On generalized principally quasi–Baer modules

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Let R be an associative ring with identity. A right R–module M is called *generalized principally quasi–Baer* if for any $m \in M$, $r_R(m\,R)$ is left s–unital as an ideal of R and the ring R is said to be *right (left) generalized principally quasi–Baer* if R is a generalized principally quasi–Baer right (left) R–module. In this paper, we investigate properties of generalized principally quasi–Baer modules and right (left) generalized principally quasi–Baer rings.

Keywords: generalized principally quasi–Baer modules, right (left) generalized principally quasi–Baer rings,

Sea R un anillo asociativo con identidad. Se dice que un módulo derecho M de tipo R es de tipo *generalizado principalmente de tipo cuasi–Baer* si para cualquier $m \in M$, $r_R(m\,R)$ es unitario de tipo s a la izquierda como un ideal de R y el anillo R se dice de tipo *generalizado principalmente de tipo cuasi–Baer derecho (izquierdo)* si R es un módulo generalizado principalmente de tipo cuasi–Baer derecho (izquierdo) de tipo R. En este artículo se investigan las propiedades de los módulos generalizados principalmente de tipo cuasi–Baer y los anillos derechos (izquierdos) generalizados principalmente de tipo cuasi–Baer.

Palabras claves: módulos generalizados principalmente de tipo cuasi–Baer, anillos derechos (izquierdos) generalizados principalmente de tipo cuasi–Baer.

MSC: 13C99, 16D80, 16U80.

Recibido: 2 de diciembre de 2012 Aceptado: 18 de febrero de 2013

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1 Introduction

Throughout this paper R denotes an associative ring with identity and modules will be unitary right R–modules. An ideal I of R is said to be *right (*respectively *left) s–unital* [18] if for each $a \in I$ there exist an element $x \in I$ such that $a\,x = a$ (respectively $x\,a = a$). It is well known that I is right s–unital if and only if R/I is flat as a left R–module if and only if I is pure as a left ideal of R. For a subset X of a module M, let $r_R(X) = \{r \in R \mid X\,r = 0\}$. In [8], Lee and Zhou introduced Baer modules, quasi–Baer modules, principally projective modules and reduced modules as follows: A module M is called *Baer* if for any subset X of M, $r_R(X) = e\,R$ where $e^2 = e \in R$, while M is called *quasi–Baer* if for any submodule N of M, $r_R(N) = e\,R$, where $e^2 = e \in R$ and M is called *principally projective* if for any $m \in M$, $r_R(m) = e\,R$, where $e^2 = e \in R$. The ring R is said to be *right principally projective* if R is a principally projective right R–module. A module M is said to be *reduced* if for any $m \in M$ and $a \in R$, $m\,a = 0$ implies $m\,R \cap M\,a = 0$, equivalently $m\,a^2 = 0$ implies $m\,R\,a = 0$. The ring R is called *reduced* if R is a reduced right R–module. According to Baser and Harmanci [5], a module M is called *principally quasi–Baer* if for any $m \in M$, $r_R(m\,R) = e\,R$, where $e^2 = e \in R$. Also in [12], principally quasi–Baer modules over their endomorphism rings are studied. The ring R is said to be *right principally quasi–Baer* if R is a principally quasi–Baer right R–module. Moreover, every Baer module is quasi–Baer and every quasi–Baer module is principally quasi–Baer. The concept of generalized principally quasi–Baer modules is introduced in [14] to extend the notion of principally quasi–Baer modules and principally projective modules. A module M is called *generalized principally quasi–Baer* if for any $m \in M$, $r_R(m\,R)$ is left s–unital as an ideal of R, that is, for any $a \in r_R(m\,R)$, there exist $b \in r_R(m\,R)$ such that $b\,a = a$. The left version of generalized principally quasi–Baer module can be defined similarly. A ring R is called *right generalized principally quasi–Baer* if R is a generalized principally quasi–Baer right R–module. In [9], right generalized principally quasi–Baer rings are named as right APP–rings. A right generalized principally quasi–Baer ring is a generalization of a right principally quasi–Baer ring and a left principally projective ring. The left version of a generalized principally quasi–Baer ring can be defined similarly. Finally, a module M is called *abelian* [2] if for any $m \in M$, $a \in R$ and any idempotent $e \in R$, $m\,a\,e = m\,e\,a$, while a ring R is called *abelian* if R is an abelian right R–module.

In what follows, by $\mathbb{Z}$ and $\mathbb{Z}/n\,\mathbb{Z}$ we denote, respectively, integers and the $\mathbb{Z}$–module of integers modulo n. We write $R[x]$, $R[[x]]$ and $R[x, x^{-1}]$ for the polynomial ring, the power series ring and the Laurent polynomial ring over a ring R, respectively.

2 Generalized principally quasi–Baer modules

Let R be an associative ring with identity. An R–module M is called *generalized principally quasi–Baer* if for any $m \in M$, $a \in r_R(m\,R)$, there exist $b \in r_R(m\,R)$ such that $b\,a = a$. It is obvious that every principally quasi–Baer module (ring) is a generalized principally quasi–Baer module (right generalized principally quasi–Baer ring). If R is commutative or M is abelian, then every principally projective module (ring) is a generalized principally quasi–Baer module (right generalized principally quasi–Baer ring). The converse is not true in general as the following example shows.

Example 2.1. *Consider the ring* $R = \left(\prod_{i=1}^{\infty} \mathbb{Z}/2\,\mathbb{Z}\right) / \left(\bigoplus_{i=1}^{\infty} \mathbb{Z}/2\,\mathbb{Z}\right)$. *It is clear that R is a Boolean ring. If $S = R[[x]]$, then S is a right generalized principally quasi–Baer ring by [8, Example 2.5], but it is neither principally projective nor principally quasi–Baer.*

Example 2.2. *Let R be the upper triangular matrix ring over a field F. We prove that R is a right generalized principally quasi–Baer ring. For if* $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in R$ *and for any $B \in r_R(A\,R)$ we find $C \in r_R(A\,R)$ such that $C\,B = B$. Consider the following cases for A:*

(1) $A_1 = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ *with $a \neq 0$ and $c \neq 0$. Then A_1 is invertible. So $r_R(A_1\,R) = 0$.*

(2) $A_2 = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ *with $a \neq 0$ and $b \neq 0$. Then $r_R(A_2\,R) = 0$. So, for $B \in r_R(A_1\,R)$ or $B \in r_R(A_2\,R)$, it is enough to take C to be the zero matrix.*

(3) $A_3 = \begin{bmatrix} 0 & b \\ 0 & c \end{bmatrix}$ *with $b \neq 0$ and $c \neq 0$. Then* $r_R(A_3\,R) = \begin{bmatrix} F & F \\ 0 & 0 \end{bmatrix}$. *For any $B \in r_R(A_3\,R)$ it is enough to take* $C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

(4) $A_4 = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$ *with $a \neq 0$. Then $r_R(A_4\,R) = 0$. Same as case (1).*

(5) $A_5 = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$ *with* $b \neq 0$*. Then* $r_R(A_5\,R) = \begin{bmatrix} F & F \\ 0 & 0 \end{bmatrix}$*. Same as case (3).*

(6) $A_6 = \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix}$ *with* $c \neq 0$*. Then* $r_R(A_6\,R) = \begin{bmatrix} F & F \\ 0 & 0 \end{bmatrix}$*. Same as case (3).*

It is clear that generalized principally quasi–Baer modules are closed under submodules. For the direct sum, we have the following.

Lemma 2.3. *Any direct sums of generalized principally quasi–Baer modules are generalized principally quasi–Baer.*

Proof. Let $M = \bigoplus_{i\in I} M_i$ where $\{M_i\}_{i\in I}$ is a collection of generalized principally quasi–Baer modules and $m = (m_i) \in M$ and $a \in r_R(m\,R)$. Then for all $i \in I$, $a \in r_R(m_i\,R)$. Assume that $m_{i_1}, m_{i_2}, \cdots, m_{i_t}$ are nonzero components of m. By hypothesis, there exist $b_{i_1} \in r_R(m_{i_1}\,R), b_{i_2} \in r_R(m_{i_2}\,R)\cdots, b_{i_t} \in r_R(m_{i_t}\,R)$ such that $b_{i_j}\,a = a$ where $1 \leq j \leq t$. Let $b = b_{i_1}\,b_{i_2}\cdots b_{i_t}$. Since for any $1 \leq l \leq t$, $m_{i_l}\,R\,b = m_{i_l}\,R\,b_{i_1}\,b_{i_2}\,\cdots\,b_{i_t} \leq m_{i_l}\,R\,b_{i_l}\,b_{i_{l+1}}\,\cdots\,b_{i_t} = 0$, we have $b\,a = a$ and $b \in r_R(m_{i_j}\,R)$, where $1 \leq j \leq t$. The rest is clear.

□

One may suspect that every homomorphic images of generalized principally quasi–Baer modules are generalized principally quasi–Baer, but the following example erases the possibility.

Example 2.4. *Let* F *be a field,* $R = F[x,y]$ *and the right* R*–module* $M = R$*. Consider the submodule* $N = (x^2, x\,y, y^2)$ *of* M *and the factor module* $\overline{M} = M/N$*. It is easy to check that* M *is a generalized principally quasi–Baer module. If* $\overline{m} = x + N \in M$*, then* $r_{R[x,y]}(\overline{m}R[x,y]) = (x, y^2)$*. Assume that* M/N *is generalized principally quasi–Baer. Then for* $x+y^2 \in r_{R[x,y]}(\overline{m}R[x,y])$*, there should be a* $f(x,y) \in r_{R[x,y]}(\overline{m}R[x,y])$ *such that* $f(x,y^2)(x+y^2) = x+y^2$*. This is not possible since* R *is a commutative domain.*

Now we give some characterizations of generalized principally quasi–Baer modules. In the following proposition, the equivalence of (1) and (2) is proved in [14].

Proposition 2.5. *The following conditions are equivalent for a module* M*:*

(1) M is a generalized principally quasi–Baer module.

(2) If N is a finitely generated submodule of M, then for all $a \in r_R(N)$, we have $a \in r_R(N)\,a$.

(3) If N is a cyclic submodule of M, then for all $a \in r_R(N)$, we have $a \in r_R(N)\,a$.

Proof.

(1) $\Rightarrow$ (3). Let $N = m\,R$ be a cyclic submodule of M and $x = m\,r \in N$ and $a \in r_R(x\,R)$. By (1), there exist $b \in r_R(x\,R)$ such that $b\,a = a \in r_R(x\,R)$. Since $b \in r_R(x\,R)$, we have $b\,a = a$.

(3) $\Rightarrow$ (1). Let $m \in M$ and $a \in r_R(m\,R)$. By (3), there exist $b \in r_R(m\,R)$ such that $b\,a = a$. Since $m \in m\,R$ and $m\,R$ is a cyclic submodule of M, the proof is completed.

□

Let R be a commutative domain and M a module over R. For $r \in R$ and $m \in M$, we say that m is *divisible* by r if there is some $m_1 \in M$ with $m = m_1 r$. It is said that M is *a divisible module* if each $m \in M$ is divisible by every nonzero $r \in R$.

Proposition 2.6. *Let R be a commutative domain and M a divisible generalized principally quasi–Baer module. Then M is torsion–free.*

Proof. Let $m \in M$ and $a \in R$ with $m\,a = 0$ and assume a is nonzero. Since R is commutative, $m\,R\,a = 0$. So $a \in r_R(m\,R)$. There exist $b \in r_R(m\,R)$ such that $b\,a = a$. By divisibility of M, there exist $m' \in M$ with $m = m'\,a$. Multiplying the equation $m = m'\,a$ from the right by b and using $m\,b = 0$ and $b\,a = a$, we have $m\,b = m'\,a\,b = m'\,a = m$. Hence $m = 0$.

□

Lemma 2.7. *Let R be a commutative ring and M a generalized principally quasi–Baer module. Then M is reduced.*

Proof. Let $m \in M$ and $a \in R$ with $m\,a = 0$. We prove $M\,a \cap m\,R = 0$. If $m' = m_1\,a = m\,a_1 \in M\,a \cap m\,R$ for some $m_1 \in M$, $a_1 \in R$, then $m\,R\,a = 0$ and so by hypothesis there exist $b \in r_R(m\,R)$ such that $b\,a = a$. Multiplying the equation $m' = m_1\,a = m\,a_1$ from the right by b and use $b\,a = a$, we have $m'\,b = m_1\,a\,b = m\,a_1\,b = 0$. Hence $0 = m'\,b = m_1\,a\,b = m_1\,a = m'$.

□

The next example shows that the commutativity of the ring R in the Lemma 2.7 is essential.

Example 2.8. *Let F be a field. Consider the ring $R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$ and the right R–module $M = \begin{bmatrix} 0 & F \\ F & F \end{bmatrix}$. It is elementary to check that M is a generalized principally quasi–Baer module. For $m = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \in M$ and $a = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in R$, $m\,a^2 = 0$ but $m\,a \neq 0$. Hence M is not reduced and R is not commutative either.*

A module M is called *symmetric* if whenever $a, b \in R$, $m \in M$ satisfy $m\,a\,b = 0$, we have $m\,b\,a = 0$. The ring R is called *symmetric* if R is a symmetric right R–module. The module M is said to be *semicommutative* if for any $m \in M$ and any $a \in R$, $m\,a = 0$ implies $m\,R\,a = 0$ (see [7] and [1]). The ring R is called *semicommutative* if R is a semicommutative right R–module.

In [4, Proposition 2.4], it is proven that if M is a principally quasi–Baer module, then M is a reduced module if and only if M is a semicommutative module. For generalized principally quasi–Baer modules, we have the following.

Theorem 2.9. *If M is a reduced module, then M is symmetric. The converse holds if M is a generalized principally quasi–Baer module.*

Proof. The first statement is clear. For the converse, assume that $m \in M$ and $a \in R$ with $m\,a = 0$. In order to see $M\,a \cap m\,R = 0$, let $m_1\,a = m\,a_1 \in M\,a \cap m\,R$ for some $m_1 \in M$, $a_1 \in R$. Then $m\,R\,a = 0$ and so by hypothesis there exist $b \in r_R(m\,R)$ such that $b\,a = a$. Then $m\,R\,b = 0$. Multiplying $m_1\,a = m\,a_1$ by b from the right, we have $m_1\,a\,b = m\,a_1\,b = 0$. By hypothesis, $m_1\,a\,b = 0$ implies $m_1\,b\,a = 0$. Hence $m_1\,a = 0$. Thus $M\,a \cap m\,R = 0$.

□

Recall that a ring R is called *reversible* [10] if for any $a, b \in R$, $a\,b = 0$ implies $b\,a = 0$.

Theorem 2.10. *Let R be a right generalized principally quasi–Baer ring. Then the following are equivalent.*

(1) R is a reduced ring.

(2) R is a symmetric ring.

(3) R is a reversible ring.

Proof.

(1) ⇒ (2) ⇒ (3) is always true without any condition on R.

(3) ⇒ (1) Let $a \in R$ with $a^2 = 0$. By (3), $a^2\, r = a\, a\, r = 0$ implies $a\, r\, a = 0$ for all $r \in R$. Hence $a \in r_R(a\, R)$. From the hypothesis, there exist $b \in r_R(a\, R)$ such that $b\, a = a$. Since R is reversible, $a\, b = 0$ implies $b\, a = 0$ and so $a = 0$.

□

Recall that a ring R is said to be *von Neumann regular* if for every $a \in R$ there exist $b \in R$ with $a = a\, b\, a$. The ring R is called *strongly regular* if for each element a of R there exist an element b satisfying $a = a^2\, b$.

Theorem 2.11. *If R is a strongly regular ring, then every R–module is generalized principally quasi–Baer and semicommutative.*

Proof. Let M be an R–module, $m \in M$ and $a \in R$ with $a \in r_R(m\, R)$. There exist $x \in R$ such that $a = a^2\, x$. Since strongly regular rings are reduced, $e = a\, x$ is a central idempotent and $a = a\, x\, a = e\, x = x\, e$. So $e\, a = a$ and $0 = m\, R\, a = m\, R\, a\, x = m\, R\, e$. Hence M is a generalized principally quasi–Baer module. As for the semicommutativity, let $m \in M$ and $a \in R$ with $m\, a = 0$. Since R is regular, there exist $x \in R$ such that $a = a\, x\, a$, and $e = a\, x$ and $f = x\, a$ are central idempotents. $m\, a = 0$ implies $0 = m\, a\, x = m\, e$ and so $0 = m\, e\, r = m\, r\, e = m\, r\, a\, x$ for all $r \in R$. Multiplying $m\, r\, a\, x = 0$ from the right by a we have $0 = m\, r\, a\, x\, a = m\, r\, a$ for all $r \in R$. Hence M is semicommutative.

□

Corollary 2.12. *If R is strongly regular, then R is a right generalized principally quasi–Baer ring.*

A module M is called *regular* (in the sense of Zelmanowitz [13]) if for any $m \in M$, there exist a right R–homomorphism $M \xrightarrow{\phi} R$ such that $m = m\, \phi(m)$.

Lemma 2.13. *Let M be a regular module and $m \in M$ with $m = m\, \phi(m)$. Then $r_R(m\, R) = r_R(\phi(m\, R))$.*

Proof. If $t \in r_R(m\,R)$, then $m\,R\,t = 0$ and so $\phi(m)\,R\,t = \phi(m\,R\,t) = 0$. Hence $t \in r_R(\phi(m\,R))$ and $r_R(m\,R) \leq r_R(\phi(m\,R))$. Conversely, let $t \in r_R(\phi(m\,R))$. Then $\phi(m)\,R\,t = 0$. Since $m\,R\,t = m\,\phi(m)\,R\,t = 0$, we have $t \in r_R(m\,R)$. Hence $r_R(\phi(m\,R)) \leq r_R(m\,R)$. Thus $r_R(m\,R) = r_R(\phi(m\,R))$.

□

Theorem 2.14. *Let M be a semicommutative regular module. Then M is generalized principally quasi–Baer.*

Proof. Let $m \in M$ and $a \in R$ with $a \in r_R(m\,R)$. By hypothesis, there exist a right R–homomorphism $\phi : M \to R$ such that $m = m\,\phi(m)$. Then $\phi(m)$ is an idempotent, and by Lemma 2.13, $r_R(m\,R) = r_R(\phi(m\,R))$. The semicommutativity of M and $m = m\,\phi(m)$ imply $m\,R\,(1-\phi(m)) = 0$. Hence $1-\phi(m) \in r_R(m\,R) = r_R(\phi(m\,R))$. Thus $a \in r_R(\phi(m\,R))$, that is $\phi(m)\,a = 0$. Therefore $(1-\phi(m))\,a = a$.

□

The following is a direct consequence of Theorem 2.14.

Corollary 2.15. *Let R be a commutative ring and M a regular module. Then M is generalized principally quasi–Baer.*

Let M be an R–module. Then a submodule N of M is called *relatively divisible* if $M\,r \cap N = N\,r$ for each element r of R. Next we recall a well–known result.

Lemma 2.16. *Let M be a flat right R–module. Then for every exact sequence*

$$0 \to K \to F \to M \to 0$$

where F is a free R–module, we have $(F\,I) \cap K = K\,I$ for each left ideal I of R. In particular, K is a relatively divisible submodule of F.

Next we prove

Theorem 2.17. *Consider the following statements for a ring R.*

(1) R is a right generalized principally quasi–Baer ring.

(2) Every free R–module is generalized principally quasi–Baer.

(3) Every projective R–module is generalized principally quasi–Baer.

(4) Every flat R–module is generalized principally quasi–Baer.

Then (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) *and* (4) $\Rightarrow$ (1). *If* R *is a semicommutative ring, then* (3) $\Rightarrow$ (4).

Proof.

(1) $\Rightarrow$ (2) Let $F = \bigoplus R_i$ where $R_i = R$ be a free module, $m = (m_i) \in F$ and $a \in r_R(m\,R)$. Let $m_1, m_2, \cdots, m_n$ be nonzero components of m. Then $r_R(m\,R) = \bigcap_{i=1}^{n} r_R(m_i\,R)$. Hence $a \in r_R(m_i\,R)$ for each i with $1 \le i \le n$. By (1), there exist $x_i \in r_R(m_i\,R)$ such that $x_i\,a = a$. If $x = x_n\,x_{n-1} \cdots x_2\,x_1$, then $x \in r_R(m\,R)$ and $x\,a = a$.

(2) $\Rightarrow$ (3) Let M be a projective R–module. Then M is a direct summand of a free module F. By (2) and Lemma 2.3, M is generalized principally quasi–Baer.

(3) $\Rightarrow$ (1) and (4) $\Rightarrow$ (1) are clear.

(3) $\Rightarrow$ (4) Let M be a flat R–module over a semicommutative ring R. Assume that $m \in M$ and $a \in r_R(m\,R)$. Suppose that for the epimorphism $\alpha : F \to M$ the sequence $0 \to K \to F \to M \to 0$ is exact, where F is a free R–module. Now there exist $y \in F$ such that $\alpha(y) = m$. This implies that $\alpha(y\,R\,a) = m\,R\,a = 0$. So $y\,R\,a \le K$ and therefore $y\,R\,a \le (F\,R\,a) \cap K = K\,(R\,a)$ by Lemma 2.16. Let $y\,a \in y\,R\,a$. There exist $k \in K$ such that $y\,a = k\,a$. Then $(y-k)\,a = 0$. Note that, being R semicommutative, any free module and every submodule of a free module is semicommutative. Hence $(y-k)\,R\,a = 0$ or $a \in r_R((y-k)\,R) = 0$. By (3), the projective module F is generalized principally quasi–Baer, there exist $b \in r_R((y-k)\,R)$ such that $b\,a = a$. Now $\alpha((y-k)\,R) = m\,R$. So $0 = \alpha(0) = \alpha((y-k)\,R\,b) = m\,R\,b$. Thus $b \in r_R(m\,R)$. Therefore M is generalized principally quasi–Baer.

□

In the sequel, we investigate relations between a generalized principally quasi–Baer module and its endomorphism ring. We also study properties of the endomorphism ring of a generalized principally quasi–Baer module.

Let M be an R–module with $S = \mathrm{End}_R(M)$. It is easy to show that if M is Baer, quasi–Baer, principally quasi–Baer module, then S is a left generalized principally quasi–Baer ring. We now show that the endomorphism ring of a finitely generated generalized principally quasi–Baer module is always a left generalized principally quasi–Baer ring.

Proposition 2.18. *Let M be a finitely generated R–module with $S = \mathrm{End}_R(M)$. If M is a generalized principally quasi–Baer module, then S is a left generalized principally quasi–Baer ring.*

Proof. Let $M = m_1 R + m_2 R + \cdots + m_n R$ for some $m_1, m_2, \cdots, m_n \in M$, where $n \in \mathbb{N}$ and $f \in S$. We show that for each $g \in l_S(Sf)$ there exist $h \in l_S(S f)$ such that $g h = g$. Since $g \in l_S(S f)$, we have $g \in l_S(S f m_i)$ for each $i = 1, 2, \cdots, n$. By hypothesis, there exist $h_i \in l_S(S f m_i)$ such that $g h_i = g$ for $i = 1, 2, \cdots, n$. If $h = h_1 h_2 \cdots h_n$, then $g h = g$ and $h \in l_S(S f)$. This completes the proof.

□

A module M is called *n–epiretractable* [6] if every n–generated submodule of M is a homomorphic image of M.

Proposition 2.19. *Let M be a 1–epiretractable R–module with $S = \mathrm{End}_R(M)$. If S is a left generalized principally quasi–Baer ring, then M is a generalized principally quasi–Baer module.*

Proof. Let $m \in M$ and $f \in l_S(S m)$. If $m = 0$, then the proof is clear. Assume that $m \neq 0$. Since M is 1–epiretractable, there exist $0 \neq g \in S$ with $g(M) = m R$. Then $f S g(M) = f S m R = 0$, and so $f \in l_S(S g)$. By hypothesis, there exist $h \in l_S(S g)$ such that $f h = f$. This implies that $h S g(M) = h S m R = 0$. Hence $h S m = 0$, and so $h \in l_S(S m)$. This completes the proof.

□

Let M be an R–module with $S = \mathrm{End}_R(M)$. Then the module M is called *Rickart* [11] if for any $f \in S$, $r_M(f) = e M$ for some $e^2 = e \in S$. Rickart modules are studied also by the present authors in [3]. We now show that the endomorphism ring of a Rickart module is a left generalized principally quasi–Baer ring.

Proposition 2.20. *Let M be an R–module with $S = End_R(M)$. If M is a Rickart module, then S is a left generalized principally quasi–Baer ring.*

Proof. Let $f \in S$. We show that for each $g \in l_S(S f)$ there exist $h \in l_S(S f)$ such that $g h = g$. Then $g \in l_S(S f)$ implies $S f(M) \leq r_M(g)$. Being M Rickart, $r_M(g) = e M$ where $e^2 = e \in S$. So $g e = 0$ and $e S f(M) = S f(M)$, therefore $(1 - e) S f = 0$ or $1 - e \in l_S(S f)$. Since $g (1 - e) = g$, it follows that S is a left generalized principally quasi–Baer ring.

□

We end this paper with some observations for right generalized principally quasi–Baer rings.

Proposition 2.21. *Let R be a reduced and right generalized principally quasi–Baer ring. Then R is a domain.*

Proof. Let $a, b \in R$ with $a\,b = 0$ and assume $b \neq 0$. Since R is reduced, we have $b \in r_R(a\,R)$. By hypothesis, there exist $r \in r_R(a\,R)$ such that $r\,a = a$. But $r \in r_R(a\,R)$ implies $a\,r = 0$. Hence $a = r\,a = 0$.

□

Let S denote a multiplicatively closed subset of a ring R consisting of central regular elements. Let $S^{-1}R$ be the localization of R at S.

Proposition 2.22. *If R is a right generalized principally quasi–Baer ring, then so is $S^{-1}R$.*

Proof. Note that $r/s \in S^{-1}R$ is central in $S^{-1}R$ if and only if r is central in R. Assume that R is a right generalized principally quasi–Baer ring and let $x/s \in r_{S^{-1}R}[(a/t)\,S^{-1}R]$. Then $[(a/t)\,S^{-1}R]\,(x/s) = 0$. Since S consists of central regular elements, we have $a\,R\,x = 0$, that is, $x \in r_R(a\,R)$. By hypothesis, there exist $y \in r_R(a\,R)$ such that $y\,x = x$. Then $(y/1)\,(x/s) = x/s$ and $y/1 \in r_{S^{-1}R}[(a/t)S^{-1}R]$.

□

Then we have the following result.

Corollary 2.23. *Let R be a ring. If the polynomial ring $R[x]$ is right generalized principally quasi–Baer, then the Laurent polynomial ring $R[x, x^{-1}]$ is right generalized principally quasi–Baer.*

Proof. Let $S = \{1, x, x^2, x^3, x^4, \cdots\}$. Then S is a multiplicatively closed subset of $R[x]$ consisting of central regular elements. Then the proof follows from Proposition 2.22.

□

Acknowledgements

The authors would like to thank the referee for his/her suggestions.

References

[1] N. Agayev and A. Harmanci, *On semicommutative modules and rings*, Kyungpook Math. J. **47**(1), 21 (2007).

[2] N. Agayev, G. Gungoroglu, A. Harmanci and S. Halicioglu, *Abelian Modules*, Acta Math. Univ. Comenianae **78**(2), 235 (2009).

[3] N. Agayev, S. Halicioglu and A. Harmanci, *On Rickart modules*, Bull. Iranian Math. Soc. **38**(2), 433 (2012).

[4] M. Baser and N. Agayev, *On Reduced and Semicommutative Modules*, Turkish J. Math. **30**, 285 (2006).

[5] M. Baser and A. Harmanci, *Reduced and p.q.–Baer Modules*, Taiwanese J. Math. **11**, 26 (2007).

[6] A. Ghorbani and M. R. Vedadi, *Epi–Retractable modules and some applications*, Bull. Iranian Math. Soc. **35**(1), 155 (2009).

[7] J. Lambek, *On the representation of modules by sheaves of factor modules*, Canadian Math. Bull. **14**(3), 359 (1971).

[8] T. K. Lee and Y. Q. Zhou, *Reduced modules*, Rings, modules, algebras, and abelian groups, 365377, Lecture Notes in Pure and Appl. Math. **236** (Dekker, New York, 2004).

[9] Z. K. Liu and R. Y. Zhao, *A generalization of p.p.–rings and p.q.–Baer rings*, Glasgow Math. J. **48**, 217 (2006).

[10] G. Marks, *Reversible and symmetric rings*, J. Pure App. Alg. **174**, 311 (2002).

[11] S. T. Rizvi and C. S. Roman, *On direct sums of Baer modules*, J. Algebra **321**, 682 (2009).

[12] B. Ungor, N. Agayev, S. Halicioglu and A. Harmanci, *On principally quasi–Baer modules*, Albanian J. Math. **5**(3), 165 (2011).

[13] J. M. Zelmanowitz, *Regular modules*, Trans. Am. Math. Soc. **163**, 341 (1972).

[14] L. Zhao and X. Zhu, *GPQ Modules and Generalized Armendariz Modules*, Turkish J. Math. **35**, 637 (2011).