Analytical Approximation of Fuel Consumption and Periodic Behaviors for a Vehicle that Travels Through a Traffic Light Series

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Abstract

In this paper, we present a piecewise smooth system, that describes the dynamics of a single vehicle moving through a street that has a sequence of lights that turn on and off with a specific frequency. The model presents three dynamic ways: accelerated, decelerated and zero state. Besides, we show the description of the mathematical model used to simulate the system. The simulation was developed under an event-based scheme and implemented in Matlab. To make the numerical analysis, we take as a parameter study the cycle traffic light, which provides benefits to vehicular traffic system due to its configuration is achieved implementing optimization strategies for the phenomenon of green wave and reduces the travel

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time as the vehicle minimizes the number of stops along the road. Also, the
stability was studied for the periodic orbits one and two. Finally, we made
an approximation of fuel consumption. We assume that this is proportional
to the mechanical energy produced by the motor. From this point of view,
it can be concluded that it is possible to apply modeling and simulation
strategies based on dynamic systems to understand the complex behaviors
associated with the travel of vehicles in a traffic controlled by traffic lights.

Keywords: Vehicular traffic; stability; fuel consumption; dynamical systems; modeling and simulation; non-smooth dynamics; applied mathematics.

Aproximación analítica de consumo de combustible y comportamiento periódico para un vehículo que viaja a través de una serie de semáforos

Resumen

En el siguiente artículo, se presenta un sistema suave por tramos que describe la dinámica de un vehículo que se mueve a través de una calle con semáforos que se encienden y apagan con una frecuencia específica. El modelo presenta tres comportamientos dinámicos: acelerado, desacelerado y detenido. Además, se muestra la descripción del modelo matemático utilizado para simular el sistema. La simulación se desarrolló bajo un esquema basado en eventos y se implementó en Matlab. Para realizar el análisis numérico, se toma como parámetro el ciclo de los semáforos, que mejora el sistema de tráfico vehicular debido a que con su configuración se logran implementar estrategias de optimización permitiendo que los vehículos se desplacen en ola verde y reduzcan el tiempo de viaje, minimizando así, el número de paradas a lo largo del camino y reduciendo el consumo de combustible debido a las paradas y aceleradas. Además, se estudió la estabilidad de las órbitas periódicas uno y dos que presenta el modelo de simulación, así como sus implicaciones dinámicas. Finalmente, se presenta una propuesta para calcular el consumo de combustible, asumiendo que es proporcional a la energía mecánica producida por el motor, resulta en una propuesta novedosa que permite a las secretarias de movilidad comprender los comportamientos de los vehículos en vías principales de las ciudades. Desde este punto de vista, se puede concluir que es posible aplicar estrategias de modelado y simulación basadas en sistemas dinámicos para comprender los comportamientos complejos asociados al desplazamiento de los vehículos en una via controlada por semáforos.

Palabras clave: Tráfico vehicular; estabilidad; consumo de combustible; sistemas dinámicos; modelado y simulación; sistemas no suaves; matemática aplicada.

1 Introduction

Currently, there are around 870 million automobiles in the world. This makes the road infrastructure of some overpopulated cities is not sufficient with respect to the great demand, collapsing in huge traffic jams, generating chaos and plunging the humankind into one of the most increasing problems like traffic congestion [1].

Although, automobiles represent a solution to the mobility problem, they constitute one of the major sources of pollution. Due to this supply a series of toxic substances in the environment, well known as greenhouse gases (carbon monoxides CO, hydrocarbons, among others), as a consequence of the complete non-combustion of the petroleum products necessary for the functioning of cars. In the same way, the produced noise pollution may generate a series of negative responses in human beings, such as balance disorders, ill feeling and fatigue, among others. On the other hand, the consequences caused by traffic accidents add invaluable losses because these take the lives of 1.2 million people in the world and more than 50 million individuals are injured [2],[3],[4].

Due to the vehicular problems in the cities, there is a need to carry out theoretical studies [5],[6]. This is why it is daily invested in time, human capital and development of new theories or even creation of tools that provide scientific basis to the possible strategies implemented by the government entities to improve mobility in cities. As an example, and to mention one of the most relevant authors in literature, as can be seen in Toledo [7],[8],[9], the construction of a model in which one single vehicle is considered, moving through a two-way traffic light sequence, with a specific period; the most relevant contribution of this work, is that the nontrivial dynamics depends on the finite acceleration and breaking power of vehicles for a set of parameters.

In a later work, Toledo establishes in [10], control strategies based on the synchronization of the traffic lights, which allowed to improve the vehicle traffic, moreover, the resonance in terms of travel time, speed and fuel consumption were studied in this work. An important

result of the Toledo model, apply the control strategy, is that the behavior close to resonance does not depend on finite acceleration and breaking power of the vehicle. Furthermore, in this deterministic model, it was shown that the resonance was an independent universal behavior of the geometry of the system for the case of the green wave strategies.

In 2009, Varas researched whether the universal behavior close to resonance persists, when several cars interact along the same route. To know the behavior of this dynamics, used a cellular automaton model [11] . One of the most recent works in this respect, is the work carried out by Mesa in [12], where the description and comparison of four vehicle traffic models are presented. This work explains the dynamic of one single car that moves through a sequence of traffic lights that turns on and off. See more on [13],[14].

The mathematical model, as well as the results of the numerical analysis are presented in this document. The main contribution, which is intended to be highlighted on this occasion, is quite useful at the moment of analyzing the periodic behaviors associated with the dynamic of the vehicle. Thus, the methodology used is explained to carry out the stability analysis and consequently, conclude about how the orbits in question are or are not stable. This in order to propose a formula that allows to calculate the fuel consumption, if necessary, a vehicle had one dynamic or another.

2 Mathematical model

In the Toledo's one-dimensional model [7], it is assumed that an automobile travels through a two-way (green and red) traffic light sequence, where the car presents one of the following behaviors:

- Positive acceleration a_- until the vehicle reaches the cruising speed or maximum speed v_{max}.

- Constant speed v_{max}, when the acceleration is zero.

- Deceleration $-a_-$ until the vehicle stops.

- A zero speed when the vehicle is stationary, waiting for the traffic light to turn green.

2.1 Description of the model

From the above and [8], the dynamic of the system can be represented using a piecewise soft system, four situations are presented in this model, which depend on acceleration, that is, accelerated state, zero state with maximum speed, zero state with zero speed and decelerated state. To make an appropriate description of each event, it is considered that the following state variables (x_1, y_1) are the position and velocity of the vehicle respectively, a acceleration of the vehicle in a time t, which will change depending on which state the vehicle is:

$$\begin{cases} \dfrac{dx_1}{dt} &= x_2 \\ \dfrac{dx_2}{dt} &= a \end{cases} \tag{1}$$

2.2 Configuration of the traffic light

It is considered that the nth traffic light will be modelled through the function (2).

$$f(t; \omega_n, \varphi_n) = \sin(\omega_n t + \varphi_n) \tag{2}$$

Where

- $\omega_n = \dfrac{2\pi}{T_n}$: indicates the switching frequency of the nth traffic light.

- T_n : the cycle of the nth traffic light, i.e., the time that the green light takes to turn green again.

- φ_n: denotes the phase difference between traffic lights.

Moreover, these are the following assumptions or conditions that should be complied by the traffic lights:

- If $\sin(\omega_n t + \varphi_n) > 0$, then the traffic light is green.

- If $\sin(\omega_n t + \varphi_n) > 0$, then the traffic light is red.

The normalized cycle T_{sn} of the nth traffic light was used in [2],[3]. It is necessary to highlight the importance of the Equation (3), given that this allows to obtain the real cycle of the nth traffic light in units of time, making use of the system parameters.

$$T_n = \frac{2\pi L_n}{T_{sn} v_{max}} \tag{3}$$

where L_n is defined as the distance that separates one traffic light from another and v_{max} the maximum velocity, whose conclusion may be; the cycle of the nth traffic light is obtained in units of time.

2.3 Dynamic system states

Making use of the system of Equation (1) and the traffic light conditions, it is established that each dynamic state can be expressed in the following way:

2.4 Accelerated state "S_+":

it occurs when the driver increases the speed constantly, i.e., the car has a constant and positive acceleration a_+, until the vehicle reaches the cruised speed v_{max} owed on the road. In this way, the system is as follows in the Equation (4).

$$\begin{cases} \dfrac{dx_1}{dt} = x_2 \\ \dfrac{dx_2}{dt} = a_+ \end{cases} \tag{4}$$

2.5 Zero state:

this is because the car may present zero acceleration in two situations during the tour. For this reason, the zero state is divided into two modes:

2.5.1 Zero state with maximum speed "S_{0m}":
it occurs when the vehicle reaches maximum speed allowed on the road v_{max}. For this reason, this velocity should be maintained, i.e., its acceleration is zero. Later, the system of equations is determined by the Equation (5).

$$\begin{cases} \dfrac{dx_1}{dt} = x_2 \\ \dfrac{dx_2}{dt} = 0 \end{cases} \tag{5}$$

2.5.2 Zero state with zero speed "S_0":
it occurs when the vehicle is stationary, considering the position of a traffic light, which is expected to turn green, i.e., $\sin(\omega_n t + \varphi_n) > 0$,. Then, the system of equations is as we shown in (6).

$$\begin{cases} \dfrac{dx_1}{dt} = 0 \\ \dfrac{dx_2}{dt} = 0 \end{cases} \tag{6}$$

2.6 Decelerated state "S_-":

when the evolution of the system is in a decelerated motion, the vehicle is forced to decrease its velocity constantly, due to the traffic light is red, i.e., $\sin(\omega_n t + \varphi_n) \leq 0$. Then, the equations associated with this state are those found in Equation (7).

$$\begin{cases} \dfrac{dx_1}{dt} = x_2 \\ \dfrac{dx_2}{dt} = -a_- \end{cases} \tag{7}$$

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When the vehicle approaches the *nth* traffic light with speed x_2,
the driver should decide whether to slow the car down or not, depend-
ing on the following traffic light signal. Then, the safety distance or
stopping distance is defined as the distance required by a car that
travels at a velocity x_2 to stop. Determining the safety distance is
necessary to optimize safety in vehicles, line of tracks and signaling
design [12]. For this case, $d_s = \dfrac{v_{max}^2}{2a_-}$ is defined as the safety distance
[7],[8].

3   Simulation analysis

In [8], the behavior of a vehicle was studied when it is considered
that all the traffic lights present fixed distances and have the same
switching rate with a zero phase difference; i.e., $L_n = L$, $\omega_n = \omega$
and $\varphi_n = 0$ respectively, different types of orbits were obtained under
these assumptions. The main objective of this article is to study the
stability of the periodic orbits $1T$ and $2T$ as a particular case, as well
as fuel consumption, assuming that this is proportional to the me-
chanical energy. In addition to carry out the respective comparisons
in the cases in which the dynamics of the vehicle evolves through a
green wave and the periodic orbits $1T$ and $2T$.

3.1   Bifurcation diagram

Assuming the fact that $L_n = L$, $\omega_n = \omega$ and $\varphi_n = 0$, a bifurcation
diagram was built, see Figure 1, this was developed varying the nor-
malized traffic light cycle. For the equation (3) and the assumptions
previously described, it is assumed that $T_{sn} = T_s$, the equivalence il-
lustrates that this parameter will be the same for all the traffic lights,
this is charted along the horizontal axis and the normalized position
of the car is shown in the vertical axis. Different phenomena can
be observed in this bifurcation diagram, indicating the dynamic rich-
ness of the system; with behaviors such as fractals, multiple periods,
duplicity of period, chaos, hard and soft bifurcations, among others.

In Figure 1 Bifurcation diagram, when T_s takes values between 0.7 and 1, lines that increase are observed as T_s approaches to the value of 1 and multiple periods are observed. Then, when T_s is increasing, it presents a duplicity of period route to the chaos, where this chaotic behavior is truncated and an orbit from period two appears again and finally, an orbit form period one. It is important to highlight that hard and soft transitions are also observed in Figure 1. For example, there is a soft transition when $1.0 < T_s < 1.1$ and a hard transition is observed when $1.3 < T_s < 1.4$.
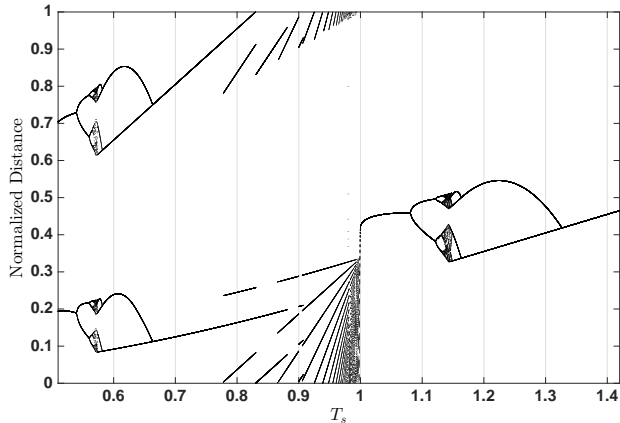
Figure 1: Bifurcation Diagram. Source: The authors.
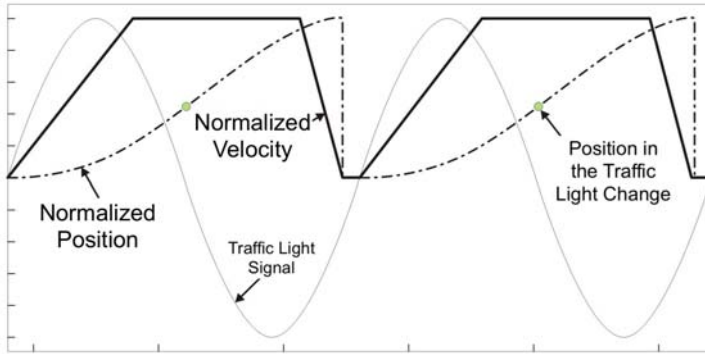
3.2 Evolution in time

Before observing the charts that show state variable evolution, it is important to highlight Table 1, given that it relates the normalized parameter cycle T_s with the cycle T_n and the time in green of a traffic light using the Equation (3), where the parameters are $v_{max} = 14 m/s$ and $L = 200 m$. In addition to the function presented in the Equation (2), which models the traffic light, it is assumed that the positive semi cycle is equal to the negative semi cycle, therefore, the time in green is equal to the time in red.

Table 1: Relationship of parameters between the normalized cycle and time in green

Normalized cycle	Traffic light cycle (sec)	Time in green (sec)
0.05	1795.15	897.57
0.1	897.57	448.39
0.2	448.78	224.39
0.5	179.52	89.75
0.9	99.73	49.86
1.0	89.78	44.89
1.2	74.80	37.39
1.4	64.11	32.05

As observed in Table 1, when T_s is close to zero, the time in green from the traffic light is long. For example: for $T_s = 0.05$, the time in green is approximately 15 minutes, which would make no sense to configure a traffic light with a cycle $T_n = 30\ mins$. Therefore, the orbits for $T_s = 1.4, T_2 = 1.2$ and $T_s = 1.0$ were obtained in [8]. The normalized speed of the vehicle is shown in a thick continuous line, the normalized position is shown in a dashed line and the traffic light signal is shown in a thin continuous line, that allow to evidence different behaviors for each value used from the parameter are shown in Figures 2, 3 and 4.

Comparing the bifurcation diagram from Figure 1 with the results in Figure 2, it is assumed that an orbit of period one is observed in this case. Moreover, with the help of the circle from Figure 2, which allows to visualize the normalized position of the vehicle when the traffic light switches. It is important to highlight that the position is always the same.

Figure 2: Evolution in Time for $T_s = 1.4$ representation. Source: The authors.
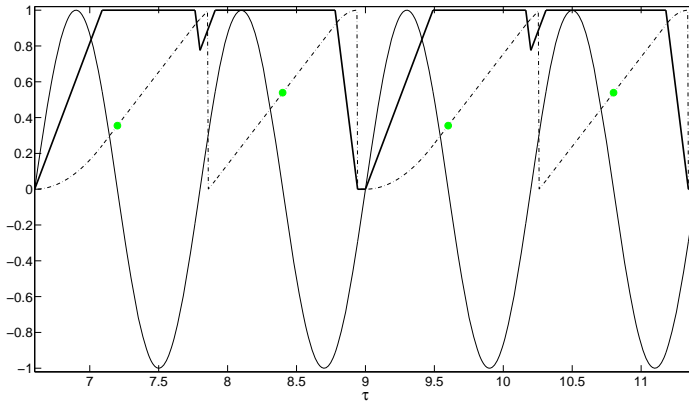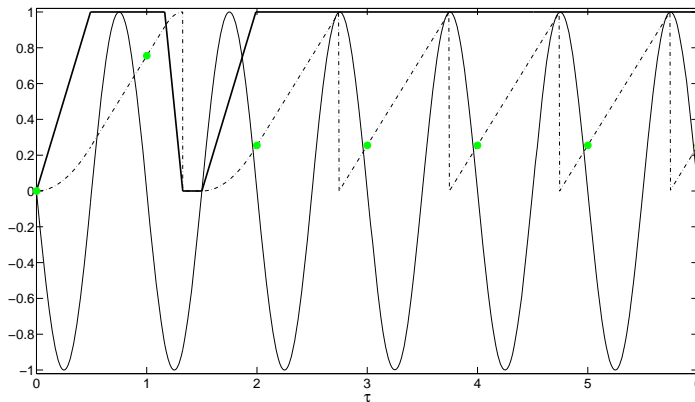

Figure 3: Evolution in Time for $T_s = 1.2$ Simulation. The dotted line refers to the normalized position, the solid dark line refers to the normalized speed, the green circular point refers to the signal change at the traffic light, and the sine signal is the periodic behavior that represents the ignition frequency and off for the traffic light. Source: The authors.

In Figure 3, an orbit from period two is observed when the normalized traffic light cycle takes the value of $T_s = 1.2$. Furthermore, the circles show the normalized position of the car when the traffic

light changes from green to red, given that the sine function changes
the positive semi cycle to negative and this coincides every two traffic
lights; in addition, the vehicle is required to stop every two traffic
lights.

In Figure 4, the thick continuous line remains at its maximum
value; i.e., the vehicle maintains its maximum speed and achieves to
cross all the traffic lights in green, this behavior is known as a green
wave. These charts were obtained and analyzed in [12].

Figure 4: Evolution in Time for $T_s = 1.0$ Simulation. The dotted line refers to
the normalized position, the solid dark line refers to the normalized speed, the
green circular point refers to the signal change at the traffic light, and the sine
signal is the periodic behavior that represents the ignition frequency and off for
the traffic light. Source: The authors.

4 Stability analysis

A stability analysis for a periodic orbit $1T$ and a periodic orbit $2T$ will
be carried out in this section. These orbits were found in [8] for the
model that describes the dynamic of a single car that travels on a road,
which has n traffic lights, where it was considered that all the traffic
lights present fixed distances, i.e., $L_n = L$. The function modeled

by the traffic light is given by the Equation (2), with $\omega_{n=\omega}$, in this way, they have the same cycle T and a zero-phase difference, $\varphi_n = 0$. This indicates that all the traffic lights have the same frequency, the switching frequency.
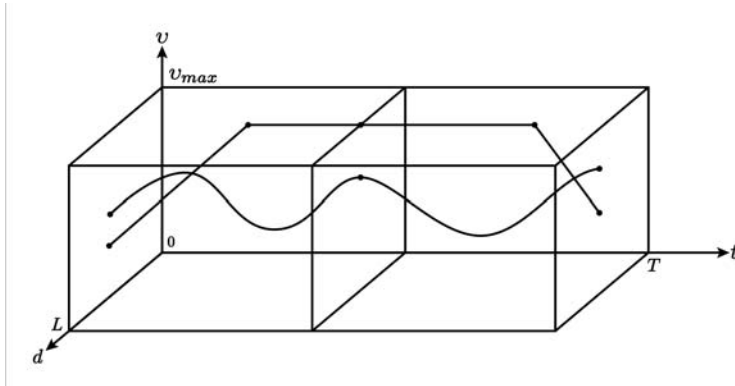
**Figure 5:** Solutions Domain. Source: The authors.

The Figure 5 explains the visualization of the model solution space. Due to the periodicity of the solutions in Figure 2, the X-axis represents the distance traveled by the vehicle. This takes values in the interval $[0, L]$, where L is the distance between two consecutive traffic lights; time t is illustrated along the Y-axis. This variable takes values in the interval $[0, T]$, where T is the traffic light cycle. Finally, the Z-axis represents the velocity v of the car, which takes values in the interval $[0, v_{max}]$. To represent the orbits to be studied, the notation explained in Table 2 will be adopted, which will depend on the state of the vehicle:

4.1 Periodic orbit 1T

The Figure 6 is obtained using Figure 2 and notation form Table

Table 2: System States

State	Notation
Zero state v_{max}	S_{0m}
Zero state $v = 0$	S_0
Decelerated state, $a_+ = 2m/s^2$	S_-
Accelerated state, $a_- = 6m/s^2$	S_+
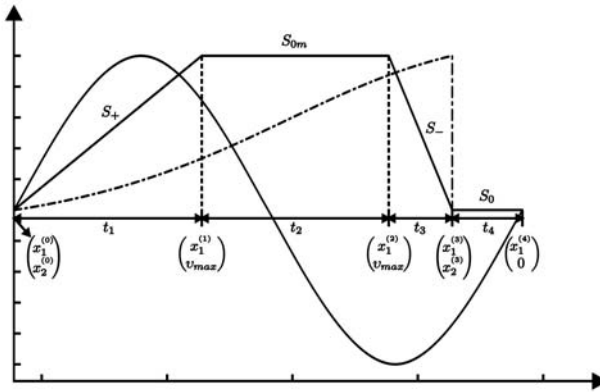
Figure 6: Periodic Orbit 1T Analysis. Source: The authors.

First of all, it is important to explain the notation used in Figure 6:

$x_1^{(i)}$:the final position in the **i-th** stretch.

$x_1^{(i)}$:the final velocity in the **i-th** stretch

$x_2^{(0)}$:initial position of the path

$x_2^{(0)}$:initial velocity of the path.

Where $i = 1, 2, 3, 4$. In addition, a stretch is considered as a part of the path, in which a state is conserved. For example, the first

stretch in Figure 6 begins when the vehicle presents a position $x_1^{(0)}$ and a speed $x_2^{(0)}$ and finishes when the vehicle has a maximum speed v_{max} and is in position $x_1^{(1)}$. In this way, the periodic orbit $1T$ has four stretches, given that four changes of state occur.

As the interest of this section is to study the stability of the orbit $1T_{s+s_{om}s-s_0}$, it is necessary to find the Jacobian matrix:

$$J_1 = \begin{pmatrix} \dfrac{dx_1^{(3)}}{dx_1^{(0)}} & \dfrac{dx_1^{(3)}}{dx_2^{(0)}} \\ \dfrac{dx_2^{(3)}}{dx_1^{(0)}} & \dfrac{dx_2^{(3)}}{dx_2^{(0)}} \end{pmatrix} \tag{8}$$

To find each component of J_1, Figure 6 is used as well as the Equations. (4),(5),(6) and (7).

By observing Figure 6, it is understood that in the fourth stretch, the car is in zero state S_0, that is, the vehicle is stationary, therefore:

$$x_1^{(4)} = x_1^{(3)}$$
$$x_2^{(4)} = x_2^{(3)} \tag{9}$$

To determine these unknown numbers, it is necessary to know the final position and velocity of each stretch.

$$x_1^{(1)} = \frac{1}{2} \frac{\left(v_{max} - x_0^{(2)}\right)^2}{a_+} + x_0^{(2)} \frac{v_{max} - x_2^{(0)}}{a_+} + x_1^{(0)} \tag{10}$$

the final position $x_1^{(2)}$ of the second section is:

$$x_1^{(2)} = x_1^{(1)} + v_{max} t_2 \tag{11}$$

Where

$$t_2 = \frac{1}{v_{max}} \left(L - x_1^{(1)} - \frac{v_{max}^2}{2a_-} \right) \tag{12}$$

and substituting $x_1^{(1)}$, the travel time from the second stretch is determined as:

$$t_2 = \frac{1}{v_{max}} \left(L - \frac{\left(v_{max} - x_2^{(0)}\right)^2}{2a_+} - x_2^{(0)} \frac{v_{max} - x_2^{(0)}}{a_+} - x_1^{(0)} - \frac{v_{max}^2}{2a_-} \right) \tag{13}$$

Finally, it is assumed that the final position $x_1^{(3)}$ in the third stretch is given by:

$$x_1^{(3)} = \frac{\left(v_{max} - x_2^{(0)}\right)^2}{2a_+} + x_2^{(0)} \frac{v_{max} - x_2^{(0)}}{a_+} + x_1^{(0)} + v_{max}t_2 + \frac{v_{max}^2}{2a_-} \tag{14}$$

In addition, by observing Figure 6, it is assumed that the car stopped when finishing the third stretch, then the final speed in the third stretch is $x_2^{(3)} = 0$. To find the first order partial derivative, the Equations. (13) and (14) are used, it also should be considered that: $x_1^{(3)} = h\left(t_2, x_1^{(0)}, x_2^{(0)}\right)$; $t_2 = k\left(x_1^{(0)}, x_2^{(0)}\right)$, therefore:

$$\frac{dx_1^{(3)}}{dx_1^{(0)}} = \frac{dx_1^{(3)}}{dt_2} \frac{dt_2}{dx_1^{(0)}} + \frac{dx_1^{(3)}}{dx_1^{(0)}} \tag{15}$$

In the same way, $\dfrac{dx_1^{(3)}}{dx_1^{(0)}}$ is found as follows:

$$\frac{dx_1^{(3)}}{dx_2^{(0)}} = \frac{dx_1^{(3)}}{dt_2} \frac{dt_2}{dx_2^{(0)}} + \frac{dx_1^{(3)}}{dx_2^{(0)}} \tag{16}$$

Then, calculating the respective derivatives, it is understood that the Jacobian matrix J_1 is the zero matrix. Therefore, it has two associated zero eigenvalues. According to [15], if the product of the eigenvalues is zero in the case of a dimensional M map, this means that the orbit $1T_{S+S_{om}S_-S_0}$ is extremely stable [16],[17].

4.2 Periodic orbit 2T

Figure 7 is obtained using Figure 3 and notation form Table 2, this is a graphic representation for the periodic orbit 2T for the model.
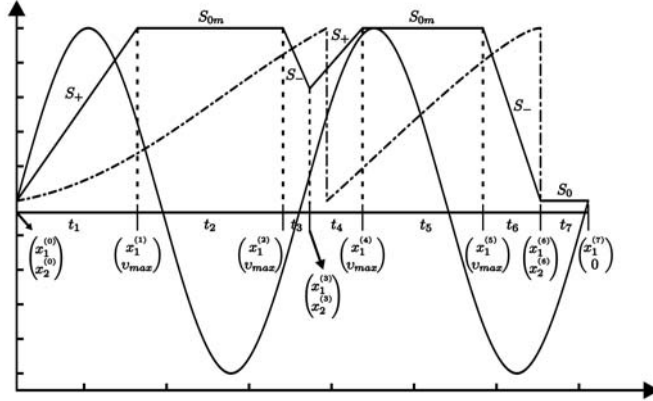
Figure 7: Periodic Orbit 2T Analysis. Source: The authors.

From Figure 7 and making use of Table 2, the map that allows to study the periodic orbit $2T$ is denoted as $2T_{S_+ S_{om} S_- S_{om} S_- S_0}$.

It should be remembered that:

$$x_1^{(i)} : \text{the final position in the } i - th \text{ stretch.}$$

$$x_2^{(i)} : \text{the final velocity in the } i - th \text{ stretch.}$$

Where $i = 1, 2, 3, 4, 5, 6, 7$.

The periodic orbit $2T$ has seven stretches, given that seven changes of state occur. As previously in the periodic orbit 1T, in this way it is necessary to determine the final position and speed in the seventh stretch $x_1^{(7)}$ and $x_2^{(7)}$ respectively, in terms of the initial conditions $x_1^{(0)}$ and $x_2^{(0)}$; for this, it is necessary to use the equations. (4),(5),(6) and (7), in order to find the final position and speed of each stretch.

The final position in the first stretch is given by:

$$x_1^{(1)} = \frac{1}{2} \frac{\left(v_{max} - x_0^{(2)}\right)^2}{a_+} + x_0^{(2)} \frac{v_{max} - x_2^{(0)}}{a_+} + x_1^{(0)} \qquad (17)$$

For the third stretch of the travel, the final position $x_1^{(3)}$ in terms of the initial conditions is:

$$x_1^{(3)} = \frac{1}{2} \frac{\left(v_{max} - x_2^{(0)}\right)^2}{a_+} + x_2^{(0)} \frac{v_{max} - x_2^{(0)}}{a_+} + x_1^{(0)} + v_{max} \left(T - t_1 - t_3\right)$$
$$- \frac{1}{2} \frac{\left(x_2^{(3)} - v_{max}\right)^2}{a_-} - v_{max} \frac{x_2^{(3)} - v_{max}}{a_-} \qquad (18)$$

In the same way, the final speed $x_2^{(3)}$, in the same stretch, is determined by:

$$\left(x_2^{(3)}\right)^2 = -2a_- \left(x_1^{(3)} - x_1^{(2)}\right) + v_{max}^2 \qquad (19)$$

Given that the work is being done with a periodic orbit $2T$ and with help of Figure 7, it is assumed that:

$$2L = x_1^{(4)} + v_{max} t_5 + \frac{v_{max}^2}{2a_-} \qquad (20)$$

Then, the travel time in the fifth stretch t_5 is given by:

$$t_5 = \frac{1}{v_{max}} \left(2L - \frac{v_{max}^2 - \left(x_2^{(3)}\right)^2}{2a_+} - \frac{v_{max}^2}{2a_-} - x_1^{(3)} \right) \qquad (21)$$

Finally, in the sixth stretch, the final position $x_1^{(6)} = x_1^{(7)}$ is determined by:

$$x_1^{(6)} = \frac{v_{max}^2}{2a_-} + v_{max}t_5 + \frac{v_{max}^2 - \left(x_2^{(3)}\right)^2}{2a_+} + \frac{v_{max}^2 - \left(x_2^{(3)}\right)^2}{2a_-} +$$

$$\frac{v_{max}^2 - \left(x_2^{(0)}\right)^2}{2a_+} + v_{max}\left(T - \frac{v_{max} - x_2^{(0)}}{a_+} + \frac{x_2^{(3)} - v_{max}}{a_-}\right) + x_1^{(0)}$$

$$(22)$$

as the vehicle stops at the end of the sixth stretch, the final speed is $x_1^{(6)} = x_1^{(7)} = 0$.

The purpose is to determine the stability of orbit $2T_{S_+S_{om}S_-S_+S_{om}S_-S_0}$, for this it is necessary to find the Jacobian matrix as follow:

$$J_2 = \begin{pmatrix} \dfrac{dx_1^{(6)}}{dx_1^{(0)}} & \dfrac{dx_1^{(6)}}{dx_2^{(0)}} \\ \dfrac{dx_2^{(6)}}{dx_1^{(0)}} & \dfrac{dx_2^{(6)}}{dx_2^{(0)}} \end{pmatrix} \qquad (23)$$

From the previous, the Jacobian matrix J_2 is determined as:

$$J_2 = \begin{pmatrix} 0 & \dfrac{v_{max}}{a_{max}} \\ 0 & 0 \end{pmatrix} \qquad (24)$$

Hence, the Jacobian matrix J_2 has two zero eigenvalues, then its product is zero and for the stability criterion for dimensional M maps, it is understood that the orbit $2T_{S_+S_{om}S_-S_{om}S_-S_0}$ is extremely stable [16],[18].

5 Fuel consumption

For this work, it is assumed that the car moves on a straight path, being subjected to an acceleration or deceleration, both constants,

reason why the weather conditions will not be considered. The fuel
consumption calculation will be especially made for the periodic or-
bits $1T$ and $2T$ and finally for the green wave. In first place, the
instantaneous power P is considered as the energy transfer portion E
in time t [19]

$$P = \frac{dE}{dt} = \overrightarrow{F} \cdot \overrightarrow{v} \tag{25}$$

then,

$$E = E(0) + \int_{t_0}^{t_f} Fvdt \tag{26}$$

Where:

t_f : final time of the stretch

t_0 :initial time of the stretch.

F :the force vector module.

v :the velocity vector module.

t :time.

According to [10], fuel consumption C can be estimated as shown
in the Equation (27).

$$C = \int_{t_0}^{t_f} Fvdt \tag{27}$$

It is important to clarify that the friction force F_r will be con-
sidered, which is opposite to movement, external forces such as aero-
dynamic drag will not be considered, and the energy spent due to
internal frictions in the mechanism of the vehicle is also omitted. The
dissipation sources in the vehicle movement, which will depend on the
state of the car (accelerated, zero and decelerated) will be described
below. To represent the states of the vehicle, the notation explained
above and summarized in Table 2 will be adapted.

Accelerated state "S_+": when the vehicle begins the route with initial speed v_0 and the driver accelerates until reaching the permissible maximum or cruising speed on the route, the kinetic energy shift is calculated in Equation (28).

$$E_{a_+} = \frac{mv_{max}^2}{2} - \frac{mv_0^2}{2} \qquad (28)$$

Zero State with Maximum Speed "S_{0m}": when the vehicle travels a distance x_r free of stops, the work carried out is opposed to the friction force and assuming a constant coefficient of friction μ along the road, we have the following Equation (29).

$$W_r = F_r x_r \qquad (29)$$

Decelerated State "S_-": when the vehicle travels with a maximum speed until reaching the safety distance and the traffic light is red, so the driver is obliged to slow down until reaching the minimum speed v_{min}; therefore, the energy lost while decelerating is given by the Equation (30).

$$E_{a_-} = \frac{mv_{max}^2 - mv_{min}^2}{2} \qquad (30)$$

Zero State with Zero Speed "S_0": when the vehicle is stationary, considering the position of a traffic light, which is expected to turn green, the idle fuel consumption C_0. The vehicle is stationary when it is in zero state, then fuel consumption is proportional to awaiting time with respect to the traffic light, i.e., $C_0 = k * t_e$, where t_e is the time waited by the driver until the traffic light turns green and k is the constant of proportionality.

With the above, fuel consumption is calculated for the cases where it displays a behavior for the periodic orbits $1T$ and $2T$ and the green wave for the model already explained.

5.1 Fuel consumption for a periodic orbit 1T

The graphic representation of the periodic orbit 1T is observed in Figure 8, in which four changes of state occur, these are: accelerated S_+, zero with maximum speed S_{0m}, decelerated S_- and zero with zero speed S_0.
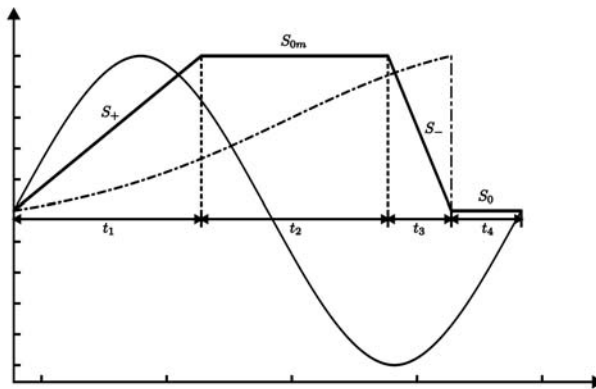
Figure 8: Periodic Orbit 1T Analysis. Source: The authors.

The fuel consumption of the vehicle applied to the orbit $1T_{S_+ S_{om} S_- S_0}$ will be carried out through each state, applying any of the Equations. (28),(29) or (30).

For this orbit, it is necessary to calculate the consumption in each state and add them algebraically. In this way, an approximation of fuel consumption between two consecutive traffic lights is obtained. When multiplying the value found by the number of stretches $n - 1$, the total of the fuel used during the travel will be obtained, where n is the number of traffic lights. In addition, it is considered that a stretch is the path between two consecutive traffic lights.

From the above, it is assumed that the total fuel consumption between two consecutive traffic lights is the sum of the consumption in each state, i.e.:

$$C \approx C_+ + C_{0m} + C_- + C_0 \tag{31}$$

As the waiting time with respect to the traffic light is small in comparison with the travel time, it is assumed that the fuel consumption will be zero, i.e., $C_0 \approx 0$. Therefore, the consumption is:

$$C \approx \frac{mv_{max}^2}{2} + F_r x_r + \frac{mv_{max}^2}{2} \tag{32}$$

Where the distance traveled by the vehicle without stopping is:

$$x_r = x_{r1} + x_{r2} \tag{33}$$

And the distance traveled in the accelerate state S_+ is:

$$x_{r1} = \frac{v_{max}^2}{2a_+} \tag{34}$$

The distance traveled in the zero state with maximum speed S_{0m} is:

$$x_{r2} = L - \left(\frac{v_{max}^2}{2a_+} + \frac{v_{max}^2}{2a_-} \right) \tag{35}$$

As the purpose is to find the fuel consumption used during the travel through the sequence of n traffic lights for the periodic orbit $1T$, then the Equation (32) is multiplied by the number of stretches $n - 1$.

Then, the total fuel consumption throughout the travel is given by:

$$C_T \approx (n - 1) \left(\frac{mv_{max}^2}{2a_+} + F_r x_r + \frac{mv_{max}^2}{2} \right) \tag{36}$$

5.2 Fuel Consumption for a Periodic Orbit 2T

The graphic representation of the periodic orbit $2T$ is observed in Figure 9.
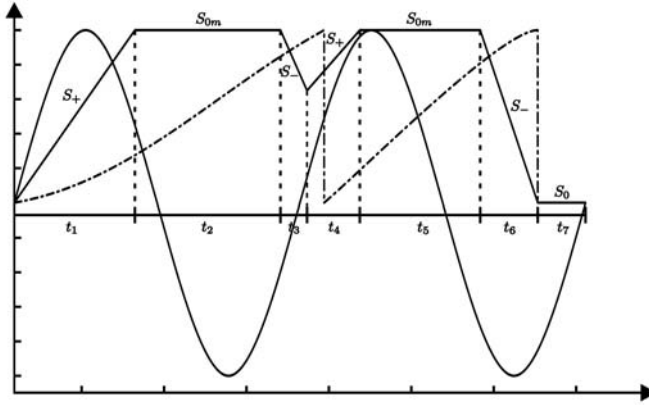
Figure 9: Periodic Orbit 2T Analysis Source: The authors.

To calculate the fuel consumption of the vehicle using the periodic orbit $2T_{S_+S_{om}S_-S_+S_{om}S_-S_0}$, it will be made again through the states. For this case, it is necessary to calculate the consumption in each state and add it. In this way, an approximation of the fuel consumption between three consecutive traffic lights would be obtained. Seven changes of states of the vehicle occur for the periodic orbit $2T$, the terms explained for the periodic orbit $1T$ will be used to calculate the fuel consumption in each state.

As the objective is to find an expression that represents the fuel consumption throughout the travel through the sequence of n traffic lights for the periodic orbit $2T$, it should be considered that a car completes a stretch when it has passed three consecutive traffic lights, and this makes that the general expression of fuel consumption for this periodic orbit $2T$ depends on whether the number n of traffic lights is even or odd.

When the number of traffic lights is odd, i.e., $n = 2k+1$ where k is the number of stretches, then the total fuel consumption is expressed as:

$$C_T\left(impar\right) \approx k\left(\frac{mv_{max}^2}{2} + F_r x_{r1} + F_r x_{r2} + mv_{max}^2 - mv_{min}^2 + F_r x_{r4} + F_r x_{r5}\right)$$
(37)

Now, n can be expressed in the following way is even: $n = (2k + 1) + 1$ where k is the number of stretches, then the total fuel consumption is expressed as:

$$C_{T(par)} \approx k\left(\frac{mv_{max}^2}{2} + F_r x_{r1} + F_r x_{r2} + mv_{max}^2 - mv_{min}^2 + F_r x_{r4} + F_r x_{r5}\right)$$
$$+ \left(\frac{mv_{max}^2}{2} + F_r x_{r1} + F_r x_{r2} + mv_{max}^2 - mv_{min}^2\right)$$
(38)

5.3 Fuel consumption for the green wave

The graphic representation of the green wave is observed in Figure 10.
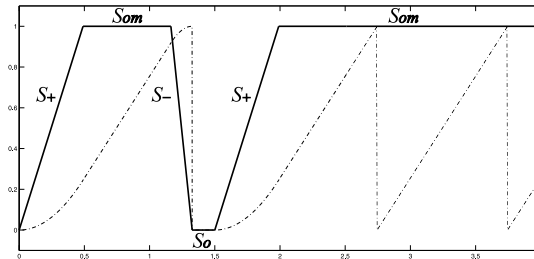
Figure 10: Green Wave Analysis. The dotted line refers to the normalized position and the solid dark line refers to the normalized speed. Source: The authors.

To calculate the fuel consumption of the vehicle when the green wave phenomenon occurs, it is necessary to consider that the driver

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stops once during the travel and achieves to pass all the traffic lights during the rest of the travel with the maximum permissible speed.

See in Figure 10 that the vehicle begins its path, but it is forced to stop in the following traffic light given that this is red, as indicated by the sine function, due to it is in its negative semi cycle, see also Figure 4. Then, the driver waits until the green light appears and thus, he/she can continue with his/her path through the road. For this case, it is necessary to calculate the fuel consumption used until beginning again its travel and add the consumption used in the rest of the sequence of traffic lights.

For the first part, the fuel consumption used is calculated between the first two traffic lights, which is similar to the calculated for the periodic orbit $1T$, Equation (32).

Hence, the fuel consumption used between the first two traffic lights is determined by:

$$C_1 \approx mv_{max}^2 + F_r x_r \tag{39}$$

For the rest of the path, it is important to observe in Figure 10, that the automobile only shows two states S_0 and S_+. Then, the fuel consumption C_2 is determined by the equation (40).

$$C_2 \approx \frac{mv_{max}^2}{2} + F_r \left(n - 2\right) L \tag{40}$$

Thus, the total fuel consumption for the green wave can be expressed as:

$$C_t \approx C_1 + C_2 \tag{41}$$

Equations such as the shown (34),(35) and (38) can be formulated through modeling, simulation and dynamic analysis tools and use of maps; which will allow the scenario evaluation and analysis in future works facilitating the comprehension of emerging behaviors like those studied in this work.

6 Conclusions

A modeling diagram and a piecewise smooth system for a traffic system have been presented. To simulate this system, it is necessary to know the equations that describe its flow in each state and the conditions in the transition limits between the dynamic states. With this information and the bifurcation diagrams, it is possible to simulate a wide range of phenomena shown by this type of systems.

In the first stage, some numerical routines were illustrated in this document to simulate the hard-dynamic system. Periodic solutions, hard and soft bifurcations, fractals and chaos, just to mention some of the phenomena made evident, were obtained with these numerical simulations.

In the second stage, a stability analysis for two types of periodic orbits, which resulted being extremely stable was carried out. The fact that the periodic orbits 1T and 2T are extremely stable, they indicate that the system solutions do not differ under small modifications in the initial conditions. It is important to highlight the difference between a stable orbit and an extremely stable orbit with respect to the rate of convergence. The book Exploring Chaos: Theory and Experiment [15], establishes that a stable orbit has a linear rate of convergence and an extremely stable orbit has a quadratic rate of convergence.

Finally, an approximation of the fuel consumption was carried out through the use of a set of tools from the classical mechanics. This calculation was made for the following three types of solutions: periodic orbits $1T$ and $2T$ and the green wave. And with this, it was evidenced that the number of stops of a car through a road increases when the fuel consumption also increases. For this reason, configuring the traffic lights to obtain a green wave brings great benefits for the driver such as the reduction of travel time and fuel consumption, given that the number of stops is minimal.

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