Perturbations of Laguerre–Hahn class linear functionals by Dirac delta derivatives

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We analyze perturbations of linear functionals (both on the real line and on the unit circle) that belong to the Laguerre–Hahn class. In particular, we obtain an expression for the Stieltjes and Carathéodory functions associated with the perturbed functionals, and we show that the Laguerre–Hahn class is preserved. We also discuss the invariance of the class under the Szegő transformation.

Keywords: linear functionals, Laguerre–Hahn class,
Stieltjes functions, Carathéodory functions,
Szegő transformations, orthogonal polynomials.

Se estudian perturbaciones de funcionales lineales (tanto en la recta real como en el círculo unidad) que pertenecen a la clase de Laguerre–Hahn. En particular, se obtiene una expresión para las funciones de Stieltjes y Carathéodory asociadas con los funcionales perturbados y se muestra que se preserva la clase de Laguerre–Hahn. Finalmente, se discute que bajo la transformación de Szegő la clase se mantiene invariante.

Palabras claves: funcionales lineales, clase de Laguerre–Hahn,
funciones de Stieltjes, funciones de Carathéodory,
transformaciones de Szegő, polinomios ortogonales.

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1 Introduction

The Laguerre–Hahn class for linear functionals defined in the linear space of polynomials with complex coefficients was introduced in [2, 16]. They are linear functionals such that their corresponding Stieltjes function satisfies a Riccati differential equation with polynomial coefficients. Later on, some perturbations to linear functionals where analyzed (see [19, 20, 24, 30], among others), in regards to whether or not they preserve the Laguerre–Hahn character. In particular, [20] deals with a perturbation consisting in the addition of a Dirac delta to a linear functional. The authors determined that this kind of perturbation preserves the Laguerre–Hahn character, and they also analyzed some interesting explicit examples using associated polynomials of the first kind for the classical polynomials families.

More recently, the Laguerre–Hahn class was extended to Hermitian linear functionals defined on the linear space of Laurent polynomials. In such a case, the corresponding Carathéodory function satisfies a Riccati differential equation, and the linear functionals satisfying this condition are said to be in the Laguerre–Hahn class on the unit circle. In [25] the author studies the Laguerre–Hahn character when some perturbations to the linear functional are applied, as well as some characterizations of the class in terms of a distributional equation for the functional and matrix Sylvester differential equations, among others.

In this contribution, our aim is to analyze a perturbation consisting in the addition of Dirac delta derivatives of order N, both for linear functionals defined on the real line and on the unit circle. The structure of the article is as follows. In Section 2, we present some preliminaries and notation regarding the Laguerre–Hahn class on the real line and some spectral transformations that have been studied on the past. We introduce a perturbation consisting in the addition of a Dirac delta derivative of order N, we obtain the relation between the corresponding Stieltjes functions and, finally, we discuss if the transformation preserves the Laguerre–Hahn class. In Section 3, a similar analysis is carried on for linear functionals on the unit circle. Finally, in Section 4, we study the Laguerre–Hahn class under the Szegő transformation, *i.e.*, the correspondence between measures on the real line and measures on the unit circle. An explicit example is presented.

2 Orthogonal polynomials on the real line

2.1 Preliminaries

Let $\mathcal{L}$ be a linear functional defined in the linear space $\mathbb{P}$ of polynomials with complex coefficients such that

$$\langle \mathcal{L},\, x^n \rangle = \mu_n \,, \tag{1}$$

for $n \geqslant 0$, where $\mu_n \in \mathbb{R}$ isthe *n–th moment* associated with the linear functional $\mathcal{L}$. We define $\phi\mathcal{L}$, the left multiplication of $\mathcal{L}$ by a polynomial ϕ with real coefficients, by

$$\langle \phi\mathcal{L},\, q(x) \rangle = \langle \mathcal{L},\, \phi(x) q(x) \rangle \,, \tag{2}$$

for $q \in \mathbb{P}$, and $\mathcal{DL}$, the usual distributional derivative of $\mathcal{L}$, as

$$\langle \mathcal{DL},\, q(x) \rangle = -\left\langle \mathcal{L},\, q'(x) \right\rangle \,, \tag{3}$$

for $q \in \mathbb{P}$. Furthermore, if $q(x) = \sum_{j=0}^{n} a_j x^j$,we define

$$\begin{aligned}
(\mathcal{L}q)(x) &= \sum_{m=0}^{n} \left(\sum_{j=m}^{n} a_j \, \mu_{j-m} \right) x^m \,, \\
(\theta_0 q)(x) &= \frac{q(x) - q(0)}{x} \,.
\end{aligned}$$

Finally, the linear functional $x^{-1}\mathcal{L}$ and the product of two linear functionals are defined by

$$\begin{aligned}
\left\langle x^{-1}\mathcal{L},\, q(x) \right\rangle &= \langle \mathcal{L},\, (\theta_0 q)(x) \rangle \,, \\
\langle \mathcal{L}_1 \mathcal{L},\, q(x) \rangle &= \langle \mathcal{L}_1,\, (\mathcal{L}q)(x) \rangle \,,
\end{aligned}$$

for $q \in \mathbb{P}$. If $\mathcal{L}$ is quasi–definite, *i.e.* if the principal leading submatrices of the Gram matrix (a Hankel matrix, in this case) associated with $\{\mu_n\}_{n\geqslant 0}$ are non–singular, then we can guarantee the existence of a unique family of monic polynomials such that

$$\langle \mathcal{L},\, P_n(x)\, P_m(x) \rangle = K_n \, \delta_{n,m} \,, \tag{4}$$

for $K_n \neq 0$ and $n, m \geqslant 0$. $\{P_n\}_{n\geqslant 0}$ is said to be the sequence of monic polynomials orthogonal with respect to $\mathcal{L}$. Properties of orthogonal polynomials have been extensively studied in the past (see [8, 28], among others). They satisfy a three term recurrence relation

$$P_{n+1}(x) = (x - b_n)\, P_n(x) - d_n\, P_{n-1}(x)\,, \tag{5}$$

for $n \geqslant 0$, with the initial conditions $P_0(x) = 1$, $P_{-1}(x) = 0$, and $b_n, d_n \in \mathbb{R}$ with $d_n \neq 0$, $n \geqslant 1$. It is well known that given two sequences of arbitrary real numbers $\{b_n\}_{n\geqslant 0}$, $\{d_n\}_{n\geqslant 0}$, with $d_n \neq 0$, then $\{P_n\}_{n\geqslant 0}$ defined by (5) is orthogonal with respect to some quasi–definite linear functional $\mathcal{L}$. This result is known in the literature as Favard's theorem ([8]).

On the other hand, if the principal leading submatrices of the Hankel matrix have positive determinant, then $\mathcal{L}$ is said to be positive definite. In such a case, there exists a family of orthonormal polynomials $\{p_n\}_{n\geqslant 0}$ satisfying (4) with $K_n = 1$, $n \geqslant 0$. Furthermore, $\mathcal{L}$ has the integral representation

$$\langle \mathcal{L},\, q(x) \rangle = \int_E q(x)\, d\mu(x)\,,$$

with $q \in \mathbb{P}$, where μ is a positive Borel measure supported on an infinite subset E of the real line.

The Stieltjes function associated with μ is defined by

$$S(x) = \int_E \frac{d\mu(y)}{x - y}\,, \tag{6}$$

and has great importance in the theory of orthogonal polynomials and in approximation theory. It admits the series expansion at infinity

$$S(x) = \sum_{k=0}^{\infty} \frac{\mu_k}{x^{k+1}}\,, \tag{7}$$

where $\{\mu_n\}_{n\geqslant 0}$ are the moments given in (1). By extension, for quasi–definite linear functionals, we will define the Stieltjes function as in (7).

2.2 The Laguerre–Hahn class

A linear functional $\mathcal{L}$ (or the associated family of orthogonal polynomials $\{P_n\}_{n\geqslant 0}$) is said to be in the Laguerre–Hahn class if the corresponding Stieltjes function satisfies a Ricatti differential equation of the form

$$A_L(x)\, S'(x) = B_L(x)\, S^2(x) + C_L(x)\, S(x) + D_L(x)\,, \tag{8}$$

where A_L, B_L, C_L and D_L are polynomials with complex coefficients such that $A_L(x) \neq 0$, and $D_L(x) = [(\mathcal{D}\mathcal{L})\theta_0 A_L](x) + (\mathcal{L}\theta_0 C_L)(x) - (\mathcal{L}^2\theta_0 B_L)(x)$. Moreover,

Theorem 1. *[19] Let $\mathcal{L}$ be a normalized ($\mu_0 = 1$) quasi–definite linear functional and $\{P_n\}_{n\geqslant 0}$ the corresponding sequence of monic orthogonal polynomials. The following statements are equivalent*

1. *$\mathcal{L}$ is in the Laguerre–Hahn class.*
2. *$\mathcal{L}$ satisfies the functional equation $\mathcal{D}[A_L\mathcal{L}]+\Psi\mathcal{L}+B_L\left(x^{-1}\mathcal{L}^2\right)=0$, where $A_L(x), B_L(x)$ and $C_L(x)$ are the polynomials defined in* (8), *and*

$$\Psi(x) = -\left[A_L'(x) + C_L(x)\right] . \qquad (9)$$

3. *$\mathcal{L}$ satisfies the functional equation*

$$\mathcal{D}[x\,A_L\mathcal{L}] + (x\,\Psi - A_L)\mathcal{L} + B_L\mathcal{L}^2 = 0\,, \qquad (10)$$

with the condition $\langle \mathcal{L}, \Psi\rangle + \left\langle \mathcal{L}^2, \theta_0 B_L\right\rangle = 0$.

4. *Every polynomial $P_n(x)$, with $n \geqslant 0$, satisfies a structure relation*

$$A_L(x)\,P_{n+1}'(x) - B_L(x)\,P_n^{(1)}(x) = \sum_{k=n-s}^{n+d} \theta_{n,k}\,P_k(x)\,,$$

for $n \geqslant s+1$, where $\left\{P_n^{(1)}\right\}_{n\geq 0}$ are the associated polynomials of first kind (see the following subsection) with respect to $\{P_n\}_{n\geq 0}$, $t = \deg A_L$, $p = \deg\Psi \geqslant 1$, $r = \deg B_L$, $s = \max\{p-1, d-2\}$, $d = \max\{t, r\}$, and $\theta_{n,k}$ are some constants.

If $B_L = 0$, then $\{P_n\}_{n\geqslant 0}$ is said to belong to the affine Laguerre–Hahn orthogonal polynomials class. Equivalently, they are the semiclassical orthogonal polynomials (see [1, 21]).

Notice that the characterization $\mathcal{D}[A_L\mathcal{L}] + \Psi\mathcal{L} + B_L\left(x^{-1}\mathcal{L}^2\right) = 0$ is not unique. If fact, multiplying by a polynomial $q(x)$, $\mathcal{L}$ also satisfies $\mathcal{D}\left[\hat{A}_L\mathcal{L}\right] + \hat{\Psi}\mathcal{L} + \hat{B}_L\left(x^{-1}\mathcal{L}^2\right) = 0$, where $\hat{A}_L = qA_L$, $\hat{\Psi} = q\Psi$, and

$\hat{B}_L = qB_L$. In order to obtain uniqueness, a minimizing condition on the degrees of the polynomials is imposed. Let t, p and r be the degrees of the polynomials $\hat{A}_L, \hat{\Psi}$ and $\hat{B}_L$, respectively, and let $d = \max\{t, r\}$. Now, set $H(\mathcal{L}) = \{\max\{p-1, d-2\}$, for any polynomials $\hat{A}_L, \hat{\Psi}$ and $\hat{B}_L$ satisfying (10)$\}$. Then, the class s of the Laguerre–Hahn linear functional $\mathcal{L}$ is defined as the minimum of $H(\mathcal{L})$.

Furthermore, we have (see [20])

Theorem 2. *Let $\mathcal{L}$ be a linear functional satisfying* (8)*, and let Z be the set of zeros of A_L. Then, $\mathcal{L}$ has class s if and only if*

$$\prod_{a\in Z} (|C_L(a)| + |B_L(a)| + |D_L(a)|) \neq 0\,,$$

i.e., A_L, B_L, C_L and D_L are coprime.

2.3 Spectral transformations

Given a linear functional $\mathcal{L}$, the following perturbations to it have been studied in [4, 30].

(i) $\mathcal{L}_C = (x-\alpha)\mathcal{L}$, $\alpha \notin supp(\mu)$;

(ii) $\mathcal{L}_G = (x-\alpha)^{-1}\mathcal{L}$, $\alpha \notin supp(\mu)$;

(iii) $\mathcal{L}_U = \mathcal{L} + M_r\delta_\alpha$, $M_r \in \mathbb{R}$, $\alpha \notin supp(\mu)$,

where δ_α is defined by $\langle \delta_\alpha, q\rangle = q(\alpha)$. The above transformations are called, respectively, Christoffel, Geronimus and Uvarov transformations. In terms of the corresponding Stieltjes functions, these perturbations can be expressed (see [30]) as

$$\widetilde{S}(x) = \frac{A(x)S(x) + B(x)}{D(x)}\,, \tag{11}$$

where $\widetilde{S}(x)$ is the perturbed Stieltjes function and A, B, and D are polynomials in the variable x, which were determined in [30], where the author also shows that all perturbations of the form (11), called *linear spectral transformations*, can be expressed as a product of Christoffel and Geronimus transformations. Moreover, it can be shown (see [19]) that the Laguerre–Hahn class is closed under the transformations (i)–(iii).

On the other hand, perturbations of the form

$$\widetilde{S}(x) = \frac{A(x)S(x) + B(x)}{C(x)S(x) + D(x)}\,, \tag{12}$$

with $AD-BC \neq 0$, are called *rational spectral transformations* and were also analyzed in [30]. An example of this kind of perturbations are the so–called associated polynomials of order k, which are defined ([8]) by

$$P_{n+1}^{(k)}(x) = (x - b_{n+k})\, P_n^{(k)}(x) - d_{n+k}\, P_{n-1}^{(k)}(x)\,, \tag{13}$$

for $n \geqslant 1$, and the initial conditions $P_0^{(k)}(x) = 1$, $P_1^{(k)}(x) = x - b_k$, where $\{b_n\}_{n\geqslant 0}$ and $\{d_n\}_{n\geqslant 0}$ are the same that in (5). In other words, we obtain the associated polynomials of order k by a shift of the recurrence relation parameters. $S^{(k)}(x)$, the corresponding Stieltjes function, can be obtained using

$$S^{(k)}(x) = \frac{S_k(x)}{d_k S_{k-1}(x)}\,,$$

where $S_k(x) = P_k(x)S(x) - P_{k-1}^{(1)}(x)$.

Another example of rational spectral transformation are the anti–associated polynomials of order k, considered in [26] and defined by

$$P_{n+1}^{(-k)}(x) = (x - \tilde{b}_n)\, P_n^{(-k)}(x) - \tilde{d}_n\, P_{n-1}^{(-k)}(x)\,,$$

for $n \geqslant 0$,, with initial conditions $P_{-1}^{(-k)}(x) = 0$, $P_0^{(-k)}(x) = 1$, where $\tilde{b}_n = b_{n-k}$, $\tilde{d}_n = d_{n-k}$, $n \geqslant k$ and $\tilde{b}_0, \tilde{b}_1, \ldots, \tilde{b}_k$, $\tilde{d}_1, \ldots, \tilde{d}_k$ are arbitrary real numbers. The corresponding Stieltjes function is also given by an expression of the form (12). Furthermore,

Theorem 3. *[30] Any rational spectral transformation of the form* (12) *can be obtained as a superposition of Christoffel, Geronimus, associated, and anti–associated transformations.*

Theorem 4. *[30] The class of Laguerre–Hahn is closed under rational spectral transformation of the form* (12).

2.4 Perturbation by the addition of Dirac delta derivatives

Let $\mathcal{L}$ be a quasi–definite linear functional. Taking into account that

$$\left\langle \mathcal{D}^{(N)}\mathcal{L},\, q(x) \right\rangle = (-1)^N \left\langle \mathcal{L},\, q^{(N)}(x) \right\rangle\,,$$

consider the linear functional $\mathcal{L}_N$ such that

$$\langle \mathcal{L}_N, q(x)\rangle = \langle \mathcal{L}, q(x)\rangle + M \left\langle \mathcal{D}^{(N)}\delta_\alpha, q(x)\right\rangle , \tag{14}$$

for $M \in \mathbb{R}_+$, *i.e.*, a perturbation of $\mathcal{L}$ by the addition of a Dirac delta derivative or order N at the point $x = \alpha \notin supp(\mu)$. Our interest is to find the Stieltjes function associated with $\mathcal{L}_N$, and to determine if this kind of perturbation preserves the Laguerre–Hahn character.

Proposition 5. *Let $\mathcal{L}$ be a quasi–definite linear functional and $S(x)$ the corresponding Stieltjes function. Then, $\widetilde{S}(x)$, the Stieltjes functions associated with $\mathcal{L}_N$ defined as in* (14) *is given by*

$$\widetilde{S}(x) = S(x) + M\,(-1)^N\, N!\, \frac{1}{(x-\alpha)^{N+1}} \,. \tag{15}$$

Remark 6. *The case $N = 1$ was analyzed in [10].*

Proof. Consider

$$\widetilde{c}_k = \left\langle \mathcal{L}_N, x^k \right\rangle = c_k + M\,(-1)^N \frac{k!}{(k-N)!}\,\alpha^{k-N} ,$$

for $k \geqslant N$. Then, the Stieltjes function is

$$\begin{aligned} \widetilde{S}(x) &= \sum_{k=0}^{\infty} \frac{\widetilde{c}_k}{x^{k+1}} = \sum_{k=N}^{\infty} \left(c_k + M\,(-1)^N \frac{k!}{(k-N)!}\,\alpha^{k-N} \right) \frac{1}{x^{k+1}} \\ &= S(x) + M\,(-1)^N \sum_{k=N}^{\infty} \frac{k!}{(k-N)!}\, \frac{\alpha^{k-N}}{x^{k+1}} \\ &= S(x) + \frac{M(-1)^N}{\alpha^{N+1}} \sum_{k=N}^{\infty} \frac{k!}{(k-N)!} \left(\frac{\alpha}{x}\right)^{k+1} . \end{aligned}$$

Taking into account that

$$\left[\left(\frac{\alpha}{x}\right)^{k-N+1} \right]^{(N)} = \frac{(-1)^N}{\alpha^N}\, \frac{k!}{(k-N)!} \left(\frac{\alpha}{x}\right)^{N+1} ,$$

we have

$$\begin{aligned}
\widetilde{S}(x) &= S(x) + \frac{M(-1)^N}{\alpha^{N+1}} \sum_{k=N}^{\infty} \frac{\alpha^N}{(-1)^N} \left[\left(\frac{\alpha}{x}\right)^{k-N+1}\right]^{(N)} \\
&= S(x) + \frac{M}{\alpha} \sum_{k=N}^{\infty} \left[\left(\frac{\alpha}{x}\right)^{k-N+1}\right]^{(N)} \\
&= S(x) + \frac{M}{\alpha} \left[\sum_{k=0}^{\infty} \left(\frac{\alpha}{x}\right)^{k+1}\right]^{(N)} \\
&= S(x) + \frac{M}{\alpha} \left[\frac{\alpha}{x-\alpha}\right]^{(N)} \\
&= S(x) + M\,(-1)^N\,N!\,\frac{1}{(x-\alpha)^{N+1}}\,,
\end{aligned}$$

for $|\alpha| < |x|$. □

In other words, the addition of a derivative of order N modifies the Stieltjes function by adding a rational function with a pole of multiplicity $N+1$ at $x = \alpha$. Notice that, in general, if we consider the perturbation

$$\langle \mathcal{L}_N, q(x)\rangle = \langle \mathcal{L}, q(x)\rangle + \sum_{k=0}^{l} M_k \left\langle \mathcal{D}^{(N_k)}\delta_{\alpha_k}, q(x)\right\rangle , \qquad (16)$$

then the corresponding Stieltjes function will consist in the addition of rational functions with poles of multiplicities N_k+1 at the points $x = \alpha_k$ to $S(x)$. As a consequence,

Corollary 7. *Let $\mathcal{L}$ be a linear functional that is in the Laguerre–Hahn class. Then, $\mathcal{L}_N$ defined as in* (16) *is also Laguerre–Hahn.*

Now, assuming that $\mathcal{L}$ is Laguerre–Hahn, we consider the class $\tilde{s}$ of the Laguerre–Hahn functional defined by (14). Combining (8) and (15), we obtain

$$\begin{aligned}
\Phi(x)\left(\widetilde{S}'(x) + \frac{(-1)^N M(N+1)!}{(x-\alpha)^{N+2}}\right) &= B(x)\left(\widetilde{S}(x) - \frac{M(-1)^N N!}{(x-\alpha)^{N+1}}\right)^2 \\
&\quad + C(x)\left(\widetilde{S}(x) - \frac{M(-1)^N N!}{(x-\alpha)^{N+1}}\right) \\
&\quad + D(x)\,,
\end{aligned}$$

so that

$$\begin{aligned}\Phi(x)\,\widetilde{S}'(x) &= B(x)\,\widetilde{S}^2(x)\\ &\quad+\left(C(x)-\frac{2M(-1)^N N!B(x)}{(x-\alpha)^{N+1}}\right)\widetilde{S}(x)\\ &\quad+\frac{M^2\,(N!)^2\,B(x)}{(x-\alpha)^{2N+2}}-\frac{M(-1)^N N!C(x)}{(x-\alpha)^{N+1}}\\ &\quad-\frac{(-1)^N M(N+1)!\Phi(x)}{(x-\alpha)^{N+2}}+D(x)\,.\end{aligned}$$

Thus, if

$$\begin{aligned}\tilde{A}_L(x) &= (x-\alpha)^{2N+2}\,\Phi(x)\,,\\ \tilde{B}_L(x) &= (x-\alpha)^{2N+2}\,B(x)\,,\\ \tilde{C}_L(x) &= (x-\alpha)^{N+1}\,\left(C(x)\,(x-\alpha)^{N+1}-2\,M\,(-1)^N\,N!\,B(x)\right)\,,\\ \tilde{D}_L(x) &= M^2\,(N!)^2\,B(x)-M\,(-1)^N\,N!\,C(x)\,(x-\alpha)^{N+1}\\ &\quad+D(x)\,(x-\alpha)^{2N+2}-(-1)^N\,M\,(N+1)!\,(x-\alpha)^N\,\Phi(x)\,,\end{aligned}$$

then

$$\tilde{A}_L(x)\,\widetilde{S}'=\tilde{B}_L(x)\,\widetilde{S}^2+\tilde{C}_L(x)\,\widetilde{S}+\tilde{D}_L(x)\,. \tag{17}$$

Now, let s be the class of $\mathcal{L}$. Notice that

$$\begin{aligned}\deg\tilde{A}_L &= \tilde{t}\leqslant s+2N+4\,,\\ \deg\tilde{\Psi} &= \tilde{p}\leqslant s+2N+3\,,\\ \deg\tilde{B}_L &= \tilde{r}\leqslant s+2N+4\,.\end{aligned}$$

As a consequence, $\tilde{d}=\max\{\tilde{t},\tilde{r}\}\leqslant s+2N+4$ and the class of $\mathcal{L}_N$ is $\widetilde{s}=\max\{\tilde{p}-1,\tilde{d}-2\}\leqslant s+2N+2$. On the other hand, since $\mu=\widetilde{\mu}-M\delta_\alpha^{(N)}$, then

Proposition 8. *Let s be the class of the linear functional $\mathcal{L}$. Then, the class $\widetilde{s}$ of the linear functional $\mathcal{L}_N$ is such that*

$$s-(2\,N+2)\leqslant\widetilde{s}\leqslant s+(2\,N+2)\,.$$

Remark 9. *The cases $N=0$ and $N=1$ were studied in [20] and [10], respectively. In the latter, the authors also obtain the conditions on the* *polynomials of the differential equation that determine each specific value of the class.*

Proposition 10. *Let $\mathcal{L}_N$ be a Laguerre–Hahn linear functional whose Stieltjes function satisfies (17). Then, for every zero of $\tilde{A}_L(x)$ different from α,* (17) *cannot be simplified by division of the polynomial coefficients.*

Proof. From the assumption on $\mathcal{L}$, A_L, B_L, C_L, and D_L are coprime. Let $\tilde{A}_L$ and $\tilde{B}$ be as above and assume a is a zero of $\tilde{A}_L$ different from α. Three different situations can be analyzed

1. If $B(a) \neq 0$, then $\tilde{B}_L(a) \neq 0$.
2. If $B(a) = 0$ and $C(a) \neq 0$, then we get $\tilde{C}_L(a) \neq 0$.
3. If $B(a) = C(a) = 0$ then taking into account $D(a) \neq 0$, we get $\tilde{D}_L(a) \neq 0$ and, as a consequence,

$$\left|\tilde{B}_L(a)\right| + \left|\tilde{C}_L(a)\right| + \left|\tilde{D}_L(a)\right| \neq 0\,.$$

As a conclusion, the equation (17) cannot be simplified. □

3 Orthogonal polynomials on the unit circle

3.1 Preliminaries

Consider a linear functional $\mathcal{L}$ defined in the linear space of Laurent polynomials with complex coefficients $\Lambda = span\{z^k\}_{k\in\mathbb{Z}E}$, such that

$$c_n = \langle \mathcal{L},\, z^n\rangle = \overline{\langle \mathcal{L},\, z^{-n}\rangle} = \bar{c}_{-n}\,,$$

i.e., $\mathcal{L}$ is an Hermitian linear functional. We will denote the corresponding bilinear functional by $\langle p(z), q(z)\rangle_{\mathcal{L}} = \left\langle \mathcal{L}, p(z)\overline{q(z)}\right\rangle$, where $\overline{q(z)} = \bar{q}(z^{-1})$, and $p, q \in \mathbb{P}$. The set of complex numbers $\{c_k\}_{k\in\mathbb{Z}E}$ are called the *moments* associated with $\mathcal{L}$. The Gram matrix associated with $\mathcal{L}$ is now a Toeplitz matrix. Analogously to the real line case, studied in the previous Section, we will say that $\mathcal{L}$ is a quasi–definite (positive definite) linear functional if the determinants of the principal leading submatrices of the Toeplitz matrix are non–negative (positive). In the quasi–definite case, it can be guaranteed that there exists a family of monic polynomials $\{\Phi_n\}_{n\geqslant 0}$ satisfying

$$\left\langle \mathcal{L},\, \Phi_n(z)\,\overline{\Phi_m(z)} \right\rangle = \mathbf{k}_n\,\delta_{n,m}\,,$$

for $n,m \geqslant 0$, where $\mathbf{k}_n \neq 0$, $n \geqslant 0$. $\{\Phi_n\}_{n\geqslant 0}$ is said to be the monic orthogonal polynomial sequence (MOPS) with respect to $\mathcal{L}$.

On the other hand, any positive definite linear functional admits the integral representation

$$\langle \mathcal{L}\,, p(z)\rangle = \int_{\mathbb{T}} p(z)\,d\sigma(z)\,,$$

where $p(z)$ is a polynomial and σ is a nontrivial positive measure supported on $\mathbb{T}$.

The properties of $\{\Phi_n\}_{n\geqslant 0}$ have been extensively studied over the years ([13, 27, 28], among others). They satisfy

$$\Phi_{n+1}(z) = z\,\Phi_n(z) + \Phi_{n+1}(0)\,\Phi_n^*(z)\,, \qquad (18)$$

$$\Phi_{n+1}(z) = \left(1 - |\Phi_{n+1}(0)|^2\right) z\,\Phi_n(z) + \Phi_{n+1}(0)\,\Phi_{n+1}^*(z)\,, \qquad (19)$$

for $n \geqslant 0$, the so–called forward and backward recurrence relations, where $\Phi_n^*(z) = z^n\bar{\Phi}_n(z^{-1})$ is the reversed polynomial and the complex numbers $\{\Phi_n(0)\}_{n\geqslant 1}$ are known as Verblunsky coefficients (they are also called Schur or reflection parameters).

In terms of the moments, we can define an analytic function in a neighborhood of the origin by

$$F(z) = c_0 + 2\sum_{k=1}^{\infty} c_{-k}\,z^k\,, \qquad (20)$$

which, if $\mathcal{L}$ is positive definite, is analytic in the open unit disc $\mathbb{D}E$ with positive real part therein. It can be represented as

$$F(z) = \int_{\mathbb{T}} \frac{w+z}{w-z}\,d\sigma(w)\,,$$

where σ is the measure associated with $\mathcal{L}$. $F(z)$ is said to be the Carathéodory function associated with $\mathcal{L}$. For quasi–definite linear functionals, we will define $F(z)$ as (20).

3.2 Spectral transformations and the Laguerre–Hahn class

Given a linear functional $\mathcal{L}$, the following perturbations (analogous to the Christoffel, Geronimus and Uvarov perturbations defined in the previous Section) have been studied in the past (see [9, 11, 14, 15, 17, 18]):

$$\langle p(z),\, q(z)\rangle_{\mathcal{L}_C} = \langle (z-\alpha)\, p(z),\, (z-\alpha)\, q(z)\rangle_{\mathcal{L}}\ ,$$

for $\alpha \in \mathbb{C}$;

$$\langle p(z),\, q(z)\rangle_{\mathcal{L}_G} = \left\langle \frac{p(z)}{z-\alpha},\, \frac{q(z)}{z-\alpha}\right\rangle_{\mathcal{L}} + \boldsymbol{m}\, p(\alpha)\, \overline{q(\bar{\alpha}^{-1})} + \bar{\boldsymbol{m}}\, p(\bar{\alpha}^{-1})\, \overline{q(\alpha)}\,,$$

for $\alpha \in \mathbb{C}$, $|\alpha| \neq 1$ and $\boldsymbol{m} \in \mathbb{C}$; and

$$\langle p(z),\, q(z)\rangle_{\mathcal{L}_U} = \langle p(z),\, q(z)\rangle_{\mathcal{L}} + \boldsymbol{m}\, p(\alpha)\, \overline{q(\bar{\alpha}^{-1})} + \bar{\boldsymbol{m}}\, p\,(\bar{\alpha}^{-1})\, \overline{q(\alpha)}\,,$$

for $\boldsymbol{m} \in \mathbb{C}$ and $\alpha \neq 0$. In terms of the corresponding Carathéodory functions, they can be expressed as

$$\widetilde{F}(z) = \frac{A(z)F(z) + B(z)}{D(z)}\,, \tag{21}$$

where A, B, and D are polynomials in the variable z whose coefficients depend on $\boldsymbol{m}$ and α (see [17]). (21) is said to be a *linear spectral transformation* of $F(z)$.

There are other transformations that result from modifications of the Verblunsky parameters, namely the Aleksandrov transformation (resulting from multiplying the Verblunsky parameters by the complex number λ, with $|\lambda| = 1$), and the associated and antiassociated polynomials of order k, defined by a forward and backward shift, respectively, of the Verblunsky coefficients, in a similar way that the associated and antiassociated polynomials are defined on the real line. Once one has the sequence of modified Verblunsly coefficients, the corresponding polynomials can be constructed using (18). In terms of the Carathéodory functions, these transformations are given by

$$\widetilde{F}(z) = \frac{A(z)F(z) + B(z)}{C(z)F(z) + D(z)}\,, \tag{22}$$

where A, B, C and D, with $AD - BC \neq 0$, are polynomials in z, which are known (see [22, 27]). In general, transformations of the form (22) are called *rational spectral transformations*. In [22], the author gives necessary and sufficient conditions on the polynomials A, B, C, and D in order for $\widetilde{F}(z)$ to be a Carathéodory function, provided that $F(z)$ is.

A linear functional $\mathcal{L}$ whose corresponding Carathéodory function satisfies the Ricatti differential equation

$$z\, A_L(z)\, F'(z) = B_L(z)\, F^2(z) + C_L(z)\, F(z) + D_L(z)\,, \tag{23}$$

where A_L, B_L, C_L, and D_L are polynomials in z, is said to be in the Laguerre–Hahn class. The Laguerre–Hahn class on the unit circle can be understood as an extension of the Laguerre–Hahn class on the real line, and was studied in [25], where the author characterizes this class in terms of a distributional equation for the corresponding orthogonality linear functional, and in terms of a first order structure relation with polynomial coefficients.

If $B_L = 0$, then we obtain the affine Laguerre–Hahn class on the unit circle (see [3, 5, 23]), and if $B_L = D_L = 0$ and C_L is some specific polynomial, then we obtain the semiclassical class on the unit circle (see [3, 5]). It is important to notice that, unlike the real line case, the affine Laguerre–Hahn and semiclassical classes do not coincide. The following result can be found in [25].

Theorem 11. *Let $\mathcal{L}$ be a quasi–definite linear functional such that $F(z)$, the corresponding Carathéodory function, satisfies* (23) *for some polynomials A_L, B_L, C_L and D_L, i.e, $F(z)$ belongs to the Laguerre–Hahn class. Let $\widetilde{F}(z)$ be a rational spectral transformation of $F(z)$ of the form* (22)*. Then, $\widetilde{F}(z)$ also belongs to the Laguerre–Hahn class.*

3.3 Perturbation by the addition of Dirac delta derivatives

Consider an Hermitian linear functional $\mathcal{L}$ and the corresponding derivative functional on the unit circle, given by (see [29])

$$\langle \mathcal{D}\mathcal{L},\, p(z)\rangle = -i\,\left\langle \mathcal{L},\, z\, p'(z)\right\rangle\,,$$

Then,

$$\begin{aligned}\left\langle \mathcal{D}\mathcal{L},\, p(z)\overline{q(z)}\right\rangle &= -i\,\left\langle \mathcal{L},\, z\,[p(z)\,\overline{q(z)}]'\right\rangle \\ &= -i\,\left\langle \mathcal{L},\, z\, p'(z)\,\overline{q(z)} - z^{-1}\, p(z)\,\overline{q'(z)}\right\rangle\,.\end{aligned}$$

For the second and third derivatives, we get, respectively,

$$\begin{aligned}\left\langle \mathcal{D}^{(2)}\mathcal{L},\, p(z)\,\overline{q(z)}\right\rangle &= -i\left\langle \mathcal{D}\mathcal{L},\, z\,[p(z)\,\overline{q(z)}]'\right\rangle\\ &= (-i)^2\left\langle \mathcal{L},\, z^2\,p''(z)\,\overline{q(z)} + z\,p'(z)\,\overline{q(z)}\right.\\ &\qquad -2\,p'(z)\,\overline{q'(z)} + z^{-1}\,p(z)\,\overline{q'(z)}\\ &\qquad \left. +z^{-2}\,p(z)\,\overline{q''(z)}\right\rangle\,,\end{aligned}$$

and

$$\begin{aligned}\left\langle \mathcal{D}^{(3)}\mathcal{L},\, p(z)\,\overline{q(z)}\right\rangle &= (-i)^3\left\langle \mathcal{L},\, z^3\,p'''(z)\,\overline{q(z)} + 3\,z^2\,p''(z)\,\overline{q(z)}\right.\\ &\qquad -3\,z\,p''(z)\,\overline{q'(z)} + z\,p'(z)\,\overline{q(z)}\\ &\qquad -z^{-1}\,p(z)\,\overline{q'(z)} + 3\,z^{-1}\,p'(z)\,\overline{q''(z)}\\ &\qquad \left. -3\,z^{-2}\,p(z)\,\overline{q''(z)} - z^{-3}\,p(z)\,\overline{q'''(z)}\right\rangle\,.\end{aligned}$$

Applying additional derivatives, it is not difficult to show that

$$\begin{aligned}&\left\langle \mathcal{D}^{(N)}\mathcal{L},\, p(z)\,\overline{q(z)}\right\rangle\\ &\qquad = (-i)^N\left\langle \mathcal{L},\, \sum_{l=1}^{N}\sum_{k=0}^{l} a_{k,l-k}\,p^{(k)}(z)\,\overline{q^{(l-k)}(z)}z^{2k-l}\right\rangle\,,\end{aligned}$$

for some integers $a_{k,l}$, $0 \leqslant k \leqslant l$, $1 \leqslant l \leqslant N$ with

$$a_{k,l} = \begin{cases} a_{l,k}, & \text{if } N \text{ is even},\\ -\,a_{l,k}, & \text{if } N \text{ is odd},\end{cases}$$

for $k \neq l$. Now, for $M \in \mathbb{R}_+$, $|\alpha| = 1$, we define the linear functional $\mathcal{L}_N$ by

$$\langle \mathcal{L}_N,\, p(z)\rangle = \langle \mathcal{L},\, p(z)\rangle + M\left\langle \mathcal{D}^{(N)}\delta_\alpha,\, p(z)\right\rangle\,, \tag{24}$$

for $p \in \Lambda$, where δ_α is the Dirac delta functional. Thus, in terms of the corresponding bilinear functionals,

$$\begin{aligned}\langle p(z),\, q(z)\rangle_{\mathcal{L}_N} &= \langle p(z),\, q(z)\rangle_{\mathcal{L}} \\ &\quad + M\,(-i)^N \sum_{l=1}^{N}\sum_{k=0}^{l} a_{k,l-k}\, p^{(k)}(\alpha)\, \overline{q^{(l-k)}(\alpha)}\, \alpha^{2k-l}\,,\end{aligned} \tag{25}$$

for $p, q \in \mathbb{P}$. Notice that, if $\{\tilde{c}_n\}_{n\geqslant 0}$ is the family of moments associated with $\mathcal{L}_N$, then $\tilde{c}_0 = c_0$ and, for $m \geqslant 1$,

$$\begin{aligned}\tilde{c}_m &= \langle z^m,\, 1\rangle_{\mathcal{L}_N} = \langle z^m,\, 1\rangle_{\mathcal{L}} + M\,(-i)^N \sum_{l=1}^{N} a_{l,0}\; [z^m]^{(l)}\, \alpha^l \Big|_{z=\alpha} \\ &= c_m + M\,(-i)^N \sum_{l=1}^{N} a_{l,0}\, \frac{m!}{(m-l)!}\, \alpha^{m-l}\, \alpha^l \\ &= c_m + M\,(-i)^N\, m!\, \alpha^m \sum_{l=1}^{N} \frac{a_{l,0}}{(m-l)!}\,.\end{aligned}$$

On the other hand,

$$\begin{aligned}\tilde{c}_{-m} &= \langle 1,\, z^m\rangle_{\mathcal{L}_N} = \langle 1,\, z^m\rangle_{\mathcal{L}} + M\,(-i)^N \sum_{l=1}^{N} a_{0,l}\; \overline{[z^m]^{(l)}}\, \alpha^{-l} \Big|_{z=\alpha} \\ &= c_{-m} + M\,(-i)^N \sum_{l=1}^{N} a_{0,l}\, \frac{m!}{(m-l)!}\, \bar{\alpha}^{m-l}\, \alpha^{-l} \\ &= c_{-m} + M\,(-i)^N\, m!\, \bar{\alpha}^m \sum_{l=1}^{N} \frac{a_{0,l}}{(m-l)!} = \overline{\tilde{c}_m}\,,\end{aligned}$$

so that $\mathcal{L}_N$ is an hermitian linear functional.

Next, we will find the Carathéodory function associated with $\mathcal{L}_N$, and will determine if this transformation preserves the Laguerre–Hahn character.

Proposition 12. *Let $\mathcal{L}$ be a quasi–definite linear functional and denote by $F(z)$ the corresponding Carathéodory function. Then, $\widetilde{F}(z)$, the Carathéodory function associated with $\mathcal{L}_N$ defined as in* (25)*, is given by*

$$\widetilde{F}(z) = F(z) + 2\,M\,(-i)^N \sum_{l=1}^{N} a_{0,l}\, z^l \, \frac{l!}{\bar{\alpha}\,(\alpha - z)^{l+1}}\,. \tag{26}$$

Remark 13. *The case $N = 1$ was analyzed in [6], where the authors obtain an expression for $\widetilde{F}(z)$ equivalent to* (26)*.*

Proof. The Carathéodory function $\widetilde{F}$ is given by

$$\begin{aligned}
\widetilde{F}(z) &= \tilde{c}_0 + 2\sum_{k=1}^{\infty} \tilde{c}_{-k}\, z^k \\
&= c_0 + 2\sum_{k=1}^{\infty}\left(c_{-k} + M\,(-i)^N\, k!\,\bar{\alpha}^k \sum_{l=1}^{N} \frac{a_{0,l}}{(k-l)!}\right) z^k \\
&= F(z) + 2\,M\,(-i)^N \sum_{k=1}^{\infty}\left(k!\,\bar{\alpha}^k \sum_{l=1}^{N} \frac{a_{0,l}}{(k-l)!}\right) z^k\,.
\end{aligned}$$

For a given m such that $1 \leqslant m \leqslant N$, notice that

$$\begin{aligned}
\left((\bar{\alpha}\,z)^k\right)^{(m)} &= \left(\bar{\alpha}^m\, k\,(k-1)\,(k-2)\,\cdots\,(k-(m-1))\right)\,(\bar{\alpha}\,z)^{k-m} \\
&= \frac{\bar{\alpha}^m k!}{(k-m)!}\,(\bar{\alpha}\,z)^{k-m}\,.
\end{aligned}$$

Therefore,

$$\frac{k!}{(k-m)!} = \frac{\left((\bar{\alpha}z)^k\right)^{(m)}}{\alpha^m(\bar{\alpha}z)^{k-m}}\,,$$

and, as a consequence,

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{a_{0,m}k!}{(k-m)!} (\bar{\alpha}\, z)^k &= \sum_{k=1}^{\infty} \frac{a_{0,m}}{\bar{\alpha}^m} \left((\bar{\alpha}\, z)^k\right)^{(m)} (\bar{\alpha}\, z)^m \\
&= a_{0,m}\, z^m \sum_{k=1}^{\infty} \left((\bar{\alpha}\, z)^k\right)^{(m)} \\
&= a_{0,m}\, z^m \left(\frac{\bar{\alpha}\, z}{1-\bar{\alpha}\, z}\right)^{(m)},
\end{aligned}$$

for $|z| < 1$. Thus,

$$\begin{aligned}
\widetilde{F}(z) &= F(z) + 2\, M\, (-i)^N \sum_{l=1}^{N} a_{0,l}\, z^l \left(\frac{\bar{\alpha}\, z}{1-\bar{\alpha}\, z}\right)^{(l)} \\
&= F(z) + 2\, M\, (-i)^N \sum_{l=1}^{N} a_{0,l}\, z^l \frac{l!}{\bar{\alpha}\, (\alpha - z)^{l+1}},
\end{aligned}$$

which completes the proof. □

Then, $\widetilde{F}(z)$ is a perturbation of $F(z)$ that consists in adding a rational function with a pole at $z = \alpha$ of multiplicity $N + 1$. Therefore,

Corollary 14. *If $\mathcal{L}$ belongs to the Laguerre–Hahn class, then $\mathcal{L}_N$ defined as in* (25) *also belongs to the Laguerre–Hahn class.*

Remark 15. *As in the real line case, this result can be generalized to perturbations consisting in the addition of l Dirac delta derivatives of different orders, as each one of them adds a rational function to the Carathéodory function.*

3.4 Semiclassical linear functionals and their class

A linear functional $\mathcal{L}$ is said to be semiclassical if it satisfies the functional equation

$$\mathcal{D}[A_s(z)\mathcal{L}] = B_s(z)\, \mathcal{L}\,,$$

where A_s, B_s are polynomials. In such a case, the corresponding orthogonal polynomials satisfy (see [29])

$$A_s'(z)\,\Phi_n(z)=\sum_{j=0}^{p}\gamma_{n,j}\,\Phi_{n-1+j}(z)+\sum_{j=2}^{p'}\eta_{n,j}\,z^{j-2}\,\Phi_{n-j}^{*}(z)\,, \tag{27}$$

for $n\geqslant p+1$, where $p=\deg A_s$, $p'=\max\{p,\deg B_s\}$, and $\gamma_{n,j},\eta_{n,j}\in\mathbb{C}E$. The pair (p,q), where $q=\max\{p-1,\deg[(p-1)A_s+iB_s]\}$, is defined as the class of $\mathcal{L}$. q is the maximum number of terms of the form $z^j\Phi_{n-j-2}^{*}(z)$ that appear in (27). Furthermore (see [25]), if the corresponding Carathéodory function satisfies

$$z\,A_s(z)\,F'(z)=[z\,A_s'(z)-i\,B_s(z)]\,F(z)+D_L(z)\,,$$

for $|z|\neq 1$, then $\mathcal{L}$ is semiclassical satisfying $\mathcal{D}[A_s(z)\mathcal{L}]=B_s(z)\mathcal{L}$ if, and only if,

$$\begin{aligned} D_L(z) &= -z\left[A_s'(z)+\left\langle \mathcal{L},\,2\,w\sum_{k=2}^{p}\frac{A_s^{(k)}(z)}{k!}\,(w-z)^{k-2}\right\rangle\right]\\ &\quad -i\left\langle \mathcal{L},\,\frac{w+z}{w-z}\,[B_s(w)-B_s(z)]\right\rangle . \end{aligned}$$

Now, we will show that the perturbation $\mathcal{L}_N$ does not, in general, preserve the semiclassical character of $\mathcal{L}$. To see this, set $B_L(z)=zA_s'(z)-iB_s(z)$. Taking into account that $\widetilde{F}(z)=F(z)+Q(z)$, where $Q(z)$ is the rational function that appears in (26), we have

$$\begin{aligned} z\,A_s(z)\,[\widetilde{F}(z)-Q(z)]' &= B_L(z)\,[\widetilde{F}(z)-Q(z)]+D_L(z)\,,\\ z\,A_s(z)\,\widetilde{F}'(z) &= B_L(z)\,\widetilde{F}(z)+z\,A_s(z)\,Q'(z)\\ &\quad -B_L(z)\,Q(z)+D_L(z)\,. \end{aligned}$$

Thus, $\mathcal{L}_N$ will be a semiclassical linear functional if and only if

$$\begin{aligned} &z\,A_s(z)\,Q'(z)-B_L(z)\,Q(z)+D_L(z)\\ &\qquad = -z\left[A_s'(z)+\left\langle \mathcal{L}_N,\,2\,w\sum_{k=2}^{p}\frac{A_s^{(k)}(z)}{k!}\,(w-z)^{k-2}\right\rangle\right]\\ &\qquad\quad -i\left\langle \mathcal{L}_N,\,\frac{w+z}{w-z}\,[B_s(w)-B_s(z)]\right\rangle , \end{aligned}$$

and, since $\mathcal{L}_N = \mathcal{L} + M\mathcal{D}^{(N)}\delta_\alpha$, the condition becomes

$$\begin{aligned} z\,A_s(z)\,Q'(z) - B_L(z)\,Q(z) & \\ = -z\,M\,\left\langle \mathcal{D}^{(N)}\delta_\alpha,\, 2\,w \sum_{k=2}^{p} \frac{A_s^{(k)}(z)}{k!}\,(w-z)^{k-2} \right\rangle & \\ -i\,M\,\left\langle \mathcal{D}^{(N)}\delta_\alpha,\, \frac{w+z}{w-z}\,[B_s(w) - B_s(z)] \right\rangle\,, & \qquad (28) \end{aligned}$$

As a consequence,

Proposition 16. *Let $\mathcal{L}$ be a semiclassical linear functional. Then, $\mathcal{L}_N$ defined as in* (24) *is semiclassical if and only if $Q(z)$ satisfies* (28)*. In such a case, $\mathcal{L}$ and $\mathcal{L}_N$ have the same class.*

As an example, if $\mathcal{L}_\theta$ is the functional associated with the Lebesgue measure, then (see [29]) it satisfies $\mathcal{D}[A_s(z)\mathcal{L}_\theta] = B_s(z)\mathcal{L}_\theta$ with $A_s = C$ (constant) and $B_s = 0$. Thus, (28) becomes $CzQ'(z) = 0$, and thus $\mathcal{L}_N$ is semiclassical is and only if $Q(z)$ is constant. Therefore, $\mathcal{L}_N$ applied to $\mathcal{L}_\theta$ is not semiclassical.

4 The Laguerre–Hahn class and the Szegő transformation

Given a positive Borel measure μ with support in $[-1, 1]$, we can define another nontrivial positive Borel measure, σ, supported in $[-\pi, \pi]$, by (see [28])

$$d\sigma(\theta) = \frac{1}{2}\,|d\mu(\cos\theta)|\,. \qquad (29)$$

We will refer to (29) as *Szegő transformation.* If μ is a probability measure (*i.e.* $\mu_0 = 1$) of the form $d\mu(x) = \omega(x)dx$, then we have

$$d\sigma(\theta) = \frac{1}{2}\,\omega(\cos\theta)\,|\sin\theta|\,d\theta\,,$$

and σ is also a probability measure supported on $\mathbb{T}$, with an associated family of orthogonal polynomials, which can be related to the family of polynomials orthogonal with respect to μ. Also, there is a relation between the coefficients of the recurrence relation (13) and the Verblunsky coefficients, which in this case are real. Furthermore, if $S(x)$ and $F(z)$ are the Stieltjes and Carathéodory functions associated with μ and σ, respectively, then ([22])

$$F(z) = \frac{1-z^2}{2z}\, S(x)\,, \tag{30}$$

where $z = x - \sqrt{x^2-1}$ and $x = \frac{z+z^{-1}}{2}$.

It has been shown in [12] that if one applies the Christoffel transformation defined in Section 2.3 to a probability measure μ (supported in $[-1,1]$) and then apply the Szegő transformation to the perturbed measure $\tilde{\mu}$, then we obtain a probability measure $\tilde{\sigma}$ supported on the unit circle, which is a Christoffel transformation of σ, defined from μ as in (29). The same occurs for the Geronimus and Uvarov transformations. In other words, transformations defined in Section 2.3 are preserved under the Szegő transformation. Our objective in this Section is to show that the Szegő transformation also preserves the Laguerre–Hahn class. Indeed,

Theorem 17. *Let μ be a positive Borel measure supported in $[-1,1]$ which belongs to the Laguerre–Hahn class and let σ be the corresponding measure supported on $\mathbb{T}$ defined by the Szego transformation. Then, σ belongs to the Laguerre–Hahn class.*

Proof. Denote by $S(x)$ the Stieltjes function associated with μ, satisfying

$$A(x)\, S'(x) = B(x)\, S^2(x) + C(x)\, S(x) + D(x)\,,$$

for some polynomials A, B, C, and D. Using (30), we have

$$\begin{aligned} S'(x) &= \frac{2z^2+2}{(1-z^2)^2}\left(1-\frac{x}{\sqrt{x^2-1}}\right) F(z) \\ &\quad + \frac{2z}{1-z^2}\, F'(z)\left(1-\frac{x}{\sqrt{x^2-1}}\right) \\ &= \left(1-\frac{x}{\sqrt{x^2-1}}\right)\left[\frac{2z^2+2}{(1-z^2)^2}\, F(z) + \frac{2z}{1-z^2}\, F'(z)\right] \\ &= \frac{-2z^2}{1-z^2}\left[\frac{2z^2+2}{(1-z^2)^2}\, F(z) + \frac{2z}{1-z^2}\, F'(z)\right], \end{aligned}$$

since $1 - x(x^2-1)^{-1/2} = -2z^2/(1-z^2)$. Thus,

$$\frac{-2z^2A(x)}{1-z^2}\left[\frac{2z^2+2}{(1-z^2)^2}\,F(z)+\frac{2z}{1-z^2}\,F'(z)\right]$$
$$= B(x)\left(\frac{2z}{1-z^2}\,F(z)\right)^2+\frac{2zC(x)}{1-z^2}\,F(z)+D(x)\,,$$

and, rearranging the terms,

$$\begin{aligned}\frac{-4z^3}{(1-z^2)^2}\,A(x)\,F'(z) &= B(x)\left(\frac{2z}{1-z^2}\right)^2 F^2(z)\\ &\quad+\left[\frac{2zC(x)}{1-z^2}+\frac{4z^2(z^2+1)A(x)}{(1-z^2)^3}\right] F(z)\\ &\quad+D(x)\,,\\ -4\,z^3\,\left(1-z^2\right)\,A(x)\,F'(z) &= 4\,z^2\,\left(1-z^2\right)\,B(x)\,F^2(z)\\ &\quad+\left(1-z^2\right)^3\,D(x)+\Big(2\,z\,\left(1-z^2\right)^2\,C(x)\\ &\quad+4\,z^2\,(z^2+1)\,A(x)\Big)\,F(z)\,.\end{aligned}$$

Taking into account that any polynomial $Q(x)$ can be expressed in terms of z as

$$Q(x)=Q\left(\frac{z+z^{-1}}{2}\right)=\frac{Q_n(z)}{Q_d(z)}\,,$$

where $Q_n(z)$ and $Q_d(z)$ are polynomials with complex coefficients, then it follows that $F(z)$ satisfies

$$z\,A_L(z)\,F'(z)=B_L(z)\,F^2(z)+C_LF(z)+D_L(z)\,,$$

with

$$\begin{aligned}A_L(z) &= -4\,z^2\,(1-z^2)\,A_n(z)\,B_d(z)\,C_d(z)\,D_d(z)\,,\\ B_L(z) &= 4\,z^2\,(1-z^2)\,A_d(z)\,B_n(z)\,C_d(z)\,D_d(z)\,,\\ C_L(z) &= 2\,z\,(1-z^2)^2\,A_d(z)\,B_d(z)\,C_n(z)\,D_d(z)\\ &\quad+4\,z^2\,(z^2+1)\,A_n(z)\,B_d(z)\,C_d(z)\,D_d(z)\,,\\ D_L(z) &= (1-z^2)^3\,A_d(z)\,B_d(z)\,C_d(z)\,D_n(z)\,,\end{aligned}$$

so $F(z)$ is in the Laguerre–Hahn class. □

4.1 Example

We consider the first kind associated polynomials for the Jacobi ones. They belong to the Laguerre–Hahn class, satisfying (see [20])

$$A(x)\, S'(x) = B(x)\, S^2(x) + C(x)\, S(x) + D(x)\,,$$

with

$$\begin{aligned} A(x) &= x^2 - 1\,, \\ B(x) &= \frac{4(\alpha+1)(\beta+1)(\alpha+\beta+1)}{(\alpha+\beta+3)(\alpha+\beta+2)^2}\,, \\ C(x) &= (\alpha+\beta+2)\, x - \frac{\alpha^2-\beta^2}{\alpha+\beta+2}\,, \\ D(x) &= \alpha+\beta+3\,. \end{aligned}$$

Notice that

$$A(x) = A\left(\frac{z+z^{-1}}{2}\right) = \frac{(1-z^2)^2}{4z^2} := \frac{A_n(z)}{A_d(z)}\,.$$

In a similar way, we have

$$\begin{aligned} B_n(z) &= B(x)\,, \\ B_d(z) &= 1\,, \\ C_n(z) &= (\alpha+\beta+2)^2\,(z^2+1) - 2\,(\alpha^2-\beta^2)\, z\,, \\ C_d(z) &= 2\,(\alpha+\beta+2)\, z\,, \\ D_n(z) &= D(x)\,, \\ D_d(z) &= 1\,, \end{aligned}$$

and thus $F(z)$, the Carathéodory function associated to $S(x)$ by means of the Szegő transformation, satisfies

$$z\, A_L(z)\, F'(z) = B_L(z)\, F^2(z) + C_L F(z) + D_L(z),$$

with

$$
\begin{aligned}
A_L(z) &= -8\,(\alpha+\beta+2)\,z^3\,(1-z^2)^3\,,\\
B_L(z) &= \left(\frac{128(\alpha+1)(\beta+2)(\alpha+\beta+1)}{(\alpha+\beta+3)(\alpha+\beta+2)}\right) z^7\,(1-z^2)\,,\\
C_L(z) &= 8\,z^3\,(1-z^2)^2\,[(\alpha+\beta+2)^2\,(z^2-1)-2\,(\alpha^2-\beta^2)\,z]\\
&\quad +16\,(\alpha+\beta+2)\,z^5\,(z^2+1)\,,\\
D_L(z) &= 8\,(\alpha+\beta+2)\,(\alpha+\beta+3)\,z^3\,(1-z^2)^3\,.
\end{aligned}
$$

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