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STOPPING DOMAINS

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Resumen. Se estudian algunos conceptos básicos y resultados no-estándar sobre dominios, líneas y puntos de parada en análisis estocástico con dos parámetros.

Abstract. When we are working in a two parameter stochastic analysis, we do not have only stopping points. We state some basic nonstandard concepts and results about stopping domains, stopping lines and stopping points.

Keywords. Stochastic analysis, parameters, stopping.

1. Introduction

For a good introduction of nonstandard analysis we refer to [1]. The main features that we need in our work are the following.

We assume the existence of a set ${}^*\mathbb{R} \supseteq \mathbb{R}$, called the set of nonstandard real numbers and a mapping $* : V(\mathbb{R}) \longrightarrow V({}^*\mathbb{R})$, (where $V_1(S) = S$, $V_{n+1}(S) = V_n(S) \cup \mathfrak{P}(V_n(S))$ and $V(S) = \cup_{n\in\mathbb{N}} V_n(S)$) with three basic properties. To state the properties we give the following notions.

An *elementary statement* is a statement Φ built up from " $=$ ", " $\in$ ", relations: $u = v$, $u \in v$, the connectives "*and*","*or*", "*not*" and "*implies*", bounded quantifiers $(\forall u \in v)$, $(\exists u \in v)$. An *internal object* A is an element of $V({}^*\mathbb{R})$ such that $A = {}^*S$, $S \in V(\mathbb{R})$. A set in $V({}^*\mathbb{R})$ which is not internal is called external.

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(1) *Extension Principle.* $^*\mathbb{R}$ is a proper extension of $\mathbb{R}$ and $* : V(\mathbb{R}) \to V(^*\mathbb{R})$ is an embedding such that $^*r = r$ for all $r \in \mathbb{R}$.

(2) *Saturation Property.* Let $\{R_n : n \in \mathbb{N}\}$ be a sequence of internal objects and $\{S_m : m \in \mathbb{N}\}$ be a sequence of internal sets. If for each $m \in \mathbb{N}$ there is an $N_m \in \mathbb{N}$ such that for all $n \geq N_m$, $R_n \in S_m$, then $\{R_n : n \in \mathbb{N}\}$ can be extended to an internal sequence $\{R_\eta : \eta \in {}^*\mathbb{N}\}$ such that $R_\eta \in \cap_m S_m$ for every $\eta \in {}^*\mathbb{N} - \mathbb{N}$.

(2') *General Saturation Principle*: Let κ be an infinite cardinal. A nonstandard extension is called κ-saturated if for every family $\{X_i\}_{i\in I}$, $card(I) < \kappa$, with the infinite intersection property, the intersection $\cap_{i\in I} X_i$ is nonempty, i.e. this intersection contains some internal object.

(3) *Transfer Principle*: Let $\Phi(X_1, \cdots, X_m, x_1, \cdots, x_n)$ be an elementary statement im $V(\mathbb{R})$. Then, for any $A_1, \cdots, A_m \subseteq \mathbb{R}$ and $r_1, \cdots, r_n \in \mathbb{R}$, $\Phi(A_1, \cdots, A_m, r_1, \cdots, r_n)$ is true in $V(\mathbb{R})$ if and only if $\Phi(^*A_1, \cdots, {}^*A_m, {}^*r_1, \cdots, {}^*r_n)$ is true in $V(^*\mathbb{R})$.

$(^*\mathbb{R}, {}^*+, {}^*\cdot, {}^*\leq)$ extends $\mathbb{R}$ as an ordered field. In general we will omit the $*$ for the operations and the order relation.

In $^*\mathbb{R}$ we can distinguish three kinds of numbers:

(a) $x \in {}^*\mathbb{R}$ is infinitesimal, if $|x| < r$ for each $r \in \mathbb{R}^+$.

(b) $x \in {}^*\mathbb{R}$ is a finite number, if there is a real number $r \in \mathbb{R}^+$ such that $|x| < r$.

(c) $x \in {}^*\mathbb{R}$ is infinite number, if $|x| > r$ for each $r \in \mathbb{R}^+$.

For each finite number $x \in {}^*\mathbb{R}$ we can associate a unique real $r := st(x) := {}^o x$, such that $x = r + \epsilon$, where ϵ is infinitesimal. We say that x is infinitely close to y, denoted by $x \approx y$ if and only if $x - y$ is infinitesimal.

In general we use capital letters H, F, X, etc. for internal functions and processes, while h, f, x, etc. are used for standard ones. For stopping times we will always use capital letters, and specify whether standard or nonstandard is meant.

For a given set A, $^*A$ stands for the elementary extension of A, and $ns(^*A)$ denotes the nearstandard points in $^*A$. If s is an element in $ns(^*A)$, the standard part of s is written as $st(s)$, or ${}^o s$. For a given function f, $^*f$ means the elementary extension of f.

We say that the set T is S-dense if $\{{}^o\underline{t} : \underline{t} \in T, {}^o\underline{t} < \infty\} = [0, \infty)$, and we define $ns(T) := \{\underline{t} \in T : {}^o\underline{t} < \infty\}$. With T we denote an internal S-dense subset of $^*[0, \infty)$. The elements of T, or more generally, of $^*[0, \infty)$, are denoted with $\underline{s}$, $\underline{t}$, $\underline{u}$, etc... . The real numbers in $[0, \infty)$ are denoted by s, t, u, etc... We will work with different sets T, so we will always specify the definition of such T.

With $\mathbb{N}$ we denote the set of nonzero natural numbers $\{1, 2, 3, \cdots\}$, and $\mathbb{N}_o = \mathbb{N} \cup \{0\}$. Elements of $\mathbb{N}_o$ are denoted with n, m, l, etc...while elements in $^*\mathbb{N} - \mathbb{N}$ will be denoted with η, N, etc... .

If $(\Omega, \mathfrak{A}, \mu)$ is an internal measure space, the corresponding Loeb space is $\underline{\Omega} = (\Omega, L(\mathfrak{A}), L(\mu))$, and $L(\mu)$ will be the unique measure extending $^o\mu$ to the σ-algebra $\sigma(\mathfrak{A})$ generated by $\mathfrak{A}$. $L(\mathfrak{A})$ will stand for the $L(\mu)$ completion of $\sigma(\mathfrak{A})$.

When we say that $F : A \to B$ is an internal function, we mean that the domain, the range and the graph of the function are internal concepts.

In order to simplify the notation and some of the proofs, in this paper we will consider stochastic processes defined on $[0, 1]^2$ with values in $\mathbb{R}$ instead of processes defined on $[0, \infty)^2$ with values on $\mathbb{R}^d$. In general we should consider nearstandard points on $^*[0, \infty)^2$. If T is an S-dense set on $[0, \infty)$, then an internal stochastic process $X : T^2 \times \Omega \to {}^*R^d$ should have a property if and only if each of its components has. Therefore we may reduce the proofs to the one dimensional case.

The set $[0, 1]^2$ is equipped with the partial orders:

$$(s_1, t_1) \leq (s_2, t_2) \Leftrightarrow s_1 \leq s_2 \text{ and } t_1 \leq t_2,$$

$$(s_1, t_1) \underline{\triangle} (s_2, t_2) \Leftrightarrow s_1 \leq s_2 \text{ and } t_1 \geq t_2;$$

we will use the notation $(s_1, t_1) < (s_2, t_2)$ to express that $(s_1, t_1) \leq (s_2, t_2)$ and $s_1 < s_2$ or $t_1 < t_2$, whereas $(s_1, t_1) \wedge (s_2, t_2)$ will mean $(s_1, t_1) \underline{\triangle} (s_2, t_2)$ and $s_1 < s_2$ or $t_1 > t_2$ and $(s_1, t_1) << (s_2, t_2)$ means that $s_1 < s_2$ and $t_1 < t_2$.

Let $(\Omega, \mathfrak{F}, P)$ a measure space. A standard filtration in two parameters is a filtration that satisfies the following conditions:

F1 - For (s, t), (s', t') in $[0, 1]^2$ such that $s \leq s'$, $t \leq t'$, then $\mathfrak{F}_{(s,t)} \subseteq \mathfrak{F}_{(s',t')}$.

F2 - $\mathfrak{F}_{(0,0)}$ is P- complete.

F3 - For each (s, t), $\mathfrak{F}_{(s,t)} = \bigcap_{(s',t') >> (s,t)} \mathfrak{F}_{(s',t')}$.

Additionally we say that the filtration satisfies $F4$, or Cairoli-Walsh condition, if for (s, t) and (s', t') such that $s \leq s'$ and $t \geq t'$ then $\mathfrak{F}_{(s,t)}$ and $\mathfrak{F}_{(s',t')}$ are conditionally independent. For conditional independence we will use the equivalent condition: if (s, t) and (s', t') are such that $s \leq s'$ and $t \geq t'$ and X is an $\mathfrak{F}_{(s',t)}$ - measurable random variable, then $E(X|\mathfrak{F}_{(s,t)}) = E(X|\mathfrak{F}_{(s,t')})$.

Condition F4 is equivalent to each one of the following:

(a) If $(s, t) \underline{\triangle} (s', t')$ and X is a random variable, then

$$E(E(X|\mathfrak{F}_{(s,t)})|\mathfrak{F}_{(s',t')}) = E(E(X|\mathfrak{F}_{(s',t')})|\mathfrak{F}_{(s,t)}) = E(X|\mathfrak{F}_{(s,t')}).$$

(b) If $(s, t) \underline{\triangle} (s', t')$ and X is an $\mathfrak{F}_{(s',t')}$-measurable random variable, then

$$E(X|\mathfrak{F}_{(s,t)}) = E(X|\mathfrak{F}_{(s,t')}).$$

Given an internal probability space $(\Omega, \mathfrak{B}, \overline{P})$, $(\Omega, L(\mathfrak{B}), P)$ denotes the corresponding Loeb space; that is, $L(\mathfrak{B})$ is the external complete σ-algebra generated by $\mathfrak{B}$ and P is the unique σ-aditive extension of $st(\overline{P})$ to $L(\mathfrak{B})$.

1. Definition.

(i) Let $L \in {}^*\mathbb{N} - \mathbb{N}$, $N = L!$, $\delta t = 1/N$. The hyperfinite line is
$T = \{0, \delta t, 2\delta t, ..., (N-1)\delta t, 1\}$.

(ii) Let $\Omega = \{-1,1\}^{T^2} = \{w : T^2 \longrightarrow \{-1,1\} |\ w \text{ is internal }\}$. The internal hyperfinite cardinality of Ω is $2^{(N+1)^2}$.

(iii) Given $(\underline{s}, \underline{t}) \in T^2$, we define on Ω the equivalence relation:

$$w \approx_{(\underline{s},\underline{t})} w' \Leftrightarrow w(\underline{s}', \underline{t}') = w'(\underline{s}', \underline{t}')$$

for all $(\underline{s}', \underline{t}') \leq (\underline{s}, \underline{t})$, $(\underline{s}', \underline{t}') \in T^2$, where $w, w' \in \Omega$. We denote by $[w]_{(\underline{s},\underline{t})}$ the equivalence class of w with respect to this equivalence relation.

(iv) Using the last equivalence relation we define for $(\underline{s}, \underline{t}) \in T^2$,

$$\mathfrak{B}_{(\underline{s},\underline{t})} = \{A \subseteq \Omega | A \text{ is internal and closed under } \approx_{(\underline{s},\underline{t})}\}.$$

This is an internal ${}^*\sigma$-algebra.

(v) An internal two parameter filtration is an internal family $\{\mathfrak{B}_{(\underline{s},\underline{t})}) : (\underline{s}, \underline{t}) \in T^2\}$ of internal *sub-σ-algebras of $\mathfrak{B}$ that satisfy property $\overline{F1}$ (that is, the corresponding property F1 in the nonstandard sense).

The filtration is $\overline{P}$-complete if $\mathfrak{B}_{(0,0)}$ is complete.

2. Definition. Let $(\Omega, \mathfrak{A}, \overline{P})$ denote an internal probability space and let

$$(\Omega, \mathfrak{F}, P) = (\Omega, L(\Omega), L(\overline{P})).$$

As we have seen in (v) of the above definition, an internal filtration on T^2 is a collection of *sub-σ-algebras of $\mathfrak{A}$: $\{\mathfrak{B}_{(\underline{s},\underline{t})} : (\underline{s}, \underline{t}) \in T^2\}$ such that, whenever $(\underline{s}, \underline{t}) \leq (\underline{s}', \underline{t}')$, then $\mathfrak{B}_{(\underline{s},\underline{t})} \subseteq \mathfrak{B}_{(\underline{s}',\underline{t}')}$.

The standard part of $\{\mathfrak{B}_{(\underline{s},\underline{t})}\}$ is the filtration $\{\mathfrak{F}_{(s,t)} : (s,t) \in [0,1]^2\}$ defined by

$$\mathfrak{F}_{(s,t)} = \left(\bigcap_{\substack{{}^\circ(\underline{s},\underline{t}) >> (s,t) \\ (\underline{s},\underline{t}) \in T^2}} \sigma\left(\mathfrak{B}_{(\underline{s},\underline{t})}\right) \right) \bigvee \mathfrak{N},$$

where $\mathfrak{N}$ is the class of P-null sets of $\mathfrak{F}$.

3. Proposition. *The standard filtration* $\{\mathfrak{F}_{(s,t)}\}_{(s,t)\in[0,1]^2}$ *satisfies properties F1 to F4.*

Proof. From the definition F1 and F2 are obvious, and the proof of F4 can be found in Dalang [4]. Let us show F3, i.e., that $\mathfrak{F}(s,t) = \bigcap_{(s',t')>>(s,t)} \mathfrak{F}_{(s',t')}$.

By F1, $\mathfrak{F}_{(s,t)} \subseteq \mathfrak{F}_{(s',t')}$ for all $(s',t') >> (s,t)$. Then, $\mathfrak{F}_{(s,t)} \subseteq \bigcap_{(s',t')>>(s,t)} \mathfrak{F}_{(s',t')}$.

On the other hand, if $A \in \bigcap_{(s',t')>>(s,t)} \mathfrak{F}_{(s',t')}$, it follows that $A = B \cap C$, where

$$B \in \bigcap_{^o(\underline{s}',\underline{t}')>>(s',t')} \sigma\left(\mathfrak{B}_{(\underline{s}',\underline{t}')}\right), \ (\underline{s}',\underline{t}') \in T^2$$

and $C \in \mathfrak{N}$ for all $(s',t') >> (s,t)$. Then $B \in \sigma(\mathfrak{B}_{(\underline{s}',\underline{t}')})$ for all $^o(\underline{s}',\underline{t}') >> (s',t')$, provided $(s',t') >> (s,t)$. Then $B \in \sigma(\mathfrak{B}_{(\underline{s}',\underline{t}')})$ for all $^o(\underline{s}',\underline{t}') >> (s,t)$ which implies that

$$B \in \bigcap_{^o(\underline{s}',\underline{t}')>>(s,t)} \sigma\left(\mathfrak{B}_{(\underline{s}',\underline{t}')}\right),$$

$(\underline{s}',\underline{t}') \in T^2$. Finally $A = B \cap C \in \mathfrak{F}_{(s,t)}$, and thus we have F3.

4. Definition. A function $x : [0,1]^2 \longrightarrow \mathbb{R}$ is a *larc* in $[0,1]^2$, if for each $(s_o, t_o) \in [0,1]^2$ the quadrantal limits exist and satisfy:

$$\lim_{\substack{s\to s_o^+\\ t\to t_o^+}} x(s,t) = x(s_o,t_o), \qquad \lim_{\substack{s\to s_o^+\\ t\to t_o^-}} x(s,t) = x(s_o,t_o^-),$$

$$\lim_{\substack{s\to s_o^-\\ t\to t_o^+}} x(s,t) = x(s_o^-,t_o), \qquad \lim_{\substack{s\to s_o^-\\ t\to t_o^-}} x(s,t) = x(s_o^-,t_o^-).$$

We denote with D^2 the set of all larcs in $[0,1]^2$.

Note: the points in $[0,1]^2$ will be denoted in general by (s,t), (s_1,t_1), $\cdots$ and the points in $^*[0,1]^2$ by $(\underline{s},\underline{t})$, $(\underline{s}_1,\underline{t}_1)$, $\cdots$.

For each point $(\underline{s},\underline{t}) \in {}^*[0,1]^2$ let us consider the following sets:

$$Q^1_{(\underline{s},\underline{t})} = \{(\underline{u},\underline{v}) \in {}^*[0,1]^2 : \underline{u} \geq \underline{s} \text{ and } \underline{v} \geq \underline{t}\},$$

$$Q^2_{(\underline{s},\underline{t})} = \{(\underline{u},\underline{v}) \in {}^*[0,1]^2 : \underline{u} < \underline{s} \text{ and } \underline{v} \geq \underline{t}\},$$

$$Q^3_{(\underline{s},\underline{t})} = \{(\underline{u},\underline{v}) \in {}^*[0,1]^2 : \underline{u} < \underline{s} \text{ and } \underline{v} < \underline{t}\},$$

$$Q^4_{(\underline{s},\underline{t})} = \{(\underline{u},\underline{v}) \in {}^*[0,1]^2 : \underline{u} \geq \underline{s} \text{ and } \underline{v} < \underline{t}\}.$$

5. Definition. Let $F \in {}^*D^2$ be such that $F(\underline{s},\underline{t}) \in ns({}^*\mathbb{R})$, for $(\underline{s},\underline{t}) \in {}^*[0,1]^2$.

(a) F is of class SD^2 if for each $(s,t) \in [0,1]^2$ there are points $(\underline{s}_1,\underline{t}_1) \approx (\underline{s}_2,\underline{t}_2) \approx (\underline{s}_3,\underline{t}_3) \approx (\underline{s}_4,\underline{t}_4) \approx (s,t)$ such that:

(i) If $(\underline{u}_1,\underline{v}_1) \approx (s,t), (\underline{u}_1,\underline{v}_1) \in Q^1_{(\underline{s}_1,\underline{t}_1)}$, then $F(\underline{u}_1,\underline{v}_1) \approx F(\underline{s}_1,\underline{t}_1)$.
(ii) If $(\underline{u}_2,\underline{v}_2) \approx (s,t), (\underline{u}_2,\underline{v}_2) \in Q^2_{(\underline{s}_2,\underline{t}_2)}$, then $F(\underline{u}_2,\underline{v}_2) \approx F(\underline{s}_2^-,\underline{t}_2)$.
(iii) If $(\underline{u}_3,\underline{v}_3) \approx (s,t), (\underline{u}_3,\underline{v}_3) \in Q^3_{(\underline{s}_3,\underline{t}_3)}$, then $F(\underline{u}_3,\underline{v}_3) \approx F(\underline{s}_3^-,\underline{t}_3^-)$.
(iv) If $(\underline{u}_4,\underline{v}_4) \approx (s,t), (\underline{u}_4,\underline{v}_4) \in Q^4_{(\underline{s}_4,\underline{t}_4)}$, then $F(\underline{u}_4,\underline{v}_4) \approx F(\underline{s}_4,\underline{t}_4^-)$.

(b) F is of class SD^2J, or a *larc lift*, if (a) holds with $(\underline{s}_1,\underline{t}_1) = (\underline{s}_2,\underline{t}_2) = (\underline{s}_3,\underline{t}_3) = (\underline{s}_4,\underline{t}_4)$ and $F(\underline{s},\underline{t}) \approx F(0,0)$ for all $(\underline{s},\underline{t}) \approx (0,0)$ in ${}^*[0,1]^2$.
(c) F is S-continuous (SC) if $F(\underline{s},\underline{t}) \approx F(\underline{u},\underline{v})$ whenever $(\underline{s},\underline{t}) \approx (\underline{u},\underline{v})$; $(\underline{s},\underline{t}),(\underline{u},\underline{v}) \in T^2$, where $T = \{k\delta t : \delta t = \frac{1}{N!}, N \in {}^*\mathbb{N} - \mathbb{N}, k = 0,1,\cdots,N!\}$.

A function $F : T^2 \longrightarrow {}^*\mathbb{R}$ is of class SD^2 (SD^2J, SC) in T^2 if it is the restriction to T^2 of an SD^2 (SD^2J, SC) function F on ${}^*[0,1]^2$.

6. Definition. The standard part of an SD^2 function F on T^2 is the function $st(F)$ defined by:

$$st(F)(s,t) = \lim_{{}^o(\underline{s},\underline{t}) \downarrow (s,t)} {}^oF(\underline{s},\underline{t}), \qquad (\underline{s},\underline{t}) \in T^2.$$

We say that $\overline{X}$ is a lifting of X if $st(X) = X$ a.s.

2. Stopping Domains

We first recall the definitions from Cairoli and Walsh [3].

7. Definition.

A set $A \subseteq [0,1]^2 \times \Omega$ is *adapted* with respect to $\mathfrak{F}_{(s,t)}$ if A is measurable and for each $(s,t) \in [0,1]^2$ the set $A_{(s,t)} = \{w : (s,t,w) \in A\}$ is $\mathfrak{F}_{(s,t)}$-measurable.

A process $x : [0,1]^2 \times \Omega \to \mathbb{R}$ is adapted if $x(s,t,\cdot)$ is $\mathfrak{F}_{(s,t)}$-measurable for each $(s,t) \in [0,1]^2$.

A process $\{X_{(s,t)}; (s,t) \in \mathbb{R}^2_+\}$ is *progressive* (or progressively measurable) if, for all $(s,t) \in \mathbb{R}^2_+$, the map $(\zeta,w) \to X_\zeta(w) I_{\{\zeta < (s,t)\}}$ is $\mathfrak{B} \times \mathfrak{F}_{(s,t)}$-measurable, where $\mathfrak{B}$ is the Borel σ-algebra in $\mathbb{R}^2_+$ and $\{\mathfrak{F}_{(s,t)}\}$ is the filtration in Ω.

Let $A : w \to A(w)$ be a mapping from Ω to $\mathfrak{P}(\mathbb{R}^2_+)$. A is a random set if, for all $(s,t) \in \mathbb{R}^2_+$, $I_A(s,t)$ is a random variable. A random set A is adapted (resp.

progressive) if the process $\{I_A(s,t), (s,t) \in \mathbb{R}^2_+\}$ is adapted (resp. progressive). If A is progressive, $\{w : A(w) \cap R_{(s,t)} \neq \emptyset\} \in \mathfrak{F}_{(s,t)}$, where $R_{(s,t)}$ denotes the rectangle $[(0,0),(s,t)]$.

8. Definition. A random variable $Z : \Omega \to \mathbb{R}^2_+ \cup \{\infty\}$ is a *stopping point* if, for all $(s,t) \in \mathbb{R}^2_+$, the set $\{Z < (s,t)\} \in \mathfrak{F}_{(s,t)}$.

9. Definition. C is a *stopping domain* if:

(1) C is a progressive random set
(2) $Z \in C$ for $\{C \neq \emptyset\}$, where $Z = \inf C$
(3) If $(s,t) \in C$, then $[Z,(s,t)] \subseteq C$.

If C is a stopping domain, define the set $int(C)$ by

$$int(C) = \{(s,t) : \exists (s',t') \in C \text{ with } (s,t) << (s',t')\},$$

and the set L by

$$L = C - int(C).$$

L is the *stopping* line associated with C.

10. Definition. Let Z be a stopping point. We say that C is a stopping neighborhood of Z, if:

(1) C is a stopping domain and $Z = \inf C$
(2) $C = \overline{int(C)}$; that is, C is the closure of $int(C)$.

11. Proposition. *Let C be a random set. Take $Z = \inf C$ and suppose that:*

(i) *$int(C)$ is dense in C*
(ii) *$Z \in C$ on $\{C \neq \emptyset\}$*
(iii) *If $(s,t),(s',t') \in C$ and $(s,t) < (s',t')$, then $[(s,t),(s',t')] \subseteq C$*
(iv) *For all $(s,t) \in \mathbb{R}^2_+$, $\{(s,t) \in C\} \in \mathfrak{F}_{(s,t)}$.*

Then, C is a stopping neighborhood of Z.

The proof is in [3].

For an internal set A we can transfer all the above definitions, so that we can speak about an internal random set or about a nonanticipanting random set, where nonanticipanting is the internal version of adapted.

We mention without proof the following result in [7], Theorem 2.11.

12. Theorem. *(Keisler) A stochastic process x is progressively measurable if and only if it has a lifting X which is nonanticipanting.*

Remark 1. If A is nonanticipanting with respect to the internal filtration $\{\mathfrak{B}_{(\underline{s},\underline{t})}\}$ we have that $\{w : A(w) \cap R_{(s,t)} \neq \emptyset\} \in \mathfrak{B}_{(\underline{s},\underline{t})}$. In fact, A is nonanticipanting means that $I_{A(w)}(\underline{s},\underline{t}) = I_{A(w')}(\underline{s},\underline{t})$, when $w \approx_{(\underline{s},\underline{t})} w'$ which means that

$$\{w : (\underline{s},\underline{t}) \in A(w)\} = [w]_{(\underline{s},\underline{t})} \in \mathfrak{B}_{(\underline{s},\underline{t})}.$$

If $A(w) \cap R_{(\underline{s},\underline{t})} \neq \emptyset$, let $(\underline{s}',\underline{t}') \in A(w) \cap R_{(\underline{s},\underline{t})}$. Then $I_{A(w)}(\underline{s}',\underline{t}') = 1 = I_{A(w')}(\underline{s}',\underline{t}')$, where $w \approx_{(\underline{s}',\underline{t}')} w'$, which implies that $\{w : A(w) \cap R_{(\underline{s},\underline{t})} \neq \emptyset\}$ is a hyperfinite union of equivalent classes $[w]_{(\underline{s}',\underline{t}')}$ and therefore, an element of $\mathfrak{B}_{(\underline{s},\underline{t})}$.

13. Definition. Let $C : w \to C(w)$ be an internal map from Ω into the internal subsets of T^2. An internal two parameter stopping domain C is an internal, random, nonanticipanting subset of T^2, such that:

(i) If $\overline{Z} = \inf C$, then $\overline{Z} \in C$

(ii) If $(\underline{s},\underline{t}) \in C$, $[\overline{Z}, (\underline{s},\underline{t})] \cap T^2 \subseteq C$.

Let $A \subseteq T^2$ and $int(A) = \{(\underline{s},\underline{t}) \in T^2 : \exists (\underline{s}',\underline{t}') \in A, (\underline{s},\underline{t}) << (\underline{s}',\underline{t}')\}$. Let $L = C - int(C)$ be the stopping line associated with C. L and C are said to be bounded if there exists a $(s,t) \in \mathbb{R}^2_+$ such that $C \subseteq {}^*R_{(s,t)}$ a.s.

14. Definition. An internal random variable $\underline{Z} : \Omega \to T^2$ is an internal stopping point if when $(\underline{s},\underline{t}) \in T^2$, $\{\underline{Z} \leq (\underline{s},\underline{t})\} \in \mathfrak{B}_{(\underline{s},\underline{t})}$.

Note that if $\underline{Z} = \inf C$ then $\{\underline{Z} \leq (\underline{s},\underline{t})\} = \{C \cap R_{(\underline{s},\underline{t})} \neq \emptyset\} \in \mathfrak{B}_{(\underline{s},\underline{t})}$, so that $\underline{Z}$ is an internal stopping point.

15. Definition. Let $\underline{Z}$ be an internal stopping point. We say that C is an internal stopping neighborhood of $\underline{Z}$ if C is an internal stopping domain and $\underline{Z} = \inf C$.

16. Proposition. *$U : \Omega \to [0,1]^2$ is an $\mathfrak{F}_{(s,t)}$-stopping point if and only if $U = {}^\circ V$ a.s. for some $\mathfrak{B}_{(\underline{s},\underline{t})}-$ stopping point V.*

Proof. $\Rightarrow$) Let $z : [0,1]^2 \times \Omega \to \{0,1\}$ be defined by

$$z(s,t,w) = \begin{cases} 1, & (s,t) \geq U(w), \\ 0, & \text{otherwise}. \end{cases}$$

Then $z(s,t)$ is $\mathfrak{F}_{(s,t)}$-adapted. In fact,

$$\{w : z(s,t)(w) = 1\} = \{w : U(w) \leq (s,t)\} \in \mathfrak{F}_{(s,t)}$$

and has sample paths in D^2. Then, by Theorem 2.2.6 in [9], there exists an SD^2J lifting

$$\overline{Z} : (T')^2 \times \Omega \to \{0,1\}$$

such that $T' = \{k\Delta' t : k \in {}^*\mathbb{N}, k\Delta' t \leq 1\} \cup \{1\}$ for some $\Delta' t \in T$, $\Delta' t \approx 0$, and for all $(\underline{s},\underline{t}) \in (T')^2$, $\overline{Z}_{(\underline{s},\underline{t})}$ is $\mathfrak{B}_{(\underline{s}\vee\Delta' t, \underline{t}\vee\Delta' t)}$-nonanticipanting. Let $A(w) = \{(\underline{s},\underline{t}) : \overline{Z}(\underline{s},\underline{t},w) = 1\}$; A is nonanticipanting. Since $\overline{Z}$ is SD^2J, there exists $V'(w) = \inf\{(\underline{s},\underline{t}) : \overline{Z}(\underline{s},\underline{t},w) = 1\}$ ($\inf \emptyset = (1,1)$), and

$$\begin{aligned} \{V' < (\underline{s}',\underline{t}')\} &= \{w : \inf\{(\underline{s},\underline{t}) : \overline{Z}(\underline{s},\underline{t},w) = 1\} < (\underline{s}',\underline{t}')\} \\ &= \{w : A(w) \cap R_{(\underline{s}',\underline{t}')} \neq \emptyset\}. \end{aligned}$$

Since $A(w)$ is nonanticipanting, then $\{w : A(w) \cap R_{(\underline{s},\underline{t})} \neq \emptyset\} \in \mathfrak{B}_{(\underline{s}\vee\Delta' t, \underline{t}\vee\Delta' t)}$. We also have ${}^oV' = U$ a.s., so that $V = V' \vee \Delta' t$ is the desired *stopping point.

$\Leftarrow$) If $U = {}^oV$ a.s. for some internal $\mathfrak{B}_{(\underline{s},\underline{t})}$ stopping point V. Then U is an $\mathfrak{F}_{(s,t)}$-stopping point for $(s,t) \approx (\underline{s},\underline{t})$, $(s,t) \in [0,1]^2$. In fact:

$$\{w : {}^oV(w) \leq {}^o(\underline{s},\underline{t})\} = \bigcap_{n=1}^{\infty} \{w : V(w) \leq (\underline{s},\underline{t}) + (1/n, 1/n)\}$$
$$\in \bigcap_{n=1}^{\infty} \sigma(\mathfrak{B}_{(\underline{s},\underline{t})+(1/n,1/n)}) = \mathfrak{F}_{{}^o(\underline{s},\underline{t})}.$$

17. Proposition. *Suppose that $X : T^2 \times \Omega \to {}^*\mathbb{R}$ is an internal SD^2J stochastic process, $x = st(X)$ a.s. and $U : \Omega \to [0,1]^2$ is $\mathfrak{F}$-measurable. Then, there is an internal $\mathfrak{A}$-measurable map $V : \Omega \to (T')^2$ and a P-null set N such that, if $w \notin N$, then ${}^oV(w) = U(w)$, and if $(\underline{s},\underline{t}) \approx U(w)$ and $(\underline{s},\underline{t}) \geq V(w)$, then ${}^oX(\underline{s},\underline{t},w) = x(U(w),w)$. If U is an $\mathfrak{F}_{(s,t)}$-stopping point, V may be chosen to be an internal $\mathfrak{B}_{(\underline{s},\underline{t})}$ – *stopping point, and if U is a constant, then V may be chosen to be a constant.*

Proof. Extend X to ${}^*[0,1]^2 \times \Omega$ by setting

$$X(\underline{u},\underline{v},w) = X(\underline{s},\underline{t},w),\ (\underline{u},\underline{v}) \in [(\underline{s},\underline{t}),(\underline{s}+\Delta t,\underline{t}+\Delta t)),\ (\underline{s},\underline{t}) \in T^2.$$

Then $x(U(w),w))$ is $\mathfrak{F}$-measurable, so that there are a lifting Y, of $x(U)$, and $U' : \Omega \to (T')^2$, a lifting of U. We have that ${}^oY = st(X)({}^oU')$ a.s., so that we may choose a sequence $\{\epsilon_n\}_{n\in\mathbb{N}}$ and $n_o \in \mathbb{N}$ such that $\frac{1}{n+n_o} < {}^o\epsilon_n < 1/n$ and

$$P\left(\sup_{0<{}^o\epsilon\leq{}^o\epsilon_n} {}^o|Y - X(U' + (\epsilon,\epsilon))| \geq 1/n\right) < 1/n.$$

Let $h = n_o + n$ and

$$D = \left\{k \in {}^*\mathbb{N} : \overline{P}\left(\sup_{0<\frac{1}{k+h}\leq\epsilon_n} \left|Y - X\left(U' + (\frac{1}{k+h}, \frac{1}{k+h})\right)\right| \geq 1/n\right) < 1/n\right\}.$$

Then $D \supseteq \mathbb{N}$, so that by the Overflow there exists an infinitesimal $\underline{\delta}_n \in T$ such that

$$\overline{P}\left(\sup_{\underline{\delta}_n\leq\epsilon\leq\epsilon_n} |Y - X(U' + (\epsilon,\epsilon))| \geq 1/n\right) < 1/n.$$

We can extend the sequence $\{\underline{\delta}_n : n \in \mathbb{N}\}$ to ${}^*\mathbb{N}$, and we can find $\nu \in {}^*\mathbb{N} - \mathbb{N}$ such that $\underline{\delta} = \max_{n\leq\nu} \underline{\delta}_n \approx 0$, and $\underline{\delta} \in T$, such that

$$\overline{P}\left\{w : \sup_{\underline{\delta}\leq\epsilon\leq\epsilon_\nu} |Y - X(U' + (\epsilon,\epsilon))| > 0\right\} \approx 0.$$

Therefore,

$$P\left\{w : \sup_{\underline{\delta} \leq \epsilon \leq \epsilon_\nu} |x(U) - {}^oX(U' + (\epsilon, \epsilon))| > 0\right\} = 0.$$

Then $N_1 = \{w : \sup_{\delta \leq \epsilon \leq \epsilon_\nu} {}^o|X(U + (\epsilon, \epsilon)) - Y| > 0\}$ is a P-null set.

Let $V = U' + (\delta, \delta)$. Then $V : \Omega \to (T')^2$ is the desired lifting. In fact,

$$N = N_1 \cup \{w : {}^oV \neq U \vee {}^oY \neq x(U)\}$$

is a null set. Let $w \notin N$, $(\underline{s}, \underline{t}) \approx U(w)$ and $(\underline{s}, \underline{t}) \geq V(w)$. Then

$$\begin{aligned} |x(U) - {}^oX(\underline{s}, \underline{t})| &= {}^o|Y - X(\underline{s}, \underline{t})| \\ &\leq \sup_{\delta \leq \epsilon \approx 0} |{}^oY - X(U' + (\epsilon, \epsilon))| = 0 \text{ a.s} \end{aligned}$$

If U is a stopping point, from Proposition 16 U' may be chosen to be an internal *stopping point. Therefore, $V = U' + (\delta, \delta)$ is a $\mathfrak{B}_{(\underline{s}+\delta, \underline{t}+\delta)}$-*stopping point. Similarly, if U is constant, V may be chosen to be a constant mapping.

Remark 2. Let C be an internal stopping neighborhood and $\underline{Z} = \inf C$. Then, from the properties of internal sets and the Propositions above, we obtain that:

(1) C is a random nonanticipanting set and $st(C)$ is a closed random progressive set
(2) $Z = \inf st(C) = st(\inf C) \in st(C)$
(3) If $(u, v) \in st(C)$, $[Z, (u, v)] \subseteq st(D)$
(4) $int(st(C)) = \{(s, t) \in st(C) : \exists(u, v) \in st(C),\ (s, t) << (u, v)\}$, $st(C)$ is closed and $\overline{int(st(C))} = st(C)$. Then it follows that $st(C)$ is a stopping neighborhood.

18. Proposition. *A mapping $U : w \to U(w) \subseteq [0, 1]^2$ is a stopping neighborhood of Z if and only if $U = st(C)$ a.s. for some internal $\mathfrak{B}_{(\underline{s}, \underline{t})}$ *neighborhood C of $\underline{Z}$, an internal *stopping point, where $st(\underline{Z}) = Z$.*

Proof. $\Rightarrow$) Define $y : [0, 1]^2 \times \Omega \to \{0. 1\}$ by

$$y(s, t, w) = \begin{cases} 1, & (s, t) \in U(w), \\ 0, & (s, t) \notin U(w). \end{cases}$$

Then $y(s, t, w) = I_{U(s,t)}(w) = I_{U(w)}(s, t)$ is a progressive and measurable process with paths in D^2, so that it has a nonanticipanting lifting

$$Y : (T')^2 \times \Omega \to \{0, 1\}.$$

Let $C'(w) = \{(\underline{s}, \underline{t}) : Y(\underline{s}, \underline{t}, w) = 1\}$. Then

$$I_{C'(\underline{s},\underline{t})}(w) = I_{C'(w)}(\underline{s}, \underline{t}) = Y(\underline{s}, \underline{t}, w),$$

so that C' is nonanticipanting. Define $\overline{Z}(w) = \inf C'(w)$.

Let
$$C(w) = \cup\{[\overline{Z}, (\underline{s}, \underline{t})] \cap T^2 : (\underline{s}, \underline{t}) \in C'(w)\}.$$
Then $\overline{Z} \in C$ and $[\overline{Z}, (\underline{s}, \underline{t})] \subseteq C$ for all $(\underline{s}, \underline{t}) \in C$. Since C' is nonanticipanting then $\overline{Z}$ is a stopping point, and therefore C is an internal stopping neighborhood, and ${}^oC = U$.

$\Leftarrow$) Is obvious from the Remark 2.

19. Proposition. *Let $X : T^2 \times \Omega \to {}^*\mathbb{R}$ be a 2S-integrable internal $\Delta't$-martingale or an internal bounded variation stochastic process of class SD^2, $x = st(X)$ a.s. and $U : w \to U(w)$ be a random set. Then, there is an internal random set $V : w \to V(w) \subseteq (T')^2$ and a P-null set N such that if $w \notin N$ then ${}^oV(w) = U(w)$. If, in addition, $(\underline{s}, \underline{t}) \approx (s,t) \in U(w)$, $(\underline{s}, \underline{t}) \geq (s,t)$, and $(\underline{s}, \underline{t}) \in V(w)$, then ${}^oX((\underline{s}, \underline{t}, w) = x(s,t,w)$. If U is a stopping neighborhood, V may be chosen to be an internal stopping neighborhood. If U is a constant set, V may be chosen to be an internal constant set.*

Proof. Extend X to ${}^*[0,1]^2 \times \Omega$ by setting

$$X(\underline{u}, \underline{v}, w) = X(\underline{s}, \underline{t}, w),\ (\underline{u}, \underline{v}) \in [(\underline{s}, \underline{t}), (\underline{s} + \Delta t, \underline{t} + \Delta t)),\ (\underline{s}, \underline{t}) \in T^2.$$

Define $h(s,t,w) = I_{U(w)}(s,t)$. Then h is a stochastic process. Let H be a lifting of h. Then

$${}^oH(\underline{s}, \underline{t}, w) = h({}^o\underline{s}, {}^o\underline{t}, w), \quad L(T^2 \times \Omega) \text{ a.s.},$$

by the Fubini Theorem (Keisler Theorem 1.14 (b) (i) in [7]). If $(\underline{s}, \underline{t}) \in T' \subseteq T^2$, T' with measure 1, then $H(\underline{s}, \underline{t}, \cdot)$ is Loeb measurable in Ω and ${}^oH(\underline{s}, \underline{t}, w) = h({}^o\underline{s}, {}^o\underline{t}, w)$ $L(\Omega)$-a.s.

Define $U'(w) = \{(\underline{s}, \underline{t}) \in T' : H(\underline{s}, \underline{t}, w) = 1\}$. Then ${}^oU'(w) = U(w)$ for almost all w. In fact, if $(\underline{s}, \underline{t}) \in U'(w)$, ${}^oH(\underline{s}, \underline{t}, w) = h({}^o\underline{s}, {}^o\underline{t}, w) = 1$, so that $({}^o\underline{s}, {}^o\underline{t}) \in U(w)$. Then, so ${}^oU'(w) = U(w)$ a.s.

Let Y be a lifting of $x \cdot h$. We may choose as in Proposition 17, $\underline{\delta} \approx 0$, $\underline{\delta} \in T'$, such that

$$N_1 = \{w : \sup_{\underline{\delta} \leq \epsilon \approx 0} {}^o|Y - (X \cdot H)((\underline{s}, \underline{t}) + (\epsilon, \epsilon))| > 0|\}$$

is a P-null set. Let $V = U' + (\underline{\delta}, \underline{\delta})$. Then

$$N = N_1 \cup \{w : {}^oV(w) \neq U(w) \vee {}^oY \neq x \cdot h\}$$

is a P-null set. Let $w \notin N$ and $(\underline{s}, \underline{t}) \approx (s,t) \in U(w)$, $(\underline{s}, \underline{t}) \in V(w)$. Then

$$|x(s,t) - {}^{o}X(\underline{s},\underline{t})| = |x \cdot h(s,t) - ({}^{o}X \cdot H)(\underline{s},\underline{t})| = {}^{o}|Y(\underline{s},\underline{t}) - (X \cdot H)(\underline{s},\underline{t})|$$
$$\leq \sup_{\underline{\delta} \leq \epsilon \approx 0} {}^{o}|Y(\underline{s},\underline{t}) - (X \cdot H)((\underline{s},\underline{t}) + (\epsilon,\epsilon))| = 0.$$

Thus V is the required set.

If U is a stopping neighborhood, then by the Proposition 18, U' may be chosen to be a $\mathfrak{B}_{(\underline{s},\underline{t})}$-*stopping neighborhood. Then $V = U' + (\underline{\delta},\underline{\delta})$ also is a $\mathfrak{B}_{(\underline{s},\underline{t})}$-*stopping neighborhood (T' is closed under addition). If U is a constant set, V may be chosen to be an internal constant set.

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