y)p(x,y),$$

hence by theorem 3.1, T has a unique fixed point $z \in M,\ z = Tz$.

Remark: If $\lambda(x,y) = k,\ 0 \leq k < 1$ and $p = d$ we get the result due to Edelstein [4].

Finally, the following result generalizes Singh's theorem [12].

Theorem 3.5. *Let $\varepsilon \in (0, +\infty)$ be and let (M,d) a complete ε-chainable metric space. If T is a mapping from M into itself satisfying the condition,*

$$d(x,y) < \varepsilon \textit{ implies } d(T^m x, T^m y) \leq \lambda(x,y)d(x,y)$$

for every $x,y \in M$, for $m \in M$ and $\lambda(x,y) \in \mathcal{F}$, then T has a unique fixed point in M.

Proof: As in theorem 3.4 we define p as follows:

$$p(x,y) = \inf\{\sum_{i=1}^{n} d(x_{i-1}, x_i)/\{x_0, \ldots, x_n\}\text{is a } \varepsilon\text{-chain joining } x \text{ and } y\}.$$

By lemma 2.2, p is a ω-distance on M satisfying $d(x,y) \leq p(x,y)$. As in theorem 3.3 we have that T^m satisfies the condition

$$p(T^m x, T^m y) \leq \lambda(x,y)p(x,y)$$

for all $x, y \in M,\ m \in \mathbb{N}$ and therefore by theorem 3.4 we conclude that T^m has a unique $z \in M$ such that $z = T^m z$. It follows that T has a unique fixed point z and moreover $p(z,z) = 0$. ■

Finally, using the ideas of M.Telci-K.Tas [18] we obtain a generalization of Rakotch's theorem on noncomplete metric spaces.

Theorem 3.6. *Let (M,d) be a noncomplete metric space and let p be a ω-distance on M. Let $T : M \to M$ be a ω-Rakotch contraction and suppose that there exists a point $u \in M$ such that*

$$\theta(u) = inf\{\theta(x)/x \in M\}$$

where $\theta(x) = p(x, Tx)$ for all $x \in M$. Then u is a fixed point of T.

PROOF: Suppose that $u \neq T(u)$, since otherwise u would be a fixed point of T. Now since T is a ω-Rakotch contraction there exists $\lambda(x,y) \in \mathcal{F}$ such that

$$p(Tx,Ty) \leq \lambda(p(x,y))p(x,y)$$

for all $x, y \in M$ and so

$$\theta(Tu) = p(Tu, T^2u) \leq \lambda(p(u,Tu))p(u,Tu) < p(u,Tu) = \theta(u)$$

which is a contradiction. ∎

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