The lattice of ordinable topologies

El retículo de las topologías ordinables

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To Professor Carlos Javier Ruiz Salguero, in memoriam

Abstract. We demonstrate that the ordinable topologies for a set X are precisely those that occupy the upper part of the lattice of topologies for X, and that they determine a lattice, not always complete or distributive. We also found the amount of complements, and principal complements, for certain ordinable topologies, generalizing a known result of P. S. Schnare.

Keywords: Ordinable element in an ordered set, lattice, ultratopologies.

Resumen. En este artículo demostramos que las topologías ordinables para un conjunto X son justamente aquellas que ocupan la parte más alta del retículo de topologías para X, y que estas topologías determinan un retículo, que no siempre es completo o distributivo. Adicionalmente encontramos la cantidad de complementos y de complementos principales para ciertas topologías ordinables, generalizando un resultado conocido de P. S. Schnare.

Palabras claves: Elemento ordinable en un conjunto ordenado, retículo, ultratopologías.

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1. Introduction

The lattice of topologies has been studied from different points of view, and all these studies show the great complexity and richness of the structure of this ordered set.

However, there are many interesting relationships between the elements of this lattice that these studies did not reveal, but which are observable through the concept of ordinable element of an ordered set. This work is an example of this.

In 2010 the author [4] does a first study about the properties of ordinable elements in this important lattice, but several questions of considerable interest were not resolved there. For example, the structure of the set of ordinable topologies is not studied.

In this article we provide information in this direction, but we also present partial results with respect to the number of complements of an ordinable topology in the case where the base set is infinite. The main purposes of this paper are:

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1. To show that the ordinable topologies for a set X are precisely those that occupy the upper part of the lattice of topologies for X.

2. To show that the set of ordinable topologies for a set is a lattice, not always complete or distributive.

3. To determine the number of complements, and principal complements, of certain ordinable topologies, obtaining a generalization of a known result of Schnare, [5].

These results constitute a small contribution to the knowledge of the structure of the lattice of topologies.

2. Preliminaries

This section will briefly mention some of the important results that the author presented in [4], which will be very useful in developing this article. We assume that the reader is familiar with the basic concepts of ordered sets. For further references the reader may consult [1] and [3].

Let $(A, \leq)$ be a partial ordered set. We associate to each ordinal number α a subset $(A, \leq)_\alpha$ of A, as follows:

For $\alpha = 0$, $(A, \leq)_0$ is the set of maximal elements of $(A, \leq)$. And for $\alpha > 0$, $(A, \leq)_\alpha$ is the set of maximal elements of $A \setminus \bigcup\limits_{\beta<\alpha} (A, \leq)_\beta$, with the induced order of $\leq$.

Definition 2.1. An element $a \in A$ is **ordinable** if there is an ordinal α (necessarily unique) such that $a \in (A, \leq)_\alpha$. In this case we write $O\,(a) = \alpha$.

The least ordinal number α such that $(A, \leq)_\alpha = \varnothing$ will be denoted by $O\,(A, \leq)$.

It is easy to verify that if a is ordinable in $(A, \leq)$ and $a \leq b$, then b is ordinable in $(A, \leq)$. Furthermore, if $a \in (A, \leq)_\alpha$ and $b \in (A, \leq)_\beta$ then $\beta \leq \alpha$. In addition, if $a \in (A, \leq)_\alpha$ and $\delta \leq \alpha$ is an ordinal number, then there is $c \in (A, \leq)_\delta$ such that $a \leq c$.

We also have that $O\,(A, \leq) < card\,(\mathcal{P}(A))$, where $\mathcal{P}(A)$ is the set of all the subsets of A.

Another result that is easily proved is: if $(A, \leq)$ is an ordered set, and $a \in A$ is such that the set of *succesors* of a, $\{b \in A : a \leq b\}$, is a finite set, then a is ordinable in $(A, \leq)$, and $O(a) < \omega$, where ω is the least infinite ordinal number.

The particular and important case that interests us is the lattice $(Top(X), \subseteq)$ of topologies for a set X, with the inclusion order.

In this case $(Top(X), \subseteq)_0 = \{\mathcal{P}(X)\}$, where $\mathcal{P}(X)$ is the discrete topology for X, and $(Top(X), \subseteq)_1$ is the set of ultratopologies for X, which has the form $\mathcal{P}(X \setminus \{a\}) \cup \mathcal{U}$, where $\mathcal{U}$ is an ultrafilter on X, $a \in X$ and $\{a\} \notin \mathcal{U}$, see [2].

With respect to other ordinable elements in this ordered set, the author has demonstrated the three following results.

Remark 2.2. If (X,τ) is a topological space,

$$A_\tau := \{x \in X : \{x\} \notin \tau\},$$

and if $A \subseteq X$,

$$\mathcal{N}_\tau(A) := \bigcap \{V \in \tau : A \subseteq V\}.$$

If $A = \{a\}$ we write $\mathcal{N}_\tau(a)$ instead of $\mathcal{N}_\tau(\{a\})$. The set $\mathcal{N}_\tau(A)$ is often called **the nucleus** of A in the space (X,τ). Note that if τ and β are topologies for X then $A_{\tau\cap\beta} = A_\tau \cup A_\beta$.

Proposition 2.3 (N.R.Pachón, [4])**.** *If Φ is ordinable in $(Top(X),\subseteq)$ then the set A_Φ is finite and for each $x \in A_\Phi$, the set $\mathcal{N}_\Phi(x)$ is finite.*

Recall some definitions of lattice theory that we need throughout the article. A lattice $(A,\leq)$ is **distributive** if $a\wedge(b\vee c) = (a\wedge b)\vee(a\wedge c)$, for all $a,b,c \in A$. If $(A,\leq)$ is a lattice with a minimum element 0 and maximum element 1, and if for $a \in A$ there exists an element $b \in A$ such that $a\wedge b = 0$ and $a \vee b = 1$, it is said that b is a **complement** of a. A lattice is called **Boolean** if it is distributive with 0 and 1, and every element has a complement (necessarily unique).

If a and b are elements in a lattice $(A,\leq)$, with $a \leq b$, and if c and d are elements in the closed interval $[a,b]$, it is said that d is a **relative complement** of c in $[a,b]$ if $c\wedge d = a$ and $c \vee d = b$. A lattice is **relatively complemented** if each of its elements has a relative complement in any closed interval containing it.

Making two changes in the structure of ultratopologies, the author obtained the ordinable elements mentioned in the two following theorems, which will be significantly generalized in this work.

Theorem 2.4 (N.R.Pachón [4])**.** *Let $\mathcal{U}_1,\mathcal{U}_2,\ldots,\mathcal{U}_n$ be ultrafilters on X, and let $a \in X$ such that $\{a\} \notin \bigcup_{i=1}^{n} \mathcal{U}_i$. If Φ is the topology $\mathcal{P}(X\setminus\{a\}) \cup \bigcap_{i=1}^{n} \mathcal{U}_i$ then:*

(i) Φ is ordinable in $(Top(X),\subseteq)$ and $\Phi \in (Top(X),\subseteq)_n$.

(ii) The closed interval $[\Phi,\mathcal{P}(X)]$ has cardinal 2^n.

(iii) The closed interval $[\Phi,\mathcal{P}(X)]$ is a Boolean lattice.

Remark 2.5. Let X be a set and F be a nonempty subset of X. If $\mathcal{U}$ is an ultrafilter on X such that $F \notin \mathcal{U}$, we denote by $\mathcal{U}_F$ the topology $\mathcal{P}(X\setminus F)\cup\mathcal{U}$.

Theorem 2.6 (N.R.Pachón [4])**.** *Let X be a set and F be a finite nonempty subset of X. If $\mathcal{U}$ is an ultrafilter on X such that $F \notin \mathcal{U}$, then:*

(i) $\mathcal{U}_F$ is ordinable in $(Top(X),\subseteq)$ and $\mathcal{U}_F \in (Top(X),\subseteq)_{card(F)}$.

(ii) The closed interval $[\mathcal{U}_F,\mathcal{P}(X)]$ has cardinal $2^{card(F)}$.

(iii) The closed interval $[\mathcal{U}_F,\mathcal{P}(X)]$ is a Boolean lattice.

As a consequence, if X is an infinite set then $(Top(X),\subseteq)_\alpha \neq \varnothing$, for all ordinal number $\alpha < \omega$ where ω is the least infinite ordinal number. The question is, what happens if $\alpha \geq \omega$? The answer is found in the next section.

3. Characterization of the ordinable elements in the lattice of topologies.

In this section we present necessary and sufficient conditions for an element to be ordinable in the lattice of topologies, which will lead us to determine the ordinal number $O\left(Top(X), \subseteq\right)$ when X is infinite. Specifically, we will show that the following propositions are equivalent:

(i) A topology Φ (for a set X) is ordinable.

(ii) The interval $[\Phi, \mathcal{P}(X)]$ is finite.

(iii) Φ is of finite depth in the lattice Top (X).

(iv) The collection of ultratopologies (for X) containing Φ is finite.

We also show that the converse of Proposition 2.3 is not true. In order to achieve this goal we need the following three lemmas.

Lemma 3.1. *Let $\{\mathcal{U}_i\}_{i\in I}$ be a nonempty collection of ultrafilters on X, and let $a \in X$ such that $\{a\} \notin \bigcup_{i\in I} \mathcal{U}_i$. Let Φ be the topology $\mathcal{P}(X \setminus \{a\}) \cup \bigcap_{i\in I} \mathcal{U}_i$. If $V \subseteq X$ with $V \notin \Phi$, and if λ is the topology for X generated by $\Phi \cup \{V\}$, then either $\lambda = \mathcal{P}(X)$ or there exists $J \subseteq I$, with $\varnothing \neq J \neq I$, such that $\lambda = \mathcal{P}(X \setminus \{a\}) \cup \bigcap_{j\in J} \mathcal{U}_j$.*

Proof. We have that $a \in V$ and $V \notin \bigcap_{i\in I} \mathcal{U}_i$. If $V \notin \bigcup_{i\in I} \mathcal{U}_i$ then $(X \setminus V) \cup \{a\} \in \bigcap_{i\in I} \mathcal{U}_i$, therefore $\{a\} = V \cap [(X \setminus V) \cup \{a\}] \in \lambda$ and $\lambda = \mathcal{P}(X)$.

If $V \in \bigcup_{i\in I} \mathcal{U}_i$, let $J = \{i \in I : V \in \mathcal{U}_i\}$. It is clear that $J \neq \varnothing$ and $I \setminus J \neq \varnothing$. Consider the topology $\Psi = \mathcal{P}(X \setminus \{a\}) \cup \bigcap_{j\in J} \mathcal{U}_j$, note that $\lambda \subseteq \Psi$. We will see now that $\Psi \subseteq \lambda$.

Suppose that $W \in \bigcap_{j\in J} \mathcal{U}_j$ and $a \in W$. Since we can write

$$W = (W \setminus V) \cup \left[(W \cup (X \setminus V)) \cap V\right],$$

with $W \setminus V \in \mathcal{P}(X \setminus \{a\})$ and $W \cup (X \setminus V) \in \left(\bigcap_{j\in J} \mathcal{U}_j\right) \cap \left(\bigcap_{i\in I\setminus J} \mathcal{U}_i\right) = \bigcap_{i\in I} \mathcal{U}_i$, then $W \in \lambda$. Thus $\Psi \subseteq \lambda$. □

Lemma 3.2. *Let $\{\mathcal{U}_i\}_{i\in I}$ be a nonempty collection of ultrafilters on X, and $a \in X$ such that $\{a\} \notin \bigcup_{i\in I} \mathcal{U}_i$. Let $\Phi = \mathcal{P}(X \setminus \{a\}) \cup (\bigcap_{i\in I} \mathcal{U}_i)$. If Φ is ordinable in $(Top(X), \subseteq)$ then*

$$[\Phi, \mathcal{P}(X)] = \{\mathcal{P}(X)\} \cup \left\{\mathcal{P}(X \setminus \{a\}) \cup \bigcap_{j\in J} \mathcal{U}_j : \varnothing \neq J \subseteq I\right\}.$$

Proof. It is clear that

$$\{\mathcal{P}(X)\} \cup \left\{ \mathcal{P}(X \setminus \{a\}) \cup \bigcap_{j \in J} \mathcal{U}_j : \varnothing \neq J \subseteq I \right\} \subseteq [\Phi, \mathcal{P}(X)] .$$

We prove the other inclusion by induction on $O(\Phi)$. If $O(\Phi) = 1$, is obvious since Φ is an ultratopology.

Assume the result for $O(\Phi) < \alpha$ and suppose $O(\Phi) = \alpha$. Let $\beta \in Top(X)$ with $\beta \in [\Phi, \mathcal{P}(X)] \setminus \{\Phi\}$. Let $V \in \beta \setminus \Phi$. Note that if $\langle \Phi \cup \{V\} \rangle$ is the topology for X generated by the set $\Phi \cup \{V\}$, then $\langle \Phi \cup \{V\} \rangle \subseteq \beta$. According to Lemma 3.1 either $\langle \Phi \cup \{V\} \rangle = \mathcal{P}(X)$ or there exists $J \subseteq I$, with $\varnothing \neq J \neq I$, such that $\langle \Phi \cup \{V\} \rangle = \mathcal{P}(X \setminus \{a\}) \cup (\bigcap_{j \in J} \mathcal{U}_j)$.

In the first case we conclude that $\beta = \mathcal{P}(X)$. In the second case, since $O(\langle \Phi \cup \{V\} \rangle) < O(\Phi)$, the induction hypothesis implies that either $\beta = \mathcal{P}(X)$ or there exists $L \subseteq J \subseteq I$, with $L \neq \varnothing$, such that $\beta = \mathcal{P}(X \setminus \{a\}) \cup \bigcap_{l \in L} \mathcal{U}_l$. □

The following lemma shows an interesting property of ultrafilters, which we will use in the proof of Proposition 3.4.

Lemma 3.3. *Let $\{\mathcal{U}_i\}_{i \in I}$ be an infinite collection of different ultrafilters on X, and let $a \in X$ such that $\{a\} \notin \bigcup_{i \in I} \mathcal{U}_i$. Then there exists $K \subseteq I$, with K infinite and $K \neq I$, and there exists $A \subseteq X$ with $a \in A$, such that $A \in \bigcap_{k \in K} \mathcal{U}_k \setminus \bigcap_{i \in I} \mathcal{U}_i$.*

Proof. Let $l \in I$, arbitrary. Since $\mathcal{U}_l \nsubseteq \bigcap_{i \neq l} \mathcal{U}_i$ there exists $B \in \mathcal{U}_l \setminus \bigcap_{i \neq l} \mathcal{U}_i$. Let $J = \{i \in I : B \in \mathcal{U}_i\}$. It is clear that $J \neq \varnothing$ and $I \setminus J \neq \varnothing$.

If J is infinite and $A = B \cup \{a\}$, then $A \in \bigcap_{j \in J} \mathcal{U}_j \setminus \bigcap_{i \in I} \mathcal{U}_i$. If $I \setminus J$ is infinite and $A = (X \setminus B) \cup \{a\}$, then $A \in \bigcap_{r \in I \setminus J} \mathcal{U}_r \setminus \bigcap_{i \in I} \mathcal{U}_i$. □

Proposition 3.4. *Let $\{\mathcal{U}_i\}_{i \in I}$ be an infinite collection of different ultrafilters on X, and let $a \in X$ such that $\{a\} \notin \bigcup_{i \in I} \mathcal{U}_i$. If $\Phi = \mathcal{P}(X \setminus \{a\}) \cup \bigcap_{i \in I} \mathcal{U}_i$ then $\Phi \notin \bigcup_{\alpha \leq \omega} (Top(X), \subseteq)_\alpha$.*

Proof. It is clear that $\Phi \notin (Top(X), \subseteq)_0$. According to Lemma 3.3 there exists $K \subseteq I$, with K infinite and $K \neq I$, and $A \subseteq X$ with $a \in A$, such that $A \in \bigcap_{k \in K} \mathcal{U}_k \setminus \bigcap_{i \in I} \mathcal{U}_i$.

Let n be any positive integer. Let $i_1, \ldots, i_n$ be distinct elements in K. We have that $A \in \left(\mathcal{P}(X \setminus \{a\}) \cup \bigcap_{j=1}^{n} \mathcal{U}_{i_j} \right) \setminus \Phi$ and therefore Φ is a proper subset of $\mathcal{P}(X \setminus \{a\}) \cup \bigcap_{j=1}^{n} \mathcal{U}_{i_j}$.

Now, $\mathcal{P}(X \setminus \{a\}) \cup \bigcap_{j=1}^{n} \mathcal{U}_{i_j} \in (Top(X), \subseteq)_n$ by Theorem 2.4. Thus $\Phi \notin (Top(X), \subseteq)_n$. Since n is arbitrary we have that $\Phi \notin \bigcup_{\alpha < \omega} (Top(X), \subseteq)_\alpha$. But also Φ is a proper subset of $\mathcal{P}(X \setminus \{a\}) \cup \bigcap_{k\in K} \mathcal{U}_k$ and

$$\mathcal{P}(X \setminus \{a\}) \cup \bigcap_{k\in K} \mathcal{U}_k \notin \bigcup_{\alpha<\omega} (Top(X), \subseteq)_\alpha .$$

Then we can conclude that $\Phi \notin (Top(X), \subseteq)_\omega$. □

The following proposition will allow us to find varied information of great interest in relation to the ordinable elements in the lattice of topologies. First, it implies that the converse of Proposition 2.3 is not true. Second, it explains why in the Theorem 2.4 the collection of ultrafilters must be taken as finite. On the other hand it allows us to characterize the ordinable elements in this lattice, and lastly it allows us to determine the ordinal number $O(Top(X), \subseteq)$, in the event that the set X is infinite.

Proposition 3.5. *Let $\{\mathcal{U}_i\}_{i\in I}$ be an infinite collection of different ultrafilters on X, and let $a \in X$ such that $\{a\} \notin \bigcup_{i\in I} \mathcal{U}_i$. If $\Phi = \mathcal{P}(X \setminus \{a\}) \cup \bigcap_{i\in I} \mathcal{U}_i$ then Φ is not ordinable in $(Top(X), \subseteq)$.*

Proof. Suppose that Φ is ordinable. According to Proposition 3.4, $O(\Phi) > \omega$. There exists $\Psi \in (Top(X), \subseteq)_\omega$ such that $\Phi \subseteq \Psi$. According to Lemma 3.2, there exists $J \subseteq I$ such that $\Psi = \mathcal{P}(X \setminus \{a\}) \cup \bigcap_{j\in J} \mathcal{U}_j$.

According to Theorem 2.2 J must be infinite, which contradicts Proposition 3.4. □

If β is ordinable in $(Top(X), \subseteq)$, we know that the set A_β is finite and that for each $x \in A_\beta$, the set $\mathcal{N}_\beta(x)$ is finite. The converse of this proposition is not true because if $\{\mathcal{U}_i\}_{i\in I}$ is an infinite collection of non-principal ultrafilters on X, and if $\Phi = \mathcal{P}(X\setminus\{a\}) \cup \bigcap_{i\in I} \mathcal{U}_i$, with $a \in X$, then $A_\Phi = \{a\}$ y $\mathcal{N}_\Phi(a) = \{a\}$. However, according to Proposition 3.5, Φ is not ordinable in $(Top(X), \subseteq)$.

We continue now exploring the characteristics of the ordinable elements in the lattice $(Top(X), \subseteq)$, and for this purpose the Proposition 3.5 will be very helpful.

If $\Phi \in Top(X) \setminus \{\mathcal{P}(X)\}$, in [2] it is proved that if $\{\Phi_i\}_{i\in I}$ is the collection of all ultratopologies for X containing Φ, then $\Phi = \bigcap_{i\in I} \Phi_i$.

For each $i \in I$ there is an ultrafilter $\mathcal{U}_i$ on X, and $a_i \in X$, such that $\{a_i\} \notin \mathcal{U}_i$ and $\Phi_i = \mathcal{P}(X \setminus \{a_i\}) \cup \mathcal{U}_i$. Thus $\Phi = \bigcap_{i\in I} [\mathcal{P}(X \setminus \{a_i\}) \cup \mathcal{U}_i]$. If $a \in X \setminus \{a_i : i \in I\}$ then $\{a\} \in \mathcal{P}(X \setminus \{a_i\})$, for each $i \in I$, so $\{a\} \in \Phi$.

On the other hand, if $j \in I$ then $\{a_j\} \notin \mathcal{P}(X \setminus \{a_j\}) \cup \mathcal{U}_j$, hence $\{a_j\} \notin \Phi$. Thus $A_\Phi = \{a_i : i \in I\}$.

In addition, if Φ is ordinable we can conclude that $\{a_i : i \in I\}$ is finite, according to Proposition 2.3. Let us prove that in reality I is a finite set.

There are $i_1, \ldots, i_n$ in I, such that $A_\Phi = \{a_{i_1}, \ldots, a_{i_n}\}$. Then there are non-empty sets $I_1, \ldots, I_n$, such that $I = I_1 \cup \cdots \cup I_n$ and

$$\Phi = \left[\bigcap_{j\in I_1} [\mathcal{P}(X \setminus \{a_{i_1}\}) \cup \mathcal{U}_j]\right] \cap \cdots \cap \left[\bigcap_{j\in I_n} [\mathcal{P}(X \setminus \{a_{i_n}\}) \cup \mathcal{U}_j]\right],$$

that is,

$$\Phi=\left[\mathcal{P}(X\setminus\{a_{i_1}\})\cup\bigcap_{j\in I_1}\mathcal{U}_j\right]\cap\cdots\cap\left[\mathcal{P}(X\setminus\{a_{i_n}\})\cup\bigcap_{j\in I_n}\mathcal{U}_j\right].$$

Since $\Phi\subseteq\mathcal{P}(X\setminus\{a_{i_k}\})\cup\bigcap_{j\in I_k}\mathcal{U}_j$, for all $k\in\{1,\ldots,n\}$, every topology $\mathcal{P}(X\setminus\{a_{i_k}\})\cup\bigcap_{j\in I_k}\mathcal{U}_j$ is ordinable in $(Top(X),\subseteq)$. According to Proposition 3.5, for all $k\in\{1,\ldots,n\}$, I_k is finite, and consequently I is finite.

We have proved the following theorem, which provides another interesting property of ordinable elements in the lattice $(Top(X),\subseteq)$.

Theorem 3.6. *If $\Phi\in Top(X)\setminus\{\mathcal{P}(X)\}$ and Φ is ordinable in $(Top(X),\subseteq)$, then the collection of ultratopologies for X containing Φ is finite, and Φ is the intersection of them.*

This theorem immediately leads us to a necessary and sufficient condition for a topology to be ordinable in the lattice of topologies.

Theorem 3.7. *If $\Phi\in Top(X)$ then Φ is ordinable if and only if the interval $[\Phi,\mathcal{P}(X)]$ is finite.*

Proof. If $[\Phi,\mathcal{P}(X)]$ is finite it is easy to see that Φ is ordinable.

Suppose that Φ is ordinable. If $\Phi=\mathcal{P}(X)$ there is nothing to prove. Assume $\Phi\neq\mathcal{P}(X)$. Let $\{\Phi_i\}_{i\in I}$ be the collection of all ultratopologies for X containing Φ. According to Theorem 3.6, I is finite and $\Phi=\bigcap_{i\in I}\Phi_i$.

Let $\Omega\in[\Phi,\mathcal{P}(X)]\setminus\{\mathcal{P}(X)\}$. Any ultratopology for X containing Ω is an element of the collection $\{\Phi_i\}_{i\in I}$. If Υ_Ω is the collection of all ultratopologies for X containig Ω, there is $K_\Omega\subseteq I$, nonempty, such that $\Upsilon_\Omega=\{\Phi_k\}_{k\in K_\Omega}$. Furthermore $\Omega=\bigcap_{k\in K_\Omega}\Phi_k$.

Since the function $\lambda:[\Phi,\mathcal{P}(X)]\setminus\{\mathcal{P}(X)\}\to\mathcal{P}(I)\setminus\{\varnothing\}$ defined by $\lambda(\Omega)=K_\Omega$, for all $\Omega\in[\Phi,\mathcal{P}(X)]\setminus\{\mathcal{P}(X)\}$, is inyective, we conclude that $[\Phi,\mathcal{P}(X)]$ is finite. □

Definition 3.8. Let $(A,\leq)$ be an ordered set with maximum element 1. We say that $b\in A$ is of **finite depth** if every chain in the interval $[b,1]$ is finite.

Corollary 3.9. *If $\Phi\in Top(X)$ then Φ is ordinable if and only if Φ has finite depth in the lattice $(Top(X),\subseteq)$.*

Proof. If Φ is ordinable then the interval $[\Phi,P(X)]$ is finite, and therefore Φ has finite depth in $(Top(X),\subseteq)$.

Now, if Φ has finite depth in $(Top(X),\subseteq)$ and Φ is not ordinable, then there exists a sequence $\{\Phi_n\}_{n=1}^{\infty}$ of non-ordinable topologies such that $\Phi\subsetneq\Phi_1\subsetneq\Phi_2\subsetneq\Phi_3\subsetneq\cdots$, which contradicts that Φ has finite depth. □

Remember that when X is infinite, $(Top(X),\subseteq)_\alpha\neq\varnothing$ for all ordinal number $\alpha<\omega$, hence we have the following important corollary, which gives us precise information about the ordinal number $O\,(Top(X),\subseteq)$.

Corollary 3.10. *If X is an infinite set and α is an ordinal number with $\alpha \geq \omega$, then $(Top(X), \subseteq)_\alpha = \varnothing$, and therefore $O\,(Top(X), \subseteq) = \omega$.*

The converse of Theorem 3.6 is true, as we will see in the following corollary.

Corollary 3.11. *If $\Phi \in Top(X) \setminus \{\mathcal{P}(X)\}$ and the collection of ultratopologies for X containing Φ is finite, then Φ is ordinable in $(Top(X), \subseteq)$.*

Proof. It is sufficient to note that any topology for X, other than the discrete, is the intersection of all ultratopologies for X that contain it, and that implies the interval $[\Phi, \mathcal{P}(X)]$ is finite. □

The following theorem gives information about the number of successors of an ordinable topology. If τ and β are topologies for X, we denote by $\tau \vee \beta$ the topology generated by $\tau \cup \beta$.

Theorem 3.12. *If Φ is ordinable in $(Top(X), \subseteq)$ and if the number of ultratopologies for X containing Φ is $n \geq 1$, then $card\,([\Phi, \mathcal{P}(X)]) \leq 2^n$. The equality occurs if and only if the lattice $[\Phi, \mathcal{P}(X)]$ is Boolean. Moreover, $O(\Phi) \leq n$.*

Proof. Let $\{\mathcal{P}(X \setminus \{a_i\}) \cup \mathcal{U}_i : 1 \leq i \leq n\}$ be the set of ultratopologies for X containing Φ. Let $\Phi_i = \mathcal{P}(X \setminus \{a_i\}) \cup \mathcal{U}_i$, for all $1 \leq i \leq n$.

If $\beta \in [\Phi, \mathcal{P}(X)]$ then there exists $J_\beta \subseteq \{1, \ldots, n\}$ such that the set of ultratopologies for X containing β is $\{\Phi_j : j \in J_\beta\}$. Since $\beta = \bigcap_{j \in J_\beta} \Phi_j$, the function $\lambda : [\Phi, \mathcal{P}(X)] \to \mathcal{P}(\{1, \ldots, n\})$, defined by $\lambda\,(\beta) = J_\beta$, is inyective. Thus $card\,([\Phi, \mathcal{P}(X)]) \leq 2^n$.

Furthermore, if $\beta, \mu \in [\Phi, \mathcal{P}(X)]$ then $\beta \subseteq \mu$ if and only if $\lambda\,(\mu) \subseteq \lambda\,(\beta)$. Therefore, $card\,([\Phi, \mathcal{P}(X)]) = 2^n$ if and only if λ is an anti-isomorphism of ordered sets.

Suppose that $[\Phi, \mathcal{P}(X)]$ is a Boolean lattice, and let $\varnothing \neq J \subseteq \{1, \ldots, n\}$ and $\varnothing \neq K \subseteq \{1, \ldots, n\}$, with $J \neq K$. Without loss of generality suppose that there exists $j_0 \in J \setminus K$.

If $\bigcap_{j \in J} \Phi_j = \bigcap_{k \in K} \Phi_k$ then $\bigcap_{k \in K} \Phi_k \subseteq \Phi_{j_0}$ and $\left(\bigcap_{k \in K} \Phi_k\right) \vee \Phi_{j_0} = \Phi_{j_0}$, but $\bigcap_{k \in K} (\Phi_k \vee \Phi_{j_0}) = \mathcal{P}(X)$. This is impossible since $[\Phi, \mathcal{P}(X)]$ is distributive. Thus $\bigcap_{j \in J} \Phi_k \neq \bigcap_{k \in K} \Phi_k$. In conclusion $card\,([\Phi, \mathcal{P}(X)]) = 2^n$.

Now, we will prove the last assertion of this theorem by induction on n. If $n = 1$ then Φ is an ultratopology for X and $O(\Phi) = 1$. Suppose that the conclusion is true for all $n \leq k$, and that the number of ultratopologies for X containing Φ is $k + 1$. Let $O(\Phi) = l$. There exists $\Omega \in (Top(X), \subseteq)_{l-1}$ such that $\Phi \subseteq \Omega$. If m is the number of ultratopologies for X containing Ω then $m < k + 1$. The inductive hypothesis implies that $l - 1 = O(\Omega) \leq m < k + 1$, and so $O(\Phi) \leq k + 1$. □

Another pending issue in [4] was to determine whether the topologies of the form $\mathcal{P}(X \setminus F) \cup \bigcap_{i=1}^{n} \mathcal{U}_i$, where $\{\mathcal{U}_i\}_{i=1}^n$ is a finite collection of ultrafilters on X and F is a non-empty finite subset of X, with $F \notin \bigcup_{i=1}^{n} \mathcal{U}_i$, are ordinable. With the help of Corollary 3.11 we proceed to give a positive response, but we will also obtain information with respect to the sets of successors of these topologies. For this purpose we use the following lemma, whose proof can be found in [4].

Lemma 3.13. *Let $\{\mathcal{U}_i\}_{i\in I}$ and $\{\mathcal{V}_j\}_{j\in J}$ be finite, disjoint and non-empty collections of ultrafilters on X. Let $F \subseteq X$ with $F \notin \mathcal{U}_i \cup \mathcal{V}_j$, for all $i \in I$ and $j \in J$. Then there are $A \in \bigcap_{i\in I} \mathcal{U}_i \setminus \bigcap_{j\in J} \mathcal{V}_j$ and $B \in \bigcap_{j\in J} \mathcal{V}_j \setminus \bigcap_{i\in I} \mathcal{U}_i$ such that $A \cap B = F$.*

Proposition 3.14. *Let $\{\mathcal{U}_i\}_{i=1}^{n}$ be a finite collection of ultrafilters on X, and F be a finite nonempty subset of X, with $F \notin \bigcup_{i=1}^{n} \mathcal{U}_i$. If $\Phi = \mathcal{P}(X\setminus F) \cup \bigcap_{i=1}^{n} \mathcal{U}_i$ then the number of ultratopologies for X containing Φ is $n \cdot car(F)$, and therefore Φ is ordinable in $(Top(X), \subseteq)$.*

Proof. It is sufficient to verify that if $\mathcal{C}$ is the collection of ultratopologies for X containing Φ, then

$$\mathcal{C} = \{\mathcal{P}(X \setminus \{f\}) \cup \mathcal{U}_i : i \in \{1, \ldots, n\} \text{ and } f \in F\}.$$

It is clear that $\{\mathcal{P}(X \setminus \{f\}) \cup \mathcal{U}_i : i \in \{1, \ldots, n\} \text{ and } f \in F\} \subseteq \mathcal{C}$. Let $\beta = \mathcal{P}(X \setminus \{a\}) \cup \mathcal{U}$ an ultratopology for X containing Φ. Since $\{a\} = A_\beta \subseteq A_\Phi = F$, we have $a \in F$.

Suppose $\mathcal{U} \neq \mathcal{U}_i$, for each $i \in \{1, \ldots, n\}$. According to Lemma 3.13 there exist $A \in \mathcal{U} \setminus \bigcap_{i=1}^{n} \mathcal{U}_i$ and $B \in \bigcap_{i=1}^{n} \mathcal{U}_i \setminus \mathcal{U}$ such that $A \cap B = \{a\}$. Since $B \in \Phi \subseteq \beta$ and $B \notin \mathcal{P}(X \setminus \{a\})$, we have that $B \in \mathcal{U}$, which is a contradiction. Therefore there exists $j \in \{1, \ldots, n\}$ such that $\mathcal{U} = \mathcal{U}_j$, and then $\beta \in \{\mathcal{P}(X \setminus \{f\}) \cup \mathcal{U}_i : i \in \{1, \ldots, n\} \text{ and } f \in F\}$.

Hence $\mathcal{C}$ has $n \cdot card(F)$ elements and according to Corollary 3.11 Φ is ordinable. □

We can really say more about the topologies considered in Proposition 3.14, as will be seen in the final theorem of this section, which is a generalization of Theorems 2.4 and 2.6. In Section 5 we also provide information about the number of complements of these topologies, in the lattice of topologies.

But first, we will show an important property about the collection of ultratopologies containing one of these topologies.

Lemma 3.15. *Let $\{\mathcal{U}_i\}_{i=1}^{n}$ be a finite collection of ultrafilters on X, and F be a nonempty finite subset of X, with $F \notin \bigcup_{i=1}^{n} \mathcal{U}_i$. If $\Phi = \mathcal{P}(X \setminus F) \cup \bigcap_{i=1}^{n} \mathcal{U}_i$, and if $\mathcal{A}$ and $\mathcal{B}$ are two different nonempty collections of ultratopologies for X containing Φ, then $\bigcap_{\beta\in\mathcal{A}} \beta \neq \bigcap_{\mu\in\mathcal{B}} \mu$.*

Proof. Without loss of generality suppose that there exists $\Omega \in \mathcal{A} \setminus \mathcal{B}$. According to the proof of Proposition 3.14, there exist $f \in F$ and $k \in \{1, \ldots, n\}$ such that

$$\Omega = \mathcal{P}(X \setminus \{f\}) \cup \mathcal{U}_k,$$

and $\{f_1,\dots,f_r\}\subseteq F$ and $\{\mathcal{U}_{11},\dots,\mathcal{U}_{1\alpha_1},\dots,\mathcal{U}_{r1},\dots,\mathcal{U}_{r\alpha_r}\}\subseteq\{\mathcal{U}_1,\dots,\mathcal{U}_n\}$, such that

$$\mathcal{B}=\{\mathcal{P}(X\setminus\{f_m\})\cup\mathcal{U}_{ms}:1\le m\le r,\ 1\le s\le\alpha_m\}.$$

If $f\notin\{f_1,\dots,f_r\}$ then $\{f\}\in\bigcap_{j=1}^{r}\mathcal{P}(X\setminus\{f_j\})\subseteq\bigcap_{\mu\in\mathcal{B}}\mu$, and as $\{f\}\notin\Omega$, then $\{f\}\notin\bigcap_{\beta\in\mathcal{A}}\beta$.

On the other hand, if $f=f_l$ for some $l\in\{1,\dots,r\}$, then $\mathcal{U}_k\neq\mathcal{U}_{ld}$ for all $d\in\{1,\dots,\alpha_l\}$. According to Lemma 3.13, there exist $A\in\mathcal{U}_k\setminus\bigcap_{d=1}^{\alpha_l}\mathcal{U}_{ld}$ and $B\in\bigcap_{d=1}^{\alpha_l}\mathcal{U}_{ld}\setminus\mathcal{U}_k$ such that $A\cap B=\{f\}$.

Let $W=B\setminus(\{f_1,\dots,f_r\}\setminus\{f_l\})$. We will see that $W\in\bigcap_{\mu\in\mathcal{B}}\mu\setminus\bigcap_{\beta\in\mathcal{A}}\beta$. Since $B\notin\mathcal{U}_k$ and $f\in W$ we have $W\notin\mathcal{U}_k$ and $W\notin\mathcal{P}(X\setminus\{f\})$. Thus $W\notin\Omega$ and so $W\notin\bigcap_{\beta\in\mathcal{A}}\beta$.

If $m\in\{1,\dots,r\}\setminus\{l\}$ it is clear that $W\in\mathcal{P}(X\setminus\{f_m\})$. Moreover, $B\cap(X\setminus\{f_1,\dots,f_r\})\in\bigcap_{d=1}^{\alpha_l}\mathcal{U}_{ld}$ and so $W=[B\cap(X\setminus\{f_1,\dots,f_r\})]\cup\{f_l\}\in\bigcap_{d=1}^{\alpha_l}\mathcal{U}_{ld}$. Thus $W\in\bigcap_{\mu\in\mathcal{B}}\mu$. □

Theorem 3.16. *Let $\{\mathcal{U}_i\}_{i=1}^n$ be a finite collection of ultrafilters on X, and F be a nonempty finite subset of X with $F\notin\bigcup_{i=1}^{n}\mathcal{U}_i$.*

If $\Phi=\mathcal{P}(X\setminus F)\cup\bigcap_{i=1}^{n}\mathcal{U}_i$ then $O(\Phi)=n\cdot card(F)$, $card([\Phi,\mathcal{P}(X)])=2^{n\cdot card(F)}$ and $[\Phi,\mathcal{P}(X)]$ is a Boolean lattice.

Proof. First, note that Lemma 3.15 implies that if $\mathcal{B}_1$ and $\mathcal{B}_2$ are two different collections of ultratopologies for X containing Φ, and $\mathcal{B}_1$ and $\mathcal{B}_2$ have the same cardinal, then $\bigcap_{\beta\in\mathcal{B}_1}\beta\nsubseteq\bigcap_{\delta\in\mathcal{B}_2}\delta$ and $\bigcap_{\delta\in\mathcal{B}_2}\delta\nsubseteq\bigcap_{\beta\in\mathcal{B}_1}\beta$.

This impies that, for all $k\in\{1,\dots,n\}$, any intersection of k ultratopologies for X containing Φ is in $(Top(X),\subseteq)_k$.

On the other hand, and considering that any topology for X, different from $\mathcal{P}(X)$, is the intersection of all ultratopologies for X containing it, it is concluded that

$$[\Phi,\mathcal{P}(X)]=\{\mathcal{P}(X)\}\cup\left\{\bigcap_{U\in\mathcal{B}}U:\varnothing\neq\mathcal{B}\subseteq\mathcal{C}\right\},$$

where $\mathcal{C}$ is the collection of ultratopologies for X containing Φ.

If for each $\beta\in[\Phi,\mathcal{P}(X)]$ we define $\mathcal{B}_\beta=\{\mu\in\mathcal{C}:\beta\subseteq\mu,\}$, and if we consider the function $\lambda:[\Phi,\mathcal{P}(X)]\to\mathcal{P}(\mathcal{C})$ defined by $\lambda(\beta)=\mathcal{B}_\beta$, for each $\beta\in[\Phi,\mathcal{P}(X)]$, it is immediate that λ is an anti-isomorphism of ordered sets. The rest is a consequence of Proposition 3.14 and Lemma 3.15. □

Remark 3.17. If $\{\mathcal{V}_j\}_{j=1}^n$ is a non-empty finite collection of ultrafilters on X, and if $G \subseteq X$ with $G \notin \bigcup_{j=1}^{n} \mathcal{V}_j$, then

$$\{\mathcal{V}_1, \ldots, \mathcal{V}_n\}_G := \mathcal{P}(X \setminus G) \cup \bigcap_{j=1}^{n} \mathcal{V}_j.$$

If $F \subseteq X$, the set of all topologies for X with the form $\{\mathcal{U}_1, \ldots, \mathcal{U}_m\}_F$, will be denoted by $\tau_{F,X}$.

Corollary 3.18. *Let F be a non-empty finite subset of X, and $\{\mathcal{V}_j\}_{j=1}^n$ be a non-empty finite collection of non-principal ultrafilters on X. Then the sets $\bigcup_{\Phi \in \tau_{F,X}} [\Phi, \mathcal{P}(X)]$ and $\bigcup_{G \in \mathcal{P}_{fin}(X)} [\{\mathcal{V}_1, \ldots, \mathcal{V}_n\}_G, \mathcal{P}(X)]$ are distributive lattices, relatively complemented and with maximum element. Here $\mathcal{P}_{fin}(X)$ is the collection of finite subsets of X.*

4. The lattice of ordinable topologies

In this section we will show other important consequences of Theorem 3.6, concerning the structure of the collection of ordinable topologies for a set. Specifically, we will show that the collection of ordinable topologies for a set is a lattice, not always complete. We also show that if the base set has more than two elements, this lattice is not distributive.

Remark 4.1. The set of ordinable topologies for X will be denoted by $Top_{ord}(X)$.

If β and Φ are elements of $Top_{ord}(X)$, it is clear that the topology generated by $\beta \cup \Phi$ is an element of $Top_{ord}(X)$. We will see now that $\beta \cap \Phi \in Top_{ord}(X)$.

Let $\{\mathcal{P}(X \setminus \{a_i\}) \cup \mathcal{U}_i : 1 \leq i \leq n\}$ and $\{\mathcal{P}(X \setminus \{b_j\}) \cup \mathcal{V}_j : 1 \leq j \leq m\}$ be the collections of ultratopologies for X containing β and Φ, respectively. According to Theorem 3.6, these collections are finite.

All these ultratopologies contain $\beta \cap \Phi$, but there may be other ultratopologies that contain $\beta \cap \Phi$. In any case there are not many options, as we will show immediately.

Let $\mu = \mathcal{P}(X \setminus \{c\}) \cup \mathcal{W}$ be an ultratopology for X containing $\beta \cap \Phi$. It is clear that $c \in \{a_1, \ldots, a_n, b_1, \ldots, b_m\}$.

If $\mathcal{W} \notin \{\mathcal{U}_1, \ldots, \mathcal{U}_n, \mathcal{V}_1, \ldots, \mathcal{V}_m\}$ then there exists $A \in \left[\left(\bigcap_{i=1}^{n} \mathcal{U}_i\right) \cap \left(\bigcap_{j=1}^{m} \mathcal{V}_j\right)\right]$ with $A \notin \mathcal{W}$. Since $A \in \beta \cap \Phi \subseteq \mu$ and $A \notin \mathcal{W}$ then $c \notin A$. Furthermore $A \cup \{c\} \in \beta \cap \Phi \subseteq \mu$, hence $A \cup \{c\} \in \mathcal{W}$, but this is impossible since $A \notin \mathcal{W}$, $\{c\} \notin \mathcal{W}$ and $\mathcal{W}$ is an ultrafilter. Hence $\mathcal{W} \in \{\mathcal{U}_1, \ldots, \mathcal{U}_n, \mathcal{V}_1, \ldots, \mathcal{V}_m\}$.

Therefore the set of ultratopologies containing $\beta \cap \Phi$ is finite. According to Corollary 3.11 $\beta \cap \Phi \in Top_{ord}(X)$.

We have proved the following theorem.

Theorem 4.2. *If X is a set, then $(Top_{ord}(X), \subseteq)$ is a sublattice of $(Top(X), \subseteq)$.*

When X is infinite, this lattice is not complete. In fact, if $a \in X$ and $\{\mathcal{U}_i\}_{i \in I}$ is an infinite collection of ultrafilters on X, with $\{a\} \notin \bigcup_{i \in I} \mathcal{U}_i$, then each topology $\mathcal{P}(X \setminus \{a\}) \cup \mathcal{U}_i$ is ordinable, but the topology $\bigcap_{i\in I} (\mathcal{P}(X \setminus \{a\}) \cup \mathcal{U}_i) = \mathcal{P}(X \setminus \{a\}) \cup \bigcap_{i\in I} \mathcal{U}_i$ is not, according to Proposition 3.5.

Corollary 4.3. *If X is infinite then the lattice $(Top_{ord}(X), \subseteq)$ has no minimal elements.*

Proof. If $\beta \in Top_{ord}(X)$ then, according Theorem 3.6, the collection of ultratopologies for X containing β is finite. If μ is an ultratopology for X such that $\beta \nsubseteq \mu$, then β contains strictly $\beta \cap \mu$ and $\beta \cap \mu \in Top_{ord}(X)$. This implies that β cannot be minimal in $(Top_{ord}(X), \subseteq)$. □

Now we will show that the lattice of ordinable topologies is not distributive if the base set has more than two elements.

Theorem 4.4. *If X is a set with more than two elements then the lattice $(Top_{ord}(X), \subseteq)$ is not distributive.*

Proof. Let a, b and c be three elements in X. Consider the topology $\Phi = \mathcal{P}(X \setminus \{a,b\}) \cup \{V \subseteq X : \{a,c\} \subseteq V\}$. If Φ_1 is the topology $\mathcal{P}(X \setminus \{b\}) \cup \{V \subseteq X : \{a,b,c\} \subseteq V\}$, then Φ is a proper subset of Φ_1, and $\Phi_1 \in (Top(X), \subseteq)_2$, as shown in [4]. Thus $\Phi \notin \bigcup_{j=0}^{2} (Top(X), \subseteq)_j$.

Suppose that $\mu \in Top(X)$ and that μ contains strictly Φ. Let $A \in \mu \setminus \Phi$. Then $A \cap \{a,b\} \neq \varnothing$ and $\{a,c\} \nsubseteq A$. If $A \cap \{a,b,c\} = \{a,b\}$, or if $A \cap \{a,b,c\} = \{a\}$, then $A \cap \{a,c\} = \{a\} \in \mu$, thus $\Phi_1 \subseteq \mu$.

On the other hand, if $A \cap \{a,b,c\} = \{b,c\}$ then $\{b,c\} \in \mu$, and therefore $\mathcal{P}(X \setminus \{a,b\}) \cup \{V \subseteq X : c \in V\} \subseteq \mu$. Furthermore, $\mathcal{P}(X \setminus \{a,b\}) \cup \{V \subseteq X : c \in V\} \in (Top(X), \subseteq)_2$, according to Theorem 2.6.

If $A \cap \{a,b,c\} = \{b\}$ then $\{b\} \in \mu$, hence $\mathcal{P}(X \setminus \{a\}) \cup \{V \subseteq X : c \in V\} \subseteq \mu$. Since $\mathcal{P}(X \setminus \{a\}) \cup \{V \subseteq X : c \in V\}$ is an ultratopology for X, $\mathcal{P}(X \setminus \{a\}) \cup \{V \subseteq X : c \in V\} \in (Top(X), \subseteq)_1$.

Consequently $\Phi \in (Top(X), \subseteq)_3$ and the interval $[\Phi, \mathcal{P}(X)]$ consists of the seven topologies: Φ, $\mathcal{P}(X)$, Φ_1,

$$\begin{aligned}
\Phi_2 &= \mathcal{P}(X \setminus \{a,b\}) \cup \{V \subseteq X : c \in V\},\\
\Phi_3 &= \mathcal{P}(X \setminus \{a\}) \cup \{V \subseteq X : c \in V\},\\
\Phi_4 &= \mathcal{P}(X \setminus \{b\}) \cup \{V \subseteq X : c \in V\},\\
\Phi_5 &= \mathcal{P}(X \setminus \{b\}) \cup \{V \subseteq X : a \in V\}.
\end{aligned}$$

Note that $\Phi_1 \subseteq \Phi_4$, $\Phi_1 \subseteq \Phi_5$, $\Phi_2 \subseteq \Phi_3$ and $\Phi_2 \subseteq \Phi_4$. Since Φ_2 and Φ_3 are two relative complements of Φ_5 in the interval $[\Phi, \mathcal{P}(X)]$, we conclude that $(Top_{ord}(X), \subseteq)$ is not distributive. □

The topology Φ of Theorem 4.4 also makes clear that the lattice of successors of an ordinable topology may not be complemented. In fact, observe that the topology Φ_4 has no complement in the lattice $[\Phi, \mathcal{P}(X)]$.

Moreover, we note that Φ also shows that the conclusion in Lemma 3.15 is not true for arbitrary ordinable topologies. In fact $\Phi_3 \cap \Phi_4 \cap \Phi_5 = \Phi = \Phi_3 \cap \Phi_5$.

If $(A, \leq)$ is a partially ordered set, the *Dedekind-MacNeille completion* of $(A, \leq)$ is a complete lattice $(A^*, \leqslant)$ containing an isomorphic copy of $(A, \leq)$, and is such that if $(B, \preceq)$ is any complete lattice containing an isomorphic copy of $(A, \leq)$, then $(B, \preceq)$ contains an isomorphic copy of $(A^*, \leqslant)$.

What is the Dedekind-MacNeille completion of the lattice $(Top_{ord}(X), \subseteq)$? If X is infinite then the lattice $(Top_{ord}(X), \subseteq)$ is not complete, but if Υ is any non-empty subset of $Top_{ord}(X)$, then there exists the least upper bound of Υ in $(Top_{ord}(X), \subseteq)$.

Consequently, if $Top^*_{ord}(X) = \{\{\varnothing, X\}\} \cup Top_{ord}(X)$ then the Dedekind-MacNeille completion of $(Top_{ord}(X), \subseteq)$ is $(Top^*_{ord}(X), \subseteq)$.

Observe that all elements in the lattice $(Top^*_{ord}(X), \subseteq)$ are ordinable, and $O\left(Top^*_{ord}(X), \subseteq\right) = \omega + 1$. Furthermore, the lattice $(Top^*_{ord}(X), \subseteq)$ is not complemented since if $\tau, \beta \in Top_{ord}(X)$ then $\tau \cap \beta \neq \{\varnothing, X\}$.

5. About the number of complements of an ordinable topology

Of all the questions related to the lattice of topologies, the complementation has been among the most outstanding. Schnare [5] showed that any proper topology for an infinite set X has at least $card(X)$ complements (resp., principal complements) and at most $2^{2^{card(X)}}$complements (resp., $2^{card(X)}$ principal complements), and that these bounds are the best possible. One result of this article called our attention: Any ultratopology for an infinite set X has exactly $2^{2^{card(X)}}$ complements, and $2^{card(X)}$ principal complements. The interesting part of this result is that the ultratopologies are ordinable topologies.

Naturally we asked for the cardinality of the set of complements of an ordinable topology for an infinite set, and the purpose of this section is to present partial answers for it. We obtain valuable information concerning the number of complements of some particular ordinable topologies, among which are those mentioned in Proposition 3.14. The following lemma is important for this purpose.

Lemma 5.1. *Let X be an infinite set and F a finite subset of X. If $\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_n$ are ultrafilters on X then there exists $Y \subseteq X$ such that $F \subseteq Y$, $card(Y) = card(X \setminus Y) = card(X)$ and $Y \in \bigcap_{i=1}^{n} \mathcal{U}_i$. Equivalently, there exists $V \subseteq X \setminus F$ such that $card(V) = card(X \setminus V) = card(X)$ and $V \notin \bigcup_{i=1}^{n} \mathcal{U}_i$.*

Proof. By induction on n. For $n = 1$. Let $V \subseteq X$ such that $F \subseteq V$ and $card(V) = card(X \setminus V) = card(X)$. If $V \in \mathcal{U}_1$ then $Y = V$. If $X \setminus V \in \mathcal{U}_1$ then $Y = (X \setminus V) \cup F$.

Assume that the result is true for $n = k$. If $\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_{k+1}$ are ultrafilters for X, then there exists $Y \subseteq X$ such that $F \subseteq Y$, $card(Y) = card(X \setminus Y) = card(X)$ and $Y \in \bigcap_{i=1}^{k} \mathcal{U}_i$.

Let $\{X_1, X_2\}$ be a partition of $X \setminus Y$ such that $card(X_1) = card(X_2) = card(X \setminus Y)$. Only one of the sets X_1 and $X \setminus X_1$ is in $\mathcal{U}_{k+1}$. We call X^* to it. Let $Y^* = Y \cup X^*$. Since $X \setminus Y^* = X_1$ or $X \setminus Y^* = X_2$, then $card\,(Y^*) = card\,(X) = card\,(X \setminus Y^*)$. Moreover $F \subseteq Y^*$ and $Y^* \in \bigcap_{i=1}^{k+1} \mathcal{U}_i$. □

A topology is **principal** if and only if it is closed under arbitrary intersections. If τ and β are topologies for a set X, we denoted by $\tau \vee \beta$ the topology generated by the set $\tau \cup \beta$. A base for $\tau \vee \beta$ is the set $\{U \cap V : U \in \tau \text{ and } V \in \beta\}$.

It is well known that on an infinite set X, there are $2^{2^{card(X)}}$ topologies and $2^{card(X)}$ principal topologies for X. In the following theorem we obtain the number of complements, and principal complements, for a great collection of ordinable topologies.

Theorem 5.2. *Let X be an infinite set and $\{x_1, x_2, \ldots, x_r\} \subseteq X$. Let $\mathcal{U}_{11}$, $\mathcal{U}_{12}, \ldots, \mathcal{U}_{1n_1}, \mathcal{U}_{21}, \mathcal{U}_{22}, \ldots, \mathcal{U}_{2n_2}, \ldots, \mathcal{U}_{r1}, \mathcal{U}_{r2}, \ldots, \mathcal{U}_{rn_r}$ ultrafilters on X, not necessarily distinct, such that $\{x_1, x_2, \ldots, x_r\} \notin \bigcup_{i=1}^{r} \bigcup_{j=1}^{n_i} \mathcal{U}_{ij}$. If Φ is the topology $\bigcap_{i=1}^{r} \left[\mathcal{P}\,(X \setminus \{x_i\}) \cup \bigcap_{j=1}^{n_i} \mathcal{U}_{ij} \right]$ then Φ has exactly $2^{2^{card(X)}}$ complements and $2^{card(X)}$ principal complements in the lattice $(Top\,(X)\,, \subseteq)$.*

Proof. If $F = \{x_1, x_2, \ldots, x_r\}$ then Lemma 5.1 guarantees that there is $V \subseteq X \setminus F$ such that $card(V) = card(X \setminus V) = card(X)$ and $V \notin \bigcup_{i=1}^{r} \bigcup_{j=1}^{n_i} \mathcal{U}_{ij}$.

Let $\beta \in Top(V)$, arbitrary. Consider the following topology for X:

$$\beta^* = \{U \cup F : U \in \beta\} \cup \{\varnothing, X\}\,.$$

If $i \in \{1, 2, \ldots, r\}$ then $(X \setminus F) \cup \{x_i\} \in \bigcap_{i=1}^{r} \bigcap_{j=1}^{n_i} \mathcal{U}_{ij} \subseteq \Phi$, and since $F \in \beta^*$ we have that $\{x_i\} = [(X \setminus F) \cup \{x_i\}] \cap F \in \Phi \vee \beta^*$. And since $\mathcal{P}(X \setminus F) \subseteq \Phi$, we have that $\Phi \vee \beta^* = \mathcal{P}\,(X)$.

Now, it is clear that, for all $U \in \beta$ and $i \in \{1, 2, \ldots, r\}$, we have that $U \cup F \notin \mathcal{P}\,(X \setminus \{x_i\}) \cup \bigcap_{j=1}^{n_i} \mathcal{U}_{ij}$, and then $U \cup F \notin \Phi$. Thus $\Phi \cap \beta^* = \{\varnothing, X\}$, and β^* is a complement of Φ.

On the other hand, if $\beta_1, \beta_2 \in Top(V)$ then $\beta_1^* = \beta_2^*$ if and only if $\beta_1 = \beta_2$, and consequently Φ has exactly $2^{2^{card(V)}} = 2^{2^{card(X)}}$ complements in the lattice $(Top(X), \subseteq)$.

Now, if β is a principal topology then β^* is a principal topology, and since there are $2^{card(V)} = 2^{card(X)}$ principal topologies for V, then Φ has exactly $2^{card(X)}$ principal complements in the lattice $(Top(X), \subseteq)$. □

The topology Φ of this theorem is an ordinable topology in the lattice $(Top(X), \subseteq)$, because it is the intersection of a finite number of elements of the lattice $Top_{ord}\,(X)$.

Corollary 5.3. *If Φ is an ordinable topology for the infinite set X, with $\Phi \neq \mathcal{P}\,(X)$, and none of the ultratopologies for X that contain Φ is principal, then Φ has exactly $2^{2^{card(X)}}$ complements and $2^{card(X)}$ principal complements in the lattice $(Top\,(X)\,, \subseteq)$.*

The next corollary provides additional information about the ordinable topologies presented in Proposition 3.14. This corollary generalizes the result of Schnare, concerning the number of complements of an ultratopology, mentioned previously.

Corollary 5.4. *Let X be an infinite set, F be a non-empty finite subset of X and $\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_r$ be ultrafilters on X such that $F \notin \bigcup_{i=1}^{r} \mathcal{U}_i$. If Φ is the topology $\mathcal{P}(X \setminus F) \cup \bigcap_{i=1}^{r} \mathcal{U}_i$ then Φ has exactly $2^{2^{card(X)}}$ complements and $2^{card(X)}$ principal complements in the lattice $(Top(X), \subseteq)$.*

The natural question is: the result of Theorem 5.2 is applicable to any ordinable topology? The answer is no, as we will see in the following proposition.

If X is a set and $a \in X$, then we denote by $\langle a \rangle$ the principal ultrafilter on X generated by $\{a\}$.

Proposition 5.5. *If X is an infinite set and $a, b \in X$, with $a \neq b$, and if $\Phi = [\mathcal{P}(X \setminus \{a\}) \cup \langle b \rangle] \cap [\mathcal{P}(X \setminus \{b\}) \cup \langle a \rangle]$ then Φ has exactly $2^{card(X)}$ complements in the lattice $(Top(X), \subseteq)$.*

Proof. Let β be a complement of Φ. If $W \in \beta$ and $\varnothing \neq W \neq X$ then $card(W \cap \{a, b\}) = 1$. There exist $A, B \in \Phi$ and $U, V \in \beta$ such that $\{a\} = A \cap U$ and $\{b\} = B \cap V$. We have that $U \cap V = \varnothing$, since otherwise $U \cap V \in (\Phi \cap \beta) \setminus \{\varnothing, X\}$, which is absurd. Moreover, as $U \cup V \in \Phi \cap \beta$ then $U \cup V = X$. Thus $V = X \setminus U$. Hence we conclude easily that $\beta = \{\varnothing, U, X \setminus U, X\}$.

On the other hand, if $Z \subseteq X$ and $card(Z \cap \{a, b\}) = 1$, then the topology $\beta_Z = \{\varnothing, Z, X \setminus Z, X\}$ is a complement of Φ.

All this allows us to conclude that Φ has exactly $2^{card(X)}$ complements in the lattice $(Top(X), \subseteq)$. □

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