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Analytical solutions of Fokker-Planck equation in ranking processes

Soluciones analíticas para ecuaciones de Fokker-Planck en procesos de *ranking*

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ABSTRACT

We consider the Fokker-Planck equation of the dynamics of ranking processes (N. Blumm *et al*. 2012 Dynamics of Ranking Processes in Complex Systems Phys. Rev. Lett. 109). On the one hand, we have generalized and enhanced the cases for the solutions of the steady-state. On the other, we have calculated some particular solutions for the transient regime. Moreover, we discuss the consistency of a normalization parameter in a special case.

KEYWORDS: *Ranking processes, Fokker-Planck equation.*

RESUMEN

Se considera la ecuación de Fokker-Planck que aparece en la dinámica de un proceso de *ranking* (N. Blumm *et al*. 2012 Dynamics of Ranking Processes in Complex Systems Phys. Rev. Lett. 109). Por un lado, se generalizan y se amplían los casos para la solución del estado estacionario. Por otro, se calculan algunas soluciones particulares para el estado transitorio. Asimismo, se discute la consistencia de un parámetro de normalización para un caso especial.

PALABRAS CLAVE: *Procesos de ranking, ecuación de Fokker-Planck*.

INTRODUCTION

Ranking lists appear in a large variety of contexts: from sports leagues to scientific journal citations. In 2012, Blumm *et al*. [1], developed a mathematical model for the ranking dynamics. Recently, this model has been applied to human microbiota so as to evaluate microbial temporal stability and to relate it with health status [2]. Furthermore, it can be applied to the rank dynamics of word usage in different languages [3].

To understand Blumm's model, let's consider that we have a list of n numbered items $i = 1,..., n$, each one with an assigned score $X_i(t) \geq 0$ at time t. Then, the normalized score is defined as

$$x_i(t) = \frac{X_i(t)}{\sum_{i=1}^{n} X_i(t)} \geq 0. \tag{1}$$

Notice that the stability of an item's score does not imply the stability in ranking, since the rank is a relative measure of the score of all items in the system. Blumm's model assumes that the dynamic of the normalized score follows Langevin's equation,

$$\dot{x}_i = f(x_i) + g(x_i)\xi_i(t) - \phi(t)x_i. \tag{2}$$

The drift term $f(x_i)$ represents the deterministic mechanisms that drive the score of item i. The stochastic term $g(x_i)\ \xi_i(t)$ represents the randomness in the system, where $\xi_i(t)$ is a Gaussian noise with zero mean

$$\langle \xi_i(t) \rangle = 0, \tag{3}$$

and with variance uncorrelated in $<\xi_i(t)\xi_i(t')> = \delta(t - t')$. Also, $g(x_i)$ denotes the noise amplitude. The variable $\phi(t)$ ensures the normalization of the scores. Indeed, from (1) we have $\forall\, t$ that

$$\sum_{i=1}^{n} x_i = 1, \tag{4}$$

thereby, from (2), we obtain

$$\phi(t) = \phi_0 + \eta(t),$$

where the global drift is

$$\phi_0 = \sum_{i=1}^{n} f(x_i),$$

and the global noise

$$\eta(t) = \sum_{i=1}^{n} g(x_i)\xi_i(t).$$

Also, Blumm's model considers two additional assumptions suggested by empirical data. On the one hand, the drift term can be written as

$$f(x_i) = A_i x_i^{\alpha}, \tag{5}$$

where α is identical $\forall\, i$, and $A_i > 0$ can be interpreted as the ability or *fitness* of each item i to increase its normalized score [4]. On the other hand, according to Taylor's power law [5], the noise amplitude has the form

$$g(x_i) = B x_i^{\beta}. \tag{6}$$

Considering (5), (6), and taking a constant value $\phi(t) = \phi_0$ (i. e. small global noise), the Langevin equation (2) becomes

$$\dot{x}_i = A_i x_i^{\alpha} + B x_i^{\beta} \xi_i(t) - \phi_0 x_i. \tag{7}$$

Taking the temporal average in (7), and applying (3), we obtain two solutions for the steady-state:

$$x_i^{*} = \begin{cases} \left(\dfrac{\phi_0}{A_i}\right)^{1/(\alpha-1)} \\ 0 \end{cases} \tag{8}$$

where the vanishing solution is stable $\forall\, \alpha$, and the non-vanishing solution is stable $\forall\, \alpha < 1$. Applying the normalization condition (4) to the non-vanishing solution, we have

$$\phi_0 = \left(\sum_{i=1}^{n} A_i^{1/(1-\alpha)} \right)^{1-\alpha}. \tag{9}$$

The Langevin equation (7) can be interpreted in probabilistic terms by means of the Fokker-Planck equation [6]. If $P(x_i, t)$ denotes the probability that item i has a normalized score x_i at time t, then the following EDP is satisfied

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x_i}\left\{ \left[A_i x_i^{\alpha} - \phi_0 x_i \right] P \right\} + \frac{B^2}{2} \frac{\partial^2}{\partial x_i^2}\left(x_i^{2\beta} P \right).$$

The steady-state solution is given by [1]

$$P(x_i)=\begin{cases} C\,x_i^{-2\beta}\exp\left(\dfrac{2A_i}{B^2}\dfrac{x_i^{\alpha-2\beta+1}}{\alpha-2\beta+1}-\dfrac{\phi_0}{B^2}\dfrac{x_i^{2-2\beta}}{1-\beta}\right), & \beta\neq\dfrac{1+\alpha}{2} \\ C\,x_i^{2A/B^2-\alpha-1}\exp\left(-\dfrac{2\phi_0}{B^2}\dfrac{x_i^{1-\alpha}}{1-\alpha}\right), & \beta=\dfrac{1+\alpha}{2}\neq 1 \\ C\,x_i^{2(A-\phi_0)/B^2-2}, & \beta=\alpha=1 \end{cases} \tag{11}$$

where C denotes a normalization constant. However, the solution given in (11) is not complete, since the case $\beta = 1 \neq \alpha$, that would be included in the case $\beta \neq (1 + \alpha)/2$, singular for $\beta = 1$. Since a Taylor's power law with $\beta = 1$ is possible [7], we have calculated the steady-state for this case in the Appendix. In fact, a more general solution for the the steady-state solution in the cases $\beta \neq (1 + \alpha)/2$ and $\beta = (1 + \alpha)/2 \neq 1$ can be derived (see the Appendix for details). We do not consider the case $\alpha = \beta = 1$ because, from (9), the limit $\alpha \to 1$ does not exist for ϕ_0, except for the case that all the fitness values are equal, i.e. $A_1 = \cdots = A_n$.

The scope of this paper is two-folded. First, in the next section, we will justify the aforementioned assertion wherein the limit $\alpha \to 1$ does not exist for ϕ_0. Secondly, in the following two sections, we calculate the transient regime of the Fokker-Planck equation in the context of Blumm's model, i. e. (10), for some cases of the parameters α and β. In order to solve (10), we will follow a similar approach to the one given in [8, Sect. 15.12]. For this purpose, in Section 3, we will consider the case $B = 0$ (i. e. we neglect the noise term). Next, in Section 4, from the general solution found for $B = 0$, we will calculate some particular solutions for $B \neq 0$. Finally, we collect our conclusions in the last section.

INEXISTENCE OF Φ_0 AS $A \to 1$

In order to prove that ϕ_0 is not well defined as $\alpha \to 1$, we need the following lemma.

Lemma If $p > r > 0$, and $\vec{x} = (x_1, \ldots, x_n)$, then

$$\|\vec{x}\|_p \leq \|\vec{x}\|_r .$$

Proof. Consider the function $\forall p > 0$

$$f(p) = \|\vec{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \geq 0. \tag{12}$$

Performing the logarithmic derivative and rearranging terms, we arrive at

$$p^2 \frac{f'(p)}{f(p)} \sum_{i=1}^{n} |x_i|^p = p \sum_{i=1}^{n} |x_i|^p \log|x_i| - \log\left(\sum_{i=1}^{n} |x_i|^p \right) \sum_{i=1}^{n} |x_i|^p. \tag{13}$$

Performing the substitution $y_i = |x_i|^p \geq 0$, we rewrite (13) as

$$p^2 \frac{f'(p)}{f(p)} \sum_{i=1}^{n} y_i = \sum_{i=1}^{n} y_i \left[\log y_i - \log\left(\sum_{j=1}^{n} y_j \right) \right] \leq 0,$$

where the inequality holds because the log function is a monotonically increasing function. Note that the equality holds for $n = 1$. Therefore,

$$f'(p) \leq 0, \tag{14}$$

hence if $p > r$ then $f(p) \leq f(r)$, as we wanted to prove. ■

Corollary For $0 < p < \infty$, the following inequality holds true:

$$\|\vec{x}\|_p \geq \|\vec{x}\|_\infty = \max\left(|x_1|, \ldots, |x_n|\right). \tag{15}$$

Now we derive the inexistence of ϕ_0 as $\alpha \to 1$. For this purpose, we define the vector $\vec{A} = (A_1, \ldots, A_n)$, where $A_i > 0$. Thereby, according to (9) and (12), we have

$$\phi_0(\alpha) = \left(\sum_{i=1}^{n} A_i^{1/(1-\alpha)} \right)^{1-\alpha} = \|\vec{A}\|_{\frac{1}{1-\alpha}}.$$

On the one hand, taking $p = 1/(1-\alpha) > 0$ in (15), we arrive at

$$\phi_0(\alpha) \geq \max(A_1, \ldots, A_n), \quad \forall \alpha < 1. \tag{16}$$

According to (14),

$$\phi_0'(\alpha) = f'\left(\frac{1}{1-\alpha} \right) \leq 0, \tag{17}$$

thus, from (16), we have

$$\lim_{\alpha\to 1^-} \phi_0(\alpha) = \max(A_1,\ldots,A_n). \tag{18}$$

On the other hand, taking $p = 1/(\alpha - 1) > 0$ in (15), we obtain

$$\left(\sum_{i=1}^{n} A_i^{1/(\alpha-1)}\right)^{\alpha-1} \geq \max(A_1,\ldots,A_n), \quad \forall \alpha > 1. \tag{19}$$

We can rewrite (19) as

$$\left(\sum_{i=1}^{n} A_i^{-1/(\alpha-1)}\right)^{\alpha-1} \geq \max(A_1^{-1},\ldots,A_n^{-1}) = \min(A_1,\ldots,A_n),$$

since $\forall\, i$, $A_i > 0$. Therefore,

$$\phi_0(\alpha) = \left(\sum_{i=1}^{n} A_i^{1/(1-\alpha)}\right)^{1-\alpha} \leq \min(A_1,\ldots,A_n), \quad \forall \alpha > 1,$$

and thereby, applying (17), we have

$$\lim_{\alpha\to 1^+} \phi_0(\alpha) = \min(A_1,\ldots,A_n). \tag{20}$$

From, (18) and (20), we conclude that the limit $\alpha \to 1^-$ in general is different from the limit $\alpha \to 1^+$; thus, such limit does not exist. The only case when the limit exists is when all the fitness values are equal, i.e. $A_1 = \cdot\cdot\cdot = A_n$.

GENERAL SOLUTION FOR $B = 0$

Despite the fact that there are numerical methods in the literature to solve the Fokker-Planck equation [9, 10], we will adopt here an analytical approach. As mentioned in the Introduction, first we will calculate the Fokker-Planck equation (10) setting $B = 0$; thus we have the following equation for $P_0(x_i,t)$

$$\frac{\partial P_0}{\partial t} = -\frac{\partial}{\partial x_i}\left\{\left[A_i x_i^{\alpha} - \phi_0 x_i\right] P_0\right\}. \tag{21}$$

Stationary regime

Notice that (21) has got a stationary solution $P_0(x_i)$ very easy to calculate,

$$0 = -\frac{\partial}{\partial x_i}\left\{\left[A_i x_i^{\alpha} - \phi_0 x_i\right] P_0\right\},$$

thus

$$P_0\left(x_i\right) = \frac{C}{A_i x_i^{\alpha} - \phi_0 x_i}, \tag{22}$$

where C is an integration constant.

Transient solution

Rewrite (21) as

$$\frac{\partial P_0}{\partial t} + \left(A_i x_i^{\alpha} - \phi_0 x_i\right)\frac{\partial P_0}{\partial x_i} = \left(\phi_0 - \alpha A_i x_i^{\alpha-1}\right) P_0. \tag{23}$$

Multiply (23) by the integrating factor λ

$$\lambda \frac{\partial P_0}{\partial t} + \lambda\left(A_i x_i^{\alpha} - \phi_0 x_i\right)\frac{\partial P_0}{\partial x_i} = \lambda\left(\phi_0 - \alpha A_i x_i^{\alpha-1}\right) P_0,$$

and compare the result to the exact form:

$$\frac{\partial P_0}{\partial t} dt + \frac{\partial P_0}{\partial x_i} dx_i = dP_0,$$

thereby

$$dt = \lambda,$$
$$dx_i = \lambda\left(A_i x_i^{\alpha} - \phi_0 x_i\right),$$
$$dP_0 = \lambda\left(\phi_0 - \alpha A_i x_i^{\alpha-1}\right) P_0,$$

and

$$\frac{dx_i}{dt} = \lambda\left(A_i x_i^{\alpha} - \phi_0 x_i\right), \tag{24}$$

$$\frac{dP_0}{dt} = \lambda\left(\phi_0 - \alpha A_i x_i^{\alpha-1}\right) P_0. \tag{25}$$

Notice that (24) is precisely Langevin equation (7) for B= 0. Its solution can be found by direct integration as

$$x_i\left(t\right) = \left(\frac{A_i + k\, e^{\phi_0(\alpha-1)t}}{\phi_0}\right)^{\frac{1}{1-\alpha}}, \tag{26}$$

where k is an integration constant. Note that (26) has got the following asymptotic solutions:

$$\lim_{t\to\infty} x_i\left(t\right) = \begin{cases} \left(\dfrac{A_i}{\phi_0}\right)^{\frac{1}{1-\alpha}} & \alpha < 1 \\ 0 & \alpha > 1 \end{cases}$$

which are the same as (8) for the steady-state solution. Now, let us solve (25), taking into account (26),

$$\frac{dP_0}{dt} = \left(\phi_0 - \frac{\alpha A_i}{A_i + k\exp\left(\phi_0\left(\alpha-1\right)t\right)}\right) P_0.$$

Integrating, we arrive at

$$P_0 = Q\, e^{\phi_0(1-\alpha)t}\left(A_i + k\, e^{\phi_0(\alpha-1)t}\right)^{\frac{\alpha}{\alpha-1}}, \tag{27}$$

where Q is another integration constant. The above result (27) suggests the following change of variables (variation of constants),

$$P_0\left(x_i, t\right) = Q\left(k, t\right) F\left(k, t\right),$$

where we have set

$$F(k,t) = e^{\phi_0(1-\alpha)t}\left(A_i + k\,e^{\phi_0(\alpha-1)t}\right)^{\frac{\alpha}{\alpha-1}}, \tag{28}$$

and, according to (26),

$$k = k(x_i,t) = e^{\phi_0(1-\alpha)t}\left(\phi_0 x_i^{1-\alpha} - A_i\right). \tag{29}$$

Therefore, the dependent variable P_0 is changed into Q, and the independent variable x_i is changed into k. Thereby, we have

$$P_0 = Q\,F, \tag{30}$$

$$\frac{\partial P_0}{\partial t} = Q\left(\frac{\partial F}{\partial t} + \frac{\partial F}{\partial k}\frac{\partial k}{\partial t}\right) + F\left(\frac{\partial Q}{\partial t} + \frac{\partial Q}{\partial k}\frac{\partial k}{\partial t}\right), \tag{31}$$

$$\frac{\partial P_0}{\partial x_i} = \left(Q\frac{\partial F}{\partial k} + \frac{\partial Q}{\partial k}F\right)\frac{\partial k}{\partial x_i}. \tag{32}$$

Inserting (30)-(32) in (23) and grouping terms, we arrive at

$$\begin{aligned} 0 = Q&\left[\frac{\partial F}{\partial t} + \frac{\partial F}{\partial k}\frac{\partial k}{\partial t} + \left(A_i x_i^{\alpha} - \phi_0 x_i\right)\frac{\partial F}{\partial k}\frac{\partial k}{\partial x_i} + \left(\alpha A_i x_i^{\alpha-1} - \phi_0\right)F\right] \\ &+ \frac{\partial Q}{\partial k}\left[F\frac{\partial k}{\partial t} + \left(A_i x_i^{\alpha} - \phi_0 x_i\right)F\frac{\partial k}{\partial x_i}\right] + \frac{\partial Q}{\partial t}F. \end{aligned}$$

Taking into account (28) and (29), the above equation is reduced to

$$\frac{\partial Q}{\partial t} = 0,$$

hence

$$Q(k,t) = f(k),$$

where f is an arbitrary function. Undoing the change of variables performed, we finally arrive at

$$P_0(x_i,t) = x_i^{-\alpha}e^{\phi_0(1-\alpha)t} f\left(e^{\phi_0(1-\alpha)t}\left[\phi_0 x_i^{1-\alpha} - A_i\right]\right).$$

We can recover the stationary solution given in (22), taking $f(z) = C\,z^{-1}$.

PARTICULAR SOLUTIONS FOR $B \neq 0$

Taking $f(z) = C$, we obtain the particular solution,

$$P_0^*\left(x_i,t\right) = C\, x_i^{-\alpha} e^{\phi_0(1-\alpha)t}.$$

Let us try solutions for (10) of the following type,

$$P\left(x_i,t\right) = P_0^*\left(x_i,t\right) + C\, g\left(x_i\right), \tag{33}$$

thereby, inserting (33) in (10), recalling that $P_0^*\,(x_i,t)$ is a solution of (21), we have

$$0 = -\frac{\partial}{\partial x_i}\left(\left[A_i x_i^{\alpha} - \phi_0 x_i\right] g\left(x_i\right)\right) + \frac{B^2}{2} e^{\phi_0(1-\alpha)t} \frac{\partial^2}{\partial x_i^2}\left(x_i^{2\beta-\alpha}\right) + \frac{B^2}{2}\frac{\partial^2}{\partial x_i^2}\left(x_i^{2\beta} g\left(x_i\right)\right).$$

If $\beta = (1 + \alpha)/2$, $\alpha/2$, the above equation is reduced to

$$0 = -\frac{\partial}{\partial x_i}\left(\left[A_i x_i^{\alpha} - \phi_0 x_i\right] g\left(x_i\right)\right) + \frac{B^2}{2}\frac{\partial^2}{\partial x_i^2}\left(x_i^{2\beta} g\left(x_i\right)\right). \tag{34}$$

Case $\beta = (1 + \alpha)/2$

A first integration of (34) yields

$$\left[\phi_0 x_i + \left((\alpha+1)\frac{B^2}{2} - A_i\right) x_i^{\alpha}\right] g\left(x_i\right) + \frac{B^2}{2} x_i^{\alpha+1} g'\left(x_i\right) = C,$$

which is a first order linear ODE. Fortunately, the solution of this ODE can be obtained in closed-form as

$$g\left(x_i\right) = x_i^{2A_i/B^2-1-\alpha} \exp\left(s_\alpha x_i^{1-\alpha}\right)\left[K_1 + K_2\,\Gamma\left(r_\alpha, s_\alpha x_i^{1-\alpha}\right)\right],$$

where K_1 and K_2 are arbitrary constants, $\Gamma(a, z)$ denotes the upper incomplete gamma function [11, Eqn. 8.2.2],

$$\Gamma\left(a,z\right) = \int_z^{\infty} u^{a-1} e^{-u}\, du, \tag{35}$$

and where we have defined

$$s_\alpha = \frac{2\phi_0}{(\alpha-1)B^2}, \tag{36}$$

$$r_\alpha = \frac{B^2-2A_i}{B^2(1-\alpha)}. \tag{37}$$

Therefore, redefining the integration constants, we have

$$P(x_i,t) = x_i^{-\alpha}\left\{e^{\phi_0(1-\alpha)t} + x_i^{2A_i/B^2-1}\exp\left(s_\alpha x_i^{1-\alpha}\right)\left[C_1 + C_2\,\Gamma\left(r_\alpha, s_\alpha x_i^{1-\alpha}\right)\right]\right\}.$$

Notice that performing the limit $t \to \infty$ with $\alpha > 1$, we recover the steadystate solution given in (41).

Case $\beta = \alpha/2$

In this case, a first integration of (34) yields,

$$\left(2\phi_0 x_i - 2A_i x_i^\alpha + B^2\alpha x_i^{\alpha-1}\right)g(x_i) + B^2 x_i^\alpha g'(x_i) = C,$$

which is a first order linear ODE. In this case, we obtain the solution in integral form as follows:

$$g(x_i) = x_i^{-\alpha}\exp\left(\frac{2x_i}{B^2}\left[A_i + \frac{\phi_0}{2-\alpha}x_i^{1-\alpha}\right]\right)\left\{K_1 + K_2\,\mathrm{I}_\alpha(x_i)\right\},$$

where we have defined

$$\mathrm{I}_\alpha(x_i) := \int \exp\left(-\frac{2x_i}{B^2}\left[A_i + \frac{\phi_0}{2-\alpha}x_i^{1-\alpha}\right]\right)dx_i. \tag{38}$$

The integral given in (38) can be calculated in some special cases, (i.e. $\alpha = 0, 1, 2$):

$$\mathrm{I}_0(x_i) = \frac{B}{2}\sqrt{\frac{\pi}{\phi_0}}\exp\left(\frac{A_i^2}{B^2\phi_0}\right)\mathrm{erf}\left(\frac{A_i+\phi_0 x_i}{B\sqrt{\phi_0}}\right),$$
$$\mathrm{I}_1(x_i) = -\frac{B^2}{2(A_i+\phi_0)}\exp\left(-\frac{2(A_i+\phi_0)}{B^2}x_i\right),$$

and

$$\lim_{\alpha\to 2} \mathrm{I}_\alpha\left(x_i\right)=0.$$

Therefore, redefining the integration constants, we have,

$$P\left(x_i,t\right)=x_i^{-\alpha}\left\{e^{\phi_0(1-\alpha)t}+\exp\left(\frac{2x_i}{B^2}\left[A_i+\frac{\phi_0}{2-\alpha}x_i^{1-\alpha}\right]\right)\left[C_1+C_2\,\mathrm{I}_\alpha\left(x_i\right)\right]\right\}.$$

Notice that setting the integration constant $C_2 = 0$ and performing the limit $t \to \infty$ with $\alpha > 1$, we recover the steady-state solution given in (11) for $\beta = \alpha/2$.

CONCLUSIONS

We have considered the Blumm's *et al*. model for the dynamics of the ranking processes. This model assumes that the probability $P(x_i, t)$ of an item i having a normalized score x_i at time t follows the Fokker-Planck equation given in (10). First, we have enhanced the solutions given in Blumm's *et al*. paper [1] for the steady-state solution $P(x_i)$ in the Appendix. Secondly, we have proved that the constant ϕ_0 is not well defined for $\alpha \to 1$. Next, we have found a general solution for the Fokker-Planck equation (10) in the case $B = 0$. Finally, we have calculated particular solutions for $\beta = (1 + \alpha)/2$ and $\beta = \alpha/2$ when $B \neq 0$. As a consistency test, from the solutions found, we have recovered the steady-state solution performing the limit $t \to \infty$.

APPENDIX. GENERALIZATION OF THE STEADY-STATE SOLUTION

The steady-state of the Fokker-Planck equation given in (10) is

$$\frac{d}{dx_i}\left\{\left[A_i x_i^\alpha-\phi_0 x_i\right]P\right\}=\frac{B^2}{2}\frac{d^2}{dx_i^2}\left(x_i^{2\beta}P\right).$$

A first integration leads to the following linear ODE,

$$\left(A_i x_i^\alpha-\phi_0 x_i-B^2\beta x_i^{2\beta-1}\right)P+C=\frac{B^2}{2}x_i^{2\beta}P', \tag{39}$$

where C is an integration constant. The solution of the homogeneous equation of (39) (i. e. taking $C = 0$) is

$$P_h\left(x_i\right)=K\,x_i^{-2\beta}\exp\left(\frac{2A_i}{B^2}\frac{x_i^{\alpha-2\beta+1}}{\alpha-2\beta+1}-\frac{\phi_0}{B^2}\frac{x_i^{2(1-\beta)}}{1-\beta}\right),$$

where K is an integration constant. By using variation of constants technique, the general solution of (39) is

$$P(x_i) = x_i^{-2\beta} e^{h(x_i)} \left[C_1 + C_2 \int e^{-h(x_i)} dx_i \right], \tag{40}$$

where we have defined,

$$h(x_i) = \frac{2A_i}{B^2} \frac{x_i^{\alpha-2\beta+1}}{\alpha-2\beta+1} - \frac{\phi_0}{B^2} \frac{x_i^{2(1-\beta)}}{1-\beta}.$$

Notice that setting the integration constant $C_2 = 0$ in (40), we recover the solution given in (11) for $\beta \neq (1 + \alpha)/2$. Notice as well that (40) is divergent for $\beta = (1 + \alpha)/2$ and $\beta = 1$. Next, we consider these two special cases.

Case $\beta = (1 + \alpha)/2$

In this case, the linear ODE given in (39) is reduced to

$$\left(\left[A_i - \frac{B^2}{2}(1+\alpha) \right] x_i^{\alpha} - \phi_0 x_i \right) P + C = \frac{B^2}{2} x_i^{1+\alpha} P',$$

whose solution can be written in terms of the upper gamma function [see (35)],

$$P(x_i) = x_i^{2A_i/B^2-\alpha-1} \exp\left(s_\alpha x_i^{1-\alpha}\right) \left[C_1 + C_2 \Gamma\left(r_\alpha, s_\alpha x_i^{1-\alpha}\right) \right], \tag{41}$$

where the constants s_α and r_α are given in (36) and (37), respectively. Note that setting the integration constant $C_2 = 0$ in (41), we recover the solution given in (11) for $\beta = (1 + \alpha)/2$. Note as well that the constants s_α and r_α are divergent for $\alpha = 1$ (thus also $\beta = 1$). However, we will not consider this latter case because ϕ_0 is not well defined $\forall\, \alpha = 1$, as mentioned in the Introduction.

Case $\beta = \alpha/2$

In this case, the linear ODE given in (39) is reduced to

$$\left(A_i x_i^{\alpha} - \left[\phi_0 + B^2 \right] x_i \right) P + C = \frac{B^2}{2} x_i^2 P',$$

whose solution can be written in terms of the upper gamma function as follows:

$$P\left(x_i\right)=x_i^{2A_i/B^2-2}\exp\left(\sigma_\alpha x_i^{\alpha-1}\right)\left[C_1+C_2\,\Gamma\left(\rho_\alpha,\sigma_\alpha x_i^{\alpha-1}\right)\right],$$

where we have defined the constants

$$\sigma_\alpha=\frac{2A_i}{B^2\left(\alpha-1\right)},$$
$$\rho_\alpha=\frac{B^2+2\phi_0}{B^2\left(\alpha-1\right)}.$$

Note that the constants σ_α and ρ_α are divergent for $\alpha = 1$, but as mentioned before, we will not consider this case, since ϕ_0 is not well defined $\forall\,\alpha = 1$.

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