Revista de Matemática: Teoría y Aplicaciones 3(1): 21–26 (1996)

THE MULTIPLICATIVE PRODUCTS BETWEEN THE DISTRIBUTION $(P \pm i0)^\lambda$ AND THE OPERATORS $L^r\{\delta\}$ AND $K^r\{\delta\}$

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Abstract

In this note we give a sense to some multiplicative products of distributions:

i) $(P \pm i0)^\lambda . L^r\{\delta\}$

ii) $(P \pm i0)^\lambda . K^r\{\delta\}$

where $(P \pm i0)^\lambda$ is the distribution defined by the formula (2), P is the quadratic form defined by the formula (1), L^r is the ultrahyperbolic operator defined by (5) and K^r is the Klein-Gordon operator iterated r-times defined by the formula (15).

Resumen

En esta nota damos sentido a algunos productos multiplicativos de distribuciones:

i) $(P \pm i0)^\lambda . L^r\{\delta\}$

ii) $(P \pm i0)^\lambda . K^r\{\delta\}$

donde $(P \pm i0)^\lambda$ es la distribución definida en la fórmula (2), P es la forma cuadrática definida en la fórmula (1), L^r es el operador ultrahiperbólico definido por (5) y K^r es el operador de Klein-Gordon iterado r veces y definido por la fórmula (15).

1 Introduction

Let $x = (x_1, \ldots, x_n)$ be a point of the n-dimensional Euclidean space R^n. Consider a non-degenerate quadratic form in n variables of the form,

$$P = P(x) = x_1^2 + \cdots + x_\mu^2 - x_{\mu+1}^2 - \cdots - x_{\mu+\nu}^2 \tag{1}$$

where $n = \mu + \nu$.

The distributions $(P \pm i0)^\lambda$ are defined by

$$(P \pm i0)^\lambda = \lim_{\varepsilon \to 0} (P \pm i\varepsilon |x|^2)^\lambda \quad ([3],\ \text{page } 275) \tag{2}$$

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where $\varepsilon > 0$,

$$|x|^2 = x_1^2 + \cdots + x_n^2 \,, \tag{3}$$

and $\lambda \in \mathbf{C}$.

A. González Domínguez proves the validity of the following formula (cf.[4], 1980, page 192, formula 9.2):

$$(P \pm i0)^{-k} \,.\, L^r\{\delta\} = a(k,n,r)\, L^{k+r}\{\delta\} \,, \tag{4}$$

$k = 0,1,2,\ldots,\ r = 0,1,2,\ldots$, and $k \leq n/2$.

Here L^r is the ultrahyperbolic operator iterated r-times:

$$L^r = \left\{ \frac{\partial^2}{\partial X_1^2} + \cdots + \frac{\partial^2}{\partial X_{\mu+1}^2} - \frac{\partial^2}{\partial X_{\mu+1}^2} - \cdots - \frac{\partial^2}{\partial X_{\mu+\mu}^2} \right\}^r \tag{5}$$

$$\text{and} \quad a(k,n,r) = 4^k (r+1)\ldots(r+k)\left(\frac{n}{2}+r\right)\ldots\left(\frac{n}{2}+r+k-1\right) \tag{6}$$

The following formulae were established ([1], 1984),

$$(P \pm i0)^{-k} \,.\, L\{\delta\} = 0 \text{ if } \frac{n}{2} \leq k \text{ for } n \text{ even}\,; \tag{7}$$

$$(P \pm i0)^{-k} \,.\, L^r\{\delta\} = a(k,n,r)\, L^{r+k}\{\delta\} \text{ if } \frac{n}{2} \leq k \text{ for } n \text{ odd}\,; \tag{8}$$

$$\begin{aligned} (P \pm i0)^{k} \,.\, L^r\{\delta\} &= b(k,n,r)\, L^{r-k}\{\delta\} \ \text{ if } r \geq k, \\ & \quad k = 0,1,\ldots;\ r = 0,1,\ldots\,. \end{aligned} \tag{9}$$

$$\text{where,} \quad b(k,n,r) = \frac{4^k\, r!\, \Gamma\left(\frac{n}{2}+r\right)}{(r-k)!\, \Gamma\left(\frac{n}{2}+r-k\right)} \,; \tag{10}$$

$a(k,n,r)$ is defined by the formula (6) and

$$\Gamma(\alpha) = \int_0^\infty e^{-x}\, x^{\alpha-1}\, dx I,1,11 \tag{11}$$

Also, we know, (cf. [2], 1988) that

$$(P \pm i0)^{\frac{n}{2}+k-1} \,.\, L^k\{\delta\} = 0 \tag{12}$$

([2], page 5, formula (17)) for $n \neq 2$ with $r = 0,1,2,\ldots$.

In this paper, we intend to give a sense to certain multiplicative products:

$$(P \pm i0)^{\lambda} \,.\, L^r\{\delta\} \tag{13}$$

and

$$(P \pm i0)^{\lambda} \,.\, K^r\{\delta\} \tag{14}$$

where λ is a complex number,

$$K^r\{\delta\} = (L - m^2)^r\{\delta\} \tag{15}$$

(Klein-Gordon operator iterated r-times) and L^k is defined by the formula (5).

2 The multiplicative product $(P\pm i0)^{\lambda} \,.\, L^r\{\delta\}$ and $(P\pm i0)^{\lambda} \,.\, K^r\{\delta\}$

THEOREM 1.

Let λ be a complex number such that $\lambda \neq \pm k,\ k = 0, 1, 2, \ldots$; then the following formula is valid:

$$(P \pm i0)^{\lambda} \,.\, L^r\{\delta\} = 0 \tag{16}$$

where the distribution $(P \pm i0)^{\lambda}$ is defined by the formula (2) and L^k by the formula (5).
Proof: From [1], page 7, formula (17), we have,

$$H_{\alpha} \,.\, H_{\beta} = \mathcal{C}_{\alpha,\beta,n}\; H_{\alpha+\beta-n} \tag{17}$$

where

$$H_{\gamma} = H_{\gamma}(P \pm i0, n) = \frac{(P \pm i0)^{\frac{\gamma-n}{2}}}{D_n(\gamma)} \tag{18}$$

$$D_n(\gamma) = \frac{e^{\gamma\pi i/2}\, e^{\pm q\pi i/2}\, \Gamma\left(\frac{n-\gamma}{2}\right)}{2^{\gamma}\, \pi^{n/2}\, \Gamma(\gamma/2)} \tag{19}$$

$$\mathcal{C}(\alpha, \beta, n) = \frac{\Gamma\left(\frac{\alpha+\beta-n}{2}\right)\Gamma\left(\frac{n-\alpha}{2}\right)\Gamma\left(\frac{n-\beta}{2}\right) e^{(n\pm q)\frac{\pi i}{2}}}{2^n \Pi^{n/2}\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\beta}{2}\right)\Gamma\left(\frac{n-\alpha+n-\beta}{2}\right)} \tag{20}$$

From (17), (20) and taking into account the formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin\ z\pi} \tag{21}$$

we have

$$\begin{aligned}(P \pm i0)^{\frac{\alpha-n}{2}} \,.\, H_{\beta} \quad &= \quad \Gamma\left(\frac{n-\beta}{2}\right)\Gamma\left(1-\frac{\beta}{2}\right)\pi^{-1}\sin\frac{\beta}{2}\pi \\ &\quad .\, e^{\pm q\pi i/2} e^{\beta\pi i/2} (P \pm i0)^{\frac{\alpha+\beta-n-n}{2}}\end{aligned} \tag{22}$$

By making $\beta = -2r$ and $\lambda = \frac{\alpha-n}{2}$ in (22) we have,

$$\begin{aligned}(P \pm i0)^{\lambda} \,.\, H_{-2r} \quad &= \quad \pi^{-1}\Gamma\left(\frac{n}{2}+k\right)\Gamma(1+k)e^{\pm q\pi i/2}\,. \\ &\quad .\ \sin\ k\pi \,.\, (P \pm i0)^{\lambda-\frac{n}{2}-k}\end{aligned} \tag{23}$$

On the other hand, from [5], page 40, formula (II,3,7), we have

$$L^r\{\delta\} = H_{-2r}\,, \quad r = 0, 1, 2, \ldots \tag{24}$$

The distribution $(P \pm i0)^{\gamma}$ ([3], page 284) have poles at the points

$$\lambda = -\frac{n}{2} - k\,, \quad k = 0, 1, 2, \ldots$$

Therefore, substituting (24) into (23) for

$$\lambda - \frac{n}{2} - k \neq -\frac{n}{2}\,,\ -\frac{n}{2} - 1\,, \ldots$$

we have

$$(P \pm i0)^{\lambda} \,.\, L^{r}\{\delta\} = 0 \tag{25}$$

where $\lambda \neq \pm k, \quad k = 0, 1, 2, \ldots$

From (25) we conclude the proof of theorem 1.

In particular from (16), when $\lambda = \frac{n}{2} + r - 1$ and n odd, we obtain the product (12).

THEOREM 2. Let the distribution $(P \pm i0)^{\lambda}$ and $K^{r}\{\delta\}$ be the Klein-Gordon operator iterated r times defined by the formula (15); then the following formulae are valid:

$$(P \pm i0)^{\lambda} \,.\, K^{r}\{\delta\} = 0 \quad \text{when } \lambda \neq \pm k,\ k = 0, 1, 2, \ldots \tag{26}$$

$$(P \pm i0)^{-k} \,.\, K^{r}\{\delta\} = 0 \quad \text{if } k \geq \frac{n}{2} \text{ for } n \text{ even.} \tag{27}$$

$$(P \pm i0)^{-k} \,.\, K^{r}\{\delta\} = \sum_{\nu=0}^{r} \binom{r}{\nu} (-m^{2})^{r-\nu} a(k,n,\nu)\, L^{\nu+k}\{\delta\} \quad \text{if} \quad k \leq \frac{n}{2} \tag{28}$$

$$(P \pm i0)^{-k} \,.\, K^{r}\{\delta\} = \sum_{\nu=0}^{r} \binom{r}{\nu} (-m^{2})^{r-\nu}\, L^{\nu+k}\{\delta\} \quad \text{if} \quad k \geq \frac{n}{2} \quad \text{for } n \text{ odd}\,. \tag{29}$$

$$(P \pm i0)^{k} \,.\, K^{r}\{\delta\} = \sum_{\nu=0}^{r} \binom{r}{\nu} (-m^{2})^{r-\nu} b(k,n,\nu)\, L^{\nu-k}\{\delta\} \quad \text{if } \nu \geq k \tag{30}$$

and

$$(P \pm i0)^{\frac{n}{2}+k-1} \,.\, K^{r}\{\delta\} = 0 \quad \text{if } n \neq 2 \text{ with } r = 0, 1, 2, \ldots \tag{31}$$

where $a(k,n,\nu)$ is defined by the formula (6) and $b(k,n,\nu)$ by the formula (10).

Proof: From (15) we have

$$K^{r}\{\delta\} = \sum_{\nu=0}^{r} \binom{r}{\nu} (-m^{2})^{r-\nu} L^{\nu}\{\delta\} \tag{32}$$

where L^{ν} is defined by the formula (15).

From (32) we have,

$$(P \pm i0)^{\lambda} \, . \, K^{r}\{\delta\} = \sum_{\nu=0}^{r} \binom{r}{\nu} (-m^{2})^{r-\nu} (P \pm i0)^{\lambda} \, . \, L^{\nu}\{\delta\} \qquad (33)$$

From (33) and taking into account (16) we obtain the result (26).

From (33) and taking into account the formulae (7), (8) and (9), we obtain the results (27), (29) and (30).

Finally from (33) and taking into account the formulae (12) and (16) we obtain the results (31) and (26).

References

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