Matrix methods in Horadam sequences

Gamaliel Cerda[1]

Instituto de Matemáticas
Universidad Católica de Valparaíso
Valparaíso

Given the generalized Fibonacci sequence $\{W_n(a, b; p, q)\}$ we can naturally associate a matrix of order 2, denoted by $W(p, q)$, whose coefficients are integer numbers. In this paper, using this matrix, we find some identities and the Binet formula for the generalized Fibonacci–Lucas numbers.

Keywords: generalized Fibonacci numbers,
matrix methods, Binet formula.

Dada la sucesión generalizada de Fibonacci $\{W_n(a, b; p, q)\}$ podemos asociar naturalmente una matriz de orden 2, denotada por $W(p, q)$, cuyos coeficientes son números enteros. En este trabajo, usando esta matriz, encontramos algunas identidades y la fórmula de Binet para los números generalizados de Fibonacci–Lucas.

Palabras claves: números generalizados de Fibonacci,
métodos matriciales, fórmula de Binet.

MSC: 11B39, 11C20, 15A24

Recibido: 12 de abril de 2012 Aceptado: 17 de julio de 2012

[1] gamaliel.cerda.m@mail.pucv.cl

1 Introduction

Let $\{W_n(a,b;p,q)\}$ be a sequence defined by the recurrence relation [1]

$$W_n = p\,W_{n-1} - q\,W_{n-2}\,, \tag{1}$$

for $n \geq 2$, with $W_0 = a$, $W_1 = b$, where a, b, p and q are integer numbers with $p > 0$, $q \neq 0$.

We are interested in the following two special cases of $\{W_n\}$:

(i) $\{U_n\}$ is defined by $U_0 = 0$, $U_1 = 1$; and

(ii) $\{V_n\}$ is defined by $V_0 = 2$, $V_1 = p$.

Then $\{U_n\}$ and $\{V_n\}$ can be expressed in the form

$$\begin{aligned} U_n &= \frac{\alpha^n - \beta^n}{\alpha - \beta}\,, \\ V_n &= \alpha^n + \beta^n\,, \end{aligned} \tag{2}$$

where $\alpha = \frac{p+\sqrt{\Delta}}{2}$, $\beta = \frac{p-\sqrt{\Delta}}{2}$ and the discriminant is denoted by $\Delta = p^2 - 4q$. If $p = 1$, $q = -1$, then $\{U_n\}$ and $\{V_n\}$ are the usual Fibonacci and Lucas sequences.

In this study we define the generalized Fibonacci–Lucas matrix W by

$$W(p,\,q) = \begin{bmatrix} p & -q \\ 1 & 0 \end{bmatrix}. \tag{3}$$

Then we can write $(U_{n+1}U_n)^T = W(p,q)(U_nU_{n-1})^T$, where $\{U_n\}$ is the n–th generalized Fibonacci sequence and v^T is the transpose of the vector v. Similarly, the n–th generalized Fibonacci–Lucas sequence $(V_{n+1}V_n)^T$ is $W(p,q)(V_nV_{n-1})^T$. Using these representations, we obtain the determinants and elements of $W^n(p,q)$, and we get the Cassini formula for the generalized Fibonacci–Lucas numbers.

2 Generalized Fibonacci–Lucas matrix $W(p, q)$

We calculate the generalized characteristic roots and the Binet formula for the matrix $W^n(p, q)$, with $n \geq 1$.

Theorem 2.1. *Let $W(p, q)$ be a matrix as in (3). Then*

$$W^n(p,\, q) = \begin{bmatrix} U_{n+1} & -q\,U_n \\ U_n & -q\,U_{n-1} \end{bmatrix}, \qquad (4)$$

where n is a positive integer number.

Proof. We will use mathematical induction. When $n = 1$,

$$W(p,\, q) = \begin{bmatrix} p & -q \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} U_2 & -q\,U_1 \\ U_1 & -q\,U_0 \end{bmatrix}.$$

So the result is true for $n = 1$. We assume the result is true for any positive integer number $n = k$. Now, we show that the result is true for $n = k + 1$. Then we can write:

$$W^{k+1}(p,\, q) = W^k(p,\, q)\, W(p,\, q) = \begin{bmatrix} U_{k+1} & -q\,U_k \\ U_k & -q\,U_{k-1} \end{bmatrix} \begin{bmatrix} U_2 & -q\,U_1 \\ U_1 & -q\,U_0 \end{bmatrix},$$

and the result follows. ■

Corollary 2.2. *For every positive integer number n:*

(i) $\det(W^n(p, q)) = q^n$*; and*

(ii) $U_{n+1}U_{n-1} - U_n^2 = -q^{n-1}$ *(Cassini formula).*

Proof. We have that $\det(W(p, q)) = q$. Then we can write $\det(W^n(p, q))$ as the product of n times $\det(W(p, q))$ equal to q^n. The determinant $\det(W^n(p, q))$ in (4) follows from (ii). ■

Theorem 2.3. *Let n be a positive integer number. The Binet formula for the generalized Fibonacci numbers is*

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}\,, \tag{5}$$

where $\alpha = \frac{p+\sqrt{\Delta}}{2}$ *and* $\beta = \frac{p-\sqrt{\Delta}}{2}$.

Proof. Let the matrix $W(p,q)$ be as in (4). The eigenvalues and eigenvectors of the matrix W are $\alpha = \frac{p+\sqrt{\Delta}}{2}$ and $\beta = \frac{p-\sqrt{\Delta}}{2}$, which are roots of the characteristic polynomial $x^2 - px + q$, and $v_1 = (\alpha, 1)$ and $v_2 = (\beta, 1)$, respectively. Then we can diagonalize the matrix W by $D = P^{-1}W(p,q)P$, where $P = (v_1^T, v_2^T)$ and $D = \mathrm{diag}(\alpha, \beta)$. From the properties of the similar matrices, we can write $D^n = P^{-1}W^n(p,q)P$, where n is any integer number. Furthermore, we can write $W^n(p,q) = PD^nP^{-1}$, where

$$W^n(p,\, q) = \frac{1}{\alpha - \beta}\begin{bmatrix} \alpha^{n+1} - \beta^{n+1} & -q(\alpha^n - \beta^n) \\ \alpha^n - \beta^n & -q(\alpha^{n-1} - \beta^{n-1}) \end{bmatrix}. \tag{6}$$

Thus the proof is complete. ∎

Consequently, the limit ratio of successive generalized Fibonacci numbers is

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \lim_{n\to\infty} \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha^n - \beta^n} = \alpha\,,$$

with $\alpha = \frac{p+\sqrt{\Delta}}{2}$ for $W(p,q)$.

Theorem 2.4. *The characteristic roots of* $W^n(p,q)$ *are* $\lambda_{1,2} = \frac{V_n \pm (\alpha-\beta)U_n}{2}$, *where* $\lambda_1 = \alpha^n$ *and* $\lambda_2 = \beta^n$*. Then,* $V_n = \alpha^n + \beta^n$.

Proof. From the characteristic polynomial of $W^n(p,q)$ we get $\det(W^n(p,q) - \lambda I_2) = \lambda^2 - (U_{n+1} - qU_{n-1})\lambda - q(U_{n+1}U_{n-1} - U_n^2) = \lambda^2 - V_n\lambda + q^n$, by identities $U_{n+1} - qU_{n-1} = V_n$ and $U_{n+1}U_{n-1} - U_n^2 = -q^{n-1}$. Thus, the characteristic equation of $W^n(p,q)$ is $\lambda^2 - V_n\lambda + q^n = 0$ and we get the generalized characteristic roots as

$$\lambda_{1,2} = \frac{V_n \pm \sqrt{V_n^2 - 4q^n}}{2}.$$

Since $V_n^2 - 4q^n = (\alpha - \beta)^2 U_n^2$, we can write $\lambda_{1,2} = \frac{V_n \pm (\alpha-\beta)U_n}{2}$. Consequently, $\alpha^n = \frac{V_n + (\alpha-\beta)U_n}{2}$ and $\beta^n = \frac{V_n - (\alpha-\beta)U_n}{2}$. ■

We define

$$R_n = \frac{W^n(p,q)}{U_{n-1}} = \left[\begin{array}{cc} \frac{U_{n+1}}{U_{n-1}} & -q\,\frac{U_n}{U_{n-1}} \\ \frac{U_n}{U_{n-1}} & -q \end{array} \right].$$

Since $\lim\limits_{n\to\infty} \dfrac{U_{n+1}}{U_n} = \alpha$, it follows that

$$R_n(\alpha) = \lim_{n\to\infty} R_n = \left[\begin{array}{cc} p\,\alpha - q & -q\,\alpha \\ \alpha & -q \end{array} \right]. \quad (7)$$

Theorem 2.5. *Let $R_n(\alpha)$ be a 2×2 matrix as in (7). If $\alpha = p$, then*

$$R_n(p) = \left[\begin{array}{cc} U_{2n+1} & -q\,U_{2n} \\ U_{2n} & -q\,U_{2n-1} \end{array} \right],$$

for any $n \geq 1$.

Proof. It can be done by mathematical induction. ■

Corollary 2.6. *For every positive integer number n, we have:*

(i) $\det(R_n(p)) = q^{2n}$*; and*

(ii) $U_{2n+1}U_{2n-1} - U_{2n}^2 = -q^{2n-1}$.

Proof. The proofs is similar to Corollary (2.2). ■

3 Sums of generalized Fibonacci numbers

When $n = 1$, the equation $\lambda^2 - V_n\lambda + q^n = 0$ becomes $\lambda^2 - p\lambda + q = 0$, which is the characteristic equation for the generalized Fibonacci mtix

$W(p,q)$. Notice that $W^2(p,q) - pW(p,q) + qI = 0$ (from the Cayley–Hamilton theorem), with I the identity matrix of order 2. Now, we have the following equation

$$(I + G + G^2 + \cdots + G^n)\,(G - I) = G^{n+1} - I\,. \tag{8}$$

Since $W^2(p,q) - pW(p,q) = -qI$, we can write $W(p,q)(W(p,q) - pI) = -qI$. Thus, $W^{-1}(p,q) = \frac{-p}{q}(\frac{1}{p}W(p,q) - I)$. Multiplying both sides of equation (8) by the inverse of $(G - I)$, with $G = \frac{1}{p}W(p,q)$, we get $I + G + \cdots + G^n$ times $(G^{n+1} - I)\frac{-p}{q}W(p,q)$, and

$$\sum_{k=0}^{n} \frac{W^k(p,q)}{p^k} = \frac{-1}{p^n q}\, W^{n+2}(p,\, q) + \frac{p}{q}\, W(p,\, q)\,. \tag{9}$$

Equating the (2, 1)–entry of both side, we obtain the following result.

Theorem 3.1. *For any integer number $n \geq 1$,*

$$\sum_{k=0}^{n} \frac{U_k}{p^k} = \frac{p}{q} - \frac{U_{n+2}}{p^n q}\,.$$

A particular case of the previous theorem is $p = 1$ and $q = -2$, known as the Jacobsthal succession. Then we can write $\sum_{k=0}^{n} U_k = \frac{1}{2}(U_{n+2} - 1)$, and if $p =$, $q = -1$, the Fibonacci sequence, we obtain $\sum_{k=0}^{n} U_k = U_{n+2} - 1$.

Theorem 3.2. *Let n and m be positive integer numbers. Then we have the following relation between the generalized Fibonacci and generalized Fibonacci–Lucas numbers*

$$V_{n+m} = U_{m+1}\, V_n - q\, U_m\, V_{n-1}\,. \tag{10}$$

Proof. From the definition of the generalized Fibonacci and Fibonacci–Lucas numbers we can write an expression for $(V_{n+1}V_n)^T = W(p,q)(V_nV_{n-1})^T$. Multiplying both sides of equation (10) by $W^n(p,q)$, we get $W^n(p,q)(V_{n+1}V_n)^T = W^{n+1}(p,q)(V_nV_{n-1})^T$. Using (4) we obtain

$$\begin{bmatrix} V_{n+m+1} \\ V_{n+m} \end{bmatrix} = \begin{bmatrix} U_{m+2}\,V_n - q\,U_{m+1}\,V_{n-1} \\ U_{m+1}\,V_n - q\,U_m\,V_{n-1} \end{bmatrix}. \tag{11}$$

Thus the proof is complete. ■

Let n and m be positive integer numbers. Since $W^{n+m} = W^n W^m$, we can write

$$\begin{bmatrix} U_{(n+m)+1} & -q\,U_{n+m} \\ U_{n+m} & -q\,U_{(n+m)-1} \end{bmatrix} = \begin{bmatrix} U_{n+1} & -q\,U_n \\ U_n & -q\,U_{n-1} \end{bmatrix} \begin{bmatrix} U_{m+1} & -q\,U_m \\ U_m & -q\,U_{m-1} \end{bmatrix},$$

then, $U_{n+m} = U_n U_{m+1} - qU_{n-1}U_m$. In particular, if $n = m$, we get

$$U_{2n} = U_n\left(U_{n+1} - q\,U_{n-1}\right),$$

i.e., $U_{2n} = U_n V_n$. Furthermore, if $n = m+1$, $U_{2m+1} = U_{m+1}^2 - qU_m^2$. For W^{-n} we get

$$W^{-n}(p,\,q) = \frac{1}{q^n}\begin{bmatrix} -qU_{n-1} & qU_n \\ -U_n & U_{n+1} \end{bmatrix}.$$

Since $W^{n-m} = W^n W^{-m}$, we can write:

$$\begin{bmatrix} U_{(n-m)+1} & -q\,U_{n-m} \\ U_{n-m} & -q\,U_{(n-m)-1} \end{bmatrix} = \frac{1}{q^m}\begin{bmatrix} U_{n+1} & -q\,U_n \\ U_n & -q\,U_{n-1} \end{bmatrix} \begin{bmatrix} -q\,U_{m-1} & q\,U_m \\ -U_m & U_{m+1} \end{bmatrix}.$$

Definition 3.3. *We define the generalized Fibonacci–Lucas matrix S by*

$$S = S(p,\, q) = \left[\begin{array}{ll} p^2 - 2\,q & -p\,q \\ p & -2\,q \end{array} \right]. \tag{12}$$

We can write $(V_{n+2}V_{n+1})^T = S(U_{n+1}U_n)^T$, where U_n and V_n are the nth generalized Fibonacci and Fibonacci–Lucas numbers, respectively. Furthermore, by (10) we get $V_{n+1} = V_1U_{n+1} - qV_0U_n$, for all $n \geq 1$. Then

$$S(p,\, q) = \left[\begin{array}{ll} V_2 & -q\,V_1 \\ V_1 & -q\,V_0 \end{array} \right],$$

in function of the succession $\{V_n\}$.

4 The matrix S representation

In this section we will get some properties of the generalized Fibonacci–Lucas matrix S. Moreover, using this matrix, we will obtain the Cassini and the Binet formulae for the generalized Fibonacci and Fibonacci–Lucas numbers.

Theorem 4.1. *Let $S(p,q)$ be a matrix as in (12). Then, for all integer numbers n, the following matrix power is given by*

$$S^n = S^n(p,\, q) = \begin{cases} (\sqrt{\Delta})^n \left[\begin{array}{ll} U_{n+1} & -q\,U_n \\ U_n & -q\,U_{n-1} \end{array} \right], & \text{for even } n \\ (\sqrt{\Delta})^{n-1} \left[\begin{array}{ll} V_{n+1} & -q\,V_n \\ V_n & -q\,V_{n-1} \end{array} \right], & \text{for odd } n \end{cases} \tag{13}$$

where $\Delta = p^2 - 4q$.

Proof. We will use mathematical induction for odd and even n, separately. For $n = 1$ we get

$$S^1(p,\, q) = \left[\begin{array}{ll} p^2 - 2\,q & -p\,q \\ p & -2q \end{array} \right] = \left[\begin{array}{ll} V_2 & -q\,V_1 \\ V_1 & -q\,V_0 \end{array} \right].$$

Therefore, for $n = 1$ the result is true. We assume that the result is correct for odd $n = k$. Now we show that the result is correct for $n = k+2$. We can write $S^{k+2} = S^k S^2$, where

$$\begin{aligned} S^k &= (\sqrt{\Delta})^{k-1} \begin{bmatrix} V_{k+1} & -q\,V_k \\ V_k & -q\,V_{k-1} \end{bmatrix}, \\ S^2 &= (\sqrt{\Delta})^2 \begin{bmatrix} p^2-q & -p\,q \\ p & -q \end{bmatrix}. \end{aligned}$$

By multiplying those two expressions we obtain

$$S^{k+2} = (\sqrt{\Delta})^{k+1} \begin{bmatrix} V_{k+3} & -q\,V_{k+2} \\ V_{k+2} & -q\,V_{k+1} \end{bmatrix}. \tag{14}$$

When $n = 2$ and using the previous equality, we obtain that the result for $S^2(p,q)$ is correct. We assume that the result is correct for even $n = k$. Finally, we show that the result is correct for $n = k+2$. We get

$$\begin{aligned} \frac{S^{k+2}}{(\sqrt{\Delta})^{k+2}} &= \begin{bmatrix} U_{k+1} & -q\,U_k \\ U_k & -q\,U_{k-1} \end{bmatrix} \begin{bmatrix} p^2-q & -p\,q \\ p & -q \end{bmatrix} \\ &= \begin{bmatrix} U_{k+3} & -q\,U_{k+2} \\ U_{k+2} & -q\,U_{k+1} \end{bmatrix}. \end{aligned}$$

The proof is complete. ■

Let $S^n(p,q)$ be as in (13). For all positive integer numbers n, the determinant of S^n is $(-q\Delta)^n$, given that $\det(S(p,q)) = -q\Delta$. Furthermore, $U_{n+1}U_{n-1} - U_n^2$ is $(-1)^n(-q)^{n-1}$.

The identity

$$U_{n+1}^2 - q\,U_n^2 = U_{2n+1}\,, \tag{15}$$

has as its Lucas counterpart

$$V_{n+1}^2 - q\,V_n^2 = \Delta\,U_{2n+1}\,. \tag{16}$$

Indeed, since $V_{n+1} = U_{n+2} - qU_n = pU_{n+1} - 2qU_n$ and $V_n = 2U_{n+1} - pU_n$, the equation (16) follows from (15). We define $R(p, q)$ as the 2×2 matrix

$$R(p,\, q) = \frac{1}{2}\begin{bmatrix} p & \Delta \\ 1 & p \end{bmatrix}. \tag{17}$$

Then for an integer number n, $R^n(p, q)$ has the form

$$R^n(p,\, q) = \frac{1}{2}\begin{bmatrix} V_n & \Delta\, U_n \\ U_n & V_n \end{bmatrix}. \tag{18}$$

Theorem 4.2. $V_n^2 - \Delta U_n^2 = 4q^n$, *for all* $n \in \mathbb{Z}$.

Proof. Since $\det(R(p, q)) = q$, we get

$$\det(R^n(p,\, q)) = (\det(R(p,\, q)))^n = q^n\,.$$

Furthermore, from (18), we get $\det(R^n(p, q)) = \frac{1}{4}(V_n^2 - \Delta U_n^2)$. The proof is complete. ■

References

[1] A. F. Horadam, *Basic properties of a certain generalized sequence of numbers*, Fibonacci Quart. **3**, 161 (1965).

[2] A. F. Horadam, *Jacobsthal representation numbers*, Fibonacci Quart. **34**, 40 (1996).

[3] F. Koken and D. Bozkurt, *On the Jacobsthal numbers by matrix methods*, Int. J. Contemp. Math. Sci. **3**(13), 605 (2008).

[4] E. Karaduman, *On determinants of matrices with general Fibonacci numbers entries*, Appl. Math. Comp. **167**, 670 (2005).

[5] T. Koshy, *Fibonacci and Lucas Numbers with Applications* (Wiley, 2001).

[6] A. Stakhov, *Fibonacci matrices, a generalization of the Cassini formula and a new coding theory*, Chaos, Solitons and Fractals **30**, 56 (2006).