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Arbitrage conditions with no short selling

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A key assumption to prove the *Fundamental Theorem of Mathematical Finance* is the possibility of *short selling* the risky assets of the market. In this article we exhibit a simple geometric condition to handle the arbitrage opportunities when short selling is not possible. Moreover, this approach provides a pedagogical tool to visualize the consistency of the model when shorting is allowed for only some of the assets. Some examples are presented, both in analytical and graphical ways.

Keywords: Fundamental Theorem of Asset Pricing, arbitrage conditions, short sales prohibition.

Condiciones de arbitraje sin venta corta

Una hipótesis clave para demostrar el *Teorema Fundamental de la Matemática Financiera* es la de utilizar la *venta corta* de los activos del mercado. En este artículo mostramos una condición geométrica simple que controla las oportunidades de arbitraje cuando la venta corta no está permitida. Por otra parte, este enfoque nos proporciona una herramienta pedagógica para visualizar la consistencia del modelo en los casos en que la venta corta pueda realizarse para unos activos y no para otros. Se presentan algunos ejemplos desde un punto de vista analítico y gráfico.

Palabras claves: Teorema Fundamental de Valoración Financiera, condiciones de arbitraje, prohibición de ventas cortas.

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1 Introduction

1.1 Arbitrage without short selling

The *Fundamental Theorem of Finance* provides the equivalence between the no–arbitrage condition (briefly, the one that states that we cannot make money without assuming risks) and the existence of a *risk neutral measure*. A very important assumption to prove this theorem is the availability of *short selling* the assets in the market. This implies that the weight of each asset in a portfolio is given by a real number. Under this assumption, elegant proofs of this theorem are provided in the textbooks for simple market models; see, for instance, [1]. The precise conditions of no arbitrage without short selling are usually not covered in elementary courses. Why should we avoid short positions as a basic hypothesis? On the one hand, we believe that short selling could be a confusing concept for newcomers in quantitative finance. In fact, we will exhibit some examples where intuition contradicts the precise definition of arbitrage. On the other hand, for simple models, the characterization of no arbitrage opportunities can also be obtained with elementary tools, even if short selling is forbidden. As we will see, there is no reason to believe that the proof is much more involved that the one of the classical *Fundamental Theorem*. For the sake of clarity, in what follows we will call *Arbitrage Theorem* to any mathematical statement providing precise conditions equivalent to no–arbitrage opportunities in a market model.

Understanding negative positions in a portfolio requires an extra effort if one is not used to business practice. It is easy to identify a negative bank account position (with loans or borrowing money), but it is rather difficult to explain the intuitive aspects of *owning minus one unit of stock*. Not all the textbooks pay proper attention to this difficulty. For instance, in Björk ([1], page 6) a negative position is identified with the sale of the asset. This interpretation involves only part of the concept: If I sell a stock that I do own, I will have a positive amount of money in my portfolio, but no liabilities in the stock. The essential aspect of short selling is the fact that I am able to sell an asset *without actually owning it*, introducing a positive position in the bank account and a negative in the stock, due to the acquired liabilities. Selling a stock *that we do not own* is something hard to digest for a layman, and it is of course a very strange statement!

In practice, the process of short selling is supplemented by certain restrictions. As explained in Luenberger ([4], Chapter 6): "short selling is considered quite risky by many investors because of the unlimited potential loss". For this reason, short selling is purposely avoided as a policy by many institutions. Luenberger also mentions that the mere definition of a *rate of return* associated with the idealized shorting is "a bit strange", because there is no *initial commitment of resources.*

John Hull [3] devotes the whole section 5.1 of his book to the concept of short selling. Using also the slogan of "selling something that we do not own", he remarks that (short selling) is something that is possible for some —but not all— investment assets. In the same chapter, while finding the forward price of an asset, Hull makes an effort to answer an important question: *What if short sales are not possible?* (page 104, Sect. 5.4). In this case, the typical valuation procedure cannot be carried out. He then suggests another interesting way to find the correct forward price, *assuming that there is at least one investor that holds the asset as an investment.* He shows that, if the forward price were below the correct value, any investor possessing the underlying asset may follow a simple strategy: **1.** enter the forward, sell the underlying, put the money in the bank; **2.** at maturity, use the forward to buy the asset and keep the difference. Eventually, the investor would have, for every possible market scenario, the original asset plus some positive amount of money. It seems reasonable to identify this situation with an arbitrage opportunity, but this is again only part of the truth. Under the standard definition, an arbitrage opportunity is a strategy that allows an investor to start with no money at all and end up with a positive amount for some future scenario, with no risk of losses. In Hull's example, if the forward mispricing does not compensate the possible fall of the asset price, our portfolio (asset + forward contract) does not fulfill the conditions for an arbitrage opportunity. If the initial price of the asset is much higher than the price at maturity, there is no guarantee that the investor will end with a portfolio of a greater value. Of course, Hull's example captures some kind of arbitrage that is not included in the standard definition, but contributes to the confusion of the reader.

When shorting is not possible, the no–arbitrage (or consistency) condition of a market model is seldom considered in basic texts. An exception is Buchanan [2] who presents the Fundamental Theorem in the language of *wagers*, avoiding negative bet positions. In other words: gamblers cannot play the role of the bookmaker, they can only buy bets but they are not allowed to *make* them. In this book, the problem is written in terms of the duality theory of linear programming and then related to an optimization problem. Unfortunately, the theorem stated on page 86 therein (the existence of the risk neutral probability) is not true if short selling is prohibited.

Recently, Pulido (see [6] and references therein) studied the *Fundamental Theorem of Asset Pricing* under short sales prohibitions in the abstract setting of continuous–time financial models. What he actually shows is that the following sets are the same:

A. the set of measures under which the values of admissible portfolios are supermartingales;

B. the set of the measures under which the prices of the assets that cannot be *shorted* are supermartingales and the prices of assets that can be sold short are local martingales.

The classical result is obtained as a particular case.

If shorting is forbidden in a market with no arbitrage opportunities, the existence of a probability measure can still be proved, but the expected value of the discounted future prices is not necessarily equal to the prices seen today. Instead, they must satisfy an inequality condition to avoid arbitrage.

1.2 Objectives and outline

In this article, our main purpose is to exhibit a simple geometric condition of no arbitrage when short selling is not allowed. On a basic level, the proof is only a bit more involved than the one of the classical Fundamental Theorem, because we have to deal with nonnegative solutions to systems of inequalities. Nevertheless, this approach has at least two pedagogic advantages: **1.** there is no need to introduce the concept of short selling from the outset; and **2.** portfolios with non–negative positions on the risky assets are more natural to deal with, at least in the first approach to the subject.

The paper is organized as follows. In the next section, we define a simple market model without using probabilities. Risk is identified with the availability of several future market scenarios. We consider also two classical examples: **1.** the binomial model, where the lack of a risk neutral measure (with the usual properties) is evident if both short selling and arbitrage opportunities are forbidden; and **2.** the case of *wagers*, where we can easily identify the no–arbitrage conditions without recourse to the general theory of inequalities. In the following sections, we state the general result and show graphical examples, exploring the consequences when shorting is allowed only for certain assets. This provides a more general view of the classical *Fundamental Theorem of Finance*, which can be recovered once short selling is allowed in every risky asset. For completeness, we provide an elementary proof of the main theorem in the Appendix.

2 Market assumptions

Our market model $\mathcal{M}$ consists of n assets with positive prices $X_1, X_2, \cdots, X_n$. An investor may buy some non–negative units $u_1, u_2, \cdots, u_n$ of each asset to form his own portfolio or investment strategy. Decisions are taken at time $t = 0$ and the portfolio value is computed at a future time T. The units $u_j \geq 0$ are held fixed during the interval $[0, T]$. The initial asset prices are known, given by $x_j := X_j(0)$ but their future values depend on the market scenario. To formalize this statement, we assume that the market can reach m possible states at time T. The positive numbers X_{ij} are the prices of the j–th asset in the i–th market scenario. With these assumptions, the value of this strategy at time T in the i–th market scenario is given by:

$$V_i := \sum_{j=1}^{n} u_j X_{ij} , \tag{1}$$

while the initial value is given by:

$$v := \sum_{j=1}^{n} u_j x_j . \tag{2}$$

We introduce also a special asset, the bank account X_0, with the following values:

$$\begin{aligned} X_0(0) &= 1 , \\ X_0(T) &= 1 + r_0 . \end{aligned} \tag{3}$$

The bank account has the following features: **1.** its future value is deterministic, that is, independent of the market scenarios; and **2.** we can hold negative units of X_0 , corresponding to a loan. The value of the debt increases in absolute value in the same amount as a deposit. The return r_0 is the *risk free interest* corresponding to the interval $[0, T]$ and is fixed (and known) at time $t = 0$.

2.1 First example: The binomial model with one risky asset

Let us consider the classical binomial model with no shorting in one risky asset denoted by X. We have only two future market scenarios, so we simplify a bit the notation denoting the future states by + and −. The values of X at the end of the interval are given by:

$$X(T) = \begin{cases} X^+ , \\ X^- . \end{cases} \tag{4}$$

Without loss of generality we assume that $X^- < X^+$ to ensure that we have at least one risky asset. Absence of arbitrage means that it is not possible to select a portfolio

$$V = u_0 \, X_0 + u \, X \, , \tag{5}$$

for $u > 0$, such that: **1.** $V(0) = 0$; **2.** $V(T) \geq 0$ for every future scenario; and **3.** $V(T) > 0$ for *at least* one scenario. The first condition implies that:

$$u_0 = -u \, X(0) \, . \tag{6}$$

That means that we are necessarily *short* in the bank account. The second condition implies that

$$u_0 \, (1 + r_0) + u \, X(T) \geq 0 \, , \tag{7}$$

and taking (6) into account we have, for this arbitrage opportunity:

$$X_0 \leq \frac{X^\pm}{1 + r_0} \, . \tag{8}$$

If (8) holds, then the third condition is guaranteed by the assumption $X^- < X^+$. The alternative to (8) yields the no–arbitrage condition for this simple model:

$$X(0) > \frac{X^-}{1 + r_0} \, . \tag{9}$$

That is, the initial price must be greater to at least one of the discounted future prices. If $X_0 > X^+/(1+r_0)$ is also valid, then it is clearly not possible to write X_0 as a convex combination of $X^-/(1 + r_0)$ and

$X^+/(1+r_0)$. In other words, the existence of the risk neutral measure is not guaranteed when short selling of the risky asset is forbidden. Of course, we may rule out this possibility by imposing a preference condition: nobody would buy a risky asset that offers a return lower than the risk free interest for every future scenario. Even if this is a natural condition to add to this simple model, it is not enough to guarantee the existence of the risk neutral measure for markets with more than one risky asset.

2.2 Second example: Wagers

Wagers provide a nice example of a very special market where the assets behave like the *Arrow–Debreu* prices. For this case we take $r_0 = 0$, that is, there is no interest in the bank account. Consider a game with n possible outcomes. A unit bet on the outcome j for $j = 1, \cdots, n$ has the following pay–off:

$$X_j = 1\,, \tag{10}$$

for $t = 0$, and

$$X_{ij} := \begin{cases} R_j & \text{when outcome } j \text{ wins, } i.e.\ i = j, \\ 0 & \text{in other cases.} \end{cases} \tag{11}$$

for $t = T$, where X_{ij} is the price of X_j in scenario i. The amount R_j is the total reward (including the initial unit payment) received when j wins. If we are not allowed to sell wagers (that we did not buy), we may assume that we have some initial money or that we are able to ask for a loan. A betting strategy of n non–negative numbers $u_1, \cdots, u_n$ is an arbitrage opportunity if:

$$\begin{aligned} \sum_{j=1}^{n} u_j &> 0\,, \\ \sum_{j=1}^{n} u_j\, X_{ij} &\geq \sum_{j=1}^{n} u_j\,. \end{aligned} \tag{12}$$

for all $i = 1, \cdots, n$, and

$$\sum_{j=1}^{n} u_j \, X_{ij} > \sum_{j=1}^{n} u_j \, . \tag{13}$$

for at least one $i = 1, \cdots, n$.

The special form of the market prices (11) gives the condition for an arbitrage opportunity:

$$u_i \, R_i \geq \sum_{j=1}^{n} u_j \, , \tag{14}$$

with strict inequality for at least one i. Dividing by $\sum_{j=1}^{n} u_j$ (we assume that we are betting some positive amount of money) we obtain:

$$\pi_i \, R_i \geq 1 \, , \tag{15}$$

$$\pi_i := \frac{u_i}{\sum_{j=1}^{n} u_j} \, , \tag{16}$$

for some probability vector $\boldsymbol{\pi} := (\pi_1, \cdots, \pi_n)$, that is:

$$\sum_{i=1}^{n} \pi_i = 1 \, . \tag{17}$$

Taking into account (15) (with strict inequality for one i) and (17) we obtain the following consequence for the existence of an arbitrage opportunity:

$$\sum_{i=1}^{n} \frac{1}{R_i} < 1 \, . \tag{18}$$

Then, we have proved that there could be no arbitrage opportunities if the rewards satisfy the inequality:

$$\sum_{i=1}^{n} \frac{1}{R_i} \geq 1 \, . \tag{19}$$

On the other hand, it is also possible to prove that if (19) is not valid we can find an arbitrage opportunity that wins with every outcome. Let us assume that (18) holds and we have a unit amount of money to distribute among the different outcomes. Define:

$$\varepsilon := 1 - \sum_{i=1}^{n} \frac{1}{R_i}\,, \tag{20}$$

and take the betting strategy:

$$u_i = \frac{1}{R_i} + \frac{\varepsilon}{n}\,. \tag{21}$$

Then the total bet adds to 1 and:

$$u_i\, R_i = 1 + \frac{\varepsilon R_i}{n} > 1 = \sum_{j=1}^{n} u_j\,. \tag{22}$$

This is the arbitrage condition given by (14), with strict inequality. We will also obtain this simple result, together with a geometric interpretation, as a consequence of the more general setting given in the next section.

3 Arbitrage theorem without shorting: A more general case

We look for conditions that guarantee the absence of arbitrage in a market with no shorting of the risky assets. Let us assume that this opportunity exists in the context defined in Section 2. In that case, we would be able to obtain a loan of, say, $C > 0$ units of money and buy a portfolio such that its value in every future scenario will be not less than the bank deposit of the initial price, and will be strictly higher *for at least one of them*. We give the general definition that *includes the short selling case*.

Definition: An arbitrage opportunity is an investment strategy defined by the units $u_j \in \mathbb{R}$ for $j = 0, \cdots, n$, such that:

$$\begin{aligned} \sum_{j=1}^{n} u_j\, x_j &= 0\,, \\ \sum_{j=1}^{n} u_j\, X_{ij} &\geq 0\,, \end{aligned} \tag{23}$$

for $i = 1, \cdots, m$, with

$$\sum_{j=1}^{n} u_j\, X_{kj} > 0\,, \tag{24}$$

for at least one scenario k. Let us observe that $X_{i0} = u_0(1 + r_0)$ for all $i = 1, \cdots, m$. When short selling is not allowed, we have $u_j \geq 0$ for every $j \geq 1$, u_0 being always negative. In this case, its absolute value corresponds to the borrowed quantity C.

The main result is the following:

Theorem (arbitrage without short selling). *Assume that the market model $\mathcal{M}$, with m future scenarios, does not allow for short selling of the risky assets. Then, the model has no arbitrage opportunities if and only if there exists a probability vector $\boldsymbol{\pi} := (\pi_1, \cdots, \pi_n)$, such that the initial prices x_j are greater or equal to the discounted expected value of the future prices in that probability measure:*

$$x_j \geq \frac{1}{1 + r_0} \sum_{i=1}^{m} \pi_i X_{ij}\,, \tag{25}$$

for $j = 1, \cdots, n$. Moreover, if short selling were allowed for some asset X_k then the probability measure can be taken such that (25) must hold for every asset, and the equality is achieved for that index k:

$$x_k = \frac{1}{1 + r_0} \sum_{i=1}^{m} \pi_i\, X_{ik}\,. \tag{26}$$

The proof is given in the Appendix.

Before considering some graphical examples, let us say a word about the *preference condition* mentioned in the binomial model example. In that case, the fact that no risky portfolio is allowed to have a lower return than the bank account in every future scenario allowed us to guarantee the existence of the risk neutral measure. Let us analyze the case with more than one risky asset in the light of the general result. If we forbid the possibility that one risky portfolio had a lower return than the risk free interest in every possible future scenario, we will have the opposite inequality of the one defining an arbitrage opportunity. Then, as we show in the Appendix, there must exist a probability vector such as the one in (25), but satisfying the opposite inequality. This fact does not imply the existence of a risk neutral measure, because the probability that satisfies (25) (obtained through no arbitrage conditions) and the probability satisfying the opposite inequality (obtained through *preference conditions*) need not be the same. We exhibit this case graphically in the following section (see Figure 2).

3.1 Two risky assets

We write the asset prices using two–dimensional vectors (each component being the discounted price of one of the two assets). The number of vectors depends on the number of future *scenarios* of the model:

$$
\begin{aligned}
\mathbf{x} &= (x_1,\, x_2)\,, \\
\mathbf{s}_i &= \left(\frac{X_{i1}}{1+r_0},\, \frac{X_{i2}}{1+r_0}\right)\,.
\end{aligned} \tag{27}
$$

Let us observe that $\mathbf{x}$ contains the initial prices and $\mathbf{s}_i$ are the rows of the matrix representing the discounted prices in the different scenarios. The Arbitrage Theorem gives the conditions to be satisfied by the discounted future prices in order to avoid arbitrage: the vector of initial prices should be contained in a region such that, for each point inside this region, there exists a convex combination of future discounted prices with both components below the initial prices. In other words, consider, for each convex combination of the vectors $\mathbf{s}_i$, the set of points (a, b) that have their components above them:

$$
\mathcal{A} = \bigcup_{\boldsymbol{\pi}:\sum \pi_i = 1} \left\{(a,\, b) \in \mathbb{R}^2 : (a,\, b) \geq \pi_1\, \mathbf{s_1} + \pi_2\, \mathbf{s_2}\right\}\,. \tag{28}
$$

This is the *admissible set* for the vector of initial prices to avoid arbitrage. In what follows we consider the graphical interpretation of several cases. In all the figures the gray set indicates the admissible initial prices for the market model. Models with initial prices outside this set would have arbitrage opportunities. The points indicating different scenarios are the vectors $\mathbf{s}_i$ for $i = 1, 2$ and 3 in some cases.

Figure 1. Admissible set for two independent discounted future prices.
Figure 2. The admissible set depends only on scenario 1, but the *Preference condition* depends on Scenario 2. This shows that any initial price in the square between both scenarios is compatible with arbitrage and with the preference condition. It does not need to be a convex combination of both prices.

Figure 3. The case of wagers. If one reward is too high, the other must be close to 1 so that the (fixed) initial price $(1, 1)$ lies inside the admissible set.
Figure 4. No short selling.

Figure 5. Short selling in asset 2 but not in asset 1. The initial price of asset 2 must be a convex combination of the discounted prices.
Figure 6. Short selling in every asset. In this case the initial price should lie inside the convex hull of the discounted future prices, recovering the *Fundamental Theorem*.

4 Concluding remarks

The *Arbitrage (or Fundamental) Theorem* is the groundwork of the modern theory of financial valuation. Its formulation involves the definition of short selling which, as we discussed above, may not be an easy concept to handle, and may also lead to some confusing interpretations. As we have shown in a simple context, an *Arbitrage Theorem* can be easily obtained without recourse to this concept (for a general result *cf.* [6]). In our opinion, it seems more natural and pedagogically attractive to consider only non–negative positions on the risky assets, as in the Markowitz's work in portfolio theory [5]. The definition of short selling may be postponed and considered when the concepts of hedging, valuation and replication are introduced. This allows the instructor to focus in the concept of arbitrage, which has a primary importance in itself.

To conclude this note, we suggest a way to approach the definition of short positions without appealing to a market intermediate or broker. Shorting an asset is equivalent to *selling* a derivative contract with the same asset as *the underlying*. The pay–off of this contract is the value of the asset in every future scenario. If trading this kind of derivatives were allowed in our market, it would be easy to price them invoking no–arbitrage opportunities.

Of course, the price of the contract turns to be identical to the initial price of the asset, but the seller *does not need to own* the asset to do the trade.

He must be paid for it at the beginning of the interval and at maturity he must face the future contract payments, which are equal to the values of the underlying asset for the different market scenarios. With this view, the concept of shorting an asset is similar to the one of issuing a bond, where the asset being "shorted" is money. Bonds allow any investor to play the role of a bank account, guaranteeing the deposit to the owner of the money. In a similar way, short selling allows any investor to issue a contract that, instead of paying a fixed amount of money in future time, it pays the market price of the traded asset.

Appendix

With the notation introduced in Section 3, we define the following matrix:

$$\begin{aligned} A &= \{a_{ij}\}, \\ a_{ij} &= \frac{X_{ij}}{1+r_0} - x_j, \end{aligned} \tag{29}$$

with $i = 1, \cdots, m$ and $j = 1, \cdots, n$, given by the difference between the discounted future prices and the initial prices for all assets in every possible scenario. For brevity, we use the notation of linear programming:

a. $\mathbf{v} \leqq \mathbf{w}$ means $v_k \leqq w_k$ for all k;

b. $\mathbf{v} \leq \mathbf{w}$ means $\mathbf{v} \leqq \mathbf{w}$, and $v_j < w_j$ for <u>some</u> j.

We define also the vectors:

$$\mathbf{c}_j := (a_{1j}, \cdots, a_{mj})^T, \tag{30}$$

with the columns of the matrix, where ^{T} means *transpose*. We collect also the units defining the portfolio in a single column vector:

$$\mathbf{u} := (u_1, \cdots, u_n)^T. \tag{31}$$

If short selling is not allowed, we must have $\mathbf{u} \geqq \mathbf{0}$.

In this setting, an arbitrage opportunity in a market without short selling is an investment strategy defined by a vector $\mathbf{u} \geq \mathbf{0}$ with as many components as the number of risky assets, such that:

$$\mathbf{A}\,\mathbf{u} \geqq \mathbf{0}\,. \tag{32}$$

That is, at least one of its components must be greater than zero.

We use now the basic theory of inequalities developed in Strang's book [7]. Inequality (32) can be transformed into an equation by means of the *slack variables.* Consider an m–dimensional vector $\mathbf{w} \geq \mathbf{0}$ such that:

$$\mathbf{A}\,\mathbf{u} - \mathbf{w} = \mathbf{0}\,. \tag{33}$$

Now, we can pose the problem as follows: an arbitrage opportunity is given by an $(n+m)$–dimensional vector $[\mathbf{u}, \mathbf{w}]$ such that $\mathbf{u} \geq \mathbf{0}$, $\mathbf{w} \geq \mathbf{0}$ and:

$$[\mathbf{A}\ \ -\mathbf{I}] \begin{bmatrix} \mathbf{u} \\ \mathbf{w} \end{bmatrix} = \mathbf{0}\,. \tag{34}$$

The existence of an arbitrage opportunity implies that, for some $\varepsilon > 0$, $\mathbf{0}$ belongs to a closed convex set $\mathcal{C}_\varepsilon \subset \mathbb{R}^m$ generated by the columns $\mathbf{c}_j$ and by the canonical vectors $\mathbf{e}_j$ for $j = 1, \cdots, n$ and $k = 1, \cdots, m$. Precisely:

$$\mathcal{C}_\varepsilon := \left\{ \mathbf{x} \in \mathbb{R}^m : \mathbf{x} = \sum_{j=1}^{n} \lambda_j\, \mathbf{c}_j - \sum_{k=1}^{m} \mu_k\, \mathbf{e}_k, \text{ for } \boldsymbol{\lambda},\, \boldsymbol{\mu} \geqq \mathbf{0},\, \sum_{k=1}^{m} \mu_k \geqq \varepsilon \right\}. \tag{35}$$

No arbitrage opportunities mean that $\mathbf{0}$ is outside $\mathcal{C} := \cup_{\varepsilon>0}\mathcal{C}_\varepsilon$. So, for each $\varepsilon > 0$, we have that $\mathbf{0}$ does not belong to $\mathcal{C}_\varepsilon$, which is a closed and convex set. Therefore, we can apply the *theorem of the separating hyperplane* in the following terms:

> *If $\mathcal{C} \subset \mathbb{R}^m$ is a non–empty closed convex set, then $\mathbf{0} \notin \mathcal{C}$ if and only if there exist $\mathbf{y} \in \mathbb{R}^m$ with $\langle \mathbf{x}, \mathbf{y} \rangle > 0$ for all $\mathbf{x} \in \mathcal{C}$.*

Here $\langle \cdot, \cdot \rangle$ is the scalar product in m–dimensional Euclidean space. If we apply the theorem to each convex set given in (35) we obtain, for each $\varepsilon > 0$, a vector $\mathbf{y}_\varepsilon$ which, without loss of generality, can be chosen with $\|\mathbf{y}_\varepsilon\| = 1$, and such that:

$$\langle \mathbf{x}, \mathbf{y}_\varepsilon \rangle > 0\,, \tag{36}$$

for all $\mathbf{x} \in \mathcal{C}_\varepsilon$. Due to the compactness of the unit ball in m–dimensional space, we can prove that the absence of arbitrage implies the existence of an m–dimensional vector $\mathbf{y} \neq \mathbf{0}$, such that:

$$\langle \mathbf{x}, \mathbf{y} \rangle > 0\,, \tag{37}$$

for all $\mathbf{x} \in \mathcal{C}$. Now, let us assume that (37) holds and let us show that arbitrage opportunities are not possible. If such an opportunity exists, then $\mathbf{0}$ would belong to some $\mathcal{C}_\varepsilon$ for $\varepsilon > 0$. Let us observe that (37) implies

$$\begin{aligned} \langle \mathbf{c}_j, \mathbf{y} \rangle &\geq 0\,, \\ \langle \mathbf{e}_k, \mathbf{y} \rangle &\geq 0\,, \end{aligned} \tag{38}$$

for $j = 1, \cdots, n$ and $k = 1, \cdots, m$. Therefore, if arbitrage exists, we can find $\boldsymbol{\lambda} \geqq \mathbf{0}$ and $\boldsymbol{\mu} \geqq \mathbf{0}$ with $\sum_{k=1}^{m} \mu_k \geq \varepsilon$ such that:

$$\sum_{j=1}^{n} \lambda_j \, \mathbf{c}_j - \sum_{k=1}^{m} \mu_k \, \mathbf{e}_k = \mathbf{0}\,, \tag{39}$$

and then, taking the scalar product with $\mathbf{y}$ we obtain

$$\sum_{j=1}^{n} \lambda_j \, \langle \mathbf{c}_j, \mathbf{y} \rangle - \sum_{k=1}^{m} \mu_k \, \langle \mathbf{e}_k, \mathbf{y} \rangle = 0\,. \tag{40}$$

At least one of the μ_k's must be different from zero, say $\mu_{k^*} \neq 0$. If we take a new set $\{\mu'_k\}_{k=1\cdots,m}$ such that $\mu'_k = \mu_k$ for $k \neq k^*$ and $0 < \mu'_{k^*} < \mu_{k^*}$ we will have that $0 < \varepsilon' = \sum_{k=1}^{m} \mu'_k$, then we would have found an $\mathbf{x} \in \mathcal{C}_{\varepsilon'}$, such that:

$$\langle \mathbf{x}, \mathbf{y} \rangle < 0\,, \tag{41}$$

contradicting (37). □

Absence of arbitrage is therefore equivalent to the existence of a vector $\mathbf{y} \neq \mathbf{0}$ satisfying (37). The second group of inequalities in (38) implies that $\mathbf{y} \leqq \mathbf{0}$, while the first group (*cf.* also (29)–(30)) implies that, for each $j = 1, \cdots, n$

$$\sum_{i=1}^{m} \left(\frac{X_{ij}}{1+r_0} - x_j \right) y_i \geq 0 \Rightarrow \sum_{i=1}^{m} \frac{X_{ij}}{1+r_0}\, y_i \geq x_j \left(\sum_{i=1}^{m} y_i \right) . \tag{42}$$

Taking into account that $\sum_{i=1}^{m} y_i < 0$, we obtain:

$$x_j \geq \frac{1}{1+r_0} \sum_{i=1}^{m} X_{ij}\, \frac{y_i}{\sum_{i=1}^{m} y_i} , \tag{43}$$

Then, the vector with components

$$\pi_i := \frac{y_i}{\sum_{i=1}^{m} y_i} , \tag{44}$$

satisfies:

$$\begin{aligned} \boldsymbol{\pi} &\geq \mathbf{0} , \\ \sum_{i=1}^{m} \pi_i &= 1 . \end{aligned} \tag{45}$$

Therefore, we have proved the following result:

The market model $\mathcal{M}$ with no short selling does not have arbitrage opportunities if and only if there exist an m–dimensional probability vector $\boldsymbol{\pi}$ such that:

$$\mathbf{x} \geq \frac{1}{1+r_0} \sum_{i=1}^{m} \pi_i\, \mathbf{s}_i , \tag{46}$$

where $\mathbf{x}$ is an n–dimensional vector containing the initial prices of the n risky assets, and $\mathbf{s}_i$ are the discounted n–dimensional price vectors in each market scenario i, for $1 \leq i \leq m$.

Now, let us assume that short selling is *allowed* for some asset k. In that case, the geometrical condition is exactly the same but we must take bigger sets in (35). Let λ_k be any real number in (35) (not only non–negative), keeping the rest of conditions unchanged. No arbitrage still means that the null vector does not belong to the union of the bigger sets, and we can follow exactly the same proof as above. From (37) we obtain the following inequality for the index k:

$$\lambda_k \langle \mathbf{c}_j, \mathbf{y} \rangle \geq 0 \,, \tag{47}$$

with $\lambda_k \in \mathbb{R}$, which easily leads to the identity (*cf.* (43)):

$$\langle \mathbf{c}_k, \mathbf{y} \rangle = 0 \Rightarrow x_k = \frac{1}{1+r_0} \sum_{i=1}^{m} \pi_i X_{ik} \,. \tag{48}$$

In other words, the equality is valid for every k corresponding to an asset that can be shorted. If the market allows short selling for all the assets, we recover the *Fundamental Theorem of Finance*, that is: There exists a probability vector $\boldsymbol{\pi}$ such that the initial prices are the discounted expected values of the future prices:

$$\mathbf{x} = \frac{1}{1+r_0} \sum_{i=1}^{m} \pi_i \, \mathbf{s}_i \,. \tag{49}$$

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