A Class of Abelian Rings

Una clase de anillos abelianos

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Abstract. Let R be a ring with identity and $J(R)$ denote the Jacobson radical of R. A ring R is called J-*abelian* if $ae - ea \in J(R)$ for any $a \in R$ and any idempotent e in R. In this paper, many characterizations of J-abelian rings are given. We prove that every J-Armendariz ring is J-abelian. We show that the class of J-abelian rings lies strictly between the class of abelian rings and the class of directly finite rings.

Keywords: Abelian ring, J-abelian ring, ring extension.

Resumen. Sea R un anillo dotado de la identidad y sea $J(R)$ el radical de Jacobson de R. Un anillo R se llama J-*abeliano* si $ae - ea \in J(R)$ para todo $a \in R$ y algún e idempotente en R. En este artículo, se dan muchas caracterizaciones de anillos J-abelianos. Demostramos que cada anillo J-Armendariz es J-abeliano. Mostramos que la clase de los anillos J-abelianos se ubica estrictamente entre la clase delos anillos abelianos y la clase de los anillos directamente finitos.

Palabras claves: anillos abelianos, anillos J-abelianos, extensiones de anillos.

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1. Introduction

Throughout this paper, all rings considered are associative with an identity unless otherwise stated. In what follows, $\mathbb{Z}$ and $\mathbb{Q}$ denote the ring of integers and the ring of rational numbers and for a positive integer n, $\mathbb{Z}_n$ is the ring of integers modulo n. We write $M_n(R)$ for the ring of all $n \times n$ matrices and $T_n(R)$ for the ring of all $n \times n$ upper triangular matrices over R. Also we write $R[x]$, $R[[x]]$, $U(R)$ and $J(R)$ for the polynomial ring, the power series ring over a ring R, the set of all invertible elements and the Jacobson radical of R, respectively.

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A ring is said to be *abelian* [1] if all of its idempotents are central. A ring (possibly without identity) is called *reduced* if it has no nonzero nilpotent elements. Reduced rings are easily shown to be abelian, but there are many commutative (hence abelian) rings which are not reduced.

In this paper, we introduce a class of rings so-called J-abelian rings which generalizes abelian rings. A ring R is called J*-abelian* if $ae - ea \in J(R)$ for any $a \in R$ and any idempotent e in R. Clearly, abelian rings are J-abelian, but the converse is not true in general. We supply an example to show that J-abelian rings need not be abelian (Example 2.2). We investigate characterizations of J-abelian rings and present some families of J-abelian rings. J-abelian rings are abundant. It is well known that all commutative rings, reduced rings, symmetric rings, reversible rings, semicommutative rings are abelian and so J-abelian. We show that all J-clean rings, Armendariz rings, J-quasipolar rings and local rings are J-abelian.

2. J-Abelian Rings

In this section we introduce a class of rings, called J-abelian rings, which is a generalization of abelian rings. We investigate which properties of abelian rings hold for this general setting. We give relations between J-abelian rings and some related rings.

We now give our main definition.

Definition 2.1. A ring R is called J*-abelian* if for any $a \in R$ and $e^2 = e \in R$, $ae - ea \in J(R)$.

One may suspect that J-abelian rings are abelian, but the following example erases this possibility.

Example 2.2. Let F be a field and consider the ring $R = \begin{bmatrix} F & F & F \\ 0 & F & F \\ 0 & 0 & F \end{bmatrix}$. Then $J(R) = \begin{bmatrix} 0 & F & F \\ 0 & 0 & F \\ 0 & 0 & 0 \end{bmatrix}$. For any $A, B \in R$, the main diagonal entries of $AB - BA$ are zero and so $AB - BA \in J(R)$. Hence R is J-abelian. It is well known that R is not abelian.

In [13], Rege and Chhawchharia introduced the notion of an Armendariz ring. A ring R is called *Armendariz* if for any $f(x) = \sum_{i=0}^{n} a_i x^i$, $g(x) = \sum_{j=0}^{s} b_j x^j \in R[x]$, $f(x)g(x) = 0$ implies that $a_i b_j = 0$ for all i and j. The name of the ring was given due to Armendariz who proved that reduced rings satisfy this condition. Recently, J-Armendariz rings were defined by the present authors in [5], that is, a ring R is called J*-Armendariz* if for any $f(x) = \sum\limits_{i=0}^{n} a_i x^i$, $g(x) = \sum\limits_{j=0}^{m} b_j x^j \in R[x]$, $f(x)g(x) = 0$ implies that $a_i b_j \in J(R)$ for all

$0 \leq i \leq n$, $0 \leq j \leq m$. In [6, Lemma 7], Kim and Lee proved that Armendariz rings are abelian. Then we obtain the similar result for J-abelian rings.

Lemma 2.3. *If R is a J-Armendariz ring, then it is J-abelian.*

Proof. Let $a, e^2 = e \in R$. Consider the polynomials $f_1(x) = e - ea(1-e)x$, $g_1(x) = (1-e)+ea(1-e)x$, $f_2(x) = (1-e)-(1-e)ax$ and $g_2(x) = e+(1-e)aex \in R[x]$. Then $f_1(x)g_1(x) = f_2(x)g_2(x) = 0$. Since R is J-Armendariz, we have $ea - eae \in J(R)$ and $ae - eae \in J(R)$. Therefore $ea - ae \in J(R)$. □

Clean rings were introduced by Nicholson in [10] as a class of exchange rings. A ring R is called *strongly clean* if every element of R is the sum of a unit and an idempotent which commute. The notion of clean ring is studied by many authors. In [3], strongly J-clean rings are introduced, that is, a ring R is called *strongly J-clean* if for every $x \in R$, there exists an idempotent $e \in R$ such that $x = e + j$ and $ej = je$ with $j \in J(R)$. Similar to the definition of strongly J-clean rings, one can define J-clean rings. A ring R is called *J-clean* if for every $x \in R$, there exist an idempotent $e \in R$ and $j \in J(R)$ such that $x = e + j$. Then we have the following result.

Lemma 2.4. *Every J-clean ring is J-abelian.*

Proof. Assume that R is J-clean. Let $a \in R$ and e be an idempotent in R. The ring R being J-clean implies $a = f + b$ where $b \in J(R)$ and $f^2 = f \in R$. Then $ae - eae = (fe - efe) + (be - ebe)$. Note that $be - ebe \in J(R)$. On the other hand, $(fe - efe)^2 = 0$ and $fe - efe = g + j$ for some $j \in J(R)$ and $g^2 = g \in R$. Hence $(fe-efe)^2 = (g+j)^2 = g+gj+jg+j^2 = 0$ yields $g \in J(R)$. So we can conclude that $fe-efe \in J(R)$. Thus $ae-eae \in J(R)$. Analogously, $ea - eae \in J(R)$. Therefore $ae - ea \in J(R)$ and so R is J-abelian. □

Let R be a ring. According to Koliha and Patricio [7], the *commutant* and *double commutant* of $a \in R$ are defined by $comm(a) = \{x \in R \mid xa = ax\}$, $comm^2(a) = \{x \in R \mid xy = yx \text{ for all } y \in comm(a)\}$, respectively. In [4], the element a is called *J-quasipolar* if there exists $p^2 = p \in comm^2(a)$ such that $a + p \in J(R)$. Then R is called *J-quasipolar* if every element of R is J-quasipolar.

Proposition 2.5. *Let R be a ring. If R is J-quasipolar, then it is J-abelian.*

Proof. The proof is similar to the proof of Lemma 2.4. □

Let $J^{\#}(R)$ denote the subset $\{x \in R \mid \exists\, n \in \mathbb{N} \text{ such that } x^n \in J(R)\}$ of R. It is obvious that $J(R) \subseteq J^{\#}(R)$, but the converse is not true in general. Consider the ring $R = M_2(\mathbb{Z}_2)$. Then

$$J^{\#}(R) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\},$$

while $J(R) = 0$.

Recall that a ring R is called *local* if it has only one maximal left ideal (equivalently, maximal right ideal). It is well known that a ring R is local if and only if $a+b=1$ in R implies that either a or b is invertible if and only if $R/J(R)$ is a division ring. In [12, Lemma 1], it is proved that if R is a local ring, then $J(R)=J^{\#}(R)$. From the following result we can say that every local ring is J-abelian.

Proposition 2.6. *Let R be a ring. If $J(R)=J^{\#}(R)$, then R is J-abelian.*

Proof. Assume that $J(R)=J^{\#}(R)$ and $a, e^2=e\in R$. Since $(ae-eae)^2=0$ and $(ea-eae)^2=0$, $ae-eae, ea-eae\in J^{\#}(R)$. Being $J(R)=J^{\#}(R)$ implies $ae-ea\in J(R)$. □

Note that $R/J(R)$ is J-abelian if and only if $R/J(R)$ is abelian, since $J(R/J(R))=0$. Then we have the following result.

Lemma 2.7. *If $R/J(R)$ is an abelian ring, then R is J-abelian. The converse holds if idempotents lift modulo $J(R)$.*

Proof. One direction is clear. Assume that R is J-abelian. Let $\overline{a}\in R/J(R)$ and $\overline{e}$ be an idempotent in $R/J(R)$. By hypothesis, there exists $f^2=f\in R$ such that $\overline{e}=\overline{f}$. Since R is J-abelian, $af-fa\in J(R)$, and so $\overline{ae}=\overline{a}\overline{f}=\overline{f}\overline{a}=\overline{ea}$. Therefore $R/J(R)$ is abelian. □

Recall that a ring R is called *directly finite* whenever $a,b\in R$, $ab=1$ implies $ba=1$. Then we have the following.

Lemma 2.8. *Every J-abelian ring is directly finite.*

Proof. Let R be a J-abelian ring and assume that $ab=1$ for $a,b\in R$ and $e=1-ba$. Since R is J-abelian, $ea-ae=a-ba^2\in J(R)$. Then $ab-ba^2b\in J(R)$, and so $1-ba\in J(R)$. Hence $1-ba=0$, so $ba=1$. □

Every J-abelian ring is directly finite, but directly finite rings may not be J-abelian as the following example shows. On the other hand, this example also shows that the class of J-abelian rings is not closed under homomorphic images.

Example 2.9. Let $S=\mathbb{Z}_{(3)}$ be the localization of $\mathbb{Z}$ at $3\mathbb{Z}$ and let Q be the set of quaternions over the ring S. It is well-known that Q is a noncommutative domain. So it is J-abelian and directly finite. Further, $J(Q)=3Q$ and $R=Q/J(Q)$ is isomorphic to 2×2 full matrix ring over $\mathbb{Z}_3$. It is clear that R is not abelian. Since $J(R)=0$, R is not J-abelian.

In [11], Nicholson asked whether a strongly clean ring is directly finite. Since every J-clean ring is clean, we provide a partial answer for this question, namely, we prove that every J-clean ring is directly finite.

Theorem 2.10. *If R is a J-clean ring, then it is directly finite.*

Proof. It follows from Lemma 2.4 and Lemma 2.8. □

A left R-module M has *finite exchange property* if for every left R-module A and any two decompositions $A = M' \oplus N = \oplus_{i\in I} A_i$ with $M' \cong M$ and I a finite set, there exists $A_i' \subseteq A_i$ such that $A = M' \oplus (\oplus_{i\in I} A_i')$. In [14], a ring R is called *exchange* if ${}_RR$ has a finite exchange property. This definition is left-right symmetric. In [10], Nicholson studied exchange rings and introduced clean rings. He showed that clean rings are exchange. Later, some authors investigate under which conditions exchange rings are clean. It is well known that abelian exchange rings are clean. We now give a more general result.

Theorem 2.11. *Let R be a ring. If R is clean, then it is exchange. The converse holds if R is J-abelian.*

Proof. One direction is clear. Assume that R is exchange and J-abelian. Since the exchange property implies that idempotents lift modulo $J(R)$ (see [10, Corollary 2.4]), and R is J-abelian, $R/J(R)$ is abelian by Lemma 2.7. Thus $R/J(R)$ is clean from [10]. Therefore R is clean, by [2, Proposition 7]. □

Let I be an ideal of a ring R. One may suspect that if R/I is J-abelian, then R is J-abelian. But the following example shows that this is not true in general.

Example 2.12. Let R be a ring, $S = M_2(R)$ and I denote the ideal generated by the commutators of S. Then S/I is commutative and so it is J-abelian. On the other hand, by Example 3.7, S is not J-abelian.

We will prove that if I is a nil ideal of a ring R and R/I is J-abelian, then so is R.

Lemma 2.13. *Let R be a ring and I a nil ideal of R. If R/I is J-abelian, then R is J-abelian.*

Proof. Let $a \in R$ and e be an idempotent in R. Then $(ae - ea) + I \in J(R/I)$. $1 - (ea - ae)x + I \in U(R/I)$, for any $x \in R$. So there exists $y + I \in R/I$ such that $(1 - (ea - ae)x)y + I = 1 + I$. Then, $1 - (1 - (ea - ae)x)y \in I$. Since I is nil, $(1 - (ea - ae)x)y \in U(R)$. Hence, $1 - (ea - ae)x$ is right invertible. Similarly, it can be shown that $1 - (ea - ae)x$ is left invertible. So we have $1-(ea-ae)x \in U(R)$. Therefore $ea-ae \in J(R)$, which completes the proof. □

Note that a direct product of J-abelian ring is again J-abelian.

Proposition 2.14. *Let $\{R_i\}_{i\in I}$ be a class of rings for an index set I. Then $\prod\limits_{i\in I} R_i$ is J-abelian if and only if for each $i \in I$, R_i is J-abelian.*

Proof. Let R_i be J-abelian for all $i \in I$, $(a_i)_{i\in I} \in \prod\limits_{i\in I} R_i$ and $(e_i)_{i\in I}$ be an idempotent of $\prod\limits_{i\in I} R_i$. By hypothesis $e_ia_i - a_ie_i \in J(R_i)$ for all $i \in I$. Hence

$(e_i)(a_i) - (a_i)(e_i) \in J(\prod_{i\in I} R_i) = \prod_{i\in I} J(R_i)$. Therefore $\prod_{i\in I} R_i$ is J-abelian. The converse is clear. $\square$

The following result is a direct consequence of Proposition 2.14.

Corollary 2.15. *Let R be a ring. Then eR and $(1-e)R$ are J-abelian for some central idempotent $e \in R$ if and only if R is J-abelian.*

If R is J-abelian and $e^2 = e \in R$, we now show that the corner ring eRe is again J-abelian.

Lemma 2.16. *If R is a J-abelian ring, then the corner ring eRe is J-abelian for any $e^2 = e \in R$.*

Proof. Let $eae \in eRe$ and efe be an idempotent in eRe. By hypothesis $a(efe) - (efe)a \in J(R)$ for $a \in R$. Then $(eae)(efe) - (efe)(eae) \in eJ(R)e$. Since $eJ(R)e = J(eRe)$, $(eae)(efe) - (efe)(eae) \in J(eRe)$. Thus eRe is J-abelian. $\square$

Let R be a ring and I an ideal of R. Then I is called *J-abelian* if for any $a, e^2 = e \in I$, $ea - ae \in J(I)$. Then we have the following result.

Proposition 2.17. *Let R be a ring and I an ideal of R. If R is J-abelian, then I is J-abelian.*

Proof. Let $a \in I$ and e an idempotent in I. Since R is J-abelian, $ea - ae \in J(R)$. So $ea - ae \in J(I) = J(R) \cap I$. $\square$

Theorem 2.18. *Let I be an ideal of a J-abelian ring T and R a subring of T with $I \subseteq R$. If R/I is J-abelian, then R is J-abelian.*

Proof. Assume that $a, e^2 = e \in R$. This implies that $ae - ea \in J(T)$. Then for every $r \in R$, $1 - (ae - ea)r \in U(T)$. Hence there exists $t \in T$ such that $t(1 - (ae - ea)r) = 1$. Since $\overline{e}$ is idempotent in R/I and R/I is J-abelian, $\overline{ae} - \overline{ea} \in J(R/I)$ and $\overline{1} - (\overline{ae} - \overline{ea})\overline{r} \in U(R/I)$. Thus there exists $\overline{s} \in R/I$ such that $(\overline{1} - (\overline{ae} - \overline{ea})\overline{r})\overline{s} = \overline{1}$. Therefore $1 - (1 - (ae - ea)r)s \in I$. So $t(1 - (1 - (ae - ea)r)s) = t - s \in I$. It follows that $t \in R$. Being $t(1 - (ae - ea)r) = 1$ implies $1 - (ae - ea)r$ is left invertible in R. Similarly, $1 - (ae - ea)r$ is also right invertible in R. Finally $1 - (ae - ea)r \in U(R)$ and so $ae - ea \in J(R)$. $\square$

Corollary 2.19. *Every finite subdirect product of J-abelian rings is J-abelian.*

Proof. Let R be a ring such that R/I and R/K are J-abelian where I, K are ideals of R and $I \cap K = 0$. To show that R is J-abelian consider the map $f : R \to R/I \oplus R/K$ which is defined by $f(r) = (r + I, r + K)$. Then, $R \cong Im(f)$ since $I \cap K = 0$. By hypothesis, $R/I \oplus R/K$ and $Im(f)/f(I) \cong R/I$ are J-abelian. Since $f(I) \subseteq Im(f) \subseteq R/I \oplus R/K$, R is J-abelian by Theorem 2.18. $\square$

Corollary 2.20. *Let R be a J-abelian ring, I be an ideal of R. If S is a J-abelian subring of R, then $I+S$ is J-abelian.*

Proof. We have $I \subseteq I+S \subseteq R$. Since $(I+S)/I$ is J-abelian, $I+S$ is J-abelian by Theorem 2.18. □

Corollary 2.21. *Let I and K be ideals of a ring R. If R/I and R/K are J-abelian, then $R/(I \cap K)$ is J-abelian.*

Proof. Let $f : R/(I\cap K) \to R/I$ and $g : R/(I\cap K) \to R/K$ with $f(r+I\cap K) = r+I$ and $g(r+I\cap K) = r+K$. Thus f and g are epimorphisms where $ker(f) \cap ker(g) = 0$. Then $R/(I \cap K)$ is the subdirect product of R/I and R/K. Also it is known that $R/I \cong (R/(I\cap K))/(I/(I\cap K))$ and $I/(I\cap K) \subseteq R/(I\cap K) \subseteq R/I$. From Corollary 2.19, $R/(I\cap K)$ is J-abelian. □

Let R be a ring. Then $S = \{(x, y) \in R \times R \ : \ x - y \in J(R)\}$ is a subring of $R \times R$ with $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ and $(x_1, y_1)(x_2, y_2) = (x_1x_2, y_1y_2)$.

Theorem 2.22. *The following are equivalent for a ring R.*

1. *R is J-abelian.*
2. *$S = \{(x, y) \in R \times R \ : \ x - y \in J(R)\}$ is J-abelian.*

Proof. (1) $\Rightarrow$ (2) It is clear that S is a subring of $R \times R$. Consider the ideals $I = 0 \times J(R)$ and $J = J(R) \times 0$ of S. Then, $I \cap J = 0$ and $S/I \cong R \cong S/J$. Hence, S is a subdirect product of R and so the proof is completed by Corollary 2.19.

(2) $\Rightarrow$ (1) Let e be an idempotent in R and $a \in R$. Since S is J-abelian, $(x, y)(f, g) - (f, g)(x, y) \in J(S)$ for any $(x, y) \in S$ and $(f, g)^2 = (f, g) \in S$. In particular, for $(a, a) \in S$ and $(e, e)^2 = (e, e) \in S$, we have $(a, a)(e, e) - (e, e)(a, a) \in J(S)$. Hence, for every $x \in R$, $(1, 1) - (ae - ea, ae - ea)(x, x) \in U(S)$. Thus, we have $(1 - (ae - ea)x, 1 - (ae, ea)x) \in U(S)$. This implies that $1 - (ae - ea)x \in U(R)$. Consequently, $ae - ea \in J(R)$, as asserted. □

3. Extensions of J-abelian rings

In this section, we study some extensions of J-abelian rings. In particular, we investigate under what conditions the polynomial ring $R[x]$, the power series ring $R[[x]]$, the Dorroh extension of R, the formal triangular matrix ring and some subrings of the ring of all $n \times n$ matrices $M_n(R)$ are J-abelian.

By [6, Lemma 8], a ring R is abelian if and only if the polynomial ring $R[x]$ is abelian. It is clear that if R is abelian, then $R[x]$ is abelian and so $R[x]$ is J-abelian. For the converse we have the following.

Proposition 3.1. *Let R be a ring. If $R[x]$ is J-abelian, then R is J-abelian.*

Proof. Assume that $R[x]$ is J-abelian. Let $e^2 = e \in R$ and $a \in R$ be an arbitrary element of R. Since $R[x]$ is J-abelian, $ae - ea \in J(R[x])$. Then $1 - (ae - ea)r$ is invertible in $R[x]$ for all $r \in R$ and so $ae - ea \in J(R)$. Therefore R is J-abelian. □

Proposition 3.2. *Let R be a ring. Then the power series ring $R[[x]]$ is J-abelian if and only if R is J-abelian.*

Proof. It can be easily seen by the fact that $J(R[[x]]) = J(R) + < x >$. □

Let R be a ring and $D(\mathbb{Z}, R)$ denote the *Dorroh extension* of R by the ring of integers $\mathbb{Z}$. Then $D(\mathbb{Z}, R)$ is the ring defined by the direct sum $\mathbb{Z} \oplus R$ with componentwise addition and multiplication $(n, r)(m, s) = (nm, ns + mr + rs)$ where $(n, r),\ (m, s) \in D(\mathbb{Z}, R)$. By [9], $J(D(\mathbb{Z}, R)) = (0, J(R))$.

Theorem 3.3. *Let R be a ring. Then R is J-abelian if and only if $D(\mathbb{Z}, R)$ is J-abelian.*

Proof. Assume that R is J-abelian. Let $(n, r) \in D(\mathbb{Z}, R)$ and (m, s) be an idempotent in $D(\mathbb{Z}, R)$. Then $(m, s)^2 = (m, s)$ implies $m^2 = m$ and $2ms + s^2 = s$. Hence $m = 0$ or $m = 1$. The case $m = 0$ implies s is an idempotent. If $m = 1$, then $-s$ is an idempotent and $(n, r)(1, s) - (1, s)(n, r) = (0, s^2 r - rs^2)$. Since $s^2 r - rs^2 \in J(R)$, $(n, r)(m, s) - (m, s)(n, r) \in J(D(\mathbb{Z}, R))$. Thus $D(\mathbb{Z}, R)$ is J-abelian.

Conversely, suppose that $D(\mathbb{Z}, R)$ is J-abelian. Since R is isomorphic to the ideal $\{(0, r) \mid r \in R\} \subseteq D(\mathbb{Z}, R)$, R is J-abelian. □

Let R be a ring and S a subring of R. Consider the set

$$T[R, S] = \{(r_1, r_2, \cdots, r_n, s, s, \cdots) : r_i \in R,\ s \in S, n \in \mathbb{Z}^+,\ 1 \leq i \leq n\}.$$

Then $T[R, S]$ is a ring under the componentwise addition and multiplication. In [8], it is shown that $J(T[R, S]) = T[J(R), J(R) \cap J(S)]$. In the following we give necessary and sufficient conditions for $T[R, S]$ to be J-abelian.

Proposition 3.4. *Let R be a ring and S a subring of R. Then the following are equivalent.*

1. *$T[R, S]$ is J-abelian.*
2. *R and S are J-abelian.*

Proof. (1) $\Rightarrow$ (2) Let $a \in R$, $e^2 = e \in R$. Set $X = (a, 0, 0, \cdots)$ and $Y = (e, 0, 0, \cdots)$. Then $Y^2 = Y$. By (1), $AE - EA \in J(T[R, S])$. Hence $ae - ea \in J(R)$ and R is J-abelian. Let s, $f^2 = f \in S$. Set $X_1 = (0, 0, s, s, s, s, \cdots) \in T[R, S]$ and $Y_1 = (0, 0, f, f, f, \cdots) \in T[R, S]$. Then Y_1 is an idempotent in $T[R, S]$. By (1) $X_1 Y_1 - Y_1 X_1 \in J(T[R, S]) = T[J(R), J(R) \cap J(S)]$. It follows that $sf - fs \in J(S)$. Hence R and S are J-abelian.

(2) $\Rightarrow$ (1) Let $a = (a_1, a_2, \cdots, a_n, b, b, \cdots)$ be an element of $T[R, S]$ and $c = (c_1, c_2, \cdots, c_m, d, d, \cdots)$ be an idempotent in $T[R, S]$. Then all components $c_1, c_2, \ldots, c_m$ of c are idempotents in R and d is an idempotent in S.

We divide the proof into some cases:

Case I: $n \leq m$. By (2) $a_ic_i - c_ia_i \in J(R)$ for $1 \leq i \leq n$, $bc_j - c_jb \in J(R)$ for $n+1 \leq j \leq m$ and $bd - db \in J(R) \cap J(S)$. Then $ac - ca \in J(T[R, S])$.

Case II: $n > m$. By (2) $a_ic_i - c_ia_i \in J(R)$ for $1 \leq i \leq m$, $a_jd - da_j \in J(R)$ for $m+1 \leq j \leq n$ and $bd - db \in J(R) \cap J(S)$. So $ac - ca \in J(T[R, S])$. $\square$

Let S and T be any rings, M an S-T-bimodule and R the formal triangular matrix ring $\begin{bmatrix} S & M \\ 0 & T \end{bmatrix}$. It is well-known that $J(R) = \begin{bmatrix} J(S) & M \\ 0 & J(T) \end{bmatrix}$.

Proposition 3.5. *Let $R = \begin{bmatrix} S & M \\ 0 & T \end{bmatrix}$. Then R is J-abelian if and only if S and T are J-abelian.*

Proof. The necessity is obvious by Lemma 2.16. Assume that S and T are J-abelian. Let $A = \begin{bmatrix} a & m \\ 0 & b \end{bmatrix}$, $E = \begin{bmatrix} e & n \\ 0 & f \end{bmatrix} \in R$ such that $E^2 = E$. Then e and f are idempotent elements of S and T, respectively. Hence we have $ae - ea \in J(S)$ and $bf - fb \in J(T)$. Hence $AE - EA \in J(R)$. Thus R is J-abelian. $\square$

The following result is a direct consequence of Proposition 3.5.

Corollary 3.6. *Let R be a ring. R is J-abelian if and only if $T_2(R)$ is J-abelian.*

For any positive integer n, the ring of all $n \times n$ matrices $M_n(R)$ need not be J-abelian as the following example shows.

Example 3.7. Consider the ring $R = M_2(\mathbb{Z})$, let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in R$ and $E = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ be an idempotent in R. Since $AE - EA = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \notin J(R)$, R is not a J-abelian ring.

In spite of the fact that $M_n(R)$ is not J-abelian for any positive integer n, we end this paper to show that there are J-abelian subrings of $M_n(R)$. For any ring R, let $T_n(R)$ be the ring of $n \times n$ upper triangular matrices over R, and $D_n(R)$ the subring $\{(a_{ij}) \in T_n(R) \mid$ all diagonal entries of (a_{ij}) are equal$\}$ and $V_n(R)$ the subring of $T_n(R)$ where n is a positive integer:

$$V_n(R) = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & \ldots & a_{n-1} & a_n \\ 0 & a_1 & a_2 & \ldots & a_{n-2} & a_{n-1} \\ 0 & 0 & a_1 & \ldots & a_{n-3} & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & a_1 & a_2 \\ 0 & 0 & 0 & \ldots & 0 & a_1 \end{bmatrix} \mid a_i \in R, 1 \leq i \leq n \right\}.$$

The Jacobson radicals of $V_n(R)$, $D_n(R)$ and $T_n(R)$ are given by

$$
\begin{aligned}
J(V_n(R)) &= \{(a_i) \in V_n(R) \mid a_1 \in J(R)\}, \\
J(D_n(R)) &= \{(a_{ij}) \in D_n(R) \mid a_{ii} \in J(R)\}, \\
J(T_n(R)) &= \{(a_{ij}) \in T_n(R) \mid a_{ii} \in J(R)\},
\end{aligned}
$$

respectively.

Theorem 3.8. *Let R be a ring. For any positive integer n, the following statements are equivalent.*

(1) R is J-abelian.

(2) $T_n(R)$ is J-abelian.

(3) $D_n(R)$ is J-abelian.

(4) $V_n(R)$ is J-abelian.

Proof. (1) $\Rightarrow$ (2) Suppose that R is J-abelian. In order to prove $T_n(R)$ is J-abelian, let $A = (a_{ij})$, $E^2 = E = (e_{ij}) \in T_n(R)$. Then $AE - EA \in J(T_n(R))$ since $a_{ii}e_{ii} - e_{ii}a_{ii} \in J(R)$ for all i with $0 \leq i \leq n$ by (1).

(2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) are clear.

(4) $\Rightarrow$ (1) Let $a \in R$ and e be an idempotent in R. Assume that A is the matrix having main diagonal entries a elsewhere 0 and E is the matrix having main diagonal entries e elsewhere 0. Then $AE - EA \in J(V_n(R))$. Hence $ae - ea \in J(R)$. □

Corollary 3.9. *The following are equivalent for a ring R.*

1. R is J-abelian.

2. $R[x]/(x^n)$ is J-abelian for any $n \geq 2$.

Proof. It is well-known that $R[x]/(x^n) \cong V_n(R)$. So the proof is clear, by Theorem 3.8. □

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