On (k, n)-closed second submodules

Sobre sub-módulos (k,n)-cerrados de segunda clase

H. Ansari-Toroghy[1,a], **F. Farshadifar**[2,b], **S. Maleki-Roudposhti**[3,c]

Abstract. In this paper we introduce the concepts of semi n-absorbing second and (k, n)-closed second submodules of modules over a commutative ring and obtain some related results.

Keywords: Strongly 2-absorbing second submodule, semi n-absorbing second submodule, (k, n)-closed second submodule.

Resumen. En este artículo, introducimos los conceptos de sub-módulos semi-ausorbentes segundos y (k, n)-cerrados segundos de módulos sobre un anillo conmutativo y obtenemos algunos resultados relacionados.

Palabras claves: submódulos fuertemente 2-absorventes de segunda clase, submódulos semi n-absorventes de segunda clase, submódulos (k, n)-cerrados de segunda clase.

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1. Introduction

Throughout this paper, R will denote a commutative ring with identity and n, k are positive integers. Further, $\mathbb{Z}$ will denote the ring of integers.

Let M be an R-module. A proper submodule P of M is said to be *prime* if for any $r \in R$ and $m \in M$ with $rm \in P$, we have $m \in P$ or $r \in (P :_R M)$ [11]. A non-zero submodule S of M is said to be *second* if for each $a \in R$, the homomorphism $S \xrightarrow{a} S$ is either surjective or zero [14].

The concept of 2-absorbing ideals was introduced in [7] and then extended to n-absorbing ideals in [1]. A proper ideal I of R is a *2-absorbing ideal* of R if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. Let I be a proper ideal of R and n a positive integer. I is called an *n-absorbing ideal* of

[1]Department of pure Mathematics, Faculty of Mathematical Sciences, University of Guilan, P. O. Box 41335-19141, Rasht, Iran.
[2]Assistant Professor, Department of Mathematics, Farhangian University, Tehran, Iran.
[3]Department of pure Mathematics, Faculty of Mathematical Sciences, University of Guilan, P. O. Box 41335-19141, Rasht, Iran.
[a]ansari@guilan.ac.ir
[b]f.farshadifar@cfu.ac.ir
[c]Sepidehmaleki.r@gmail.com

R if whenever $x_1 \cdots x_{n+1} \in I$ for $x_1, \dots, x_{n+1} \in R$, then there are n of the x_i's whose product is in I. A proper ideal I of R is said to be a (k, n)-*closed ideal* of R if $x^k \in I$ for $x \in R$ implies $x^n \in I$ [2].

A proper submodule N of M is called n-absorbing submodule of M if whenever $a_1 \dots a_n m \in N$ for $a_1, \dots, a_n \in R$ and $m \in M$, then either $a_1 \dots a_n \in (N :_R M)$ or there are $n-1$ of a_i's whose product with m is in N [10]. A proper submodule N of M is called a (k, n)-*closed submodule* of M if whenever $r \in R$, $m \in M$ with $r^k m \in N$, then $r^n \in (N :_R M)$ or $r^{n-1} m \in N$. In particular, we call N as a *semi n-absorbing submodule* of M if whenever $r \in R$, $m \in M$ with $r^n m \in N$, then $r^n \in (N :_R M)$ or $r^{n-1} m \in N$ [15]. It is clear that a semi n-absorbing submodule is (n, n)-closed.

In [3], the authors introduced the notion of strongly 2-absorbing second submodules as a the dual notion of 2-absorbing submodules and then extended to strongly n-absorbing second submodules in [6]. A non-zero submodule N of M is said to be a *strongly 2-absorbing second submodule of* M if whenever $a, b \in R$, K is a submodule of M, and $abN \subseteq K$, then $aN \subseteq K$ or $bN \subseteq K$ or $ab \in Ann_R(N)$. A non-zero submodule N of M is said to be a *strongly n-absorbing second submodule* of M if whenever $a_1 \dots a_n N \subseteq K$ for $a_1, \dots, a_n \in R$ and a submodule K of M, then either $a_1 \dots a_n \in Ann_R(N)$ or there are $n-1$ of a_i's whose product with N is a subset of K.

The purpose of this paper is to introduce the concepts of semi n-absorbing second and (k, n)-closed second submodules of modules over a commutative ring as dual notions of the concepts of semi n-absorbing and (k, n)-closed submodules, respectively and investigate their basic properties.

2. Main Results

Let M be an R-module. A proper submodule N of M is said to be *completely irreducible* if $N = \bigcap_{i \in I} N_i$, where $\{N_i\}_{i \in I}$ is a family of submodules of M, implies that $N = N_i$ for some $i \in I$. It is easy to see that every submodule of M is an intersection of completely irreducible submodules of M [13].

We frequently use the following basic fact without further comment.

Remark 2.1. Let N and K be two submodules of an R-module M. To prove $N \subseteq K$, it is enough to show that if L is a completely irreducible submodule of M such that $K \subseteq L$, then $N \subseteq L$.

Definition 2.2. Let M be an R-module and N be a non-zero submodule of M. We say that N is a (k, n)-*closed second submodule of* M if whenever $r \in R$ and K is a submodule of M with $r^k N \subseteq K$, then $r^n \in Ann_R(N)$ or $r^{n-1} N \subseteq K$. We say that M is a (k, n)-*closed second module* if M is a (k, n)-closed second submodule of itself.

Clearly, every non-zero submodule is (k, n)-closed for $1 \leq k < n$; so we often assume that $1 \leq n \leq k$.

Definition 2.3. Let M be an R-module and N be a non-zero submodule of M. We say that N is a *semi n-absorbing second submodule of M* if whenever $r \in R$ and K is a submodule of M with $r^n N \subseteq K$, then $r^n \in Ann_R(N)$ or $r^{n-1}N \subseteq K$.

It is clear that every submodule of M is a semi n-absorbing second submodule if and only if it is a (n,n)-closed second submodule.

Theorem 2.4. *Let N be a non-zero submodule of an R-module M. Then the following statements are equivalent:*

(a) N is a (k,n)-closed second submodule of M;

(b) If $r \in R$ and L is a completely irreducible submodule of M with $r^k N \subseteq L$, then $r^n \in Ann_R(N)$ or $r^{n-1}N \subseteq L$; In particular, a non-zero submodule N of M is a semi n-absorbing second submodule of M if and only if whenever $r \in R$, L a completely irreducible submodule of M with $r^n N \subseteq L$, then $r^n \in Ann_R(N)$ or $r^{n-1}N \subseteq L$.

Proof. $(a) \Rightarrow (b)$ This is clear.

$(b) \Rightarrow (a)$ Let N be a non-zero submodule of M, $r \in R$, and K be a submodule of M with $r^k N \subseteq K$. Assume on the contrary that $r^{n-1}N \not\subseteq K$ and $r^n \notin Ann_R(N)$. Then there exists a completely irreducible submodule L of M such that $K \subseteq L$ but $r^{n-1}N \not\subseteq L$. Thus $r^k N \subseteq L$. By part (b), $r^{n-1}N \subseteq L$ or $r^n \in Ann_R(N)$ which are contradictions. □

Theorem 2.5. *Let N be a non-zero submodule of an R-module M and $k \geq n$. Then the following statements are equivalent:*

(a) N is a (k,n)-closed second submodule of M;

(b) $(K :_R r^k N) = (K :_R r^{n-1}N)$ or $r^n \in Ann_R(N)$, where $r \in R$ and K is a submodule of M;

(c) $r^k N = r^{n-1}N$ or $r^n \in Ann_R(N)$, where $r \in R$.

Proof. $(a) \Rightarrow (b)$ Let $r \in R$ and K be a submodule of M. Assume that $r^n \notin Ann_R(N)$ and $s \in (K :_R r^k N)$. Then $r^k N \subseteq (K :_R s)$. Since N is (k,n)-closed second and $r^n \notin Ann_R(N)$, we have $r^{n-1}N \subseteq (K :_M s)$. It follows that $s \in (K :_R r^{n-1}N)$. Thus $(K :_R r^k N) \subseteq (K :_R r^{n-1}N)$. The inverse inclusion is always hold since $k \geq n$.

$(b) \Rightarrow (a)$ Let $r \in R$ and K be a submodule of M with $r^k N \subseteq K$. If $r^n \in Ann_R(N)$, then we are done. So assume that $r^n \notin Ann_R(N)$. Then by part (b), $(K :_R r^k N) = (K :_R r^{n-1}N)$. Thus $1 \in (K :_R r^k N)$ implies that $1 \in (K :_R r^{n-1}N)$. Hence $r^{n-1}N \subseteq K$, as needed.

$(b) \Rightarrow (c)$ Let $r \in R$ and $r^n \notin Ann_R(N)$. Since $k \geq n$, we have $r^k N \subseteq r^{n-1}N$. Now let L be a completely irreducible submodule of M such that $r^k N \subseteq L$. Then $1 \in (L :_R r^k N)$. By part (b), $(L :_R r^k N) = (L :_R r^{n-1}N)$. Hence $1 \in (L :_R r^{n-1}N)$ and so $r^{n-1}N \subseteq L$, Thus $r^{n-1}N \subseteq r^k N$ as needed.

$(c) \Rightarrow (b)$ This is clear. □

Theorem 2.6. *Let N be a submodule of an R-module M. Then we have the following.*

(a) If N is a (k,n)-closed second submodule of M, then $(K :_R N)$ is a (k,n)-closed ideal of R for each submodule K of M with $N \not\subseteq K$.

(b) If $(K :_R N)$ is a (k,n)-closed ideal of R for each each submodule K of M with $N \not\subseteq K$, then N is a $(k,n+1)$-closed second submodule of M.

Proof. (a) Assume on the contrary that $r^k \in (K :_R N)$ and $r^n \notin (K :_R N)$ for some submodule K of M with $N \not\subseteq K$. Then $r^k N \subseteq K$ but $r^n N \not\subseteq K$ and so $r^n N \neq 0$. Now since N is a (k,n)-closed submodule of M, we have $r^{n-1} \in (K :_R N)$ and so $r^n \in (K :_R N)$, a contradiction. Thus $(K :_R N)$ is a (k,n)-closed ideal of R for each submodule K of M with $N \not\subseteq K$.

(b) Let $r^k N \subseteq K$ for some $r \in R$ and a submodule K of M. If $N \subseteq K$, we are done. So suppose that $N \not\subseteq K$. Assume that $r^{n+1} \notin Ann_R(N)$. Since $r^k \in (K :_R N)$ and $(K :_R N)$ is a (k,n)-closed ideal of R for each submodule K of M with $N \not\subseteq K$, we conclude that $r^n \in (K :_R N)$. It follows that N is a $(k,n+1)$-closed second submodule of M. □

Corollary 2.7. *Let N be a (k,n)-closed second submodule of an R-module M. Then $Ann_R(N)$ is a (k,n)-closed ideal of R.*

Proof. Take $K = 0$ in Theorem 2.6 (a). □

The following example shows that the converse of Corollary 2.7 (a) is not true in general.

Example 2.8. Consider $N = t\mathbb{Z}$ as a submodule of the $\mathbb{Z}$-module $\mathbb{Z}$, where t is a positive integer. Then clearly, $Ann_{\mathbb{Z}}(t\mathbb{Z}) = 0$ is a $(2,1)$-closed ideal of $\mathbb{Z}$. But since $2^2 t\mathbb{Z} \subseteq 4t\mathbb{Z}$, $2^0 t\mathbb{Z} \not\subseteq 4t\mathbb{Z}$, and $2^1 t\mathbb{Z} \neq 0$, we have $t\mathbb{Z}$ is not $(2,1)$-closed submodule of Z.

Proposition 2.9. *Let N a submodule of an R-module M. If N is a semi n-absorbing second submodule of M, then N is a (k,n)-closed second submodule of M for all positive integer k.*

Proof. If $k \leq n$, the the claim is clear. So suppose that $k > n$. Let $r^k N \subseteq K$ for some $r \in R$ and a submodule K of M. Assume that $r^n \notin Ann_R(N)$. Then since $r^n N \subseteq (K :_M r^{k-n})$ and N is semi n-absorbing second, we get that $r^{n-1} N \subseteq (K :_M r^{k-n})$. This implies that $r^{k-1} N \subseteq K$. So we continue with this argument and obtain that $r^{n-1} N \subseteq K$ and so N is a (k,n)-closed second submodule of M. □

Corollary 2.10. *Let N be a submodule of an R-module M and $k > n$. Then N is a (k,n)-closed second submodule of M if and only if N is a semi n-absorbing second submodule of M.*

Proof. Let N be a (k,n)-closed second submodule of M and $r^nN \subseteq K$ for $r \in R$ and a submodule K of M. Then since $k > n$, we have $r^kN \subseteq K$, and this implies that either $r^n \in Ann_R(N)$ or $r^{n-1}N \subseteq K$. Thus N is a semi n-absorbing second submodule of M. The reverse implication follows from Proposition 2.9. $\square$

An R-module M is said to be *semi-second* if $rM = r^2M$ for each $r \in R$ [4].

Theorem 2.11. *Let M be an R-module. Then we have the following.*

(a) If N is a semi-second submodule of M. Then N is a (k,n)-closed second submodule of M for all positive integers k and $n > 1$. Moreover, N is a semi n-absorbing second submodule of M for all positive integer $n > 1$.

(b) If $\{N_i\}_{i\in I}$ is a family of semi-second submodules of M, then $\sum_{i\in I} N_i$ is a (k,n)-closed submodule of M for all positive integers k and $n > 1$.

(c) If N is a strongly n-absorbing second submodule of M, then N is a semi n-absorbing second submodule of M.

(d) If N is a (k,n)-closed second submodule of M, then N is a (k_1,n_1)-closed second submodule of M for all $k_1 \leq k$ and $n_1 \geq n$.

(e) If N is a semi n-absorbing second submodule of M, then N is a semi n_1-absorbing second submodule of M for all $n_1 \geq n$.

Proof. (a), (c), and (d) are clear from the definitions.

(b) Suppose that $r^k \sum_{i\in I} N_i \subseteq K$ for some $r \in R$ and a submodule K of M. Then $r^kN_i \subseteq K$ for all $i \in I$. Since each N_i is semi-second, we conclude that $rN_i \subseteq K$ for all $i \in I$. Thus $r\sum_{i\in I} N_i \subseteq K$ which means that $r^{n-1}\sum_{i\in I} N_i \subseteq K$ for all $n > 1$, as needed.

(e) Let $t = n_1 - n$ and $r^{n_1}N \subseteq K$ for some $r \in R$ and a submodule K of M. Then $r^nN \subseteq (K :_M r^t)$. Thus by assumption, $r^nN = 0$ or $r^{n-1}N \subseteq (K :_M r^t)$. Thus $r^{n_1}N = 0$ or $r^{n_1-1}N \subseteq K$, as needed. $\square$

For a submodule N of an R-module M the the *second radical* (or second socle) of N is defined as the sum of all second submodules of M contained in N and it is denoted by $sec(N)$ (or $soc(N)$). In case N does not contain any second submodule, the second radical of N is defined to be (0) (see [9] and [5]).

Corollary 2.12. *Let N be a non-zero submodule of an R-module M. Then $sec(N)$ and $Soc_R(N)$ are (k,n)-closed second submodule of M for all integers k and n. (Here $Soc_R(N)$ denotes the sum of all minimal submodules of N.)*

Proof. Since every minimal and every second submodule is a semi-second submodule, the results follows from part (b) of Theorem 2.11. $\square$

The following example shows that the converse of part (c) in Theorem 2.11 is not true in general.

Example 2.13. Let $M = \mathbb{Z}_{30}$ as a $\mathbb{Z}$-module. Since $M \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5$ is sum of semi-second $\mathbb{Z}$-modules, it is semi 2-absorbing second submodule of M by Theorem 2.11 (b). However, M is not strongly 2-absorbing second submodule of M. In fact $2 \times 3\mathbb{Z}_{30} \subseteq \bar{6}\mathbb{Z}_{30}$ but $2\mathbb{Z}_{30} \not\subseteq \bar{6}\mathbb{Z}_{30}$ and $3\mathbb{Z}_{30} \not\subseteq \bar{6}\mathbb{Z}_{30}$ and $2 \times 3\mathbb{Z}_{30} \neq 0$.

Theorem 2.14. *Let $\{N_i\}_{i\in I}$ be a chain of (k,n)-closed second submodules of an R-module M. Then $\sum_{i\in I} N_i$ is a (k,n)-closed second submodule of M.*

Proof. Set $N = \sum_{i\in I} N_i$. Let $r^k N \subseteq K$ for some $r \in R$ and a submodule K of M. If $r^n \in Ann_R(N_i)$ for all $i \in I$, then $r^n \in \cap_{i\in I} Ann_R(N_i) = Ann_R(N)$ and we are done. So suppose that $r^n \notin Ann_R(N_j)$ for some $j \in I$. Then $r^n \notin Ann_R(N_t)$ for all $N_j \subseteq N_t$. Hence $r^{n-1}N_t \subseteq K$ for all $N_j \subseteq N_t$ since each N_t is (k,n)-closed second. Therefore $r^{n-1}\sum_{i\in I} N_i \subseteq K$ which means that N is a (k,n)-closed second submodule of M. □

The following example shows that the sum of two semi n-absorbing second submodules may not be a semi n-absorbing second submodule in general.

Example 2.15. Consider $M = \mathbb{Z}_{p^n} \oplus \mathbb{Z}_{q^n}$ as $\mathbb{Z}$-module. Clearly $\mathbb{Z}_{p^n} \oplus 0$ and $0 \oplus \mathbb{Z}_{q^n}$ both are strongly n-absorbing second submodules and so semi n-absorbing second submodules of M by Theorem 2.11 (c). However $p^n M \subseteq 0 \oplus \mathbb{Z}_{q^n}$, $p^{n-1}M \not\subseteq 0 \oplus \mathbb{Z}_{q^n}$, and $p^n M \neq 0$ implies that M is not semi n-absorbing second submodule of M.

Definition 2.16. We say that a (k,n)-closed second submodule N of an R-module M is a *maximal (k,n)-closed second submodule* of a submodule K of M, if $N \subseteq K$ and there does not exist a (k,n)-closed second submodule L of M such that $N \subset L \subset K$.

Lemma 2.17. *Let M be an R-module. Then every (k,n)-closed second submodule of M is contained in a maximal (k,n)-closed second submodule of M.*

Proof. This proved easily by using Zorn's Lemma and Theorem 2.14. □

Theorem 2.18. *Every Artinian R-module has only a finite number of maximal (k,n)-closed second submodules.*

Proof. Suppose that there exists a non-zero submodule N of M such that it has an infinite number of maximal (k,n)-closed second submodules. Let S be a submodule of M chosen minimal such that S has an infinite number of maximal (k,n)-closed second submodules. Then S is not (k,n)-closed second submodule. Thus there exist $r \in R$ and a submodule K of M such that $r^k S \subseteq K$ but $r^{n-1}S \not\subseteq K$ and $r^n S \neq 0$. Let V be a maximal (k,n)-closed second submodule of M contained in S. Then $r^{n-1}V \subseteq K$ or $r^n V = 0$. Thus $V \subseteq (K :_M r^{n-1})$ or $V \subseteq (0 :_M r^n)$. Therefore, $V \subseteq (K :_S r^{n-1})$ or $V \subseteq (0 :_S r^n)$. By the choice of S, the modules $(K :_S r^{n-1})$ and $(0 :_S r^n)$ have only finitely many maximal (k,n)-closed second submodules. Therefore, there is only a finite number of possibilities for the module S, which is a contradiction. □

Theorem 2.19. *Let M be an R-module. If N_1 is a semi n_1-absorbing second and N_2 is a semi n_2-absorbing second submodule of M, then N_1+N_2 is semi $(n+1)$-absorbing second submodule of M, where $n = max\{n_1, n_2\}$.*

Proof. Let $r \in R$ and K be a submodule of M such that $r^{n+1}(N_1 + N_2) \subseteq K$. First observe by Corollary 2.10, N_1 and N_2 are $(n+1, n_1)$-closed second and $(n+1, n_2)$-closed second submodules of M, respectively. Hence we have $r^{n_1} \in Ann_R(N_1)$ or $r^{n_1-1}N_1 \subseteq K$ and $r^{n_2} \in Ann_R(N_2)$ or $r^{n_2-1}N_2 \subseteq K$. If $r^{n_1} \in Ann_R(N_1)$ and $r^{n_2} \in Ann_R(N_2)$, then $r^n \in Ann_R(N_1) \cap Ann_R(N_2) = Ann_R(N_1+N_2)$. If $r^{n_1} \in Ann_R(N_1)$ and $r^{n_2-1}N \subseteq K$, then $r^n(N_1+N_2) \subseteq K$. Similarly, if $r^{n_2} \in Ann_R(N_2)$ and $r^{n_1-1}N \subseteq K$, then $r^n(N_1 + N_2) \subseteq K$. For the last, if $r^{n_1-1}N_1 \subseteq K$ and $r^{n_2-1}N_2 \subseteq K$, then $r^{n-1}(N_1 + N_2) \subseteq K$. Thus we conclude either $r^{n+1} \in Ann_R(N_1+N_2)$ or $r^n(N_1+N_2) \subseteq K$, as needed. □

Proposition 2.20. *Let M be a finitely cogenerated R-module such that $\cap_{i=1}^n L_i = 0$, where each L_i is a completely irreducible submodule of M for $i = 1, \dots, n$. If N is a non-zero submodule of M, $k > n$, and $(L_i :_R N)$ is a (k,n)-closed ideal of R for all $i = 1, \dots, n$, then $Ann_R(N)$ is a (k,n)-closed ideal of R.*

Proof. Assume that $(L_i :_R N)$ is a (k,n)-closed ideal of R for all $i = 1, \dots, n$. Suppose that $r^k \in Ann_R(N)$ and $r^n \notin Ann_R(N)$ for some $r \in R$. Then $r^n \notin (L_j :_R N)$ for some $j = 1, \dots, n$. Hence $r^k \notin (L_j :_R N)$, and so $r^k \notin Ann_R(N)$, which is a contradiction. Thus $Ann_R(N)$ is a (k,n)-closed ideal of R. □

Definition 2.21. Let N be a non-zero submodule of an R-module M. We say that N is a *strongly semi n-absorbing second submodule of M* if whenever I is an ideal of R and K is a submodule of M with $I^nN \subseteq K$, then $I^n \in Ann_R(N)$ or $I^{n-1}N \subseteq K$.

Definition 2.22. Let N be a non-zero submodule of an R-module M. We say that N is a *strongly (k, n)-closed second submodule of M* if whenever I is an ideal of R and K is a submodule of M with $I^kN \subseteq K$, then $I^n \in Ann_R(N)$ or $I^{n-1}N \subseteq K$.

Note that every strongly (k,n)-closed second submodule is a (k,n)-closed second submodule of M. Clearly a $(k,1)$-closed second submodule is also a strongly $(k,1)$-closed second submodule of M. Also observe that a strongly semi n-absorbing second submodule is a semi n-absorbing second submodule of M.

Theorem 2.23. *Let N be a non-zero submodule of an R-module M. Then the following statements are equivalent:*

(a) N is a strongly (k,n)-closed second submodule of M;

(b) If I is an ideal of R and L is a completely irreducible submodule of M with $I^kN \subseteq L$, then $I^n \in Ann_R(N)$ or $I^{n-1}N \subseteq L$;

(c) *For any ideal I of R and $H \subseteq N$ a submodule of M with $I^k N \subseteq H$ implies that $I^n \in Ann_R(N)$ or $I^{n-1}N \subseteq H$.*

Proof. $(a) \Rightarrow (b)$ This is clear.

$(b) \Rightarrow (a)$ Suppose that $I^k N \subseteq K$ for an ideal I of R and a submodule K of M. Assume that $I^{n-1}N \not\subseteq K$. Then there exists a completely irreducible submodule L of M such that $K \subseteq L$ but $I^{n-1}N \not\subseteq L$. Since $I^k N \subseteq L$, we have $I^n \in Ann_R(N)$ by part (b). Thus N is a strongly (k,n)-closed second submodule of M.

$(a) \Rightarrow (c)$ This is clear.

$(c) \Rightarrow (a)$ Let K be a submodule of M and I an ideal of R such that $I^k N \subseteq K$. Hence $I^k N \subseteq K \cap N$. Put $H = K \cap N$. Since N is strongly (k,n)-closed second, we conclude that either $I^n \in Ann_R(N)$ or $I^{n-1}N \subseteq H$ by part (c). Thus $I^n \in Ann_R(N)$ or $I^{n-1}N \subseteq K$ as needed. □

Proposition 2.24. *Let R be a principal ideal domain and N be a submodule of an R-module M. Then the following statements are equivalent:*

(a) *N is a (k,n)-closed second submodule of M;*

(b) *N is a strongly (k,n)-closed second submodule of M.*

Proof. This is clear. □

Proposition 2.25. *Let N be a submodule of an R-module M. If N is a (k,n)-closed second submodule of M, then IN is a (k,n)-closed second submodule of M for all ideals I of R with $I \not\subseteq Ann_R(N)$. Moreover; if N is a strongly (k,n)-closed second submodule of M, then $I^k N = I^{n-1}N$, where $k \geq n$.*

Proof. Suppose that $r^k IN \subseteq K$ for $r \in R$ and a submodule K of M. Hence $r^k N \subseteq (K :_M I)$, which implies that either $r^n \in Ann_R(N)$ or $r^{n-1}N \subseteq (K :_M I)$. Thus $r^n \in Ann_R(IN)$ or $r^{n-1}IN \subseteq K$. Thus IN is a (k,n)-closed second submodule of M for all ideals I of R. Now suppose that N is a strongly (k,n)-closed second submodule of M. Since $I^k N \subseteq I^{n-1}N$ is always true, it is sufficient to show the inverse inclusion. Let $I^k N \subseteq L$ for some completely irreducible submodule L of M. Then we have $I^n \in Ann_R(N)$ or $I^{n-1}N \subseteq L$ by Theorem 2.23. If $I^{n-1}N \subseteq L$, then we are done. So suppose that $I^n \in Ann_R(N)$. Thus $I^k \in Ann_R(N)$, as needed. □

An R-module M is said to be a *multiplication module* if for every submodule N of M there exists an ideal I of R such that $N = IM$ [8].

Corollary 2.26. *Let M be a multiplication (k,n)-closed second R-module. Then every non-zero submodule of M is a (k,n)-closed second submodule of M.*

Proof. This follows from Proposition 2.25. □

Theorem 2.27. *Let M be an R-module, N an $(k,2)$-closed second submodule of M, and I an ideal of R. Then we have the following.*

(a) If $I^k \subseteq Ann_R(N)$, then $2I^2 \subseteq Ann_R(N)$.

(b) Suppose that $2 \in U(R)$ (here $U(R)$ denotes the set of all units of R). If $I^k \subseteq Ann_R(N)$, then $I^2 \subseteq Ann_R(N)$ (i.e., $Ann_R(N)$ is a strongly $(k,2)$-closed ideal of R).

Proof. By Corollary 2.7, $Ann_R(N)$ is an $(k,2)$-closed ideal of R. Thus the result follows from [2, 2.6]. □

Let R be an integral domain. Recall that if for every element r of its field of fractions F, at least one of r or r^{-1} belongs to R, then R is called *valuation domain.*

Proposition 2.28. *Let R be a valuation domain with quotient field F. Let M be an R-module and N a non-zero submodule of M. Then N is a semi n-absorbing second submodule of M if and only if whenever $r \in F$, H a submodule of M with $r^{n+1}N \subseteq H$ implies that $r^nN \subseteq H$ or $r^{n+1} \in Ann_R(N)$.*

Proof. Suppose that N is a semi n-absorbing second submodule of M. Assume that $r^{n+1}N \subseteq H$, but $r^{n+1} \notin Ann_R(N)$, where $r \in F$, H a submodule of M. If $r \in R$, then we are done. So assume that $r \notin R$. Since R is a valuation domain, $r^{-1} \in R$. Hence we have $r^{-1}r^{n+1}N = r^nN \subseteq H$. The converse is clear. □

Theorem 2.29. *Let $f : M \to \acute{M}$ be a monomorphism of R-modules. Then we have the following.*

(a) If N is a (k,n)-closed (resp. semi n-absorbing) second submodule of M, then $f(N)$ is a (k,n)-closed (resp. semi n-absorbing) second submodule of $\acute{M}$.

(b) If $\acute{N}$ is a (k,n)-closed (resp. semi n-absorbing) second submodule of $\acute{M}$ and $\acute{N} \subseteq f(M)$, then $f^{-1}(\acute{N})$ is a (k,n)-closed (resp. semi n-absorbing) second submodule of M.

Proof. (a) Let N be a (k,n)-closed second submodule of M. Since $N \neq 0$ and f is a monomorphism, we have $f(N) \neq 0$. Let $r \in R$, $\acute{K}$ be a submodule of $\acute{M}$, and $r^kf(N) \subseteq \acute{K}$. Then $r^kN \subseteq f^{-1}(\acute{K})$. As N is (k,n)-closed second submodule, $r^{n-1}N \subseteq f^{-1}(\acute{K})$ or $r^nN = 0$. Therefore,

$$r^{n-1}f(N) \subseteq f(f^{-1}(\acute{K})) = f(M) \cap \acute{K} \subseteq \acute{K}$$

or $r^nf(N) = 0$, as needed. For semi n-absorbing second, the proof can be easily verified similar.

(b) If $f^{-1}(\acute{N}) = 0$, then $f(M) \cap \acute{N} = ff^{-1}(\acute{N}) = f(0) = 0$. Thus $\acute{N} = 0$, a contradiction. Therefore, $f^{-1}(\acute{N}) \neq 0$. Now let $r \in R$, K be a submodule of M, and $r^kf^{-1}(\acute{N}) \subseteq K$. Then

$$r^k\acute{N} = r^k(f(M) \cap \acute{N}) = r^kff^{-1}(\acute{N}) \subseteq f(K).$$

As $\acute{N}$ is (k, n)-closed second submodule, $r^{n-1}\acute{N} \subseteq f(K)$ or $r^n\acute{N} = 0$. Hence $r^{n-1}f^{-1}(\acute{N}) \subseteq f^{-1}f(K) = K$ or $r^n f^{-1}(\acute{N}) = 0$, as desired. Similarly, for semi n-absorbing second, the proof can be easily verified. $\square$

Corollary 2.30. *Let M be an R-module and $N \subseteq K$ be two submodules of M. Then we have the following.*

(a) If N is a (k, n)-closed (resp. semi n-absorbing) second submodule of K, then N is a (k, n)-closed (resp. semi n-absorbing) second submodule of M.

(b) If N is a (k, n)-closed (resp. semi n-absorbing) second submodule of M, then N is a (k, n)-closed (resp. semi n-absorbing) second submodule of K.

Proof. This follows from Theorem 2.29 by using the natural monomorphism $K \to M$. $\square$

Proposition 2.31. *Let M_1, M_2 be R-modules with $M = M_1 \oplus M_2$, and let N_1, N_2 be non-zero submodules of M_1, M_2, respectively. N_1 is a (k_1, n_1)-closed second submodule of M_1 if and only if $N_1 \oplus 0$ is a (k, n)-closed second submodule of $M_1 \oplus M_2$ for all positive integers $k_1 \leq k$ and $n \geq n_1$.*

Proof. This is straightforward. $\square$

Theorem 2.32. *Let R be an integral domain and N be a non-zero submodule of an R-module M. Let $Ann_R(N) = p^tR$, where p is prime element of R and $t > 0$. If N is a (k, n)-closed second submodule of M, then we have the following.*

(a) $t = ka + r$, where a and r are integers such that $a \geq 0$, $1 \leq r \leq n$, $a(k \bmod n) + r \leq n$, and if $a \neq 0$, then $k = n + c$ for an integer c with $1 \leq c \leq n - 1$.

(b) If $k = bn + c$ for integers b and c with $b \geq 2$ and $0 \leq c \leq n - 1$, then $t \in \{1, \ldots, n\}$. If $k = n + c$ for an integer c with $1 \leq c \leq n - 1$, then $t \in \cup_{h=1}^{n}\{ki + h : i \in \mathbb{Z}$ and $0 \leq ic \leq n - h\}$.

Proof. Suppose that N is a (k, n)-closed second submodule of M. Then $Ann_R(N)$ is a (k, n)-closed ideal of R by Corollary 2.7. Thus the result follows from [2, 3.1]. $\square$

Corollary 2.33. *Let R be an integral domain and N be a non-zero submodule of an R-module M. Let $Ann_R(N) = p^tR$, where p is prime element of R and $t > 0$. If N is a semi n-absorbing second submodule of M, then $t = na + r$, where a and r are integers such that $a \geq 0$, $1 \leq r < n$, that is $t \in \cup_{h=1}^{n}\{ni + h : i \in \mathbb{Z}$ and $0 \leq i \leq n - h\}$.*

Proof. Since a semi n-absorbing second submodule is a $(n+1, n)$-closed second submodule of M by Proposition 2.9, the result follows from Theorem 2.32. $\square$

Corollary 2.34. *Let R be an integral domain and N be a non-zero submodule of an R-module M. Let $Ann_R(N) = p^t R$ where p is prime element of R and $k > 0$. Then if N is a semi 2-absorbing second submodule of M, then $t \in \{1, 2\}$.*

Proof. This follows from Corollary 2.33. □

Example 2.35. Consider the $\mathbb{Z}$-module $M = \mathbb{Z}_8$. Then M is not a semi 2-absorbing second submodule of M since $t = 3$ by Corollary 2.34.

An element m of an R-module M is called a *torsion element* if $rm = 0$ for some non-zero element $r \in R$. The set of all torsion elements of M is denoted by $T(M) := \{m \in M | rm = 0 \ for \ some \ nonzero \ r \in R\}$.

Proposition 2.36. *Let M be an R-module. If every proper ideal of R is (k, n)-closed (resp. semi n-absorbing), then every non-zero submodule of M is (k, n)-closed (resp. semi n-absorbing) second. The converse holds if $T(M) \neq M$.*

Proof. First suppose that every proper ideal of R is (k, n)-closed. Let N be a non-zero submodule of M, $r \in R$, and K be a submodule of M such that $r^k N \subseteq K$. If $(K :_R N) = R$, then we are done. So suppose that $(K :_R N) \neq R$. Then by assumption, $r^n \in (K :_R N)$ and so $r^{n-1}N \subseteq K$. For the converse, suppose that $T(M) \neq M$. Then there exists $m \in M$ such that $Ann_R(Rm) = 0$. Now let $I \neq R$ be an ideal of R. Then $I = (Im :_R Rm)$ by [12, 3.1]. Assume that $r^k \in I$ for some $r \in R$. Then $r^k \in (Im :_R Rm)$. Hence $r^k(Rm) \subseteq Im$. By assumption, Rm is a (k, n)-closed second submodule. Thus $r^n Rm = 0$ or $r^{n-1}Rm \subseteq Im$. If $r^n Rm = 0$, then $r^n = 0 \in I$ since $Ann_R(Rm) = 0$ and we are done. If $r^{n-1}Rm \subseteq Im$, then $r^{n-1} \in I$ as needed. The proof for semi-n-absorbing is similar. □

Theorem 2.37. *Let M be an R-module. If E is an injective R-module and N is a (k, n)-closed submodule of M such that $Hom_R(M/N, E) \neq 0$, then $Hom_R(M/N, E)$ is a (k, n)-closed second R-module, where $k \geq n$.*

Proof. Let $r \in R$. Since N is a (k, n)-closed submodule of M, we can assume that $(N :_M r^k) = (N :_M r^{n-1})$ or $(N :_M r^n) = M$ by using [15, 2.7]. Since E is an injective R-module, by replacing M with M/N in [4, 3.13 (a)], we have $Hom_R(M/(N :_M r), E) = rHom_R(M/N, E)$. Therefore,

$$r^k Hom_R(M/N, E) = Hom_R(M/(N :_M r^k), E) =$$

$$Hom_R(M/(N :_M r^{n-1}), E) = r^{n-1} Hom_R(M/N, E)$$

or

$$r^n Hom_R(M/N, E) = Hom_R(M/(N :_M r^n), E) =$$

$$Hom_R(M/M, E) = 0,$$

as needed □

Theorem 2.38. *Let M be a (k, n)-closed second R-module, where $k \geq n$ and F be a right exact linear covariant functor over the category of R-modules. Then $F(M)$ is a (k, n)-closed second R-module if $F(M) \neq 0$.*

Proof. This follows from [4, 3.14] and Theorem 2.5 $(a) \Rightarrow (c)$. □

Corollary 2.39. *Let M be an R-module, S be a multiplicative subset of R and N be a (k, n)-closed second submodule of M, where $k \geq n$. Then $S^{-1}N$ is a (k, n)-closed second submodule of $S^{-1}M$ if $S^{-1}N \neq 0$.*

Proof. This follows from Theorem 2.38. □

A proper submodule N of an R-module M is said to be a *primary submodule* of M if for each $r \in R$ the homothety $M/N \xrightarrow{r} M/N$ is either injective or nilpotent. In this case $P = \sqrt{(N :_R M)}$ is a prime ideal of R, and we call N a *P-primary submodule* of M.

Theorem 2.40. *Let N be a primary submodule of an R-module M. If K is a semi n-absorbing second submodule of M such that $N + K \neq M$, then $N + K$ is a primary submodule of M.*

Proof. Let N be a P-primary submodule of M, $r \in R$, and $r(n + k) \in N + K$ for some $n \in N$ and $k \in K$. If $r \in P = \sqrt{(N :_R M)}$, then clearly $r \in \sqrt{(N + K :_R M)}$. So assume that $r \notin P$. As $r(n + k) \in N + K$, we have $r(n + k) = n_1 + k_1$ for some $n_1 \in N$ and $k_1 \in K$. It follows that $r^n n + r^n k - r^{n-1}k_1 \in N$. Since K is a semi n-absorbing second submodule of M, we have $r^n K = 0$ or $r^n K = r^{n-1}K$ by Theorem 2.5. If $r^n K = 0$, then $r^n n + r^n k - r^{n-1}k_1 = r^{n-1}(rn - k_1) \in N$. This implies that $rn - k_1 \in N$ because N is a P-primary submodule of M and $r^{n-1} \notin P$. So that $k_1 \in N$. Therefore, $rn+rk = n_1+k_1 \in N$. Thus $n+k \in N$ as needed. If $r^n K = r^{n-1}K$, then $r^{n-1}k_1 = r^n k_2$ for some $k_2 \in K$. Thus $r^n n+r^n k-r^n k_2 \in N$. This implies that $n + k - k_2 \in N$ because N is a P-primary submodule of M and $r^n \notin P$. Thus $n + k = n + k - k_1 + k_2 \in N + K$, as desired. □

Corollary 2.41. *Let N and K be two non-zero submodules of an R-module M with $N \subseteq K \neq M$. If N is a primary and K is a semi n-absorbing second submodule of M, then K is a primary submodule of M.*

Proof. This follows from Theorem 2.40. □

Theorem 2.42. *Let M_1, M_2 be R-modules, N_1 be a (k_1, n_1)-closed second submodule of M_1, and N_2 be a (k_2, n_2)-closed second submodule of M_2. Then $N_1 \oplus N_2$ is a (k, n)-closed second submodule of $M_1 \oplus M_2$ for all positive integers $k \leq min\{k_1, k_2\}$ and $n \geq max\{n_1, n_2\} + 1$.*

Proof. By Theorem 2.11 (d), N_1, N_2 are both (k, n)-closed second submodules of M_1 and M_2, respectively. Let $r \in R$. Then $r^k N_1 = r^{n-1}N_1$ or $r^n N_1 = 0$ and $r^k N_2 = r^{n-1}N_2$ or $r^n N_2 = 0$. If $r^k N_1 = r^{n-1}N_1$ and $r^k N_2 = r^{n-1}N_2$

(resp. $r^nN_1 = 0$ and $r^nN_2 = 0$), then $r^k(N_1 \oplus N_2) = r^{n-1}(N_1 \oplus N_2)$ (resp. $r^n(N_1 \oplus N_2) = 0$) so we are done. If $r^kN_1 = r^{n-1}N_1$ and $r^nN_2 = 0$, then $r^kN_2 = 0$ because $n \leq k$. Thus $r^k(N_1 \oplus N_2) = r^kN_1 \oplus 0 = r^{n-1}(N_1 \oplus N_2)$. Similarly, we are done if $r^kN_2 = r^{n-1}N_2$ and $r^nN_1 = 0$. □

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